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ORIENTED AND WEAKLY COMPLEX BORDISM OF FREE METACYCLIC ACTIONS

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Abstract. Oriented and weakly complex bordism modules of free metacyclic actions are determined up to the Kasparov formula which describes the bordism classes of generalized lens spaces in terms of a linear combination of those of the standard lens spaces. In the oriented case for $p=2$ (the dihedral case), the module structure is particularly simple because the corresponding Kasparov formula reduces to the multiplication by ± 1 . We also compute the abelian group structure of these bordisms in case $p \geq 2$ a prime and $q \geq 3$ an odd prime. Of independent interest is the canonical projections defined on these bordism modules which select a direct summand with one generator in each $2pj-1$ dimension ($j=1, 2, \dots$).

1. Introduction.

Let $Z_{q,p}$ be the metacyclic group

$$Z_{q,p} = \{x, y \mid x^q = y^p = 1, yxy^{-1} = x^r\}$$

where $p \geq 2$ is a prime integer, $q \geq 3$ is an odd integer and r is a primitive p -th root of 1 mod q such that $(r-1, q)=1$. (So $r \equiv -1 \pmod q$ when $p=2$.) By virtue of Fermat's theorem, these conditions imply $(p, q)=1$.

Obviously there is an exact sequence

$$1 \longrightarrow Z_q \xrightarrow{i} Z_{q,p} \xleftarrow[\pi]{s} Z_p \longrightarrow 1$$

with s a cross-section defined by $s(\bar{y})=y$.

Kamata—Minami [3] determined the additive structure of the weakly complex reduced bordism group of the free dihedral group actions $\tilde{\Omega}_m^U(Z_{q,2})$ in case q is an odd prime. Here we generalize their results to the cases for the oriented and weakly complex bordism modules $\tilde{\Omega}_*^{SO}(Z_{q,p})$ and $\tilde{\Omega}_*^U(Z_{q,p})$ of the free metacyclic actions.

For the basic notations and prerequisites, we refer the reader to the introductory part and §1 of Kamata—Minami [3].

Thanks are due to Professor Minoru Nakaoka for suggesting me the subject.

2. The module structure of $\tilde{\Omega}_*^L(Z_{q,p})$; $L=SO, U$

First we recall the basic fact about $\tilde{\Omega}(Z_{q,p})$ from Lazarov [5].

Lemma 2.1. (Lazarov [5]).

- (1) $i_*: \tilde{\Omega}_*^L(Z_q) \rightarrow \tilde{\Omega}_*^L(Z_{q,p})$ is surjective onto the q -torsion.
- (2) $s_*: \tilde{\Omega}_*^L(Z_p) \rightarrow \tilde{\Omega}(Z_{q,p})$ is injective onto the p -torsion (which is a direct summand as an $\tilde{\Omega}_*^L$ -module because $\pi_* \circ s = \text{id}$).

The proof is done by calculating the integral homology $H_*(Z_{q,p}; Z)$. Thanks to our assumption on p, q and r stated in the introduction, Lazarov's proof still works here in a slightly generalized situation.

Therefore it suffices to know the kernel of i_* for the determination of the module structure $\tilde{\Omega}_*^L(Z_{q,p})$ because we already know the structure of $\tilde{\Omega}_*^L(Z_m)$ (Conner-Floyd [1], Kamata [2], Shibata [6]).

Let

$$T_{(q,j)}: Z_q \times S^{2n-1} \rightarrow S^{2n-1}$$

denote the Z_q -action on the $(2n-1)$ -dimensional sphere defined by $T_{(q,j)}(x^h, z) = \rho^{hj}z$, where $\rho = \exp(2\pi\sqrt{-1}/q)$. This is a free action if j is a unit in Z_q .

Let us consider the images of the $[T_{(q,rj)}, S^{2n-1}]$ by the canonical homomorphism

$$i_*: \tilde{\Omega}_*^L(Z_q) \rightarrow \tilde{\Omega}_*^L(Z_{q,p}).$$

Lemma 2.2.

$$i_*[T_{(q,rj)}, S^{2n-1}] = [\hat{T}_{(q,rj)}, Z_p \times S^{2n-1}],$$

where $\hat{T}_{(q,rj)}(x, (\bar{y}^h, z)) = (\bar{y}^h, \rho^{rj-h}z)$ and $\hat{T}_{(q,rj)}(y, (\bar{y}^h, z)) = (\bar{y}^{h+1}, z)$.

Proof. The map i_* is the extension (see Conner-Floyd [1] page 53), and so

$$i_*[T_{(q,rj)}, S^{2n-1}] = [\tilde{T}_{(q,rj)}, Z_{q,p} \times_{Z_q} S^{2n-1}]$$

where $\tilde{T}_{(q,rj)}$ is the natural operation of $Z_{q,p}$ on $Z_{q,p} \times_{Z_q} S^{2n-1}$ from the left. There is an L -structure preserving ($L=SO$ or U), $Z_{q,p}$ -equivariant diffeomorphism

$$\phi_j: (\tilde{T}_{(q,rj)}, Z_{q,p} \times_{Z_q} S^{2n-1}) \rightarrow (\hat{T}_{(q,rj)}, Z_p \times S^{2n-1})$$

defined by $\phi_j([x^a y^b, z]) = (\bar{y}^b, \rho^{arj-b}z)$.

Hence the lemma follows.

Corollary 2.3.

$$i_*[T_{(q,1)}, S^{2n-1}] = i_*[T_{(q,r^j)}, S^{2n-1}]$$

for $j=0, 1, 2, \dots, p-1$.

Proof. From the preceding lemma, it suffices to find an L -structure preserving, $Z_{q,p}$ -equivariant diffeomorphism

$$\psi_j: (\hat{T}_{(q,1)}, Z_p \times S^{2n-1}) \rightarrow (\hat{T}_{(q,r^j)}, Z_p \times S^{2n-1}).$$

In fact the formula $\psi_j(\bar{y}^b, z) = (\bar{y}^{b+j}, z)$ defines a desired one.

Let $t: \tilde{\Omega}_*^L(Z_{q,p}) \rightarrow \tilde{\Omega}_*^L(Z_q)$ be the transfer homomorphism, i.e. the homomorphism induced by the restriction of the action on the subgroup (Conner-Floyd [1] page 52).

Lemma 2.4.

$$t \circ i_*[T_{(q,1)}, S^{2n-1}] = \sum_{j=0}^{p-1} [T_{(q,r^j)}, S^{2n-1}].$$

Proof. The lemma is obvious from Lemma 2.2 and the definition of t .

DEFINITION 2.5. We define the elements β_{2n-1} ($n=1, 2, \dots$) of $\tilde{\Omega}_{2n-1}^L(Z_q)$ as follows.

- (1) In case $(n, p) = 1$, $\beta_{2n-1} = \sum_{0 \leq j < p-1} ([T_{(q,1)}, S^{2n-1}] - [T_{(q,r^j)}, S^{2n-1}])$.
- (2) $\beta_{2pm-1} = t \circ i_*[T_{(q,1)}, S^{2pm-1}] = \sum_{0 \leq j < p-1} [T_{(q,r^j)}, S^{2pm-1}]$.

- Lemma 2.6.** (1) $i_*\beta_{2n-1} = 0$ in case $(n, p) = 1$.
 (2) $t \circ i_*\beta_{2pm-1} = p\beta_{2pm-1}$.

Proof. (1) is obvious from definition 2.5 and corollary 2.3. Also definition 2.5, corollary 2.3 and lemma 2.4 imply (2).

At this stage, we need the formula of Kasparov [4], which describes the unitary bordism classes of the generalized lens spaces as a linear combination of those of the standard lens spaces. We restate his formula only in the special case which we concern.

Theorem 2.7 (Kasparov [4]). *In $\tilde{\Omega}_*^L(Z_q)$, the class $[T_{(q,r^j)}, S^{2n-1}]$ is the coefficient of X^n in*

$$\left(\sum_{1 \leq k} [T_{(q,1)}, S^{2k-1}] X^k \right) (X/g^{-1}(r^jg(X)))^n$$

(or its image in $\tilde{\Omega}_*^{SO}(Z_q)$ by the natural homomorphism $\tilde{\Omega}_*^L(Z_q) \rightarrow \tilde{\Omega}_*^{SO}(Z_q)$), where $g(X) = \sum_{1 \leq k} ([CP_{k-1}]/h) X^k$ is the logarithm of the cobordism formal group law, i.e.

“the Miscenko series”.

Corollary 2.8.

$$\beta_{2n-1} - p[T_{(q,1)}, S^{2n-1}] \in \Omega_*^L \{ [T_{(q,1)}, S^{2k-1}]; 1 \leq k \leq n-1 \},$$

where $\Omega_*^L \{ \dots \}$ denotes the Ω_*^L -submodule of $\tilde{\Omega}_*^L(Z_q)$ generated by the elements $\{ \dots \}$.

Proof. By the Kasparov formula, we see that

$$[T_{(p,r^j)}, S^{2n-1}] - \left(\frac{1}{r^j}\right)^n [T_{(q,1)}, S^{2n-1}] \in \Omega_*^L \{ [T_{(q,1)}, S^{2j-1}]; 1 \leq j \leq n-1 \}.$$

Notice that here we can treat everything as reduced mod q since $q[T_{(q,1)}, S^{2j-1}] \in \Omega_*^L \{ [T_{(q,1)}, S^{2h-1}]; 1 \leq h < j \}$ (Shibata [6]). In case $n = kp$, $\sum_{0 \leq j \leq p-1} \left(\frac{1}{r^j}\right)^n = \sum_{0 \leq j \leq p-1} (1/(r^p)^{kj}) \equiv p$. Thus the lemma is true. Otherwise put $n = kp + t (1 \leq t \leq p-1)$. Then $\sum_{0 \leq j \leq p-1} (1/r^j)^n = \sum_{0 \leq j \leq p-1} (r^{-t})^j \equiv 0 \pmod q$ since r^{-t} is a root of the equation $x^p - 1 = (x-1)(x^{p-1} + \dots + x + 1) \equiv 0$ and $r^{-t} - 1$ is a unit in Z_q by virtue of the condition $(r-1, q) = 1$. Therefore the lemma holds also in case $(n, p) = 1$.

Corollary 2.9. $\Omega_*^L \{ [T_{(q,1)}, S^{2j-1}]; 1 \leq j \leq k \} = \Omega_*^L \{ \beta_{2j-1}; 1 \leq j \leq k \}$. In particular, $\tilde{\Omega}_*^L(Z_q) = \Omega_*^L \{ \beta_{2j-1}; 1 \leq j \}$.

Proof. Since we are assuming $(p, q) = 1$, this corollary is easily proved by induction on k by virtue of 2.8.

Now we can state the main theorem of this section as follows.

Theorem 2.10. *There are the following exact sequences of Ω_*^L -module homomorphisms*

$$\begin{aligned} (1) \quad & 0 \rightarrow \Omega_*^L \{ \beta_{2m-1}; 1 \leq m, (m, p) = 1 \} \xrightarrow{\iota \oplus 0} \\ & \rightarrow \tilde{\Omega}_*^L(Z_q) \oplus \tilde{\Omega}(Z_p) \xrightarrow{i_* + s_*} \tilde{\Omega}_*^L(Z_{q,p}) \rightarrow 0, \text{ and} \\ (2) \quad & 0 \rightarrow \tilde{\Omega}_*^L \{ \beta_{2pk-1}; 1 \leq k \} \oplus \tilde{\Omega}_*^L(Z_p) \xrightarrow{i_* + s_*} \tilde{\Omega}_*^L(Z_{q,p}) \rightarrow 0, \end{aligned}$$

where ι is the canonical inclusion as a submodule,

$$\beta_{2m-1} = \begin{cases} p[T_{(q,1)}, S^{2m-1}] - \sum_{0 \leq j \leq p-1} [T_{(q,r^j)}, S^{2m-1}] & \text{if } (m, p) = 1, \text{ and} \\ \sum_{0 \leq j \leq p-1} [T_{(q,r^j)}, S^{2m-1}] & \text{if } p | m, \end{cases}$$

and the $[T_{(q,r^j)}, S^{2m-1}]$ can be written down as a linear combination over Ω_*^L of the $[T_{(q,1)}, S^{2n-1}] (1 \leq n \leq m)$ by the Kasparov formula (Theorem 2.7).

Proof. The proof is now obvious from 2.1, 2.6 and 2.9. We only indicate the proof of the fact that $\text{Ker } i_* \subset \Omega_*^L \{\beta_{2m-1}; 1 \leq m, (m, p)=1\}$. Suppose x belongs to $\text{Ker } i_*$ and is homogeneous of dimension $2t-1$. By 2.9,

$$x = \sum_{m=1}^t \alpha_{2(t-m)} \beta_{2m-1}$$

for some $\alpha_{2(t-m)} \in \Omega_{2(t-m)}^L$. Then $0 = t \circ i_*(x) = \sum_{m=1}^{\lfloor t/p \rfloor} p \alpha_{2t-2pm} \beta_{2pm-1}$. So $\sum_{m=1}^{\lfloor t/p \rfloor} \alpha_{2t-2pm} \beta_{2pm-1} = 0$ and this implies $x = x - \sum_{m=1}^{\lfloor t/p \rfloor} \alpha_{2t-2pm} \beta_{2pm-1} = \sum_{\substack{1 \leq m \leq t \\ (m, p)=1}} \alpha_{2(t-m)} \beta_{2m-1}$ as desired.

3. The oriented case for $p=2$.

There is a special simplicity for the oriented bordism of the free dihedral actions.

Lemma 3.1. *Let s be a unit in Z_q . It holds in $\tilde{\Omega}_*^{SO}(Z_q)$ that*

$$[T_{(q,-s)}, S^{2n-1}] = (-1)^n [T_{(q,s)}, S^{2n-1}].$$

Proof. Consider the Z_q -equivariant diffeomorphism

$$c: (T_{(q,-s)}, S^{2n-1}) \rightarrow (T_{(q,s)}, S^{2n-1})$$

defined by $c(z_0, \dots, z_{n-1}) = (\bar{z}_0, \dots, \bar{z}_{n-1})$, i.e. the complex conjugation. Then c preserves the orientation when n is even and reverses when n is odd. *Q.E.D.*

It follows that, in $\tilde{\Omega}_*^{SO}(Z_q)$,

$$\begin{aligned} \beta_{4i+1} &= [T_{(q,1)}, S^{4i+1}] - [T_{(q,-1)}, S^{4i+1}] \\ &= 2[T_{(q,1)}, S^{4i+1}], \text{ and} \\ \beta_{4i-1} &= [T_{(q,1)}, S^{4i-1}] + [T_{(q,-1)}, S^{4i-1}] \\ &= 2[T_{(q,1)}, S^{4i-1}]. \end{aligned}$$

Therefore theorem 2.10 of the last section reduces to the following.

Theorem 3.2. *There are the following exact sequences of Ω_*^{SO} -module homomorphisms*

$$\begin{aligned} (1) \quad 0 \rightarrow \Omega_*^{SO} \{[T_{(q,1)}, S^{4m+1}]; 0 \leq m\} &\xrightarrow{\iota \oplus 0} \\ \rightarrow \tilde{\Omega}_*^{SO}(Z_q) \oplus \tilde{\Omega}_*^{SO}(Z_2) &\xrightarrow{i_* + s_*} \tilde{\Omega}_*^{SO}(Z_{q,2}) \rightarrow 0, \end{aligned}$$

and

$$\begin{aligned} (2) \quad 0 \rightarrow \Omega_*^{SO} \{[T_{(q,1)}, S^{4m-1}]; 1 \leq m\} \oplus \tilde{\Omega}_*^{SO}(Z_2) &\rightarrow \\ \xrightarrow{i_* + s_*} \tilde{\Omega}_*^{SO}(Z_{q,2}) &\rightarrow 0. \end{aligned}$$

REMARK 3.3. The module structures of $\tilde{\Omega}_*^{SO}(Z_q)$ (q odd) and $\tilde{\Omega}_*^{SO}(Z_2)$ are determined in Shibata [6]. According to 6.1 and 6.3 of Shibata [6], together with the fact that the natural homomorphism $\Omega_*^Y \rightarrow \Omega_*^{SO}/\text{Tor}$ kills the elements of dimension $4j+2(j=0, 1, 2, \dots)$, we see that the restriction of the Smith homomorphism

$$\Delta: \Omega_*^{SO}\{[T_{(q,1)}, S^{4m-1}]; 1 \leq m\} \rightarrow \Omega_*^{SO}\{[T_{(q,1)}, S^{4m-3}]; 1 \leq m\}$$

is an isomorphism, and thus $\tilde{\Omega}_*^{SO}(Z_q)$ is a direct sum of two isomorphic copies (with dimension shift) of Ω_*^{SO} -submodules.

REMARK 3.4. We can not expect such a simple phenomenon in the unitary bordism of the dihedral actions. For example,

$$\begin{aligned} [T_{(q,-1)}, S^3] &= [T_{(q,1)}, S^3] - 2[CP_1][T_{(q,1)}, S^1], \\ [T_{(q,-1)}, S^5] &= -[T_{(q,1)}, S^5] + 3[CP_1][T_{(q,1)}, S^3] \\ &\quad - 3[CP_1]^2[T_{(q,1)}, S^1], \end{aligned}$$

and so

$$t \circ i_*[T_{(q,1)}, S^3] = 2[T_{(q,1)}, S^3] - 2[CP_1][T_{(q,1)}, S^1], \neq 2[T_{(q,1)}, S^3],$$

and in case $q > 3$,

$$t \circ i_*[T_{(q,1)}, S^5] = 3[CP_1][T_{(q,1)}, S^3] - 3[CP_1]^2[T_{(q,1)}, S^1] \neq 0$$

in $\tilde{\Omega}_*^Y(Z_q)$.

Also when $q=3$,

$$t \circ i_*[T_{(3,1)}, S^9] = 5[CP_1][T_{(3,1)}, S^7] - [CP_1]^2[T_{(3,1)}, S^5] + \dots \neq 0.$$

REMARK 3.5. Even in the oriented case, if we take the case for $p \geq 3$, the Kasparov formula becomes complicated. The lowest dimensional example is the case for $p=3, q=7, r=2$. The computation shows that

$$\begin{aligned} t \circ i_*[T_{(7,1)}, S^5] &= 3[T_{(7,1)}, S^5] + 4[CP_2][T_{(7,1)}, S^1] \neq 3[T_{(7,1)}, S^5], \\ t \circ i_*[T_{(7,1)}, S^9] &= 5[CP_2][T_{(7,1)}, S^5] + 2[CP_2]^2[T_{(7,1)}, S^1]. \end{aligned}$$

Therefore $t \circ i_*[T_{(7,1)}, S^9] \neq 0$ in $\tilde{\Omega}_*^{SO}(Z_7)$.

4. Computation of abelian group structure of $\tilde{\Omega}_*^L(Z_{q,p})$ for q an odd prime

In this section we present a generalization of the main theorem of Kamata-Minami [3] to the case for $\tilde{\Omega}_*^L(Z_{q,p})$ with $p \geq 2$ a prime and $q \geq 3$ an odd prime. So in this section, we assume q an odd prime.

As in Kamata [2], let $\Gamma_*(q)$ be the polynomial subring of $\Omega_*^U = Z[x_1, x_2, \dots]$ which is generated by x_i ($i \neq q-1$) (unitary case) or its image in Ω_*^{SO} by the canonical homomorphism $\Omega_*^U \rightarrow \Omega_*^{SO}$ (oriented case).

Analogously to Kamata-Minami [3], proposition 3.1, we obtain;

Proposition 4.1. *The following two conditions for the elements $[M^{2(l-k)}] \in \Gamma_{\mathcal{X}(l-k)}(q)$ are equivalent;*

- (1) $\sum_{k=1}^n [M^{2(l-k)}] \beta_{2k-1} = 0$ in $\tilde{\Omega}_*^L(Z_q)$, and
- (2) $[M^{2(l-k)}] \in q^{[k-1/q-1]+1} \Gamma_{\mathcal{X}(l-k)}(q)$,

where the β_{2k-1} are the module generators of $\tilde{\Omega}_*^L(Z_q)$ defined in section 2.

Now $\tilde{\Omega}_*^L(Z_q)$ can be considered as a $\Gamma_*(q)$ -module and we denote by $\Gamma_*(q)\{\dots\}$ the $\Gamma_*(q)$ -submodule of $\tilde{\Omega}_*^L(Z_q)$ generated by the elements $\{\dots\}$.

Lemma 4.2. *There is a $\Gamma_*(q)$ -isomorphism*

$$\nu: \Gamma_*(q)\{[T_{(q,1)}, S^{2n-1}]; 1 \leq n\} \rightarrow \Gamma_*(q)\{\beta_{2n-1}; 1 \leq n\}$$

defined by $\nu[T_{(q,1)}, S^{2n-1}] = \beta_{2n-1}$.

Proof. According to proposition 4.1 and Kamata [2], proposition 2.5, the $[T_{(q,1)}, S^{2n-1}]$ and the β_{2n-1} satisfy the same $\Gamma_*(q)$ -module relations. Q.E.D.

Corollary 4.3. $\tilde{\Omega}_*^L(Z_q) = \Gamma_*(q)\{[T_{(q,1)}, S^{2n-1}]; 1 \leq n\} = \Gamma_*(q)\{\beta_{2n-1}; 1 \leq n\}$.

Proof. The first equality is a consequence of Kamata [2], proposition 2.6. So the map ν of 4.2 defines an injective endomorphism of $\tilde{\Omega}_*^L(Z_q)$ which is dimension preserving. But $\tilde{\Omega}_*^L(Z_q)$ contains only a finite number of elements in each dimension, and thus the injectivity of ν implies the surjectivity. This means $\Gamma_*(q)\{\beta_{2n-1}; 1 \leq n\} = \text{Image } \nu = \tilde{\Omega}_*^L(Z_q)$.

Corollary 4.4. $\Omega_*^L\{\beta_{2pm-1}; 1 \leq m\} = \Gamma_*(q)\{\beta_{2pm-1}; 1 \leq m\}$

Proof. It is obvious that $\Omega_*^L\{\beta_{2pm-1}; 1 \leq m\} \supset \Gamma_*(q)\{\beta_{2pm-1}; 1 \leq m\}$. Conversely let

(*) $x = \sum_{m=0}^n \alpha_{2t+2(n-m)p} \beta_{2pm-1}; \alpha_{2t+2(n-m)p} \in \Omega_*^L$. From the preceding corollary, $\Omega_*^L\{\beta_{2j-1}; 1 \leq j\} \subset \Gamma_*(q)\{\beta_{2n-1}; 1 \leq n\}$. So

$$(**) \quad x = \sum_{j=0}^{t+n/p} \gamma_{2t-2pn-2j} \beta_{2j-1}; \gamma_{2t+2pn-2j} \in \Gamma_*(q).$$

By (*), we have $t \circ i_*(x) = px$. On the other hand, (**) implies

$$t \circ i_*(x) = \sum_{m=1}^{n+[t/p]} p \gamma_{2t+2p(n-m)} \beta_{2pm-1}.$$

Therefore $x = \sum_{m=1}^{n+[t/p]} \gamma_{2t+2p(n-m)} \beta_{2pm-1}$. Q.E.D.

Now we are ready to prove the main theorem of this section.

Theorem 4.5. *The additive structure of $\tilde{\Omega}_*^L(Z_{q,p})$ with q an odd prime is determined by the following exact sequence of $\Gamma_*(q)$ -homomorphisms*

$$0 \rightarrow \Gamma_*(q)\{\{q^{[pj-1/q-1]+1}\beta_{2pj-1}; 1 \leq j\}\} \xrightarrow{\iota \oplus 0} \\ \rightarrow \Gamma_*(q)\{\{\beta_{2pj-1}; 1 \leq j\}\} \oplus \tilde{\Omega}_*^L(Z_p) \xrightarrow{i_* + s_*} \tilde{\Omega}(Z_{q,p}) \rightarrow 0,$$

where $\Gamma_*(q)\{\{\dots\}\}$ denotes the free $\Gamma_*(q)$ -module generated by $\{\dots\}$.

Proof. According to 2.10 and 4.4, $i_* + s_*$ is epimorphic. And 4.1 implies that the kernel of $i_* + s_*$ is as stated in the theorem. Q.E.D.

REMARK 4.6. Except for the case $\tilde{\Omega}_*^{SO}(Z_2)$, it holds that additively

$$\tilde{\Omega}_*^L(Z_p) \cong \Gamma_*(p)\{[T_{(p,1)}, S^{2^n-1}]; 1 \leq n\} / \\ \Gamma_*(p)\{p^{[n-1/p-1]+1}[T_{(p,1)}, S^{2^n-1}]; 1 \leq n\}$$

(Kamata [2], proposition 2.6)

And, also additively,

$$\tilde{\Omega}_*^{SO}(Z_2) = \bigoplus_{j=0}^{\infty} E^{2j+1} \mathcal{W}_*,$$

where \mathcal{W}_* is Wall's polynomial subalgebra $Z_2[X_{2k-1}, X_{2k}; k \neq 2^j, (X_{2j})^2]$ in \mathcal{R}_* and E^{2j+1} is the isomorphism of raising the dimension of each element by $2j+1$. (Shibata [6], corollary 3.3, lemma 4.1)

5. Canonical splitting for $\tilde{\Omega}_*^L(Z_q)$

According to the results of section 2, we have the following proposition.

Proposition 5.1. *Let p, q, r be as stated in the introduction. (1) There is the projection homomorphism*

$$\rho_{(p,r)}: \tilde{\Omega}_*^L(Z_q) \rightarrow \tilde{\Omega}_*^L(Z_q)$$

defined by $\rho_{(p,r)} = \iota \circ (i_* | \Omega_*\{\beta_{2pm-1}; 1 \leq m\})^{-1} \circ i_*$.

(2) *The corresponding direct sum decomposition as Ω_*^L -modules Image $\rho_{(p,r)} \oplus \text{Ker } \rho_{(p,r)}$ is*

$$\tilde{\Omega}_*^L(Z_q) = \Omega_*^L\{\beta_{2pm-1}; 1 \leq m\} \oplus \Omega_*^L\{\beta_{2n-1}; 1 \leq n, (n, p) = 1\}.$$

When $p=2$, r is necessarily equal to -1 , or equivalently, $q-1$.

Corollary 5.2. *Let q be an odd integer.*

(1) *The formulas*

$$\rho_2(\beta_{4n+1}) = 0, \rho_2(\beta_{4n+3}) = \beta_{4n+3}$$

define an Ω_*^L -homomorphism

$$\rho_2: \tilde{\Omega}_*^L(Z_q) \rightarrow \tilde{\Omega}_*^L(Z_q)$$

which is a projection operator.

(2) The corresponding direct sum splitting is;

$$\begin{aligned} \tilde{\Omega}_*^U(Z_q) &= \Omega_*^U\{\beta_{4n-1}; 1 \leq n\} \oplus \Omega_*^U\{\beta_{4n-3}; 1 \leq n\}, \\ \tilde{\Omega}_*^{SO}(Z_q) &= \Omega_*^{SO}\{[T_{(q,1)}, S^{4n-1}]; 1 \leq n\} \oplus \Omega_*^{SO}\{[T_{(q,1)}, S^{4n-3}]; 1 \leq n\}. \end{aligned}$$

I doubt if there is an analogous direct sum splitting for $\tilde{\Omega}_*^U(Z_{2^a})$; $a \geq 1$.

In the rest of this section, we assume q an odd prime and p a prime such that $p \mid q-1$.

By elementary number theory arguments we obtain the following fact.

Lemma 5.3. *The equation $x^p - 1 \equiv 0 \pmod q$ has exactly p distinct roots in Z_q . If $r \neq 1$ is one of them, then r, r^2, \dots, r^{p-1} are the primitive p -th roots mod q and $x^p - 1 \equiv \prod_{j=0}^{p-1} (x - r^j) \pmod q$.*

Theorem 5.4. *For $q \geq 3$ an odd prime and p a prime such that $p \mid q-1$, there is the canonical projection*

$$\rho_p: \tilde{\Omega}_*^L(Z_q) \rightarrow \tilde{\Omega}_*^L(Z_q)$$

which gives the canonical direct sum decomposition

$$\Omega_*^L(Z_q) = \tilde{\Omega}_*^L\{\beta_{2pm-1}; 1 \leq m\} \oplus \Omega_*^L\{\beta_{2n-1}; 1 \leq n, (n, p) = 1\},$$

and in particular for p an odd prime,

$$\begin{aligned} \tilde{\Omega}_*^{SO}(Z_q) &= \Omega_*^{SO}\{\beta_{4pm-1}; 1 \leq m\} \oplus \Omega_*^{SO}\{\beta_{4pm-2p-1}; 1 \leq m\} \\ &\oplus \Omega_*^{SO}\{\beta_{4n-1}; 1 \leq n, (n, p) = 1\} \\ &\oplus \Omega_*^{SO}\{\beta_{4n-3}; 1 \leq n, (2n-1, p) = 1\}. \end{aligned}$$

Proof. Lemma 5.3 implies that we can find primitive p -th roots in Z_q and that the definition of the β_{2n-1} does not depend on the choice of a p -th root. Hence the theorem follows from 5.1.

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