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NOTE ON A 2-DIMENSIONAL VERSAL D_8-COVER

Dedicated to Professor Makoto Namba on his sixtieth birthday

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(Received April 9, 2003)

Introduction

Let G be a finite group. Let X and Y be normal projective varieties. X is called a G-cover of Y if there exists a finite surjective morphism $\pi\colon X \to Y$ such that the induced morphism $\pi^*\colon \mathbb{C}(Y) \to \mathbb{C}(X)$ gives a Galois extension with $\mathrm{Gal}(\mathbb{C}(X)/\mathbb{C}(Y)) \cong G$, where $\mathbb{C}(X)$ and $\mathbb{C}(Y)$ denote the rational function fields of X and Y, respectively.

DEFINITION 0.1. A G-cover $\varpi\colon X \to M$ is said to be versal if it satisfies the following property:

For any G-cover $\pi\colon Y \to Z$, there exist a rational map $\nu\colon Z \cdots \to M$ and a Zariski open set U in Z such that

(i) $\nu|_U : U \to M$ is a morphism, and

(ii) $\pi^{-1}(U)$ is birational to $U \times_M X$ over U.

One could say the investigation of versal G-covers is a geometric study of generic or versal G-polynomials (see [2] and [4]). The notion of versal G-covers implicitly appeared in [7], [8], and is defined explicitly in [12], [13]. It is known that there exists a versal G-cover for any finite group G ([8, Theorem 2.4]). The dimension of the versal G-cover given by Namba, however, is equal to $\sharp(G)$. Hence it does not seem to be tractable in practical use. We need to find a tractable model for G. So far it has been done for some cases by ad-hoc methods in ([12], [13]).

In this note, we consider versal D_8-covers, where D_8 is the dihedral group of order 8, i.e., $D_8 = \langle \sigma, \tau \mid \sigma^2 = \tau^4 = (\sigma\tau)^2 = 1\rangle$. It is known that any versal D_8-cover has dimension at least 2 (see [2]), and one of such models was given in [12]. The purpose of this article is to give another new model, which is described as follows:

Let $\varphi_{141}\colon X_{141} \to \mathbb{P}^1$ be the rational elliptic surface obtained by blowing-up base

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points of the (2, 2)-pencil on $\mathbb{P}^1 \times \mathbb{P}^1$ given by

$$\left\{\lambda_0(s_0 - s_1)^2(t_0 - t_1)^2 + \lambda_1(s_0 s_1 t_0 t_1) = 0\right\}_{[\lambda_0:\lambda_1]\in\mathbb{P}^1}.$$

As for X_{141}, the following facts are well-known:

(i) X_{141} is so-called the elliptic modular surface attached to

$$\Gamma_1(4) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod 4 \right\},$$

and φ_{141} has three singular fibers and their types are of I_1^*, I_4 and I_1 (see [3, p.350]).

(ii) The group of sections, $MW(X_{141})$, is isomorphic to $\mathbb{Z}/4\mathbb{Z}$ (see [6] or [9]).

Let $\sigma_{\varphi_{141}}$ be the involution on X_{141} induced by the inversion with respect to the group law and let τ_s be the translation by a 4-torsion section s. $\sigma_{\varphi_{141}}$ and τ_s generate a finite fiber preserving automorphism group isomorphic to D_8. Let $\Sigma_{141} := X_{141}/\langle\sigma_{\varphi_{141}}, \tau_s\rangle$ be the quotient surface and we denote its quotient morphism by $\pi_{141}\colon X_{141} \to \Sigma_{141}$. Now we are in position to state our result:

Theorem 0.2. *$\pi_{141}\colon X_{141} \to \Sigma_{141}$ is a versal D_8-cover.*

In comparison with the model in [12], this model has a nice description with respect to the action of the Galois group.

1. Preliminaries

Let S be a smooth minimal projective surface, and let Λ be a pencil of curves on S such that a general member is irreducible. Let $\bar{\varphi}_\Lambda\colon S \cdots\to \mathbb{P}^1$ be the rational map determined by Λ, and let $q\colon S_\Lambda \to S$ be the resolution of the indeterminacy of $\bar{\varphi}_\Lambda$. We denote the induced morphism from S_Λ to $\mathbb{P}^1$ by φ_Λ. We may assume that φ_Λ is relatively minimal. Let us begin with the following lemma.

Lemma 1.1. *Let σ be an automorphism of S. Suppose that $\bar{\varphi}_\Lambda^\sigma = \bar{\varphi}_\Lambda$ (we regard $\bar{\varphi}_\Lambda$ as an element of $\mathbb{C}(S_\Lambda)$). Then σ gives rise to a fiber preserving automorphism of S_Λ (By abuse of notation, we also denote it by σ).*

Proof. Since $\mathbb{C}(S) \cong \mathbb{C}(S_\Lambda)$, $\bar{\varphi}_\Lambda^\sigma = \bar{\varphi}_\Lambda$ implies $\varphi_\Lambda^\sigma = \varphi_\Lambda$. Hence a general fiber of φ_Λ goes to that of φ_Λ under σ. Let $\bar{\sigma}$ be the induced birational map from S_Λ to itself induced by σ. Let $\mu\colon \hat{S}_\Lambda \to S_\Lambda$ be a succession of blowing-ups so that (i) $\hat{\sigma} := \bar{\sigma}\circ\mu$ becomes a birational morphism and (ii) the number of the blowing-ups is minimal. It is well-known that $\hat{\sigma}$ is a composition of blowing-downs. Let $E_1, \ldots, E_r$ be the exceptional divisors for μ, and let $F_1, \ldots, F_s$ be those for $\hat{\sigma}$. We may assume that F_1 is the (-1) curve for the first blow-down. Since F_1 can not be any of E_i

$(i = 1, \dots, r)$, $F_1 E_i \geq 0$ $(i = 1, \dots, r)$. Since

$$-1 = F_1 K_{\hat{S}_\Lambda} = F_1 \left(\mu^* K_{S_\Lambda} + \sum_{j=1}^{r} m_j E_j \right) = F_1 \mu^* K_{S_\Lambda} + F_1 \left(\sum_{j=1}^{r} m_j Ej \right),$$

we have $F_1 \mu^* K_{S_\Lambda} \leq -1$. Hence $\mu(F_1)$ is not any irreducible component in a fiber of φ_Λ. Since the image of $\mu(F_1)$ by σ is a point, this implies that a general fiber of φ_Λ does not go to that of φ_Λ under $\hat{\sigma}$, which leads us to a contradiction. □

EXAMPLE 1.2. Let (x, y) be an inhomogeneous coordinate of $\mathbb{P}^1 \times \mathbb{P}^1$. The pencil on $\mathbb{P}^1 \times \mathbb{P}^1$ in Introduction is given by

$$\Lambda : \left\{ \lambda_0 (x-1)^2 (y-1)^2 + \lambda_1 xy = 0 \right\}_{[\lambda_0, \lambda_1] \in \mathbb{P}^1}.$$

Let σ and τ be the automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$ given by

$$(x^\sigma, y^\sigma) = (y, x), \ (x^\tau, y^\tau) = \left(y, \frac{1}{x} \right).$$

σ and τ generate a finite automorphism group isomorphic to D_8. Since the rational function on $\mathbb{P}^1 \times \mathbb{P}^1$,

$$\frac{(x-1)^2 (y-1)^2}{xy},$$

is invariant under σ and τ, one can apply Lemma 1.1 to this case. In fact, it follows that $(\mathbb{P}^1 \times \mathbb{P}^1)_\Lambda = X_{141}$, $\varphi_\Lambda = \varphi_{141}$, and σ, τ fiber preserving automorphisms of X_{141}. By Example 4.7, Chapter III in [11], both σ and τ are the compositions of isogenies and translations. In particular, the corresponding isogenies should be isomorphisms as an elliptic curve. By Theorem 10.1, Chapter III in [11], the automorphisms of X_{141} as an elliptic curve are only the identity and $\sigma_{\varphi_{141}}$. Since $\sigma \circ \tau$ has a fixed point $(1, 0)$, the induced fiber preserving automorphism on X_{141} is non-trivial and fixes a section, O. Hence we may assume that $\sigma \circ \tau = \sigma_{\varphi_{141}}$ on X_{141}, by regarding O as the zero. Also if τ is the composition of $\sigma_{\varphi_{141}}$ and a translation, then $\tau^2 = \mathrm{id}$ on X_{141}. This implies that we may assume that τ is a translation by a 4-torsion section s.

EXAMPLE 1.3. Let $N = \mathbb{Z}^{\oplus 2}$ and let Δ_i $(i = 1, \dots, 6)$ be 2-dimensional cones in $N \otimes \mathbb{R}$ given by

$$\Delta_1 = \mathbb{R}_{\geq 0} \boldsymbol{e}_1 + \mathbb{R}_{\geq 0} (\boldsymbol{e}_1 + \boldsymbol{e}_2) \qquad \Delta_2 = \mathbb{R}_{\geq 0} (\boldsymbol{e}_1 + \boldsymbol{e}_2) + \mathbb{R}_{\geq 0} \boldsymbol{e}_2$$
$$\Delta_3 = \mathbb{R}_{\geq 0} (-\boldsymbol{e}_1) + \mathbb{R}_{\geq 0} \boldsymbol{e}_2 \qquad \Delta_4 = \mathbb{R}_{\geq 0} (-\boldsymbol{e}_1) + \mathbb{R}_{\geq 0} (-\boldsymbol{e}_1 - \boldsymbol{e}_2)$$
$$\Delta_5 = \mathbb{R}_{\geq 0} (-\boldsymbol{e}_1 - \boldsymbol{e}_2) + \mathbb{R}_{\geq 0} (-\boldsymbol{e}_2) \qquad \Delta_6 = \mathbb{R}_{\geq 0} (-\boldsymbol{e}_2) + \mathbb{R}_{\geq 0} \boldsymbol{e}_1,$$

where $\boldsymbol{e}_1 = {}^t(1,0)$, $\boldsymbol{e}_2 = {}^t(0,1)$. Let $\Sigma = \bigcup_{i=1}^{6} \Delta_i$ and $X_{D_{12}} = T_N\,\mathrm{emb}(\Sigma)$ (see [10, §1.2]). The subgroup, G, of $\mathrm{GL}(2,\mathbb{Z})$ generated by

$$\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

is isomorphic to D_{12}. Since G preserves Σ as above, D_{12} acts on $X_{D_{12}}$. An explicit description is as follows:

Let $\{f_1, f_2\}$ be the basis of $\mathrm{Hom}(N,\mathbb{Z})$ dual to $\{\boldsymbol{e}_1, \boldsymbol{e}_1 + \boldsymbol{e}_2\}$ and let $x = \boldsymbol{e}(f_1)$ and $y = \boldsymbol{e}(f_2)$. (x, y) gives an affine coordinate of the affine open set, U_{Δ_1}, determined by Δ_1. Then

$$(x^\sigma, y^\sigma) = (y, x), \quad (x^\tau, y^\tau) = \left(\frac{1}{y}, xy\right).$$

Consider the pencil on X which is given on U_{Δ_1} by

$$\{\lambda_0(xy) + \lambda_1(xy+1)(x+1)(y+1) = 0\}_{[\lambda_0,\lambda_1]\in\mathbb{P}^1}$$

on U_{σ_1}. Since

$$\frac{xy}{(xy+1)(x+1)(y+1)}$$

is D_{12}-invariant, one can apply Lemma 1.1 to this case. In this case, X_Λ coincides with the rational elliptic surface $\varphi_{6321}\colon X_{6321} \to \mathbb{P}^1$ (the notation is due to [6]). The following facts on X_{6321} are well-known:

(i) X_{6321} is so-called the elliptic modular surface attached to

$$\Gamma_1(6) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod 6 \right\},$$

and has four singular fibers and their types are of I_6, I_3, I_2 and I_1 (see [1] and [6]).

(ii) The group of sections, $MW(X_{6321})$, is isomorphic to $\mathbb{Z}/6\mathbb{Z}$ (see [6]).

In our case, $\varphi_\Lambda = \varphi_{6321}$, and σ and τ generate a fiber preserving automorphism group of X_{6321} isomorphic to D_{12}. By the same argument to that in Example 1.2, we may assume that it coincides with one given by the inversion with respect to the group law and the translation by a 6-torsion section.

We here raise a question concerning Example 1.3.

Question 1.4. Let $M_{D_{12}}$ be the quotient of $X_{D_{12}}$ with respect to the D_{12}-action in Example 1.3. Is the D_{12}-cover $X_{D_{12}} \to M_{D_{12}}$ versal? In other words, is $X_{6321} \to \Sigma_{6321}$ a versal D_{12}-cover, where Σ_{6321} is the quotient by the inversion with respect to the group law and the translation by 6-torsion?

2. Proof of Theorem 0.2

Let σ and τ be the automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$ as in Example 1.2. They give rise to a finite automorphism group isomorphic to D_8. Put $\Sigma_{D_8} := \mathbb{P}^1 \times \mathbb{P}^1/\langle \sigma, \tau \rangle$, and we denote the quotient morphism by $\varpi_{D_8} \colon \mathbb{P}^1 \times \mathbb{P}^1 \to \Sigma_{D_8}$. Theorem 0.2 follows from Example 1.2 and the proposition below.

Proposition 2.1. *$\varpi_{D_8} \colon \mathbb{P}^1 \times \mathbb{P}^1 \to \Sigma_{D_8}$ is versal.*

We need two lemmas to prove Proposition 2.1.

Lemma 2.2. *Let $\pi \colon Y \to Z$ be a D_8-cover. Then there exist non-constant rational functions ψ_1 and ψ_2 such that*
(i) $(\psi_1^\sigma, \psi_2^\sigma) = (\psi_2, \psi_1)$,
(ii) $(\psi_1^\tau, \psi_2^\tau) = (\psi_2, 1/\psi_1)$,
(iii) $\psi_1/\psi_2 \notin \mathbb{C}$ *and*
(iv) $\psi_1\psi_2 \neq 1$.

Proof. By the normal basis theorem (see [5, p.229]), there exists $\theta \in \mathbb{C}(Y)$ such that $\{\theta^{\tau^i}, \theta^{\sigma\tau^i}\}$ $(i = 0, 1, 2, 3)$ form a basis of $\mathbb{C}(Y)$ as a vector $\mathbb{C}(Z)$-space. Put

$$\psi_1 = \frac{\theta + \theta^\sigma + \theta^{\tau^3} + \theta^{\sigma\tau^3}}{\theta^{\tau^2} + \theta^{\sigma\tau^2} + \theta^\tau + \theta^{\sigma\tau}}, \quad \psi_2 = \frac{\theta + \theta^\sigma + \theta^\tau + \theta^{\sigma\tau}}{\theta^{\tau^2} + \theta^{\sigma\tau^2} + \theta^{\tau^3} + \theta^{\sigma\tau^3}}.$$

Both ψ_1 and ψ_2 are non-constant rational functions since $\{\theta^g \mid g \in D_8\}$ is a basis over $\mathbb{C}(Z)$, and the statements (i) and (ii) are straightforward. If $\psi_1 = c\psi_2$ for some $c \in \mathbb{C}$, we have $\psi_1 = \pm\psi_2$ by (i). If this happens, then we infer that $\psi_2^2 = \pm 1$ by (ii), but this is impossible as ψ_2 is non-constant. Suppose that $\psi_1\psi_2 = 1$. Then $\psi_2/\psi_1 = 1$ by (i). This contradicits to (iii). □

Lemma 2.3. *Let ψ_1 and ψ_2 be the rational functions as in* Lemma 2.2. *Then* $\mathbb{C}(Y) = \mathbb{C}(Z)(\psi_1, \psi_2)$.

Proof. Choose a rational number c not equal to ± 1 and we put $\psi = \psi_1 + c\psi_2$. It is enough to see that $\psi \neq \psi^g$ for all $g(\neq 1) \in D_8$.
(i) $\psi \neq \psi^\tau$. If $\psi = \psi^\tau$, we have

$$\psi_1 - \psi_2 = c\left(\frac{1 - \psi_1\psi_2}{\psi_1}\right).$$

By Lemma 2.2, $1 - \psi_1\psi_2 \neq 0$, $\psi_1 - \psi_2 \neq 0$. So

$$c = \frac{\psi_1 - \psi_2}{1 - \psi_1\psi_2}\psi_1 \neq 0.$$

On the other hand, we have

$$\left(\frac{\psi_1-\psi_2}{1-\psi_1\psi_2}\psi_1\right)^{\sigma}=\frac{\psi_2-\psi_1}{1-\psi_1\psi_2}\psi_2=-\frac{\psi_1-\psi_2}{1-\psi_1\psi_2}\psi_2.$$

As $c^{\sigma}=c$, we have $\psi_1=-\psi_2$, but this contradicts to Lemma 2.2, (iii).
(ii) $\psi\neq\psi^{\tau^2}$. If $\psi=\psi^{\tau^2}$, we have

$$c=\frac{(\psi_1^2-1)\psi_2}{(1-\psi_2^2)\psi_1}.$$

By Lemma 2.2, $c\neq 0$. On the other hand, we have

$$\left(\frac{(\psi_1^2-1)\psi_2}{(1-\psi_2^2)\psi_1}\right)^{\sigma}=\frac{(1-\psi_2^2)\psi_1}{(\psi_1^2-1)\psi_2}=\frac{1}{c}.$$

As $c^{\sigma}=c$, we have $c^2=1$. This contradicts to our choice of c.
(iii) $\psi\neq\psi^{\tau^3}$. If $\psi=\psi^{\tau^3}$, we have

$$c=\frac{\psi_1\psi_2-1}{\psi_1-\psi_2}\frac{1}{\psi_2}.$$

By the similar argument to the first case, we infer $\psi_1=-\psi_2$, but this is impossible.
(iv) $\psi\neq\psi^{\sigma}$. If $\psi=\psi^{\sigma}$, we have $c=1$, but this contradicts to our choice of c.
(v) $\psi\neq\psi^{\sigma\tau}$. If $\psi=\psi^{\sigma\tau}$, $\psi_1^2=1$. This contradicts to Lemma 2.2.
(vi) $\psi\neq\psi^{\sigma\tau^2}$. If $\psi=\psi^{\sigma\tau^2}$, we have $c=-\psi_1/\psi_2$. As $c=c^{\sigma}$, we have $\psi_1/\psi_2=\psi_2/\psi_1$. This implies that $\psi_1=\pm\psi_2$, but this contradicts to Lemma 2.2 (iii).
(vii) $\psi\neq\psi^{\sigma\tau^3}$. If $\psi=\psi^{\sigma\tau^3}$, we have $\psi_2^2=1$, but this is impossible. □

Proof of Proposition 2.1. Let $\pi\colon Y\to Z$ be an arbitrary D_8-cover. By Lemmas 2.2 and 2.3, there exist non-constant rational functions ψ_1 and ψ_2 such that (i) $\mathbb{C}(Y)=\mathbb{C}(Z)(\psi_1,\psi_2)$ and (ii) $(\psi_1^{\sigma},\psi_2^{\sigma})=(\psi_2,\psi_1)$ and $(\psi_1^{\tau},\psi_2^{\tau})=(\psi_2,1/\psi_1)$. Define the D_8-equivalent rational map $\Psi\colon Y\cdots\to\mathbb{P}^1\times\mathbb{P}^1$ by

$$p\in Y\mapsto(\psi_1(p),\psi_2(p))\in\mathbb{P}^1\times\mathbb{P}^1.$$

This shows Proposition 2.1. □

Now Theorem 0.2 follows from Proposition 2.1 and Example 1.2. We end this section with the following example.

EXAMPLE 2.4. Let

$$\rho\colon D_8 \to \mathrm{GL}(2, \mathbb{C})$$

be the representation given by

$$\rho(\sigma) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho(\tau) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let $\tilde{\rho} = 1_{D_8} \oplus \rho$, and define the D_8-action on $\mathbb{P}^2$ by

$$g([z_0, z_1, z_2]) = [z_0, z_1, z_2](\tilde{\rho}(g))^{-1}, \quad [z_0, z_1, z_2] \in \mathbb{P}^2.$$

Put $X_1 = \mathbb{P}^2$ and $M_1 = \mathbb{P}^2/D_8$ (note that $X_1 \to M_1$ is the versal D_8-cover by [12, Proposition 4.1]). Let u and v be the rational functions of $\mathbb{P}^2$ given by z_1/z_0 and z_2/z_0, respectively. Then one can check

$$\{\theta^g\}_{g \in D_8}, \quad \theta = \frac{1}{1 - u - 2v}$$

form a basis over $\mathbb{C}(M_1) = \mathbb{C}(u, v)^{D_8}$. To see this, let

$$A := \begin{pmatrix}
\theta & \theta^{\tau} & \theta^{\tau^2} & \theta^{\tau^3} & \theta^{\sigma} & \theta^{\sigma\tau} & \theta^{\sigma\tau^2} & \theta^{\sigma\tau^3} \\
\theta^{\tau} & \theta^{\tau^2} & \theta^{\tau^3} & \theta & \theta^{\sigma\tau^3} & \theta^{\sigma} & \theta^{\sigma\tau} & \theta^{\sigma\tau^2} \\
\theta^{\tau^2} & \theta^{\tau^3} & \theta & \theta^{\tau} & \theta^{\sigma\tau^2} & \theta^{\sigma\tau^3} & \theta^{\sigma} & \theta^{\sigma\tau} \\
\theta^{\tau^3} & \theta & \theta^{\tau} & \theta^{\tau^2} & \theta^{\sigma\tau} & \theta^{\sigma\tau^2} & \theta^{\sigma\tau^3} & \theta^{\sigma} \\
\theta^{\sigma} & \theta^{\sigma\tau} & \theta^{\sigma\tau^2} & \theta^{\sigma\tau^3} & \theta & \theta^{\tau} & \theta^{\tau^2} & \theta^{\tau^3} \\
\theta^{\sigma\tau} & \theta^{\sigma\tau^2} & \theta^{\sigma\tau^3} & \theta^{\sigma} & \theta & \theta^{\tau} & \theta^{\tau^2} & \theta^{\tau^3} \\
\theta^{\sigma\tau^2} & \theta^{\sigma\tau^3} & \theta^{\sigma} & \theta^{\sigma\tau} & \theta^{\tau^2} & \theta^{\tau^3} & \theta & \theta^{\tau} \\
\theta^{\sigma\tau^3} & \theta^{\sigma} & \theta^{\sigma\tau} & \theta^{\sigma\tau^2} & \theta^{\tau} & \theta^{\tau^2} & \theta^{\tau^3} & \theta
\end{pmatrix},$$

and check that $\det A \neq 0$. The explicit forms of ψ_1 and ψ_2 with respect to the normal basis $\{\theta^g\}_{g \in D_8}$ are as follows:

$$\begin{aligned}
\psi_1 &= -\frac{(-2 + 6u^3 + 9u - 9uv^2 - 13u^2 + 5v^2)}{(2 + 6u^3 + 9u - 9uv^2 + 13u^2 - 5v^2)} \\
&\quad \times \frac{(1 + u + 2v)(1 + 2u + v)(1 + 2u - v)(1 + u - 2v)}{(-1 + u + 2v)(-1 + 2u + v)(-1 - v + 2u)(-1 + u - 2v)} \\
\psi_2 &= -\frac{(2 + 9u^2 v - 5u^2 - 6v^3 + 13v^2 - 9v)}{(-2 + 9u^2 v + 5u^2 - 6v^3 - 13v^2 - 9v)} \\
&\quad \times \frac{(1 + u + 2v)(1 + 2u + v)(-1 - v + 2u)(-1 + u - 2v)}{(-1 + u + 2v)(-1 + 2u + v)(1 + 2u - v)(1 + u - 2v)}
\end{aligned}$$

Hence we have a D_8-equivalent rational map from $\mathbb{P}^2 \to \mathbb{P}^1 \times \mathbb{P}^1$. Note that the existence of this rational map gives another proof for Proposition 2.1.

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