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NOTE ON A 2-DIMENSIONAL VERSAL $D_8$-COVER

Dedicated to Professor Makoto Namba on his sixtieth birthday

HIRO-O TOKUNAGA

(Received April 9, 2003)

Introduction

Let $G$ be a finite group. Let $X$ and $Y$ be normal projective varieties. $X$ is called a $G$-cover of $Y$ if there exists a finite surjective morphism $\pi: X \to Y$ such that the induced morphism $\pi^*: \mathbb{C}(Y) \to \mathbb{C}(X)$ gives a Galois extension with $\text{Gal}(\mathbb{C}(X)/\mathbb{C}(Y)) \cong G$, where $\mathbb{C}(X)$ and $\mathbb{C}(Y)$ denote the rational function fields of $X$ and $Y$, respectively.

DEFINITION 0.1. A $G$-cover $\pi: X \to M$ is said to be versal if it satisfies the following property:

For any $G$-cover $\pi: Y \to Z$, there exist a rational map $\nu: Z \to M$ and a Zariski open set $U$ in $Z$ such that

(i) $\nu|_U : U \to M$ is a morphism, and

(ii) $\pi^{-1}(U)$ is birational to $U \times_M X$ over $U$.

One could say the investigation of versal $G$-covers is a geometric study of generic or versal $G$-polynomials (see [2] and [4]). The notion of versal $G$-covers implicitly appeared in [7], [8], and is defined explicitly in [12], [13]. It is known that there exists a versal $G$-cover for any finite group $G$ ([8, Theorem 2.4]). The dimension of the versal $G$-cover given by Namba, however, is equal to $\sharp(G)$. Hence it does not seem to be tractable in practical use. We need to find a tractable model for $G$. So far it has been done for some cases by ad-hoc methods in ([12], [13]).

In this note, we consider versal $D_8$-covers, where $D_8$ is the dihedral group of order 8, i.e., $D_8 = \langle \sigma, \tau \mid \sigma^2 = \tau^4 = (\sigma\tau)^2 = 1 \rangle$. It is known that any versal $D_8$-cover has dimension at least 2 (see [2]), and one of such models was given in [12]. The purpose of this article is to give another new model, which is described as follows:

Let $\varphi_{141}: X_{141} \to \mathbb{P}^1$ be the rational elliptic surface obtained by blowing-up base
points of the \((2,2)\)-pencil on \(\mathbb{P}^1 \times \mathbb{P}^1\) given by
\[
\{ \lambda_0 (s_0-t_1)^2 (t_0-t_1)^2 + \lambda_1 (s_0 s_1 t_0 t_1) = 0 \} \mid [\lambda_0, \lambda_1] \in \mathbb{P}^1.
\]

As for \(X_{141}\), the following facts are well-known:
(i) \(X_{141}\) is so-called the elliptic modular surface attached to
\[
\Gamma_1(4) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod 4 \right\},
\]
and \(\varphi_{141}\) has three singular fibers and their types are of \(I^*_1, I_4\) and \(I_1\) (see [3, p.350]).
(ii) The group of sections, \(MW(X_{141})\), is isomorphic to \(\mathbb{Z}/4\mathbb{Z}\) (see [6] or [9]).

Let \(\sigma_{141}\) be the involution on \(X_{141}\) induced by the inversion with respect to the group law and let \(\tau_s\) be the translation by a 4-torsion section \(s\). \(\sigma_{141}\) and \(\tau_s\) generate a finite fiber preserving automorphism group isomorphic to \(D_8\). Let \(\Sigma_{141} := X_{141}/\langle \sigma_{141}, \tau_s \rangle\) be the quotient surface and we denote its quotient morphism by \(\pi_{141} : X_{141} \to \Sigma_{141}\). Now we are in position to state our result:

**Theorem 0.2.** \(\pi_{141} : X_{141} \to \Sigma_{141}\) is a versal \(D_8\)-cover.

In comparison with the model in [12], this model has a nice description with respect to the action of the Galois group.

1. Preliminaries

Let \(S\) be a smooth minimal projective surface, and let \(\Lambda\) be a pencil of curves on \(S\) such that a general member is irreducible. Let \(\varphi_{\Lambda} : S \to \mathbb{P}^1\) be the rational map determined by \(\Lambda\), and let \(q : S_\Lambda \to S\) be the resolution of the indeterminacy of \(\varphi_{\Lambda}\). We denote the induced morphism from \(S_\Lambda\) to \(\mathbb{P}^1\) by \(\varphi_{\Lambda}\). We may assume that \(\varphi_{\Lambda}\) is relatively minimal. Let us begin with the following lemma.

**Lemma 1.1.** Let \(\sigma\) be an automorphism of \(S\). Suppose that \(\varphi_{\Lambda}^\sigma = \varphi_{\Lambda}\) (we regard \(\varphi_{\Lambda}\) as an element of \(\mathbb{C}(S_\Lambda)\)). Then \(\sigma\) gives rise to a fiber preserving automorphism of \(S_\Lambda\) (By abuse of notation, we also denote it by \(\sigma\)).

Proof. Since \(\mathbb{C}(S) \cong \mathbb{C}(S_\Lambda)\), \(\varphi_{\Lambda}^\sigma = \varphi_{\Lambda}\) implies \(\varphi_{\Lambda}^\sigma = \varphi_{\Lambda}\). Hence a general fiber of \(\varphi_{\Lambda}\) goes to that of \(\varphi_{\Lambda}\) under \(\sigma\). Let \(\tilde{\sigma}\) be the induced birational map from \(S_\Lambda\) to itself induced by \(\sigma\). Let \(\mu : S_\Lambda \to S_\Lambda\) be a succession of blowing-ups so that
(i) \(\tilde{\sigma} := \tilde{\sigma} \circ \mu\) becomes a birational morphism and (ii) the number of the blowing-ups is minimal. It is well-known that \(\tilde{\sigma}\) is a composition of blowing-downs. Let \(E_1, \ldots, E_r\) be the exceptional divisors for \(\mu\), and let \(F_1, \ldots, F_s\) be those for \(\tilde{\sigma}\). We may assume that \(F_1\) is the \((-1)\) curve for the first blow-down. Since \(F_1\) can not be any of \(E_i\)
(i = 1, \ldots, r), \ F_i E_i \geq 0 (i = 1, \ldots, r). \ Since
\[
-1 = F_1 K_{S_\lambda} = F_1 \left( \mu^* K_{S_\lambda} + \sum_{j=1}^r m_j E_j \right) = F_1 \mu^* K_{S_\lambda} + F_1 \left( \sum_{j=1}^r m_j E_j \right),
\]
we have \( F_1 \mu^* K_{S_\lambda} \leq -1 \). Hence \( \mu(F_1) \) is not any irreducible component in a fiber of \( \varphi_\Lambda \). Since the image of \( \mu(F_1) \) by \( \sigma \) is a point, this implies that a general fiber of \( \varphi_\Lambda \) does not go to that of \( \varphi_\Lambda \) under \( \hat{\sigma} \), which leads us to a contradiction. \qed

**Example 1.2.** Let \((x, y)\) be an inhomogeneous coordinate of \( \mathbb{P}^1 \times \mathbb{P}^1 \). The pencil on \( \mathbb{P}^1 \times \mathbb{P}^1 \) in Introduction is given by
\[
\Lambda: \{ \lambda_0(x - 1)^2(y - 1)^2 + \lambda_1 xy = 0 \}_{(\lambda_0, \lambda_1) \in \mathbb{P}^1}.
\]
Let \( \sigma \) and \( \tau \) be the automorphisms of \( \mathbb{P}^1 \times \mathbb{P}^1 \) given by
\[
(x^\sigma, y^\sigma) = (y, x), \quad (x^\tau, y^\tau) = \left(y, \frac{1}{x}\right).
\]
\( \sigma \) and \( \tau \) generate a finite automorphism group isomorphic to \( D_8 \). Since the rational function on \( \mathbb{P}^1 \times \mathbb{P}^1 \),
\[
\frac{(x - 1)^2(y - 1)^2}{xy},
\]
is invariant under \( \sigma \) and \( \tau \), one can apply Lemma 1.1 to this case. In fact, it follows that \( (\mathbb{P}^1 \times \mathbb{P}^1)_\Lambda = X_{141} \), \( \varphi_\Lambda = \varphi_{141} \), and \( \sigma, \tau \) fiber preserving automorphisms of \( X_{141} \). By Example 4.7, Chapter III in [11], both \( \sigma \) and \( \tau \) are the compositions of isogenies and translations. In particular, the corresponding isogenies should be isomorphisms as an elliptic curve. By Theorem 10.1, Chapter III in [11], the automorphisms of \( X_{141} \) as an elliptic curve are only the identity and \( \sigma_{\varphi_{141}} \). Since \( \sigma \circ \tau \) has a fixed point \((1, 0)\), the induced fiber preserving automorphism on \( X_{141} \) is non-trivial and fixes a section, \( O \). Hence we may assume that \( \sigma \circ \tau = \sigma_{\varphi_{141}} \) on \( X_{141} \), by regarding \( O \) as the zero. Also if \( \tau \) is the composition of \( \sigma_{\varphi_{141}} \) and a translation, then \( \tau^2 = \text{id} \) on \( X_{141} \).

This implies that we may assume that \( \tau \) is a translation by a 4-torsion section \( s \).

**Example 1.3.** Let \( N = \mathbb{Z}^2 \) and let \( \Delta_i (i = 1, \ldots, 6) \) be 2-dimensional cones in \( N \otimes \mathbb{R} \) given by
\[
\begin{align*}
\Delta_1 &= \mathbb{R}_{\geq 0} e_1 + \mathbb{R}_{\geq 0} (e_1 + e_2) \\
\Delta_2 &= \mathbb{R}_{\geq 0} (e_1 + e_2) + \mathbb{R}_{\geq 0} e_2 \\
\Delta_3 &= \mathbb{R}_{\geq 0} (-e_1) + \mathbb{R}_{\geq 0} e_2 \\
\Delta_4 &= \mathbb{R}_{\geq 0} (-e_1) + \mathbb{R}_{\geq 0} (-e_1 - e_2) \\
\Delta_5 &= \mathbb{R}_{\geq 0} (-e_1 - e_2) + \mathbb{R}_{\geq 0} (-e_2) \\
\Delta_6 &= \mathbb{R}_{\geq 0} (-e_2) + \mathbb{R}_{\geq 0} e_1,
\end{align*}
\]
where \( e_1 = (1, 0), e_2 = (0, 1) \). Let \( \Sigma = \bigcup_{i=1}^{6} \Delta_i \) and \( X_{D_{12}} = T_N \text{emb}(\Sigma) \) (see [10, §1.2]). The subgroup, \( G \), of \( \text{GL}(2, \mathbb{Z}) \) generated by

\[
\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}
\]

is isomorphic to \( D_{12} \). Since \( G \) preserves \( \Sigma \) as above, \( D_{12} \) acts on \( X_{D_{12}} \). An explicit description is as follows:

Let \( \{ f_1, f_2 \} \) be the basis of \( \text{Hom}(\mathcal{N}, \mathbb{Z}) \) dual to \( \{ e_1, e_1 + e_2 \} \) and let \( x = e(f_1) \) and \( y = e(f_2) \). \((x, y)\) gives an affine coordinate of the affine open set, \( U_{\Delta_i} \), determined by \( \Delta_i \). Then

\[
(x^\sigma, y^\sigma) = (y, x), \quad (x^\tau, y^\tau) = \left( \frac{1}{y}, xy \right).
\]

Consider the pencil on \( X \) which is given on \( U_{\Delta_i} \) by

\[
\{ \lambda_0(xy) + \lambda_1(xy + 1)(x + 1)(y + 1) = 0 \}_{\lambda_0, \lambda_1 \in \mathbb{P}^1}
\]
on \( U_{\sigma_1} \). Since

\[
\frac{xy}{(xy + 1)(x + 1)(y + 1)}
\]
is \( D_{12}\)-invariant, one can apply Lemma 1.1 to this case. In this case, \( X_\Delta \) coincides with the rational elliptic surface \( \varphi_{6321}: X_{6321} \to \mathbb{P}^1 \) (the notation is due to [6]). The following facts on \( X_{6321} \) are well-known:

(i) \( X_{6321} \) is so-called the elliptic modular surface attached to

\[
\Gamma_1(6) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{6} \right\},
\]

and has four singular fibers and their types are of \( I_5, I_3, I_2 \) and \( I_1 \) (see [1] and [6]).

(ii) The group of sections, \( MW(X_{6321}) \), is isomorphic to \( \mathbb{Z}/6\mathbb{Z} \) (see [6]).

In our case, \( \varphi_\Delta = \varphi_{6321} \), and \( \sigma \) and \( \tau \) generate a fiber preserving automorphism group of \( X_{6321} \) isomorphic to \( D_{12} \). By the same argument to that in Example 1.2, we may assume that it coincides with one given by the inversion with respect to the group law and the translation by a 6-torsion section.

We here raise a question concerning Example 1.3.

**Question 1.4.** Let \( M_{D_{12}} \) be the quotient of \( X_{D_{12}} \) with respect to the \( D_{12}\)-action in Example 1.3. Is the \( D_{12}\)-cover \( X_{D_{12}} \to M_{D_{12}} \) versal? In other words, is \( X_{6321} \to \Sigma_{6321} \) a versal \( D_{12}\)-cover, where \( \Sigma_{6321} \) is the quotient by the inversion with respect to the group law and the translation by 6-torsion?
2. Proof of Theorem 0.2

Let $\sigma$ and $\tau$ be the automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$ as in Example 1.2. They give rise to a finite automorphism group isomorphic to $D_8$. Put $\Sigma_{D_8} := \mathbb{P}^1 \times \mathbb{P}^1 / (\sigma, \tau)$, and we denote the quotient morphism by $\pi_{D_8} : \mathbb{P}^1 \times \mathbb{P}^1 \to \Sigma_{D_8}$. Theorem 0.2 follows from Example 1.2 and the proposition below.

**Proposition 2.1.** $\pi_{D_8} : \mathbb{P}^1 \times \mathbb{P}^1 \to \Sigma_{D_8}$ is versal.

We need two lemmas to prove Proposition 2.1.

**Lemma 2.2.** Let $\pi : Y \to Z$ be a $D_8$-cover. Then there exist non-constant rational functions $\psi_1$ and $\psi_2$ such that

(i) $(\psi_1^2, \psi_2^2) = (\psi_2, \psi_1)$,

(ii) $(\psi_1^2, \psi_2^2) = (\psi_2, 1/\psi_1)$,

(iii) $\psi_1/\psi_2 \notin \mathbb{C}$ and

(iv) $\psi_1 \psi_2 \neq 1$.

**Proof.** By the normal basis theorem (see [5, p.229]), there exists $\theta \in \mathbb{C}(Y)$ such that $\{\theta^{\sigma^f}, \theta^{\sigma^f} \}$ $(f = 0, 1, 2, 3)$ form a basis of $\mathbb{C}(Y)$ as a vector $\mathbb{C}(Z)$-space. Put

$$
\psi_1 = \frac{\theta + \theta^{\sigma} + \theta^{\sigma^3} + \theta^{\sigma^3}}{\theta^{\sigma^2} + \theta^{\sigma^2} + \theta^\tau + \theta^\tau}, \quad \psi_2 = \frac{\theta + \theta^{\sigma} + \theta^\tau + \theta^\tau}{\theta^{\sigma^2} + \theta^{\sigma^2} + \theta^\tau + \theta^\tau}.
$$

Both $\psi_1$ and $\psi_2$ are non-constant rational functions since $\{\theta^g \mid g \in D_8\}$ is a basis over $\mathbb{C}(Z)$, and the statements (i) and (ii) are straightforward. If $\psi_1 = c\psi_2$ for some $c \in \mathbb{C}$, we have $\psi_1 = \pm \psi_2$ by (i). If this happens, then we infer that $\psi_2^2 = \pm 1$ by (ii), but this is impossible as $\psi_2$ is non-constant. Suppose that $\psi_1 \psi_2 = 1$. Then $\psi_2/\psi_1 = 1$ by (i). This contradicts to (iii). \[ \square \]

**Lemma 2.3.** Let $\psi_1$ and $\psi_2$ be the rational functions as in Lemma 2.2. Then $\mathbb{C}(Y) = \mathbb{C}(Z)(\psi_1, \psi_2)$.

**Proof.** Choose a rational number $c$ not equal to $\pm 1$ and we put $\psi = \psi_1 + c\psi_2$. It is enough to see that $\psi \neq \psi^g$ for all $g(\neq 1) \in D_8$.

(i) $\psi \neq \psi^\tau$. If $\psi = \psi^\tau$, we have

$$
\psi_1 - \psi_2 = c \left( \frac{1 - \psi_1 \psi_2}{\psi_1} \right).
$$

By Lemma 2.2, $1 - \psi_1 \psi_2 \neq 0$, $\psi_1 - \psi_2 \neq 0$. So

$$
c = \frac{\psi_1 - \psi_2}{1 - \psi_1 \psi_2} \neq 0,
$$
On the other hand, we have
\[
\left( \frac{\psi_1 - \psi_2}{1 - \psi_1 \psi_2} \right)^\sigma \psi_2 = \frac{\psi_2 - \psi_1}{1 - \psi_1 \psi_2} \psi_2 = -\frac{\psi_1 - \psi_2}{1 - \psi_1 \psi_2} \psi_2.
\]
As \( c^\sigma = c \), we have \( \psi_1 = -\psi_2 \), but this contradicts to Lemma 2.2, (iii).

(ii) \( \psi \neq \psi^{\tau^2} \). If \( \psi = \psi^{\tau^2} \), we have
\[
c = \frac{(\psi_1^2 - 1) \psi_2}{(1 - \psi_2^2) \psi_1}.
\]
By Lemma 2.2, \( c \neq 0 \). On the other hand, we have
\[
\left( \frac{(\psi_1^2 - 1) \psi_2}{(1 - \psi_2^2) \psi_1} \right)^\sigma = \frac{(1 - \psi_2^2) \psi_1}{(\psi_1^2 - 1) \psi_2} = \frac{1}{c}.
\]
As \( c^\sigma = c \), we have \( c^2 = 1 \). This contradicts to our choice of \( c \).

(iii) \( \psi \neq \psi^3 \). If \( \psi = \psi^3 \), we have
\[
c = \frac{\psi_1 \psi_2 - 1}{\psi_1 - \psi_2} \psi_2.
\]
By the similar argument to the first case, we infer \( \psi_1 = -\psi_2 \), but this is impossible.

(iv) \( \psi \neq \psi^\sigma \). If \( \psi = \psi^\sigma \), we have \( c = 1 \), but this contradicts to our choice of \( c \).

(v) \( \psi \neq \psi^{\sigma \tau} \). If \( \psi = \psi^{\sigma \tau} \), \( \psi_1^2 = 1 \). This contradicts to Lemma 2.2.

(vi) \( \psi \neq \psi^{\sigma \tau^2} \). If \( \psi = \psi^{\sigma \tau^2} \), we have \( c = -\psi_1/\psi_2 \). As \( c = c^\sigma \), we have \( \psi_1/\psi_2 = \psi_2/\psi_1 \). This implies that \( \psi_1 = \pm \psi_2 \), but this contradicts to Lemma 2.2 (iii).

(vii) \( \psi \neq \psi^{\sigma \tau^3} \). If \( \psi = \psi^{\sigma \tau^3} \), we have \( \psi_2^2 = 1 \), but this is impossible.

Proof of Proposition 2.1. Let \( \pi : Y \to Z \) be an arbitrary \( D_8 \)-cover. By Lemmas 2.2 and 2.3, there exist non-constant rational functions \( \psi_1 \) and \( \psi_2 \) such that

(i) \( \mathbb{C}(Y) = \mathbb{C}(Z)(\psi_1, \psi_2) \) and

(ii) \( (\psi_1^\sigma, \psi_2^\sigma) = (\psi_2, \psi_1) \) and \( (\psi_1^\tau, \psi_2^\tau) = (\psi_2, 1/\psi_1) \).

Define the \( D_8 \)-equivalent rational map \( \Psi : Y \to \mathbb{P}^1 \times \mathbb{P}^1 \) by
\[
p \in Y \mapsto (\psi_1(p), \psi_2(p)) \in \mathbb{P}^1 \times \mathbb{P}^1.
\]
This shows Proposition 2.1.

Now Theorem 0.2 follows from Proposition 2.1 and Example 1.2. We end this section with the following example.
Example 2.4. Let 

\[ \rho: D_8 \rightarrow \text{GL}(2, \mathbb{C}) \]

be the representation given by

\[ \rho(\sigma) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho(\tau) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \]

Let \( \tilde{\rho} = 1_{D_8} \oplus \rho \), and define the \( D_8 \)-action on \( \mathbb{P}^2 \) by

\[ g([z_0, z_1, z_2]) = [z_0, z_1, z_2](\tilde{\rho}(g))^{-1}, \quad [z_0, z_1, z_2] \in \mathbb{P}^2. \]

Put \( X_1 = \mathbb{P}^2 \) and \( M_1 = \mathbb{P}^2/D_8 \) (note that \( X_1 \to M_1 \) is the versal \( D_8 \)-cover by [12, Proposition 4.1]). Let \( u \) and \( v \) be the rational functions of \( \mathbb{P}^2 \) given by \( z_1/z_0 \) and \( z_2/z_0 \), respectively. Then one can check

\[ \{\theta^g\}_{g \in D_8}, \quad \theta = \frac{1}{1 - u - 2v} \]

form a basis over \( \mathbb{C}(M_1) = \mathbb{C}(u, v)^{D_8} \). To see this, let

\[
A := \begin{pmatrix}
\theta & \theta^\tau & \theta^{\tau^2} & \theta^\sigma & \theta^{\sigma \tau} & \theta^{\sigma \tau^2} & \theta^{\sigma^2} \\
\theta^\tau & \theta^{\tau^2} & \theta^{\sigma \tau} & \theta^\sigma & \theta^{\sigma^2} & \theta^{\sigma \tau^2} & \theta^{\sigma \tau^3} \\
\theta^{\tau^2} & \theta^{\sigma \tau} & \theta^{\sigma \tau^2} & \theta^{\sigma^2} & \theta^{\sigma^3} & \theta^{\sigma \tau^3} & \theta^{\sigma \tau^4} \\
\theta^{\sigma} & \theta^{\sigma \tau} & \theta^{\sigma \tau^2} & \theta^{\sigma \tau^3} & \theta^{\sigma^2} & \theta^{\sigma^3} & \theta^{\sigma^4} \\
\theta^{\sigma \tau} & \theta^{\sigma \tau^2} & \theta^{\sigma \tau^3} & \theta^{\sigma^2} & \theta^{\sigma^3} & \theta^{\sigma^4} & \theta^{\sigma^5} \\
\theta^{\sigma^2} & \theta^{\sigma^3} & \theta^{\sigma^4} & \theta^{\sigma^5} & \theta^{\sigma^6} & \theta^{\sigma^7} & \theta^{\sigma^8} \\
\theta^{\sigma^3} & \theta^{\sigma^4} & \theta^{\sigma^5} & \theta^{\sigma^6} & \theta^{\sigma^7} & \theta^{\sigma^8} & \theta^{\sigma^9} \\
\theta^{\sigma^4} & \theta^{\sigma^5} & \theta^{\sigma^6} & \theta^{\sigma^7} & \theta^{\sigma^8} & \theta^{\sigma^9} & \theta^{\sigma^{10}} 
\end{pmatrix},
\]

and check that \( \det A \neq 0 \). The explicit forms of \( \psi_1 \) and \( \psi_2 \) with respect to the normal basis \( \{\theta^g\}_{g \in D_8} \) are as follows:

\[
\psi_1 = -\frac{(-2 + 6u^2 + 9u - 9uv^2 - 13u^2 + 5v^2)}{(2 + 6u^2 + 9u - 9uv^2 + 13u^2 - 5v^2)} \times \frac{(1 + u + 2v)(1 + 2u + v)(1 + 2u - v)(1 + u - 2v)}{(-1 + u + 2v)(-1 + 2u + v)(-1 + v + 2u)(-1 + u - 2v)}
\]

\[
\psi_2 = -\frac{(-2 + 9u^2v - 5u^2 - 6u^3 + 13u^2 - 9v)}{(2 + 9u^2v - 5u^2 - 6u^3 + 13u^2 - 9v)} \times \frac{(1 + u + 2v)(1 + 2u + v)(-1 + v + 2u)(-1 + u - 2v)}{(-1 + u + 2v)(-1 + 2u + v)(1 + 2u - v)(1 + u - 2v)}
\]
Hence we have a $D_8$-equivalent rational map from $\mathbb{P}^2 \to \mathbb{P}^1 \times \mathbb{P}^1$. Note that the existence of this rational map gives another proof for Proposition 2.1.

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