Shibata, K.
Osaka J. Math.
11 (1974), 171-180

ORIENTED AND WEAKLY COMPLEX BORDISM OF FREE METACYCLIC ACTIONS

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(Received October 2, 1973)

Abstract. Oriented and weakly complex bordism modules of free metacyclic actions are determined up to the Kasparov formula which describes the bordism classes of generalized lens spaces in terms of a linear combination of those of the standard lens spaces. In the oriented case for $p=2$ (the dihedral case), the module structure is particularly simple because the corresponding Kasparov formula reduces to the multiplication by ± 1. We also compute the abelian group structure of these bordisms in case $p \geqslant 2$ a prime and $q \geqslant 3$ an odd prime. Of independent interest is the canonical projections defined on these bordism modules which select a direct summand with one generator in each $2pj-1$dimension ($j=1, 2,\cdots$).

1. Introduction.

Let $Z_{q,p}$ be the metacyclic group

$$Z_{q,p} = \{x, y \mid x^q = y^p = 1, yxy^{-1} = x^r\}$$

where $p \geqslant 2$ is a prime integer, $q \geqslant 3$ is an odd integer and r is a primitive p-th root of 1 mod q such that $(r-1, q)=1$. (So $r \equiv -1$ mod q when $p=2$.) By virtue of Fermat's theorem, these conditions imply $(p, q)=1$.

Obviously there is an exact sequence

$$1 \longrightarrow Z_q \xrightarrow{\ i\ } Z_{q,p} \underset{s}{\overset{\pi}{\rightleftarrows}} Z_p \longrightarrow 1$$

with s a corss-section defined by $s(\bar{y})=y$.

Kamata—Minami [3] determined the additive structure of the weakly complex reduced bordism group of the free dihedral group actions $\tilde{\Omega}_m^U(Z_{q,2})$ in case q is an odd prime. Here we generalize their results to the cases for the oriented and weakly complex bordism modules $\tilde{\Omega}_*^{SO}(Z_{q,p})$ and $\tilde{\Omega}_*^U(Z_{q,p})$ of the free metacyclic actions.

For the basic notations and prerequisites, we refer the reader to the introductory part and §1 of Kamata—Minami [3].

Thanks are due to Professor Minoru Nakaoka for suggesting me the subject.

2. The module structure of $\tilde{\Omega}_*^L(Z_{q,p})$; $L=SO$, U

First we recall the basic fact about $\tilde{\Omega}(Z_{q,p})$ from Lazarov [5].

Lemma 2.1. (Lazarov [5]).

(1) $i_*\colon \tilde{\Omega}_*^L(Z_q)\to\tilde{\Omega}_*^L(Z_{q,p})$ *is surjective onto the q-torsion.*

(2) $s_*\colon \tilde{\Omega}_*^L(Z_p)\to\tilde{\Omega}(Z_{q,p})$ *is injective onto the p-torsion (which is a direct summand as an* $\tilde{\Omega}_*^L$*-module because* $\pi_*\circ s=\mathrm{id}$).

The proof is done by calculating the integral homology $H_*(Z_{q,p};Z)$. Thanks to our assumption on p, q and r stated in the introduction, Lazarov's proof still works here in a slightly generalized situation.

Therefore it suffices to know the kernel of i_* for the determination of the module structure $\tilde{\Omega}_*^L(Z_{q,p})$ because we already know the structure of $\tilde{\Omega}_*^L(Z_m)$ (Conner-Floyd [1], Kamata [2], Shibata [6]).

Let

$$T_{(q,j)}\colon Z_q\times S^{2n-1}\to S^{2n-1}$$

denote the Z_q-action on the $(2n-1)$-dimensional sphere defined by $T_{(q,j)}(x^h,z)=\rho^{hj}z$, where $\rho=\exp(2\pi\sqrt{-1}/q)$. This is a free action if j is a unit in Z_q.

Let us consider the images of the $[T_{(q,rj)}, S^{2n-1}]$ by the canonical homomorphism

$$i_*\colon \tilde{\Omega}_*^L(Z_q)\to\tilde{\Omega}_*^L(Z_{q,p}).$$

Lemma 2.2.

$$i_*[T_{(q,rj)}, S^{2n-1}] = [\hat{T}_{(q,rj)}, Z_p\times S^{2n-1}],$$

where

$$\hat{T}_{(q,rj)}(x, (\bar{y}^h, z)) = (\bar{y}^h, \rho^{rj-h}z)$$

and

$$\hat{T}_{(q,rj)}(y, (\bar{y}^h, z)) = (\bar{y}^{h+1}, z).$$

Proof. The map i_* is the extension (see Conner-Floyd [1] page 53), and so

$$i_*[T_{(q,rj)}, S^{2n-1}] = [\tilde{T}_{(q,rj)}, Z_{q,p}\times_{Z_q} S^{2n-1}]$$

where $\tilde{T}_{(q,rj)}$ is the natural operation of $Z_{q,p}$ on $Z_{q,p}\times_{Z_q} S^{2n-1}$ from the left. There is an L-structure preserving ($L=$SO or U), $Z_{q,p}$-equivariant diffeomorphism

$$\phi_j\colon (\tilde{T}_{(q,rj)}, Z_{q,p}\times_{Z_q} S^{2n-1})\to(\hat{T}_{(q,rj)}, Z_p\times S^{2n-1})$$

defined by $\phi_j([x^a y^b, z])=(\bar{y}^b, \rho^{arj-b}z)$.

Hence the lemma follows.

Corollary 2.3.

$$i_*[T_{\langle q,1\rangle}, S^{2n-1}] = i_*[T_{\langle q,r^j\rangle}, S^{2n-1}]$$

for $j=0, 1, 2, \cdots, p-1$.

Proof. From the preceding lemma, it suffices to find an L-structure preserving, $Z_{q,p}$-equivariant diffeomorphism

$$\psi_j: (\hat{T}_{\langle q,1\rangle}, Z_p \times S^{2n-1}) \to (\hat{T}_{\langle q,r^j\rangle}, Z_p \times S^{2n-1}).$$

In fact the formula $\psi_j(\bar{y}^b, z) = (\bar{y}^{b+j}, z)$ defines a desired one.

Let $t: \tilde{\Omega}^L_*(Z_{q,p}) \to \tilde{\Omega}^L_*(Z_q)$ be the transfer homomorphism, *i.e.* the homomorphism induced by the restriction of the action on the subgroup (Conner-Floyd [1] page 52).

Lemma 2.4.

$$t \circ i_*[T_{\langle q,1\rangle}, S^{2n-1}] = \sum_{j=0}^{p-1} [T_{\langle q,r^j\rangle}, S^{2n-1}].$$

Proof. The lemma is obvious from Lemma 2.2 and the definition of t.

DEFINITION 2.5. We define the elements β_{2n-1} $(n=1, 2, \cdots)$ of $\tilde{\Omega}^L_{2n-1}(Z_q)$ as follows.

(1) In case $(n, p) = 1$, $\beta_{2n-1} = \sum_{0 \leq j \leq p-1} ([T_{\langle q,1\rangle}, S^{2n-1}] - [T_{\langle q,r^j\rangle}, S^{2n-1}])$.

(2) $\beta_{2pm-1} = t \circ i_*[T_{\langle q,1\rangle}, S^{2pm-1}] = \sum_{0 \leq j \leq p-1} [T_{\langle q,r^j\rangle}, S^{2pm-1}]$.

Lemma 2.6. (1) $i_*\beta_{2n-1} = 0$ *in case* $(n, p) = 1$.
(2) $t \circ i_*\beta_{2pm-1} = p\beta_{2pm-1}$.

Proof. (1) is obvious from definition 2.5 and corollary 2.3. Also definition 2.5, corollary 2.3 and lemma 2.4 imply (2).

At this stage, we need the formula of Kasparov [4], which describes the unitary bordism classes of the generalized lens spaces as a linear combination of those of the standard lens spaces. We restate his formula only in the special case which we concern.

Theorem 2.7 (Kasparov [4]). *In* $\tilde{\Omega}^L_*(Z_q)$, *the class* $[T_{\langle q,r^j\rangle}, S^{2n-1}]$ *is the coefficient of* X^n *in*

$$\left(\sum_{1 \leq k} [T_{\langle q,1\rangle}, S^{2k-1}] X^k\right) \left(X/g^{-1}(r^j g(X))\right)^n$$

(*or its image in* $\tilde{\Omega}^{SO}_*(Z_q)$ *by the natural homomorphism* $\tilde{\Omega}^U_*(Z_q) \to \tilde{\Omega}^{SO}_*(Z_q)$), *where* $g(X) = \sum_{1 \leq h} ([CP_{h-1}]/h) X^h$ *is the logarithm of the cobordism formal group law, i.e.*

"the Miscenko series".

Corollary 2.8.

$$\beta_{2n-1}-p[T_{(q,1)}, S^{2n-1}]\in\Omega_*^L\{[T_{(q,1)}, S^{2k-1}];\ 1\leqslant k\leqslant n-1\},$$

where $\Omega_*^L\{\cdots\}$ *denotes the* Ω_*^L *-submodule of* $\tilde{\Omega}_*^L(Z_q)$ *generated by the elements* $\{\cdots\}$.

Proof. By the Kasparov formula, we see that

$$[T_{(p,r^j)}, S^{2n-1}]-\left(\frac{1}{r^j}\right)^n[T_{(q,1)}, S^{2n-1}]\in\Omega_*^L\{[T_{(q,1)}, S^{2j-1}];\ 1\leqslant i\leqslant n-1\}\ .$$

Notice that here we can treat everything as reduced mod q since $q[T_{(q,1)}, S^{2j-1}]$ $\in\Omega_*^L\{[T_{(q,1)}, S^{2h-1}];\ 1\leqslant h<j\}$ (Shibata [6]). In case $n=kp$, $\sum_{0\leqslant j\leqslant p-1}\left(\frac{1}{r^j}\right)^n=$ $\sum_{0\leqslant j\leqslant p-1}(1/(r^p)^{kj})\equiv p$. Thus the lemma is true. Otherwise put $n=kp+t(1\leqslant t\leqslant p-1)$. Then $\sum_{0\leqslant j\leqslant p-1}(1/r^j)^n=\sum_{0\leqslant j\leqslant p-1}(r^{-t})^j\equiv 0$ mod q since r^{-t} is a root of the equation $x^p-1=(x-1)\ (x^{p-1}+\cdots+x+1)\equiv 0$ and $r^{-t}-1$ is a unit in Z_q by virtue of the condition $(r-1, q)=1$. Therefore the lemma holds also in case $(n, p)=1$.

Corollary 2.9. $\Omega_*^L\{[T_{(q,1)}, S^{2j-1}];\ 1\leqslant j\leqslant k\}=\Omega_*^L\{\beta_{2j-1};\ 1\leqslant j\leqslant k\}$. *In particular,* $\tilde{\Omega}_*^L(Z_q)=\Omega_*^L\{\beta_{2j-1};\ 1\leqslant j\}$.

Proof. Since we are assuming $(p, q)=1$, this corollary is easily proved by induction on k by virtue of 2.8.

Now we can state the main theorem of this section as follows.

Theorem 2.10. *There are the following exact sequences of* Ω_*^L*-module homomorphisms*

$$(1)\quad 0\to\Omega_*^L\{\beta_{2m-1};\ 1\leqslant m,\ (m, p)=1\}\xrightarrow{\iota\oplus 0}$$
$$\to\tilde{\Omega}_*^L(Z_q)\oplus\tilde{\Omega}(Z_p)\xrightarrow{i_*+s_*}\tilde{\Omega}_*^L(Z_{q,p})\to 0,\quad \textit{and}$$
$$(2)\quad 0\to\tilde{\Omega}_*^L\{\beta_{2pk-1};\ 1\leqslant k\}\oplus\tilde{\Omega}_*^L(Z_p)\xrightarrow{i_*+s_*}\tilde{\Omega}_*^L(Z_{q,p})\to 0,$$

where ι *is the canonical inclusion as a submodule,*

$$\beta_{2m-1}=\begin{cases} p[T_{(q,1)}, S^{2m-1}]-\sum_{0\leqslant j\leqslant p-1}[T_{(q,r^j)}, S^{2m-1}] \\ \qquad \textit{if } (m, p)=1, \textit{ and} \\ \sum_{0\leqslant j\leqslant p-1}[T_{(q,r^j)}, S^{2m-1}] \\ \qquad \textit{if } p\mid m, \end{cases}$$

and the $[T_{(q,r^j)}, S^{2m-1}]$ *can be written down as a linear combination over* Ω_*^L *of the* $[T_{(q,1)}, S^{2n-1}]$ $(1\leqslant n\leqslant m)$ *by the Kasparov formula (Theorem 2.7).*

Proof. The proof is now obvious from 2.1, 2.6 and 2.9. We only indicate the proof of the fact that Ker $i_* \subset \Omega_*^L\{\beta_{2m-1};\ 1 \leqslant m,\ (m, p)=1\}$. Suppose x belongs to Ker i_* and is homogeneous of dimension $2t-1$. By 2.9,

$$x = \sum_{m=1}^{t} \alpha_{2(t-m)}\beta_{2m-1}$$

for some $\alpha_{2(t-m)} \in \Omega^L_{2(t-m)}$. Then $0 = t \circ i_*(x) = \sum_{m=1}^{[t/p]} p\alpha_{2t-2pm}\beta_{2pm-1}$. So $\sum_{m=1}^{[t/p]} \alpha_{2t-2pm}$ $\beta_{2pm-1}=0$ and this implies $x = x - \sum_{m=1}^{[t/p]} \alpha_{2t-2pm}\beta_{2pm-1} = \sum_{\substack{1 \leqslant m \leqslant t \\ (m,p)=1}} \alpha_{2(t-m)}\beta_{2m-1}$ as desired.

3. The oriented case for p=2.

There is a special simplicity for the oriented bordism of the free dihedral actions.

Lemma 3.1. *Let s be a unit in Z_q. It holds in $\tilde{\Omega}_*^{SO}(Z_q)$ that*

$$[T_{(q,-s)}, S^{2n-1}] = (-1)^n [T_{(q,s)}, S^{2n-1}].$$

Proof. Consider the Z_q-equivariant diffeomorphism

$$c\colon (T_{(q,-s)}, S^{2n-1}) \to (T_{(q,s)}, S^{2n-1})$$

defined by $c(z_0, \cdots, z_{n-1}) = (\bar{z}_0, \cdots, \bar{z}_{n-1})$, *i.e.* the complex conjugation. Then c preserves the orientation when n is even and reverses when n is odd. *Q.E.D.*

It follows that, in $\tilde{\Omega}_*^{SO}(Z_q)$,

$$\begin{aligned}\beta_{4i+1} &= [T_{(q,1)}, S^{4i+1}] - [T_{(q,-1)}, S^{4i+1}] \\ &= 2[T_{(q,1)}, S^{4i+1}], \text{ and} \\ \beta_{4i-1} &= [T_{(q,1)}, S^{4i-1}] + [T_{(q,-1)}, S^{4i-1}] \\ &= 2[T_{(q,1)}, S^{4i-1}].\end{aligned}$$

Therefore theorem 2.10 of the last section reduces to the following.

Theorem 3.2. *There are the following exact sequences of Ω_*^{SO}-module homomorphisms*

$$(1)\quad 0 \to \Omega_*^{SO}\{[T_{(q,1)}, S^{4m+1}];\ 0 \leqslant m\} \xrightarrow{\iota \oplus 0}$$
$$\to \tilde{\Omega}_*^{SO}(Z_q) \oplus \tilde{\Omega}_*^{SO}(Z_2) \xrightarrow{i_* + s_*} \tilde{\Omega}_*^{SO}(Z_{q,2}) \to 0,$$

and

$$(2)\quad 0 \to \Omega_*^{SO}\{[T_{(q,1)}, S^{4m-1}];1 \leqslant m\} \oplus \tilde{\Omega}_*^{SO}(Z_2) \to$$
$$\xrightarrow{i_* + s_*} \tilde{\Omega}_*^{SO}(Z_{q,2}) \to 0.$$

REMARK 3.3. The module structures of $\tilde{\Omega}_*^{SO}(Z_q)$ (q odd) and $\tilde{\Omega}_*^{SO}(Z_2)$ are determined in Shibata [6]. According to 6.1 and 6.3 of Shibata [6], together with the fact that the natural homomorphism $\Omega_*^U \to \Omega_*^{SO}/\mathrm{Tor}$ kills the elements of dimension $4j+2$($j=0, 1, 2, \cdots$), we see that the restriction of the Smith homomorphism

$$\Delta\colon \Omega_*^{SO}\{[T_{(q,1)}, S^{4m-1}];\ 1\leqslant m\} \to \Omega_*^{SO}\{[T_{(q,1)}, S^{4m-3}];\ 1\leqslant m\}$$

is an isomorphism, and thus $\tilde{\Omega}_*^{SO}(Z_q)$ is a direct sum of two isomorphic copies (with dimension shift) of Ω_*^{SO}-submodules.

REMARK 3.4. We can not expect such a simple phenomenon in the unitary bordism of the dihedral actions. For example,

$$\begin{aligned}[T_{(q,-1)}, S^3] &= [T_{(q,1)}, S^3]-2[CP_1]\,[T_{(q,1)}, S^1],\\ [T_{(q,-1)}, S^5] &= -[T_{(q,1)}, S^5]+3[CP_1]\,[T_{(q,1)}, S^3]\\ &\quad -3[CP_1]^2\,[T_{(q,1)}, S^1],\end{aligned}$$

and so

$$t\circ i_*[T_{(q,1)}, S^3] = 2[T_{(q,1)}, S^3]-2[CP_1]\,[T_{(q,1)}, S^1],\ \neq 2[T_{(q,1)}, S^3],$$

and in case $q>3$,

$$t\circ i_*[T_{(q,1)}, S^5] = 3[CP_1]\,[T_{(q,1)}, S^3]-3[CP_1]^2[T_{(q,1)}, S^1]\neq 0$$

in $\tilde{\Omega}_*^U(Z_q)$.
Also when $q=3$,

$$t\circ i_*[T_{(3,1)}, S^9] = 5[CP_1]\,[T_{(3,1)}, S^7]-[CP_1]^2[T_{(3,1)}, S^5]+\cdots\neq 0.$$

REMARK 3.5. Even in the oriented case, if we take the case for $p\geqslant 3$, the Kasparov formula becomes complicated. The lowest dimensional example is the case for $p=3$, $q=7$, $r=2$. The computation shows that

$$\begin{aligned}t\circ i_*[T_{(7,1)}, S^5] &= 3[T_{(7,1)}, S^5]+4[CP_2]\,[T_{(7,1)},S^1]\neq 3[T_{(7,1)}, S^5],\\ t\circ i_*[T_{(7,1)}, S^9] &= 5[CP_2]\,[T_{(7,1)}, S^5]+2[CP_2]^2[T_{(7,1)}, S^1].\end{aligned}$$

Therefore $\quad t\circ i_*[T_{(7,1)}, S^9]\neq 0$ in $\tilde{\Omega}_*^{SO}(Z_7)$.

4. Computation of abelian group sturcture of $\tilde{\Omega}_*^L(Z_{q,p})$ for q an odd prime

In this section we present a generalization of the main theorem of Kamata-Minami [3] to the case for $\tilde{\Omega}_*^L(Z_{q,p})$ with $p\geqslant 2$ a prime and $q\geqslant 3$ an odd prime. So in this section, we assume q an odd prime.

As in Kamata [2], let $\Gamma_*(q)$ be the polynomial subring of $\Omega_*^U=Z[x_1, x_2, \cdots]$ which is generated by x_i $(i\neq q-1)$ (unitary case) or its image in Ω_*^{SO} by the canonical homomorphism $\Omega_*^U\rightarrow\Omega_*^{SO}$ (oriented case).

Analogously to Kamata-Minami [3], proposition 3.1, we obtain;

Proposition 4.1. *The following two conditions for the elements* $[M^{2(l-k)}]\in\Gamma_{2(l-k)}(q)$ *are equivalent;*

(1) $\sum_{k=1}^{n}[M^{2(l-k)}]\beta_{2k-1}=0$ *in* $\tilde{\Omega}_*^L(Z_q)$, *and*

(2) $[M^{2(l-k)}]\in q^{[k-1/q-1]+1}\Gamma_{2(l-k)}(q)$,

where the β_{2k-1} *are the module generators of* $\tilde{\Omega}_*^L(Z_q)$ *defined in section* 2.

Now $\tilde{\Omega}_*^L(Z_q)$ can be considered as a $\Gamma_*(q)$-module and we denote by $\Gamma_*(q)\{\cdots\}$ the $\Gamma_*(q)$-submodule of $\tilde{\Omega}_*^L(Z_q)$ generated by the elements $\{\cdots\}$.

Lemma 4.2. *Ther is a* $\Gamma_*(q)$*-isomorphism*

$$\nu\colon \Gamma_*(q)\{[T_{(q,1)}, S^{2n-1}];\ 1\leqslant n\}\rightarrow\Gamma_*(q)\{\beta_{2n-1};\ 1\leqslant n\}$$

defined by $\nu[T_{(q,1)}, S^{2n-1}]=\beta_{2n-1}$.

Proof. According to proposition 4.1 and Kamata [2], proposition 2.5, the $[T_{(q,1)}, S^{2n-1}]$ and the β_{2n-1} satisfy the same $\Gamma_*(q)$-module relations. Q.E.D.

Corollary 4.3. $\tilde{\Omega}_*^L(Z_q)=\Gamma_*(q)\{[T_{(q,1)}, S^{2n-1}];\ 1\leqslant n\}=\Gamma_*(q)\{\beta_{2n-1};\ 1\leqslant n\}$.

Proof. The first equality is a consequence of Kamata [2], proposition 2.6. So the map ν of 4.2 defines an injective endomorphism of $\tilde{\Omega}_*^L(Z_q)$ which is dimension preserving. But $\tilde{\Omega}_*^L(Z_q)$ contains only a finite number of elements in each dimension, and thus the injectivity of ν implies the surjectivity. This means $\Gamma_*(q)\{\beta_{2n-1};\ 1\leqslant n\}=\text{Image } \nu=\tilde{\Omega}_*^L(Z_q)$.

Corollary 4.4. $\Omega_*^L\{\beta_{2pm-1};\ 1\leqslant m\}=\Gamma_*(q)\{\beta_{2pm-1};\ 1\leqslant m\}$

Proof. It is obvious that $\Omega_*^L\{\beta_{2pm-1};\ 1\leqslant m\}\supset\Gamma_*(q)\{\beta_{2pm-1};\ 1\leqslant m\}$. Conversely let

(*) $x=\sum_{m=0}^{n}\alpha_{2t+2(n-m)p}\beta_{2pm-1};\ \alpha_{2t+2(n-m)p}\in\Omega_*^L$. From the preceding corollary, $\Omega_*^L\{\beta_{2j-1};\ 1\leqslant j\}\subset\Gamma_*(q)\{\beta_{2n-1};\ 1\leqslant n\}$. So

(**) $x=\sum_{j=0}^{t+np}\gamma_{2t-2pn-2j}\beta_{2j-1};\ \gamma_{2t+2pn-2j}\in\Gamma_*(q)$.

By (*), we have $t\circ i_*(x)=px$. On the other hand, (**) implies

$$t\circ i_*(x)=\sum_{m=1}^{n+[t/p]}p\gamma_{2t+2p(n-m)}\beta_{2pm-1}.$$

Therefore $x=\sum_{m=1}^{n+[t/p]}\gamma_{2t+2p(n-m)}\beta_{2pm-1}$. Q.E.D.

Now we are ready to prove the main theorem of this section.

Theorem 4.5. *The additive structure of $\tilde{\Omega}^L_*(Z_{q,p})$ with q an odd prime is determined by the following exact sequence of $\Gamma_*(q)$-homomorphisms*

$$0 \to \Gamma_*(q)\{\{q^{[pj-1/q-1]+1}\beta_{2pj-1};\ 1\leqslant j\}\} \xrightarrow{\iota\oplus 0}$$
$$\to \Gamma_*(q)\{\{\beta_{2pj-1};\ 1\leqslant j\}\}\oplus\tilde{\Omega}^L_*(Z_p) \xrightarrow{i_*+s_*} \tilde{\Omega}(Z_{q,p}) \to 0\,,$$

where $\Gamma_*(q)\{\{\cdots\}\}$ denotes the free $\Gamma_*(q)$-module generated by $\{\cdots\}$.

Proof. According to 2.10 and 4.4, i_*+s_* is epimorphic. And 4.1 implies that the kernel of i_*+s_* is as stated in the theorem. Q.E.D.

REMARK 4.6. Except for the case $\tilde{\Omega}^{SO}_*(Z_2)$, it holds that additively

$$\tilde{\Omega}^L_*(Z_p)\cong\Gamma_*(p)\{[T_{(p,1)},\ S^{2n-1}];\ 1\leqslant n\}/$$
$$\Gamma_*(p)\{p^{[n-1/p-1]+1}[T_{(p,1)},\ S^{2n-1}];\ 1\leqslant n\}$$

(Kamata [2], proposition 2.6)

And, also additively,

$$\tilde{\Omega}^{SO}_*(Z_2) = \bigoplus_{j=0}^{\infty} E^{2j+1}\mathcal{W}_*\,,$$

where $\mathcal{W}_*$ is Wall's polynomial subalgebra $Z_2[X_{2k-1},\ X_{2k};\ k\neq 2^j,\ (X_{2j})^2]$ in $\mathfrak{N}_*$ and E^{2j+1} is the isomorphism of raising the dimension of each element by $2j+1$. (Shibata [6], corollary 3.3, lemma 4.1)

5. Canonical splitting for $\tilde{\Omega}^L_*(Z_q)$

According to the results of section 2, we have the following proposition.

Proposition 5.1. *Let p, q, r be as stated in the introduction.* (1) *There is the projection homomorphism*

$$\rho_{(p,r)}\colon \tilde{\Omega}^L_*(Z_q) \to \tilde{\Omega}^L_*(Z_q)$$

defined by $\rho_{(p,r)}=\iota\circ(i_*\,|\,\Omega_*\{\beta_{2pm-1};\ 1\leqslant m\})^{-1}\circ i_*$.

(2) *The corresponding direct sum decomposition as Ω^L_*-modules* Image $\rho_{(p,r)}\oplus$ Ker $\rho_{(p,r)}$ *is*

$$\tilde{\Omega}^L_*(Z_q) = \Omega^L_*\{\beta_{2pm-1};\ 1\leqslant m\}\oplus\Omega^L_*\{\beta_{2n-1};\ 1\leqslant n,\ (n,p) = 1\}\ .$$

When $p=2$, r is necessarily equal to -1, or equivalently, $q-1$.

Corollary 5.2. *Let q be an odd integer.*
(1) *The formulas*

$$\rho_2(\beta_{4n+1}) = 0,\ \rho_2(\beta_{4n+3}) = \beta_{4n+3}$$

define an Ω^L_*-homomorphism

$$\rho_2\colon \tilde{\Omega}^L_*(Z_q) \to \tilde{\Omega}^L_*(Z_q)$$

which is a projection operator.

(2) *The corresponding direct sum splitting is;*

$$\tilde{\Omega}^U_*(Z_q) = \Omega^U_*\{\beta_{4n-1};\ 1\leqslant n\} \oplus \Omega^U_*\{\beta_{4n-3};\ 1\leqslant n\}\ ,$$
$$\tilde{\Omega}^{SO}_*(Z_q) = \Omega^{SO}_*\{[T_{(q,1)}, S^{4n-1}];\ 1\leqslant n\} \oplus \Omega^{SO}_*\{[T_{(q,1)}, S^{4n-3};\ 1\leqslant n\}\ .$$

I doubt if there is an analogous direct sum splitting for $\tilde{\Omega}^U_*(Z_{2^a})$; $a\geqslant 1$.

In the rest of this section, we assume q an odd prime and p a prime such such that $p\,|\,q-1$.

By elementary number theory arguments we obtain the following fact.

Lemma 5.3. *The equation $x^p-1\equiv 0$ mod q has exactly p distinct roots in Z_q. If $r\neq 1$ is one of them, then $r, r^2, \cdots, r^{p-1}$ are the primitive p-th roots mod q and $x^p-1\equiv\prod_{j=0}^{p-1}(x-r^j)$ mod q.*

Theorem 5.4. *For $q\geqslant 3$ an odd prime and p a prime such that $p\,|\,q-1$, there is the canonical projection*

$$\rho_p\colon \tilde{\Omega}^L_*(Z_q) \to \tilde{\Omega}^L_*(Z_q)$$

whcih gives the canonical direct sum decomposition

$$\Omega^L_*(Z_q) = \tilde{\Omega}^L_*\{\beta_{2pm-1};\ 1\leqslant m\} \oplus \Omega^L_*\{\beta_{2n-1};\ 1\leqslant n,\ (n,p)=1\}\ ,$$

and in particular for p an odd prime,

$$\begin{aligned}\tilde{\Omega}^{SO}_*(Z_q) = {} & \Omega^{SO}_*\{\beta_{4pm-1};\ 1\leqslant m\} \oplus \Omega^{SO}_*\{\beta_{4pm-2p-1};\ 1\leqslant m\} \\ & \oplus\ \Omega^{SO}_*\{\beta_{4n-1};\ 1\leqslant n,\ (n,p)=1\} \\ & \oplus\ \Omega^{SO}_*\{\beta_{4n-3};\ 1\leqslant n,\ (2n-1,p)=1\}\ .\end{aligned}$$

Proof. Lemma 5.3 implies that we can find primitive p-th roots in Z_q and that the definition of the β_{2n-1} does not depend on the choice of a p-th root. Hence the theorem follows from 5.1.

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