Rational curves on general hypersurfaces of degree 7 in $\mathbb{P}^5$
Abstract

We prove that there is no smooth irreducible reduced rational curve of degree $e$, $2 \leq e \leq 11$, on general hypersurfaces of degree 7 in $\mathbb{P}^5$.

1. Introduction

Throughout this paper we work over an algebraically closed field $k$ of characteristic 0.

Let $X_d$ be a general hypersurface in $\mathbb{P}^n$ of degree $d$. H. Clemens proved in [2] that if $d \geq 2n - 1$ and $n \geq 3$ then there is no rational curve in $X_d$. In [9, 10], C. Voisin sharpened Clemens’ lower bound for $d$ by proving that if $d \geq 2n - 2$ and $n \geq 4$ then $X_d$ contains no rational curve.

On the other hand, if $d = 2n - 3$ and $n \geq 3$, it has been classically known that there always exists a line on $X_{2n-3}$ ([7, Theorem V.4.3.]). Note that for $n = 3$ and $d = 2n - 2 = 4$, every surface of degree 4 in $\mathbb{P}^3$ contains a rational curve (although a general such surface contains no smooth rational curve). Therefore Voisin’s lower bound for $d$ and $n$ are sharp in the sense that there is no rational curve on a general hypersurface $X_d \subset \mathbb{P}^n$.

The number of lines on $X_{2n-3}$ is finite ([7, Theorem V.4.3.]). In [9, 10], C. Voisin extended this classical fact in case $n \geq 5$: If $n \geq 5$ then $X_{2n-3}$ contains at most finite number of rational curves of each degree $e \geq 1$. Note that the analogue of this result for $n = 4$ would solve Clemens’ conjecture on the finiteness of rational curves of each degree $e \geq 1$ on general quintic threefolds in $\mathbb{P}^4$.

Recently G. Pacienza extended Voisin’s result in [8] by proving that there is, in fact, no rational curve of degree $e \geq 2$ on $X_{2n-3}$ if $n \geq 6$. Therefore the only rational curves on $X_{2n-3}$ are lines if $n \geq 6$.

It is natural to raise a question about the case $n = 5$ in Pacienza’s result: Is there a rational curve of degree greater than one on general hypersurfaces of degree 7 in $\mathbb{P}^5$?

In this paper we prove
Theorem 1.1. There is no smooth irreducible reduced rational curve of degree $e$, $2 \leq e \leq 11$, on general hypersurfaces of degree 7 in $\mathbb{P}^5$.

To do so, we count the dimension of the incidence scheme $\{(C, X) \mid C \subset X\}$, where $C$ is a smooth irreducible reduced rational curve of degree $e$ and $X$ is a hypersurface of degree 7 in $\mathbb{P}^5$. We use similar techniques in [6], where the authors treat rational curves of degree at most 9 on general quintic threefolds.

We introduce some notation. For a projective variety $Y$, let $\text{Hilb}^{e+1}(Y)$ be the Hilbert scheme parametrizing subschemes with the Hilbert polynomial $e+1$. We define a subscheme $R_e(Y)$ of $\text{Hilb}^{e+1}(Y)$ to be the open subscheme parametrizing smooth irreducible reduced rational curves of degree $e$.

Let $F = \mathbb{P}H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(7))$ be the parameter space of hypersurfaces of degree 7 in $\mathbb{P}^5$, i.e., $F \cong \mathbb{P}^N$, $N = \binom{s+7}{7} - 1$. We define the incidence scheme

$$I_e := \{(C, X) \in R_e(\mathbb{P}^5) \times F \mid C \subset X\}$$

and let

$$p_R: I_e \to R_e(\mathbb{P}^5) \quad \text{and} \quad p_F: I_e \to F$$

be the projections. Note that $R_e(X) \cong p_F^{-1}(X)$ for $X \in F$.

We define $R_{e,i}(\mathbb{P}^5)$ to be the locally closed subset of $R_e(\mathbb{P}^5)$ parametrizing curves $C$ with $h^1(\mathbb{P}^5, \mathcal{I}_{C,\mathbb{P}^5}(7)) = i$ where $\mathcal{I}_{C,\mathbb{P}^5}$ is the ideal sheaf of $C$ in $\mathbb{P}^5$. Set

$$I_{e,i} := p_R^{-1}(R_{e,i}(\mathbb{P}^5)).$$

Finally let $G(k, n)$ be the Grassmannian parametrizing $k$-linear space in $\mathbb{P}^n$.

2. Proof of Theorem 1.1

Throughout this section, $X$ is a general hypersurface in $\mathbb{P}^5$ of degree 7 and $C$ is a smooth irreducible reduced rational curve of degree $e \geq 1$.

Theorem 1.1 is a consequence of the following result.

Proposition 2.1. For $e \leq 11$, $I_e$ is irreducible of dimension $1 - e + N$.

Before proving Proposition 2.1, we prove Theorem 1.1 by using the above result.

Proof of Theorem 1.1. By Proposition 2.1, if $2 \leq e \leq 11$, then $\dim I_e < \dim F = N$. So $p_F$ is not surjective. Therefore

$$R_e(X) \cong p_F^{-1}(X) = \emptyset$$

for general $X$. \qed
To prove Proposition 2.1, we need the following lemma.

**Lemma 2.2.** $R_e(\mathbb{P}^n)$ is smooth, irreducible, and of dimension $(n + 1)e + n - 3$.

**Proof.** Fix $C \in R_e(\mathbb{P}^n)$. The restricted Euler sequence

$$0 \to \mathcal{O}_C \to \mathcal{O}_C(1)^{\oplus n+1} \to \mathcal{T}_{\mathbb{P}^n}|_C \to 0$$

yields $H^1(C, \mathcal{T}_{\mathbb{P}^n}|_C) = 0$. The sequence of tangent and normal sheaves

$$0 \to \mathcal{T}_C \to \mathcal{T}_{\mathbb{P}^n}|_C \to \mathcal{N}_{C, \mathbb{P}^n} \to 0$$

yields $H^1(C, \mathcal{N}_{C, \mathbb{P}^n}) = 0$. Hence, by the functorial property of the Hilbert scheme, $R_e(\mathbb{P}^n)$ is smooth at $C$ of dimension $h^0(C, \mathcal{N}_{C, \mathbb{P}^n})$, and

$$h^0(C, \mathcal{N}_{C, \mathbb{P}^n}) = \chi(\mathcal{T}_{\mathbb{P}^n}|_C) - \chi(\mathcal{T}_C)$$

$$= \chi(\mathcal{O}_C(1)^{\oplus n+1}) - \chi(\mathcal{O}_C) - \chi(\mathcal{T}_C)$$

$$= (n + 1)(e + 1) - 1 - (2 + 1)$$

$$= (n + 1)e + n - 3.$$

Note that morphisms of degree $e$ from $\mathbb{P}^1$ to $\mathbb{P}^n$ are parametrized by a Zariski open set of the projective space $\mathbb{P}((S^e k^2)^{s+1})$, where $S^e k^2$ is the symmetric product. We denote this quasi-projective variety $\text{Mor}_e(\mathbb{P}^1, \mathbb{P}^n)$. Let $\text{RatMor}_e(\mathbb{P}^1, \mathbb{P}^n)$ be the subset of $\text{Mor}_e(\mathbb{P}^1, \mathbb{P}^n)$ consisting of all morphisms whose image is a smooth irreducible reduced rational curve. Then $\text{RatMor}_e(\mathbb{P}^1, \mathbb{P}^n)$ is an open subset of $\text{Mor}_e(\mathbb{P}^1, \mathbb{P}^n)$. Since $\text{Mor}_e(\mathbb{P}^1, \mathbb{P}^n)$ is irreducible, so is $\text{RatMor}_e(\mathbb{P}^1, \mathbb{P}^n)$. There is a surjective morphism from $\text{RatMor}_e(\mathbb{P}^1, \mathbb{P}^n)$ to $R_e(\mathbb{P}^n)$. Therefore $R_e(\mathbb{P}^n)$ is irreducible.

**Proof of Proposition 2.1.** Assume $C \in R_{e, f}(\mathbb{P}^5)$. Let

$$r: H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(7)) \to H^0(C, \mathcal{O}_C(7))$$

be the restriction map. Then $p_R^{-1}(C)$ is the projectivization of the kernel of $r$. From the standard exact sequence

$$0 \to H^0(\mathbb{P}^5, \mathcal{I}_{C, \mathbb{P}^5}(7)) \to H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(7))$$

$$\to H^0(C, \mathcal{O}_C(7)) \to H^1(\mathbb{P}^5, \mathcal{I}_{C, \mathbb{P}^5}(7)) \to 0,$$

we get

$$\dim p_R^{-1}(C) = h^0(\mathbb{P}^5, \mathcal{I}_{C, \mathbb{P}^5}(7)) - 1 = (N + 1 - (7e + 1) + i) - 1.$$
Therefore
\[ \dim I_{e,i} = \dim R_{e,i}(\mathbb{P}^5) + \dim p_R^{-1}(C) \]
\[ = \dim R_{e,i} + (N + 1 - (7e + 1) + i) - 1. \tag{1} \]

Assume that \( e \leq 9 \). By the regularity theorem in [4], \( C \) is 8-regular, i.e., \( H^1(\mathbb{P}^5, \mathcal{I}_C \mathbb{P}^5(7)) = 0 \). So \( R_{e,0}(\mathbb{P}^5) = R_e(\mathbb{P}^5) \), which has dimension \( 6e + 2 \), and the fibers \( p_R^{-1}(C) \) are irreducible of same dimension \( (N + 1 - (7e + 1)) - 1 \). Therefore \( I_e \) is irreducible of dimension \( 1 - e + N \). The proof is done in case \( e \leq 9 \).

Assume that \( e = 10 \) or \( 11 \). The following Lemma 2.3 implies that \( R_{e,0}(\mathbb{P}^5) \) is open and nonempty, and hence \( R_{e,0}(\mathbb{P}^5) \) is irreducible. So \( I_{e,0} \) is irreducible of dimension \( 1 - e + N \) since fibers \( p_R^{-1}(C) \) for \( C \in R_{e,0}(\mathbb{P}^5) \) are irreducible of same dimension \( (N + 1 - (7e + 1)) - 1 \).

Also from the following Lemma 2.3 and equation (1)
\[ \dim I_{e,i} < 1 - e + N \quad \text{for} \quad i > 0. \]

It is also clear, from the way \( I_e \) is defined, that all its components have dimension at least \( 1 - e + N \) because the corresponding incidence in \( \text{RatMor}_e(\mathbb{P}^1, \mathbb{P}^n) \times \mathbb{P}^N \) is cut out by \( 7e + 1 \) equations, so both this incidence, and \( I_e \), have codimension at most \( 7e + 1 \) (locally). Therefore the closure of \( I_{e,0} \) is \( I_e \), and hence \( I_e \) is irreducible of dimension \( 1 - e + N \). Thus Proposition 2.1 is proved if given Lemma 2.3.

**Lemma 2.3.** For \( e = 10, 11 \), if \( i > 0 \) and if \( R_{e,i}(\mathbb{P}^5) \) is nonempty, then
\[ \text{codim}(R_{e,i}(\mathbb{P}^5), R_e(\mathbb{P}^5)) > i. \]

Before proving Lemma 2.3, we begin with some general observations.

**Remark 2.4.** Suppose \( C \in R_e(\mathbb{P}^5) \).
(1) If \( e \geq 3 \), then \( C \) cannot lie in a 2-plane because its arithmetic genus is 0. Moreover, if \( e \geq 4 \), then \( C \) cannot lie in a 2-dimensional quadric cone by [5, V, Ex. 2.9].
(2) If \( C \) lies in a \( k \)-linear subspace \( H \) in \( \mathbb{P}^5 \) with the ideal sheaf \( \mathcal{I}_{C,H} \), then
\[ h^1(\mathbb{P}^5, \mathcal{I}_{C,\mathbb{P}^5}(7)) = h^1(H, \mathcal{I}_{C,H}(7)). \]

We briefly prove this formula. Consider the following exact sequence of twisted ideals
\[ 0 \rightarrow \mathcal{I}_{\mathcal{H},\mathbb{P}^5}(7) \rightarrow \mathcal{I}_{C,\mathbb{P}^5}(7) \rightarrow \mathcal{I}_{C,H}(7) \rightarrow 0, \]
where \( k + 1 \leq s \leq 5 \) and \( \mathcal{H} \) is a hyperplane \( \mathbb{P}^{s-1} \) in \( \mathbb{P}^5 \). Note that \( \mathcal{I}_{\mathcal{H},\mathbb{P}^5}(7) = \mathcal{O}_{\mathbb{P}^5}(6) \); hence we have \( h^1(\mathbb{P}^{s-1}, \mathcal{I}_{C,\mathbb{P}^{s-1}}(7)) = h^1(\mathbb{P}^4, \mathcal{I}_{C,\mathbb{P}^4}(7)) \) because \( \mathcal{O}_{\mathbb{P}^s}(6) \) has no \( H^1 \) or \( H^2 \). Using this formula \( 5 - k \) times proves the desired formula.
We recall the following useful facts which will be used when proving Lemma 2.3.

Lemma 2.5 ([4]). Let $C$ be a nondegenerate $(e + 1 - r)$-irregular curve in $\mathbb{P}^r$ $(r \geq 3)$ of degree $e$. If $e > r + 1$, then $C$ is rational, smooth with a $(e + 2 - r)$-secant line, and one of the following holds;

1. $r = 3$, $C$ is contained in a smooth quadric, and $h^1(\mathbb{P}^r, \mathcal{I}_C, \mathbb{P}^r(e - r)) = e - 3$, or
2. $r = 3$, $C$ is not contained in a smooth quadric, and $h^1(\mathbb{P}^r, \mathcal{I}_C, \mathbb{P}^r(e - r)) = 1$, or
3. $r \geq 4$ and $h^1(\mathbb{P}^r, \mathcal{I}_C, \mathbb{P}^r(e - r)) = 1$.

Lemma 2.6 ([3]). Let $C$ be an irreducible smooth curve in $\mathbb{P}^3$. Suppose $C$ is nondegenerate, of degree $e$, and of genus $g$. If $e \geq 6$ and $(e, g) \not\in \{(7, 0), (7, 1), (8, 0)\}$, then $h^1(\mathbb{P}^3, \mathcal{I}_C, \mathbb{P}^3(e - 4)) > 0$ if and only if $C$ has a $(e - 2)$-secant line.

Lemma 2.7 ([6]). Let $e \geq 4$ and $r \geq 3$. Fix $s$ with $e > s \geq 3$. In $R_e(\mathbb{P}^r)$ the subset of curves with a $s$-secant line has codimension at least $(r - 1)(s - 2) - s$.

Proof of Lemma 2.3. Assume $e = 10$. If $C$ is not contained in any hyperplane, then $C$ is 7-regular and hence 8-regular, i.e., $H^1(\mathbb{P}^5, \mathcal{I}_C, \mathbb{P}^5(7)) = 0$. Therefore $R_{10}(\mathbb{P}^5) - R_{10,0}(\mathbb{P}^5)$ is contained in the closed set $G$ of curves contained in hyperplanes in $\mathbb{P}^5$. Then

$$\text{codim}(G, R_{10}(\mathbb{P}^5)) \geq \dim R_{10}(\mathbb{P}^5) - (\dim R_{10}(\mathbb{P}^4) + \dim G(4, 5)) = (6 \times 10 + 2) - (5 \times 10 + 1 + 5) = 6.$$ 

In particular,

$$\text{codim}(R_{10,1}(\mathbb{P}^5), R_{10}(\mathbb{P}^5)) > 1,$$

as asserted.

Suppose $h^1(\mathbb{P}^5, \mathcal{I}_C, \mathbb{P}^5(7)) \geq 2$. Then $C$ must lie in a hyperplane $G$, since, if not, $C$ is 7-regular. If $C$ is nondegenerate in $G$, then $C$ is 8-regular, i.e., $h^1(\mathbb{P}^5, \mathcal{I}_C, \mathbb{P}^5(7)) = 0$, which contradicts $h^1(\mathbb{P}^5, \mathcal{I}_C, \mathbb{P}^5(7)) \geq 2$. Therefore $C$ is contained in a 3-linear space $H$ in $\mathbb{P}^5$ and, by Remark 2.4 (2),

$$h^1(H, \mathcal{I}_C, H(7)) \geq 2.$$ 

Then, by Lemma 2.5 (1),

$$h^1(H, \mathcal{I}_C, H(7)) = h^1(\mathbb{P}^5, \mathcal{I}_C, \mathbb{P}^5(7)) = 7.$$ 

Therefore if $R_{10,i}(\mathbb{P}^5)$ is nonempty then $i$ is 0, 1, or 7. So it remains to prove that

$$\text{codim}(R_{10,7}(\mathbb{P}^5), R_{10}(\mathbb{P}^5)) > 7.$$
Since $h^1(\mathbb{P}^5, \mathcal{I}_{C, \mathbb{P}^5}(7)) = 7$, from Lemma 2.5 (1), $C$ lies in a smooth quadric surface $Q$ contained in $H$. Since $C$ is smooth, rational, and of degree 10, $C$ is contained in the linear system $|9L + M|$ on $Q$ where $L$ and $M$ are two generators of Pic($Q$). Then $Q$ varies in $\mathbb{P}H^0(H, \mathcal{O}_H(2))$ and $H$ varies in $\mathbb{G}(3, 5)$. Therefore,

$$\text{codim}(\mathcal{R}_{10,7}(\mathbb{P}^5), \mathcal{R}_{10}(\mathbb{P}^5)) \geq \dim \mathcal{R}_{10,7}(\mathbb{P}^5) - (\dim |9L + M| + \dim \mathbb{P}H^0(H, \mathcal{O}_H(2)) + \dim \mathbb{G}(3, 5))$$

$$= 62 - (19 + 9 + 8) = 26 > 7.$$  

Thus Lemma 2.3 holds for $e = 10$.

Assume that $e = 11$. If $C$ is nondegenerate in $\mathbb{P}^5$, then $C$ is 8-regular, i.e., $h^1(\mathbb{P}^5, \mathcal{I}_{C, \mathbb{P}^5}(7)) = 0$. Hence $R_{11}(\mathbb{P}^5) - R_{11,0}(\mathbb{P}^5)$ is contained in the closed set $\mathcal{G}$ of curves in hyperplanes in $\mathbb{P}^5$.

$$\text{codim}(\mathcal{G}, R_{11}(\mathbb{P}^5)) \geq \dim R_{11}(\mathbb{P}^5) - (\dim R_{11}(\mathbb{P}^4) + \dim \mathbb{G}(4, 5))$$

$$= (6 \times 11 + 2) - (5 \times 11 + 1 + 5) = 7.$$  

In particular,

$$\text{codim}(R_{11,i}(\mathbb{P}^5), R_{11}(\mathbb{P}^5)) \geq 7 > i \quad \text{for} \quad i = 1, \ldots, 6$$

as asserted.

Assume $h^1(\mathbb{P}^5, \mathcal{I}_{C, \mathbb{P}^5}(7)) \geq 7$. $C$ lies in a hyperplane $G$, since, if not, $C$ is 8-regular. By Remark 2.4 (2),

$$h^1(G, \mathcal{I}_{C,G}(7)) = h^1(\mathbb{P}^5, \mathcal{I}_{C,\mathbb{P}^5}(7)) \geq 7.$$  

Suppose that $C$ is nondegenerate in $G$. Then $C$ is 8-irregular in $G$ since $h^1(G, \mathcal{I}_{C,G}(7)) \geq 7$.

However, from Lemma 2.5 (3), we know that

$$h^1(G, \mathcal{I}_{C,G}(7)) = 1,$$

which contradicts our assumption $h^1(G, \mathcal{I}_{C,G}(7)) \geq 7$. Thus $C$ is contained in a 3-linear space $H$ in $\mathbb{P}^5$. There are three possible cases;

(1) $C$ lies in $H$, and $C$ has no 9-secant line,

(2) $C$ lies in some smooth quadric surface $Q$ with ideal $\mathcal{I}_{C,Q}$.

(3) $C$ lies in $H$, but $C$ lies in no smooth quadric surface, and $C$ has a 9-secant line.

In case (1), by Lemma 2.6 and Remark 2.4,

$$h^1(H, \mathcal{I}_{C,H}(7)) = h^1(\mathbb{P}^5, \mathcal{I}_{C,\mathbb{P}^5}(7)) = 0,$$

which contradicts our assumption $h^1(G, \mathcal{I}_{C,G}(7)) \geq 7$. 

In case (2), since $C$ is rational, smooth, and of degree 11, $C$ is contained in the linear system $|10L + M|$ on $Q$ where $L$ and $M$ are two generators of $\text{Pic}(Q)$. Thus

$$\mathcal{I}_{C,Q}(7) = \mathcal{O}_Q(-3, 6).$$

Then the Künneth formula yields

$$h^1(Q, \mathcal{I}_{C,Q}(7)) = 0 \times 0 + 2 \times 7 = 14.$$ 

Note that

$$h^1(Q, \mathcal{I}_{C,Q}(7)) = h^1(H, \mathcal{I}_{C,H}(7)) = h^1(\mathbb{P}^5, \mathcal{I}_{C,F}(7)) = 14.$$ 

Indeed, the second equality is Remark 2.4 (2), and the first can be proved similarly. Therefore

$$h^1(\mathbb{P}^5, \mathcal{I}_{C,F}(7)) = 14$$

and it remains to prove that

$$\text{codim}(R_{11,14}(\mathbb{P}^5), R_{11}(\mathbb{P}^5)) > 14.$$ 

Let $\mathcal{G}$ be the subset of $R_{11}(\mathbb{P}^5)$ consisting of all curves $C$ included in the case (2). These $C$ are contained in the linear system $|10L + M|$ on $Q$ and $Q$ varies in $\mathbb{P}H^0(H, \mathcal{O}_H(2))$ and $H$ varies in $\mathcal{G}(3, 5)$. Therefore

$$\text{codim}(R_{11,14}(\mathbb{P}^5), R_{11}(\mathbb{P}^5)) \geq \dim R_{11}(\mathbb{P}^5)$$

$$- (\dim |10L + M| + \dim \mathbb{P}H^0(H, \mathcal{O}_H(2)) + \dim \mathcal{G}(3, 5))$$

$$= 68 - (21 + 9 + 8) = 30 > 14.$$ 

In particular,

$$\text{codim}(R_{11,14}(\mathbb{P}^5), R_{14}(\mathbb{P}^5)) > 14$$

as asserted.

In case (3), let $\mathcal{S}$ be the subset of $R_{11}(H)$ consisting of all $C$ satisfying the conditions in case (3). Then, by Lemma 2.7, $\mathcal{S}$ is of codimension at least 5 in $R_{11}(H)$.

Let $\mathcal{G}$ be the subset of $R_{11}(\mathbb{P}^5)$ consisting of all $C$ satisfying the conditions in case (3) for a 3-linear space $H$. Note that $H$ varies in $\mathcal{G}(3, 5)$. Therefore

$$\text{codim}(\mathcal{G}, R_{11}(\mathbb{P}^5)) \geq \dim R_{11}(\mathbb{P}^5) - (\dim R_{11}(H) + \dim \mathcal{G}(3, 5))$$

$$+ \text{codim}(\mathcal{S}, R_{11}(H))$$

$$= 68 - (44 + 8) + 5 = 21.$$ 

Therefore it suffices to prove that

\[ h^1(\mathbb{P}^5, \mathcal{I}_{C, \mathbb{P}^5}(7)) < 21, \]

or equivalently, by Remark 2.4 (2), that

\[ h^1(H, \mathcal{I}_{C, H}(7)) = h^1(\mathbb{P}^5, \mathcal{I}_{C, \mathbb{P}^5}(7)) < 21. \]

Choose a 2-plane \( U \) in \( H \) that meets \( C \) in 11 distinct points, no three of which are collinear. Such an \( U \) exists by [1, Lemma, p.109].

Let \( k \geq 5 \). These 11 points impose independent conditions on the system of curves of degree \( k \) in \( U \) by [1, Lemma, p.115]. Therefore, in the long exact sequence

\[
H^0(U, \mathcal{O}_U(k)) \to H^0(C \cap U, \mathcal{O}_{C \cap U}(k)) \\
\to H^1(U, \mathcal{I}_{C \cap U, U}(k)) \to H^1(U, \mathcal{O}_U(k)),
\]

the first map is surjective. However, the last term vanishes. Therefore

\[ H^1(U, \mathcal{I}_{C \cap U, U}(k)) = 0. \]

Consequently, the exact sequence of sheaves

\[ 0 \to \mathcal{I}_{C, H}(k - 1) \to \mathcal{I}_{C, H}(k) \to \mathcal{I}_{C \cap U, U}(k) \to 0 \]

yields

\[ h^1(H, \mathcal{I}_{C, H}(4)) \geq h^1(H, \mathcal{I}_{C, H}(5)) \geq h^1(H, \mathcal{I}_{C, H}(6)) \geq \cdots. \]

Consider the standard exact sequence of sheaves

\[ 0 \to \mathcal{I}_{C, H}(k) \to \mathcal{O}_H(k) \to \mathcal{O}_C(k) \to 0. \]

Since \( H^1(H, \mathcal{O}_H(k)) = 0 \) for \( k \geq 0 \), taking cohomology yields

\[ h^0(H, \mathcal{I}_{C, H}(k)) = \binom{k + 3}{3} - (11k + 1) + h^1(H, \mathcal{I}_{C, H}(k)). \]

Proceeding by contradiction, assume \( h^1(H, \mathcal{I}_{C, H}(7)) \geq 21 \). We will prove that

\[ h^0(H, \mathcal{I}_{C, H}(8)) \geq 78. \]

Then, by the equation (3),

\[ h^1(H, \mathcal{I}_{C, H}(8)) \geq 2. \]
However, by Lemma 2.5, \( h^1(H, \mathcal{I}_{C,H}(8)) \) must be one. Therefore there is a contradiction.

By the equation (2) and equation (3), we get

\[
h^0(H, \mathcal{I}_{C,H}(4)) \geq 11, \quad h^0(H, \mathcal{I}_{C,H}(7)) \geq 63.
\]

Note that \( h^0(H, \mathcal{I}_{C,H}(2)) = 0 \) since \( C \) cannot lie neither on a smooth quadric surface by the assumption of case (3) nor on a quadric cone by Remark 2.4. Therefore every element in \( H^0(H, \mathcal{I}_{C,H}(3)) \) is irreducible.

Suppose \( h^0(H, \mathcal{I}_{C,H}(3)) \geq 2 \). Take two independent irreducible cubics \( F_3 \) and \( F'_3 \) in \( H^0(H, \mathcal{I}_{C,H}(3)) \). Then \( \deg(F_3 \cap F'_3) = 9 \), but \( C \subset F_3 \cap F'_3 \) and \( \deg(C) = 11 \), which is impossible. Therefore \( h^0(H, \mathcal{I}_{C,H}(3)) \leq 1 \).

Suppose there exists a nonzero cubic \( F_3 \) in \( H^0(H, \mathcal{I}_{C,H}(3)) \). Let

\[
\alpha : H^0(H, \mathcal{O}_H(1)) \to H^0(H, \mathcal{I}_{C,H}(4))
\]

be the linear map defined by multiplying with \( F_3 \). The image of \( \alpha \) is a subspace of \( H^0(H, \mathcal{I}_{C,H}(4)) \) of dimension 4. Note that

\[
h^0(H, \mathcal{I}_{C,H}(1)) = h^0(H, \mathcal{I}_{C,H}(2)) = 0.
\]

Therefore there exist irreducible quartics in \( H^0(H, \mathcal{I}_{C,H}(4)) \).

Suppose \( h^0(H, \mathcal{I}_{C,H}(3)) = 0 \). Since

\[
h^0(H, \mathcal{I}_{C,H}(1)) = h^0(H, \mathcal{I}_{C,H}(2)) = h^0(H, \mathcal{I}_{C,H}(3)) = 0,
\]

every element in \( H^0(H, \mathcal{I}_{C,H}(4)) \) is irreducible.

Therefore, since \( h^0(H, \mathcal{I}_{C,H}(3)) \leq 1 \), there always exists an irreducible quartic \( F_4 \) in \( H^0(H, \mathcal{I}_{C,H}(4)) \).

Let

\[
\alpha : H^0(H, \mathcal{O}_H(3)) \to H^0(H, \mathcal{I}_{C,H}(7))
\]

be the linear map defined by multiplying with \( F_4 \). The image of \( \alpha \) is a subspace of \( H^0(H, \mathcal{I}_{C,H}(7)) \) of dimension 20. Let \( W \) be a subspace of \( H^0(H, \mathcal{I}_{C,H}(7)) \) satisfying

\[
H^0(H, \mathcal{I}_{C,H}(7)) = \text{image}(\alpha) \oplus W.
\]

Note that \( \dim W = h^0(H, \mathcal{I}_{C,H}(7)) - \dim \text{image}(\alpha) \geq 63 - 20 = 43 \).

Take a nonzero \( L \in H^0(H, \mathcal{O}_H(1)) \). Define

\[
X := \{ F_4 F : F \in H^0(H, \mathcal{O}_H(4)) \},
\]

\[
Y := \{ FL : F \in W \}.
\]
$X$ and $Y$ are subspaces of $H^0(H,\mathcal{I}_{C,H}(8))$ of dimension 35 and 43, respectively. Moreover, by the irreducibility of $F_4$ and by the choice of $W$, we have $X \cap Y = 0$. Therefore

$$h^0(H,\mathcal{I}_{C,H}(8)) \geq \dim X + \dim Y = 78,$$

as asserted.

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**References**


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