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LOCALIZATION OF EILENBERG-MACLANE G-SPACES
WITH RESPECT TO HOMOLOGY THEORY

ZEN-ICHI YOSIMURA

(Received November 17, 1981)

In [11] and [12] May and others have constructed and have characterized
equivariant localizations and completions of G-nilpotent G-spaces when G is
a compact Lie group. Let J be a set of primes and X be a based G-nilpotent
G-space. Then the equivariant localization λ: X → X_J is characterized by the
universal property that the H-fixed point space X^H_J is J-local for each closed
subgroup H of G and λ*: [X_J, Y_J]_G → [X, Y]_G is a bijection for any based G-
nilpotent G-space Y with Y^H_J J-local, and it is constructed as a based G-map
whose restriction to H-fixed point spaces is a J-localization in the non-equiv-
ariant sense. The equivariant completion γ: X → X_J is similarly characterized
and constructed.

According to Bousfield [3], each non-equivariant homology theory h_*
determines h_*-localizations of based CW-complexes. Special cases of the h_*-
localization η: X → Lh_*X are familiar if a based CW-complex X is nilpotent.
Taking H*( ; Z[J^{-1}]) as h_*, then the H*( ; Z[J^{-1}])-localization is the usual
J-localization where J^c denotes the complement of the set J. Taking
H*( ; ∐_{p ∈ J} Z[p]), then the H*( ; ∐_{p ∈ J} Z[p])-localization is the usual J-localization.

In this paper we study a localization η: X → Lh_*X = LX of a based G-
CW complex X such that its H-fixed point map η^H*: X^H* → (LX)^H* is an h_*-localiza-
tion for any closed subgroup H of G. This localization is characterized by the
universal property that η^H*: h_*(X^H) → h_*(LX)^H* is an isomorphism for each H,
and for any based G-map f: X → Y inducing isomorphisms f^H*: h_*(X^H) → h_*(Y^H)
there is a unique based G-map r: Y → LX with r ⋅ f = η ∈ [X, LX]_G.

First we investigate some relations between Bredon homology and co-
homology theories and ordinary homology and cohomology theories in § 1,
following Wilson [16]. Given a homology theory h_* and a family \( \mathcal{A}_0 = \{ A_0 \} \)
of abelian groups we define our localization in a general form in § 2, and prove
the existence theorem of our localizations in Appendix using the technique
developed by Bousfield. In § 3 we treat the special case that h_* is the or-
dinary homology theory H_*.

We proceed cocellularity the construction of the localizations of G-nilpotent G-CW complexes so that we obtain our main result (Theorem 3.3), which may be a slight generalization of main results in [11]
and [12]. Finally we compute the localizations of Eilenberg-MacLane $G$-spaces $K(N, n)$ with respect to the complex homology $K$-theory $K_\ast$ (Theorem 4.5) in § 4, as Mislin [14] did in the non-equivariant case.

1. Bredon homology and cohomology

Let $G$ be a compact Lie group and $\mathcal{C}_G$ be the category of pairs $(X, Y)$ of $G$-CW complexes, where $Y \subset X$, and $G$-maps $f: (X, Y) \to (X', Y')$. A covariant coefficient system $M$ for $G$ is a covariant functor from the category of left homogeneous spaces $G/H$ by closed subgroups $H$ and $G$-homotopy classes of $G$-maps to the category of abelian groups. A contravariant coefficient system $N$ for $G$ is similarly defined.

Bredon homology and cohomology theories written as $cH_\ast(X, Y; M)$ and $cH^\ast(X, Y; N)$ are $\mathbb{Z}$-graded equivariant homology and cohomology theories with coefficients in the covariant coefficient system $M$ and the contravariant one $N$ respectively defined on the category $\mathcal{C}_G$, both of which satisfy the dimension axiom. For any coefficient system $M$ or $N$ for $G$ and each closed subgroup $K$ of $G$, denote by $i^K_\ast M$ or $i^K_\ast N$ the induced coefficient system for $K$ which assigns to each $K/H$ the abelian group $M(G/H)$ or $N(G/H)$ respectively. The composites

$$cH_\ast(X, Y; i^K_\ast M) \to cH_\ast(G^K X, G^K Y; i^K_\ast M) \to cH_\ast(G^K X, G^K Y; M)$$

and

$$cH^\ast(G^K X, G^K Y; N) \to cH^\ast(G^K X, G^K Y; i^K_\ast N) \to cH^\ast(X, Y; i^K_\ast N)$$

are isomorphisms for every closed subgroup $K$ of $G$ and $(X, Y) \in \mathcal{C}_G$.

We now recall the useful notion in interpreting coefficient systems for $G$, introduced by Wilson [16, § 4]. Let $C(G)$ be a collection of closed subgroups of $G$ which contains precisely one subgroup from every conjugacy class of closed subgroups of $G$ and fix it. We have a partial ordering on $C(G)$, namely $H \leq K$ if and only if $H$ is subconjugate to $K$. The isotropy ring $I_G$ is defined to be the free abelian group on the set of $G$-homotopy classes of $G$-maps from $G/H$ to $G/K$ for all pairs $H \leq K$ in $C(G)$, whose ring structure is imposed by compositions of $G$-maps. When $G$ is a finite group, the ring $I_G$ has the multiplicative unit $1 = \sum_{H \in C(G)} 1_{G/H}$ where $1_{G/H}$ denotes the identity map on $G/H$. However we notice that the ring $I_G$ has in general no multiplicative unit.

We call a left $I_G$-module an abelian group $M$ together with a structure map $\Phi: I_G \times M \to M$ written as $\Phi(\lambda, x) = \lambda x$ satisfying the condition that $M \cong \bigoplus_{H \in C(G)} 1_{G/H} M$ in place of the unitary property in the usual case. A right $I_G$-module is similarly treated. According to [16, Theorem 5.1] there is a one to one correspondence between covariant coefficient systems and left $I_G$-modules and analogously between contravariant coefficient systems and right $I_G$-modules.
Write $I = I_G$ for short. Given any abelian group $A$ and each $H \in C(G)$, the abelian group $I_{1/G/H} \otimes A = \bigoplus_{K \in C(G)} I_{1/G/K} I_{1/G/H} \otimes A$ is a left $I$-module, and the abelian group $\bigoplus_{K \in C(G)} \text{Hom}(I_{1/G/K} I_{1/G/H}, A)$ denoted by $\text{Hom}(I_{1/G/H}, A)$, is a right $I$-module. The following identifications of Bredon homology and cohomology with ordinary homology and cohomology were given implicitly in [16, Theorem 7.3].

**Lemma 1.1.** Let $G$ be a compact Lie group, $H$ a closed subgroup of $G$ contained in $C(G)$ and $A$ be any abelian group. Then for any pair $(X, Y)$ of $G$-CW complexes we have natural isomorphisms
\begin{enumerate}
  \item $\mathcal{H}_*(X, Y; I_{1/G/H} \otimes A) \cong \mathcal{H}_*(X^H/W(H), Y^H/W(H)\;A)$
  \item $\mathcal{H}^*(X, Y; \text{Hom}(I_{1/G/H}, A)) \cong \mathcal{H}^*(X^H/W(H), Y^H/W(H)\;A)$
\end{enumerate}
where $W(H)$ denotes the identity component of the Weyl group $N(H)/H$ and $X^H/W(H)$ denotes the orbit space of the $H$-fixed point space $X^H$ by $W(H)$.

Let $M$ be a left $I$-module. Denote by $I(M)$ the left $I$-module defined to be
\[ I(M) = \bigoplus_{H \in C(G)} I_{1/G/H} \otimes 1_{G/H} M. \]
The map $\pi: I(M) = \bigoplus_{H \in C(G)} I_{1/G/H} \otimes 1_{G/H} M \to M$ given by $\pi(\lambda 1_{G/H} \otimes 1_{G/H} x) = \lambda 1_{G/H} x$ is a homomorphism of left $I$-modules, which is obviously epic. Let $N$ be a right $I$-module. Denote by $I(N)$ the right $I$-module defined to be
\[ I(N) = \bigoplus_{K \in C(G)} \Pi_{H \in C(G)} \text{Hom}(I_{1/G/K} I_{1/G/H}, N_{1/G/H}). \]
The map $i: N = \bigoplus_{K \in C(G)} N_{1/G/K} \to I(N) = \bigoplus_{K \in C(G)} \Pi_{H \in C(G)} \text{Hom}(I_{1/G/K} I_{1/G/H}, N_{1/G/H})$ given by $i(x 1_{1/G/K}) 1_{1/G/K} \lambda 1_{1/G/H} = x 1_{1/G/K} \lambda 1_{1/G/H}$ is a homomorphism of right $I$-modules and it is monic.

**Corollary 1.2.** Let $G$ be a compact Lie group, $M$ be a left $I$-module and $N$ a right $I$-module. Then for any pair $(X, Y)$ of $G$-CW complexes we have natural isomorphisms
\begin{enumerate}
  \item $\mathcal{H}_*(X, Y; I(M)) \cong \bigoplus_{H \in C(G)} \mathcal{H}_*(X^H/W(H), Y^H/W(H)\;1_{G/H} M)$
  \item $\mathcal{H}^*(X, Y; I(N)) \cong \Pi_{H \in C(G)} \mathcal{H}^*(X^H/W(H), Y^H/W(H)\;N_{1/G/H}).$
\end{enumerate}

By means of Corollary 1.2 we show

**Proposition 1.3.** Let $G$ be a compact Lie group, $f: (X, Y) \to (X', Y')$ be a $G$-map of pairs of $G$-CW complexes and $n \geq 0$.
\begin{enumerate}
  \item Let $M$ be a left $I$-module. If $f^* : H_i(X^K/W(K), Y^K/W(K)\;1_{G/H} M) \to H_i(X'^K/W(K), Y'^K/W(K)\;1_{G/H} M)$ is an isomorphism for each $i \leq n$ and every pair $H \leq K$ in $C(G)$, then $f^* : \mathcal{H}_i(X, Y; M) \to \mathcal{H}_i(X', Y'; M)$ is an isomorphism for each $i \leq n$.  
  \item Let $N$ be a right $I$-module. If $f^* : H^i(X^K/W(K), Y^K/W(K)\;N_{1/G/H}) \to$
\[ H^i(X^K \mid W(K))_0, \ Y^K \mid W(K)_0; \ N_1_{G/H} \] is an isomorphism for each \( i \leq n \) and every pair \( H \leq K \) in \( C(G) \), then \( f^* : \_ {\_} H^i(X', Y'; N) \to \_ {\_} H^i(X, Y; N) \) is an isomorphism for each \( i \leq n \).

**Proof.** i) Consider the exact sequence \( 0 \to M_1 \to I(M) \to M \to 0 \) of left \(_I\)-modules. Since the exact sequence \[ 0 \to 1_{G/H} M_1 \to 1_{G/H} I(M) = \bigoplus L 1_{G/L} I_{G/L} \to 1_{G/H} M \to 0 \] is split as abelian groups, our assumption is maintained for \( M_1 \) as well as \( M \).

By induction on \( i \), \( 0 \leq i \leq n \), we will show that \( f^* : \_ {\_} H^i(X, Y; M) \to \_ {\_} H^i(X', Y'; M) \) is an isomorphism. By Corollary 1.2 our assumption implies that \( f^* : \_ {\_} H^i(X, Y; I(M)) \to \_ {\_} H^i(X', Y'; I(M)) \) is an isomorphism for each \( i \leq n \). Using induction hypothesis and the weak four lemma we first verify that \( f^* : \_ {\_} H^i(X, Y; M) \to \_ {\_} H^i(X', Y'; M) \) is epic and hence \( f^* : \_ {\_} H^i(X, Y; M) \to \_ {\_} H^i(X', Y'; M) \) is epic, too. Using again induction hypothesis and the weak four lemma we next see that \( f^* : \_ {\_} H^i(X, Y; M) \to \_ {\_} H^i(X', Y'; M) \) is monic.

The case ii) is analogously shown, considering the exact sequence \( 0 \to N \to I(N) \to N_1 \to 0 \) of right \(_I\)-modules.

Let \( \_ {\_} h^* \) and \( \_ {\_} h^* \) be RO\(_G\)-graded (or Z-graded) equivariant homology and cohomology theories defined on the category \( C_G \). By definition the composites

\[ [\_ {\_} h^a_1](X, Y) \to [\_ {\_} h^a_1](G_1^X, G_1^Y) \to \_ {\_} h^a(G_1^X, G_1^Y) \]

and

\[ \_ {\_} h^a(G_1^X, G_1^Y) \to [\_ {\_} h^a_1](G_1^X, G_1^Y) \to \_ {\_} h^a(X, Y) \]

are isomorphisms for each degree \( a \in RO(G) \) (or \( e \in Z \)) and each pair \( (X, Y) \in C_G \), taking every closed subgroup \( H \) of \( G \) (see Kosniowski [10]). Applying entirely the same method adopted in [11] we obtain the following proposition regarded as a generalization of the result [11, Proposition 2].

**Proposition 1.4.** Let \( G \) be a compact Lie group, \( (X, Y) \) be a pair of \( G\)-CW complexes and \( a \in RO(G) \) (or \( e \in Z \)).

i) Let \( \_ {\_} h^* \) be an RO\(_G\)-graded (or Z-graded) equivariant homology theory. If \( \_ {\_} h^a_i = 0 \) for each \( i \geq 0 \) and every pair \( H \subset K \) of closed subgroups of \( G \), then \( \_ {\_} h^a_i(X, Y) = 0 \) for each \( i \geq 0 \).

ii) Let \( \_ {\_} h^* \) be an RO\(_G\)-graded (or Z-graded) equivariant cohomology theory. If \( \_ {\_} h^a_i = 0 \) for each \( i \geq 0 \) and every pair \( H \subset K \) of closed subgroups of \( G \), then \( \_ {\_} h^a_i(X, Y) = 0 \) for each \( i \geq 0 \).

**Proof.** We first prove the cohomology case ii). We may assume that \( \_ {\_} h^a_i(X, Y) = 0 \) for each \( i \geq 0 \) and all \( H \) in the family \( F \) of proper closed subgroups of \( G \) and that \( \_ {\_} h^a_i(X^K, Y^K) = 0 \) for each \( i \geq 0 \) and all \( H \) in the family.
of all closed subgroups of $G$. There is a commutative diagram with exact rows

$$
\begin{array}{ccc}
\mathcal{H}^{a-i}[F](X, Y) & \longrightarrow & \mathcal{H}^{a-i}[F_m, F](X, Y) \\
\downarrow & & \downarrow \\
\mathcal{H}^{a-i}[F](X^G, Y^G) & \longrightarrow & \mathcal{H}^{a-i}[F_m, F](X^G, Y^G)
\end{array}
$$

where all vertical arrows are induced by the inclusion $(X^G, Y^G) \to (X, Y)$ (see [7]). Observing exactly Segal's spectral sequence in the proof of Jackowski [9, Proposition 1.4], it is easy to check under the above assumptions that $\mathcal{H}^{a-i}(X, Y) = G^H(\mathcal{H}^{a-i}(X^G, Y^G))$ for each $i \geq 0$. On the other hand, the inclusion $(X^G, Y^G) \to (X, Y)$ induces an isomorphism $\mathcal{H}^*[F_m, F](X, Y) \cong \mathcal{H}^*[F_m, F](X^G, Y^G)$ as investigated by tom Dieck (see [7, Proposition 7.4.2]). Consequently we obtain that $\mathcal{H}^{a-i}(X, Y) = G^H(\mathcal{H}^{a-i}(X^G, Y^G)) = 0$ for each $i \geq 0$.

We next prove the homology case i) by coming back to the cohomology case ii) by duality. For any divisible abelian group $A$, consider the equivariant cohomology theory $\mathcal{H}^*(A)$ given by setting $\mathcal{H}^*(A)(X, Y) = \text{Hom}(\mathcal{H}^*(X, Y), A)$. Applying the cohomology case ii) it follows at once that $\mathcal{H}^{a-i}(X, Y) = \mathcal{H}^*(X, Y) = 0$ for each $i \geq 0$ and any divisible abelian group $A$. Taking the injective resolution $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ of the integers $\mathbb{Z}$, we have that $\text{Hom}(\mathcal{H}_{a-i}(X, Y), \mathbb{Z}) = \text{Ext}(\mathcal{H}_{a-i}(X, Y), \mathbb{Z})$. This means that $\mathcal{H}_{a-i}(X, Y) = 0$ for each $i \geq 0$.

Let $H$ be a closed subgroup of $G$ and $M$ be a covariant coefficient system for $G$ and $N$ a contravariant coefficient system for $G$. If $(X, Y)$ is a pair of trivial $H$-CW complexes, then we have natural isomorphisms

$$
\iiota_H \mathcal{H}_a(X, Y ; i^h_H M) \cong \mathcal{H}_a(X, Y ; M(G/H))
$$

and

$$
\iiota_H \mathcal{H}^a(X, Y ; i^h_H N*) \cong \mathcal{H}^a(X, Y ; N(G/H))
$$

where $i^h_H M$ and $i^h_H N$ are the induced coefficient systems for $H$. Taking Bredon homology and cohomology as $\mathcal{H}_a^*$ and $\mathcal{H}^*$ in the above proposition we have

**Corollary 1.5.** Let $G$ be a compact Lie group and $(X, Y)$ be a pair of $G$-CW complexes and $n \geq 0$.

i) Let $M$ be a left $I$-module. If $\mathcal{H}_i(X^G, Y^G ; 1_{G/H} M) = 0$ for each $i \leq n$ and every pair $H \leq K$ in $C(G)$, then $\mathcal{H}_i(X, Y ; M) = 0$ for each $i \leq n$.

ii) Let $N$ be a right $I$-module. If $\mathcal{H}^i(X^G, Y^G ; N 1_{G/H}) = 0$ for each $i \leq n$ and every pair $H \leq K$ in $C(G)$, then $\mathcal{H}^i(X, Y ; N) = 0$ for each $i \leq n$. (Cf., [11, Proposition 4]).

Let $\mathcal{H}_*$ and $\mathcal{H}^*$ be (non-equivariant) homology and cohomology theories defined on the category $C$ of pairs of CW-complexes. Putting $\mathcal{H}_*(X) = \mathcal{H}_*(X/G)$
and \( \phi^h(X) = h^*(X/G) \), \( \phi^h_\ast \) and \( \phi^h \) are equivariant homology and cohomology theories respectively defined on the category \( C_G \).

**Corollary 1.6.** Let \( G \) be a compact Lie group, \( H \) its closed subgroup and \( N(H) \) the normalizer of \( H \) in \( G \). Let \((X, Y)\) be a pair of \( G \)-CW complexes and \( n \geq 0 \).

i) If \( h_i(X^K, Y^K) = 0 \) for each \( i \leq n \) and each \( K, H \subset K \subset N(H) \), then \( h_i(X^H/W(H)_0, Y^H/W(H)_0) = 0 \) for each \( i \leq n \).

ii) If \( h_i(X^K, Y^K) = 0 \) for each \( i \leq n \) and each \( K, H \subset K \subset N(H) \), then \( h_i(X^H/W(H)_0, Y^H/W(H)_0) = 0 \) for each \( i \leq n \).

### 2. \( h_k \)-localization of \( G \)-CW complexes

For a (non-equivariant) homology theory defined on the category \( \mathcal{C} \), let \( h_\ast(\; ; A) \) denote the associated homology theory with coefficients in the abelian group \( A \). Let \( \mathcal{A}_G = \{ A_H \} \) be a family of abelian groups indexed by \( H \in C(G) \).

A based \( G \)-map \( f: X \rightarrow Y \) of based \( G \)-CW complexes is said to be an \((h_\ast, \mathcal{A}_G)\)-equivalence if its \( H \)-fixed point map \( f^H: h_\ast(X^H; A_H) \rightarrow h_\ast(Y^H; A_H) \) for every \( H \in C(G) \). A based \( G \)-CW complex \( X \) is said to be \((h_\ast, \mathcal{A}_G)\)-local if any \((h_\ast, \mathcal{A}_G)\)-equivalence \( f: X' \rightarrow Y' \) induces a bijection \( f^*: [Y', X]_G \rightarrow [X', X]_G \) where \([ \, , \]_G\) denotes the set of \( G \)-homotopy classes of based \( G \)-maps.

Using the technique developed by Bousfield [3, 5] we can show the existence of \((h_\ast, \mathcal{A}_G)\)-localizations of based \( G \)-CW complexes, although we avoid to rely on the simplicial homotopy theory.

**Theorem 2.1.** Let \( G \) be a compact Lie group, \( h_\ast \) be a homology theory defined on the category \( \mathcal{C} \) and \( \mathcal{A}_G = \{ A_H \} \) be a family of abelian groups. Given any based \( G \)-CW complex \( X \) there is a based \( G \)-map

\[
\eta_X: X \rightarrow LX
\]

of based \( G \)-CW complexes such that \( \eta_X \) is an \((h_\ast, \mathcal{A}_G)\)-equivalence and \( LX \) is \((h_\ast, \mathcal{A}_G)\)-local.

The proof is deferred to Appendix. Remark that our construction given in Appendix is not necessarily functorial.

Let \( A \) be an abelian group and \( J_A \) denote the set of primes \( p \) such that \( A \) is uniquely \( p \)-divisible. Associate with \( A \) the special abelian groups

\[
S_A = \begin{cases} 
\oplus_{p \in J_A} \mathbb{Z}/p & \text{if } A \otimes Q = 0 \\
\mathbb{Z}[1/J_A] & \text{if } A \otimes Q \neq 0
\end{cases}
\]

and
if $\text{Hom}(Q, A) = 0 = \text{Ext}(Q, A)$

\[
S^*_A = \begin{cases} 
\Pi_{p \in \mathcal{J}_A} Z/p & \text{if } \text{Hom}(Q, A) = 0 = \text{Ext}(Q, A) \\
Z[J^{-1}] & \text{if not}
\end{cases}
\]

Since each homology theory $h_*$ holds a universal coefficient sequence

\[0 \to h_*(X, Y) \otimes A \to h_*(X, Y; A) \to \text{Tor}(h_{*-1}(X, Y), A) \to 0,
\]

from [2, Proposition 2.3] it follows easily that

(2.1) \[h_*(X, Y; A) = 0 \text{ if and only if } h_*(X, Y; S_A) = 0\]

(see [13, Proposition 1.9]). Let $h^*$ be a cohomology theory of finite type, thus its coefficient group $h^*(*)$ is finitely generated for every degree $i$. Then there exists a homology theory $\nabla h_*$ related with the universal coefficient sequence [6, 18]

\[0 \to \text{Ext}(\nabla h_{*-1}(X, Y), A) \to h^*(X, Y; A) \to \text{Hom}(\nabla h_*(X, Y), A) \to 0.
\]

As a similar result we have

(2.2) \[h^*(X, Y; A) = 0 \text{ if and only if } h^*(X, Y; S^*_A) = 0.
\]

Moreover, when either $S = \bigoplus_{p \in J} Z/p$ and $\nabla S = \Pi_{p \in J} Z/p$ or $S = \nabla S = Z[J^{-1}]$ for some set $J$ of primes, we get

(2.3) \[h^*(X, Y; \nabla S) = 0 \text{ if and only if } \nabla h_*(X, Y; S) = 0.
\]

For a family $\mathcal{A}_\mathcal{G} = \{A_H\}_{H \in \mathcal{G}()}$ of abelian groups we define the family

$S_{\mathcal{A}} = \{S_H\}_{H \in \mathcal{G}()}$ of special abelian groups given by

\[
S_H = \begin{cases} 
\bigoplus_{p \in \mathcal{J}_H} Z/p & \text{if } A_H \otimes Q = 0 \\
Z[J^{-1}] & \text{if } A_H \otimes Q \neq 0
\end{cases}
\]

where $J_H$ denotes the set of primes such that $A_H$ is uniquely $p$-divisible. Clearly (2.1) implies

(2.4) A based $G$-CW complex $X$ is $(h_*, \mathcal{A}_\mathcal{G})$-local if and only if it is $(h_*, S_{\mathcal{A}})$-

local.

Given abelian groups $A$ and $B$, write $\langle A \rangle \leq \langle B \rangle$ if $J_A \supseteq J_B$ and $B \otimes Q = 0$ implies $A \otimes Q = 0$, and $\langle A \rangle^* \leq \langle B \rangle^*$ if $J_A \supseteq J_B$ and $\text{Hom}(Q, B) = 0 = \text{Ext}(Q, B)$ implies $\text{Hom}(Q, A) = 0 = \text{Ext}(Q, A)$. Obviously it follows from (2.1) that

(2.5) \[h_*(X, Y; B) = 0 \text{ implies } h_*(X, Y; A) = 0
\]

when $\langle B \rangle \leq \langle A \rangle$, and under the restriction that $h^*$ is of finite type it follows from (2.2) that

(2.6) \[h^*(X, Y; B) = 0 \text{ implies } h^*(X, Y; A) = 0
\]
A family $\mathcal{A}_G = \{A_H\}_{H \in C(G)}$ of abelian groups is said to be (homologically) order preserving if $\langle A_H \rangle \leq \langle A_K \rangle$ for every pair $H \leq K$ in the partially ordered set $C(G)$.

**Lemma 2.2.** Assume that a family $\mathcal{A}_G = \{A_H\}$ is order preserving. If a based $G$-CW complex $X$ is $(h_*, \mathcal{A}_G)$-local, then its $H$-fixed point space $X^H$ is $h_*(; A_H)$-local for every $H \in C(G)$.

**Proof.** Let $f: U \to V$ be a based map which induces an isomorphism $f_*: h_*(U; A_H) \to h_*(V; A_H)$ for a fixed $H \in C(G)$. Then $1 \wedge f: (G/H)_+ \wedge U \to (G/H)_+ \wedge V$ becomes an $(h_*, \mathcal{A}_G)$-equivalence under the hypothesis on $\mathcal{A}_G = \{A_H\}$. Therefore it induces a bijection $(1 \wedge f)_*: [(G/H)_+ \wedge V, X]_G \to [(G/H)_+ \wedge U, X]_G$. This means that $f^*: [V, X^H] \to [U, X^H]$ is certainly a bijection, too.

Owing to the existence theorem of $(h_*, \mathcal{A}_G)$-localizations we show that the converse of Lemma 2.2 is also valid.

**Proposition 2.3.** Assume that a family $\mathcal{A}_G = \{A_H\}$ is order preserving. Then a based $G$-CW complex $X$ is $(h_*, \mathcal{A}_G)$-local if and only if its $H$-fixed point spaces $X^H$ are $h_*(; A_H)$-local for all $H \in C(G)$.

**Proof.** We have to show only the "if" part. Let $\eta: X \to LX$ be an $(h_*, \mathcal{A}_G)$-localization of $X$. By the "only if" part, Lemma 2.2, $(LX)^H$ is $h_*(; A_H)$-local for each $H \in C(G)$. Since the $H$-fixed point map $\eta^H: X^H \to (LX)^H$ is an $(h_*, \mathcal{A}_G)$-equivalence, we see easily that $\eta^H: X^H \to (LX)^H$ is a homotopy equivalence for each $H \in C(G)$. Thus $\eta: X \to LX$ itself is a homotopy equivalence, and hence $X$ is $(h_*, \mathcal{A}_G)$-local as desired.

An Eilenberg-MacLane $G$-space $K(N, n)$ is a based $G$-CW complex by which the $n$-th (reduced) Bredon cohomology group $\widetilde{H}^*(X; N)$ with coefficients in $N$ is represented as $\widetilde{H}^*(X; N) = [X, K(N, n)]_G$. For Eilenberg-MacLane $G$-spaces $K(N, n)$ we can give another proof of Proposition 2.3 without use of the existence theorem of $(h_*, \mathcal{A}_G)$-localizations.

**Proposition 2.4** (as a special case of Proposition 2.3). Assume that a family $\mathcal{A}_G = \{A_H\}$ is order preserving. Let $N$ be a right $I$-module. Then an Eilenberg-MacLane $G$-space $K(N, n)$ is $(h_*, \mathcal{A}_G)$-local if and only if Eilenberg-MacLane spaces $K(N_{1/G/H}, n)$ are $h_*(; A_H)$-local for all $H \in C(G)$.

**Proof.** The "if" part: Let $f: X \to Y$ be an $(h_*, \mathcal{A}_G)$-equivalence. Then by Corollary 1.6 $f_*: h_*(X^K/W(K)_0; A_H) \to h_*(Y^K/W(K)_0; A_H)$ is an isomorphism for each pair $H \leq K$ in $C(G)$. Since the Eilenberg-MacLane space $K(N_{1/G/H}, m)$ is $h_*(; A_H)$-local for any $m \leq n$, $f^*: H^*(Y^K/W(K)_0; N_{1/G/H}) \to H^*(X^K/W(K)_0; N_{1/G/H})$ is an isomorphism for any $m \leq n$ and every pair $H \leq K$ in $C(G)$. Using
Proposition 1.3 we observe that $f^*: \sigma H^*(Y; N) \rightarrow \sigma H^*(X; N)$ is an isomorphism, thus the Eilenberg-MacLane G-space $K(N, n)$ is $(\eta^*_H, \mathcal{A}_H)$-local.

3. $H^*_\mathcal{G}$-localization of $G$-nilpotent $G$-CW complexes

Let $\mathcal{A}_H = \{A_H\}_{H \in \mathbb{C}(G)}$ be a family of abelian groups which is order preserving and $N$ be a right $I$-module. For each $H \in \mathbb{C}(G)$ put

$$E_{\mathcal{A}_H}N(G/H) = \begin{cases} \Pi_{p\in \mathbb{P}} \mathrm{Ext}(Z_{p, m}, N_{1_{G/H}}) & \text{if } A_H \otimes Q = 0 \\ N_{1_{G/H}} \otimes Z(J_{H^*}) & \text{if } A_H \otimes Q \neq 0 \end{cases}$$

and

$$H_{\mathcal{A}_H}N(G/H) = \begin{cases} \Pi_{p\in \mathbb{P}} \mathrm{Hom}(Z_{p, m}, N_{1_{G/H}}) & \text{if } A_H \otimes Q = 0 \\ 0 & \text{if } A_H \otimes Q \neq 0 \end{cases}$$

where $J_H$ denotes the set of primes $p$ such that $A_H$ is uniquely $p$-divisible and $Z_{p, m} = \lim Z[p^m]$. As is easily seen, setting $E_{\mathcal{A}_H}N = \oplus_{H \in \mathbb{C}(G)} E_{\mathcal{A}_H}N(G/H)$ it is a right $I$-module and the canonical map $\iota: N = \oplus_{H \in \mathbb{C}(G)} N_{1_{G/H}} \rightarrow E_{\mathcal{A}_H}N = \oplus_{H \in \mathbb{C}(G)} E_{\mathcal{A}_H}N(G/H)$ is a homomorphism of right $I$-modules. Similarly for $H_{\mathcal{A}_H}N$. Note that $H_{\mathcal{A}_H}N = 0$ if $N$ is torsion free as an abelian group.

Lemma 3.1. Assume that a family $\mathcal{A}_H = \{A_H\}$ is order preserving. Let $N$ be a right $I$-module such that $H_{\mathcal{A}_H}N = 0$. Then the induced $G$-map

$$\eta_N = l^*: K(N, n) \rightarrow K(E_{\mathcal{A}_H}N, n), \quad n \geq 1,$$

is an $(H^*_\mathcal{G}, \mathcal{A}_H)$-localization.

Proof. According to Bousfield [3, Proposition 4.3] the $H$-fixed point map $\eta^*_H: K(N_{1_{G/H}}, n) \rightarrow K(E_{\mathcal{A}_H}N(G/H), n)$ is an $H^*_\mathcal{G}(; A_H)$-localization. Now the result follows from Proposition 2.4 (or Proposition 2.3).

Let $M$ be a left $I$-module. We say a based $G$-map $f: X \rightarrow Y$ of based $G$-CW complexes an $\mathcal{G}_H( ; M)$-equivalence if $f^*: \mathcal{G}_H( X; M) \rightarrow \mathcal{G}_H( Y; M)$ is an isomorphism, and a based $G$-CW complex $X \in \mathcal{G}_H( ; M)$-local if any $\mathcal{G}_H( X; M)$-equivalence $f: X' \rightarrow Y'$ induces a bijection $f^*: [Y', X]_0 \rightarrow [X', X]_0$. Similarly for $\mathcal{G}_H( ; N)$-equivalence and $\mathcal{G}_H( ; N)$-local space when $N$ is a right $I$-module.

For the order preserving family $\mathcal{A}_H = \{A_H\}_{H \in \mathbb{C}(G)}$ we put $I(\mathcal{A}) = \oplus_{H \in \mathbb{C}(G)} I_{1_{G/H}} \otimes A_H$, which is a left $I$-module.

Lemma 3.2. Assume that $\mathcal{A}_H = \{A_H\}$ is order preserving. Let $N$ be a right $I$-module such that $H_{\mathcal{A}_H}N = 0$. Then the Eilenberg-MacLane G-space $K(E_{\mathcal{A}_H}N, n)$ is $\mathcal{G}_H( I(\mathcal{A}))$-local for every $n \geq 1$. 


Proof. By use of Lemma 1.1, (2.1) and (2.3) we can see that the following
four conditions are all equivalent:
i) a based $G$-map $f: X \to Y$ is an $cH_*(\ ; I(\mathcal{A}))$-equivalence,
ii) each $f^H: X^H/W(H)_0 \to Y^H/W(H)_0$ is an $H_*(\ ; A_H)$-equivalence,
iii) each $f^H$ is an $H_*(\ ; S_H)$-equivalence,
iv) each $f^H$ is an $H^*(\ ; \nabla S_H)$-equivalence
where $S_H = \oplus_{p \in J_H} Z/p$ and $\nabla S_H = \prod_{p \in J_H} Z/p$ if $A_H \otimes Q = 0$, or $S_H = \nabla S_H = Z[J_H^\infty]$ if $A_H \otimes Q \neq 0$. Obviously $\langle \nabla S_H \rangle^* \cong \langle \mathcal{E}_A \mathcal{N}(G/H) \rangle^*$, hence (2.5) says that the condition iv) implies
v) each $f^H$ is an $H^*(\ ; E_{\mathcal{A}}N(G/H))$-equivalence.
Moreover it follows from Proposition 1.3 that the condition v) implies
vi) $f: X \to Y$ is an $cH_*(\ ; I(\mathcal{A}))$-local
Therefore the Eilenberg-MacLane $G$-space $K(E_{\mathcal{A}}N, n)$ is $cH_*(\ ; I(\mathcal{A}))$-local as desired.

Following [11] we say a based $G$-$CW$ complex $X$ $G$-nilpotent if each $X^H$ is connected and nilpotent and if for every $n \geq 1$ the orders of nilpotency of the $\pi_n(X^H)$-groups $\pi_n(X^H)$ have a common bound for varying $H$. According to [11, Proposition 8] we have the following analogous result to the non-equivariant case.

(3.1) If a based $G$-$CW$ complex $X$ is $G$-nilpotent, then there is a (nilpotent) $G$-tower $\mathcal{X} = \{X_n\}$ such that
i) $X$ is weakly $G$-homotopy equivalent to the inverse limit of $X_n$;
ii) $X_0 = \{\ast\}$ and $X_{n+1}$ is the fiber of a based $G$-map $k_n: X_n \to K(N, q_n)$ where $q_n \geq 2$, and
iii) $q_{n+1} \geq q_n$ and only finitely many $q_r = r$ for each $r$.

Theorem 3.3. Let $\mathcal{A}_G = \{A_H\}_{H \in C(G)}$ be a family of abelian groups which is order preserving and denote $I(\mathcal{A}) = \oplus_{H \in C(G)} I_{G/H} \otimes A_H$. Given any $G$-nilpotent $G$-$CW$ complex $X$ there exists a based $G$-map $$\eta_X: X \to L_{\mathcal{A}}X$$ of $G$-nilpotent $G$-$CW$ complexes such that
i) its $H$-fixed point map $\eta_X^H: X^H / (L_{\mathcal{A}}X)^H$ is an $H_*(\ ; A_H)$-localization for each $H \in C(G)$, and
ii) $L_{\mathcal{A}}X$ is $cH_*(\ ; I(\mathcal{A}))$-local.

Proof. First take a right $I$-module $N$ and an exact sequence $0 \to F_2 \to F_1 \to N \to 0$ of right $I$-modules such that $F_1$ is projective. Note that both $F_1$ and $F_2$ are free as abelian groups. Denote by $L_{\mathcal{A}}K(N, n)$, $n \geq 1$, the fiber of the $G$-map $K(E_{\mathcal{A}}F_n, n+1) \to K(E_{\mathcal{A}}F_1, n+1)$. It is a 2-stage $G$-$CW$ complex with homotopy groups only in dimensions $n$ and $n+1$. We have a dotted arrow
\[ \eta_N: K(N, n) \to L_{\Delta} K(N, n) \]

making the diagram below \( G \)-homotopy commutative

\[
\begin{array}{c}
K(F_1, n) \\ \downarrow \\
K(E_\Delta F_1, n) \end{array} \rightarrow
\begin{array}{c}
K(N, n) \\ \downarrow \\
L_{\Delta} K(N, n) \end{array} \rightarrow
\begin{array}{c}
K(F_2, n+1) \\ \downarrow \\
K(E_\Delta F_1, n+1) \end{array}
\]

By use of the Serre spectral sequences and Lemma 3.1 we see easily that \( \eta_N \) is an \((H_*, \mathcal{A}_G)\)-equivalence and by Lemma 3.2 that \( L_{\Delta} K(N, n) \) is \( cH_*( ; I(\mathcal{A})) \)-local.

For a \( G \)-nilpotent \( G \)-\( CW \) complex \( X \) we may regard that it is given as the inverse limit of a nilpotent \( G \)-tower \( \mathcal{X} = \{X_n\} \). Inductively we can construct \( cH_*( ; I(\mathcal{A})) \)-local spaces \( L_{\Delta} X_{n+1} \), being the fiber of \( L_{\Delta} h_n: L_{\Delta} X_n \to L_{\Delta} K(N_n, q_n) \), and also \((H_*, \mathcal{A}_G)\)-equivalences \( \eta_{n+1}: X_{n+1} \to L_{\Delta} X_{n+1} \) such that the following diagram is \( G \)-homotopy commutative

\[
\begin{array}{c}
K(N_n, q_n-1) \\ \downarrow \\
L_{\Delta} K(N_n, q_n-1) \end{array} \rightarrow
\begin{array}{c}
X_{n+1} \\ \downarrow \\
L_{\Delta} X_{n+1} \end{array} \rightarrow
\begin{array}{c}
X_n \\ \downarrow \\
L_{\Delta} X_n \end{array} \rightarrow
\begin{array}{c}
K(N_n, q_n) \\ \downarrow \\
L_{\Delta} K(N_n, q_n) \end{array}
\]

Put as \( L_{\Delta} X \) a \( G \)-\( CW \) approximation of the inverse limit space of \( L_{\Delta} X_n \). Then the above construction shows that \( L_{\Delta} X_n \) is \( G \)-nilpotent for each \( n \) and that \( L_{\Delta} X \) is also \( G \)-nilpotent (see [8]). So we obtain a desired localization map \( \eta_X: X \to L_{\Delta} X \).

Because of the localization theorem 3.3 we have the following characterization, although it is trivial if \( G \) is a finite group.

**Proposition 3.4.** Assume that a family \( \mathcal{A}_G = \{A_H\} \) is order preserving. On a based \( G \)-map \( f: X \to Y \) the following three conditions are equivalent:

i) \( f^H: H_*(X^H; A_H) \to H_*(Y^H; A_H) \) is an isomorphism for each \( H \in C(G) \),

ii) \( f^H: H_*(X^H \cup W(H)_0; A_H) \to H_*(Y^H \cup W(H)_0; A_H) \) is an isomorphism for each \( H \in C(G) \), and

iii) \( f_*: cH_*(X; I(\mathcal{A})) \to cH_*(Y; I(\mathcal{A})) \) is an isomorphism.

(Cf., [12, Proposition 6]).

Proof. Lemma 1.1 asserts that the conditions ii) and iii) are equivalent, and Corollary 1.5 says that the condition i) implies iii). It remains to show only the implication iii) \( \to \) i).

Consider the \( G \)-homotopy commutative square

\[
\begin{array}{ccc}
S^2X & \xrightarrow{S^2f} & S^2Y \\
\eta_{S^2X} & & \downarrow \eta_{S^2Y} \\
L_{\Delta} S^2X & \longrightarrow & L_{\Delta} S^2Y.
\end{array}
\]

Then the condition iii) implies that \( L_{\Delta} S^2f: L_{\Delta} S^2X \to L_{\Delta} S^2Y \) is an \( cH_*( ; I(\mathcal{A})) \)-equivalence, and hence it becomes a \( G \)-homotopy equivalence since \( L_{\Delta} S^2X \)
and $L_dS^2Y$ are both $gH_\ast(\ ; I(\mathcal{A}))$-local. Hence $f: X \to Y$ is certainly an $(H_\ast, \mathcal{A}_G)$-equivalence.

As a dual of Proposition 3.4 we have

**Corollary 3.5.** Let $\mathcal{A}_G = \{A_H\}_{H \in C(G)}$ be a family of abelian groups such that $\langle A_H \rangle^* \leq \langle A_K \rangle^*$ for every pair $H \leq K$ in $C(G)$. On a based $G$-map $f: X \to Y$ the following three conditions are equivalent:

i) $f^*: H^\ast(Y^H; A_H) \to H^\ast(X^H; A_H)$ is an isomorphism for each $H \in C(G)$,

ii) $f^*: H^\ast(Y^H/W(H)^0; A_H) \to H^\ast(X^H/W(H)^0; A_H)$ is an isomorphism for each $H \in C(G)$, and

iii) $f^*: gH^\ast(Y; I(\mathcal{A})) \to gH^\ast(X; I(\mathcal{A}))$ is an isomorphism. Here $I(\mathcal{A})^* = \bigoplus_H \Pi_H \text{Hom}(1_{G/K}, 1_{G/H}, A_H)$.

Proof. Use Lemma 1.1, Corollary 1.5, (2.2), (2.3) and Proposition 3.4.

### 4. $K_\ast$-localization of Eilenberg-MacLane $G$-spaces

Denote by $K_\ast(\ ; A)$ and $K^\ast(\ ; A)$ respectively the complex homology and cohomology $K$-theories with coefficients in $A$. Let $BU_A$ be the connected component of the base point of the $CW$-complex which represents $K^\ast(\ ; A)$. As a consequence of [14, Theorem 1.11] we notice that

(4.1) $BU(A \otimes R)$ is $K_\ast(\ ; R)$-local when $R = \mathbb{Z}[J^{-1}]$ is a subring of the rationals $\mathbb{Q}$.

Moreover, Mislin [14, Lemma 2.1] showed that

(4.2) the Eilenberg-MacLane space $K(A, 2)$ is a factor of $BU_A$ if the abelian group $A$ is torsion free.

Combining (4.1) with (4.2) we obtain examples of $K_\ast(\ ; R)$-local spaces. If $R$ is a subring of $\mathbb{Q}$, then

(4.3) the Eilenberg-MacLane spaces $K(A \otimes R, 1)$ and $K(A/T \otimes R, 2)$ are $K_\ast(\ ; R)$-local where $T$ denotes the torsion subgroup of $A$.

By aid of the computation of Mislin [14, Theorem 2.2] we get immediately

**Proposition 4.1.** Let $A$ be an abelian group, $T$ its torsion subgroup and $R$ be a subring of $\mathbb{Q}$. Then the following induced maps are respectively $K_\ast(\ ; R)$-localizations:

i) $K(A, 1) \to K(A \otimes R, 1)$

ii) $K(A, 2) \to K((A/T) \otimes R, 2)$

iii) $K(A, n) \to K(A \otimes Q, n)$ for $n \geq 3$.

When $R = \bigoplus_{p \in J} \mathbb{Z}/p$ for some set $J$ of primes, we consider the cofibering
\[ \bigvee_{a} S^1 \to \bigvee_{\beta} S^1 \to M_R \] associated with a free resolution \[ 0 \to \bigoplus_{a} Z \to \bigoplus_{\beta} Z \to \bigoplus_{p \neq 1} Z_{p^\infty} \to 0. \] 

\( M_R \) is a Moore space of type \((\bigoplus_{p \neq 1} Z_{p^\infty}, 1)\).

**Lemma 4.2.** Let \( A \) be a torsion free abelian group and \( R = \bigoplus_{p \neq 1} Z/p \).

Putting \( E_R A = \Pi_{p \neq 1} \text{Ext}(Z_{p^\infty}, A) \), then \( BUE_R A \) is homotopy equivalent to the connected component of the constant map of the based mapping space \( F(M_R, BU A) \).

Proof. It is sufficient to show that there is a natural isomorphism between \( \tilde{K}(X; E_R A) \) and \( \tilde{K}(X, M_R; A) \) for any based CW-complex \( X \). We work in the category of \( CW \)-spectra. Let \( MA \) be a Moore spectrum of type \( A \) and \( ME_R A \) of type \( E_R A \). Consider the pairing \( u_\alpha: (\Pi MA) \wedge (\vee S^2) \to S^2 MA \) induced by the projections \( p_\alpha: (\Pi MA) \wedge (\vee S^2) \to S^2 MA \). The canonical morphism \( K \wedge \Pi MA \to \Pi K \wedge MA \) is a homotopy equivalence since \( \Pi MA \) is a Moore spectrum of type \( \Pi A \) (see [18, Lemma 4] or [1]). Thus the map

\[ T(u_\alpha)_K: \{ X, K \wedge \Pi MA \} \to \{ X \wedge (\vee S^2), K \wedge S^2 MA \} \]

defined by \( T(f) = (1 \wedge u_\alpha)(f \wedge 1) \) for any CW-spectrum \( X \), is an isomorphism. Similarly for \( u_\beta \).

Consider the homotopy commutative square

\[
\begin{array}{ccc}
(\Pi MA) \wedge (\vee S^2) & \overset{k_\wedge 1}{\longrightarrow} & (\Pi MA) \wedge (\vee S^3) \\
\downarrow_{1 \wedge j} & & \downarrow_{u_\alpha} \\
(\Pi MA) \wedge (\vee S^2) & \overset{u_\beta}{\longrightarrow} & S^2 MA
\end{array}
\]

where the map \( k: \Pi MA \to \Pi MA \) is one induced by the map \( j: \bigvee_{a} S^1 \to \bigvee_{\beta} S^1 \).

Then, by [18, Lemma 1] (or [15, Theorem 6.10]) there exists a nice map

\[ w: ME_R A \wedge M_R \to S^2 MA \]

which induces an isomorphism

\[ T(w)_K: \{ X, K \wedge ME_R A \} \overset{\cong}{\to} \{ X \wedge M_R, K \wedge S^2 MA \} \]

for any \( CW \)-spectrum \( X \). Composing the Bott isomorphism with the above map we obtain a natural isomorphism

\[ \tilde{K}^*(X; E_R A) \overset{\cong}{\to} \tilde{K}^*(X \wedge M_R; A) \]

for any \( CW \)-spectrum \( X \), and in particular for any based \( CW \)-complex \( X \).

Applying Mislin’s method [14, Corollary 2.5] with Lemma 4.2 we have

**Lemma 4.3.** Let \( A \) be a torsion free abelian group and \( R = \bigoplus_{p \neq 1} Z/p \). Then
the Eilenberg-MacLane space $K(E_R A, 2)$ is $K^*( ; R)$-local.

Proof. Let $f: X \rightarrow Y$ be a $K^*( ; R)$-equivalence. Then $f \wedge 1: X \wedge M_R \rightarrow Y \wedge M_R$ is clearly a $K^*$-equivalence. Therefore the based mapping space $F (M_R, BUA)$ is $K^*( ; R)$-local because $BUA$ is $K^*$-local by (4.1). On the other hand, by (4.2) the Eilenberg-MacLane space $K(E_R A, 2)$ is a factor of $BUE_R A$ since the abelian group $E_R A$ is torsion free. We use Lemma 4.2 to obtain that $K(E_R A, 2)$ is $K^*( ; R)$-local.

By use of Lemma 4.3 we obtain

**Proposition 4.4.** Let $A$ be an abelian group, $T$ its torsion subgroup and $R=\bigoplus_{p \in J} \mathbb{Z}/p$ for some set $J$ of primes. Then the following canonical maps are respectively $K^*( ; R)$-localizations:

i) $K(A, 1)\rightarrow L_R K(A, 1)$

ii) $K(A, 2)\rightarrow K(E_R(A/T), 2)=L_R K(A/T, 2)$

iii) $K(A, n)\rightarrow \{\ast\}$ for $n \geq 3$

where $L_R X$ denotes the $H^*( ; R)$-localization of $X$ and $E_R(A/T)=\prod_{p \in J} \text{Ext}(\mathbb{Z}/p, A/T)$. (Cf., [14, Corollaries 2.3 and 2.5]).

Proof. i) Take a free resolution $0 \rightarrow F_2 \rightarrow F_1 \rightarrow A \rightarrow 0$. Since $L_R K(A, 1)$ is the fiber of the map $K(E_R F_2, 2)\rightarrow K(E_R F_1, 2)$, it is $K^*( ; R)$-local by Lemma 4.3. The $H^*( ; R)$-localization map $\eta_A: K(A, 1)\rightarrow L_R K(A, 1)$ is obviously a $K^*( ; R)$-equivalence.

ii) In the composite map $K(A, 2)\rightarrow K(A/T, 2)\rightarrow K(E_R(A/T), 2)$ the former is a $K^*$-equivalence and the latter is an $H^*( ; R)$-equivalence, and hence the composite map is a $K^*$-equivalence.$R$-equivalence.

iii) For $n \geq 3$ the constant map $K(A, n)\rightarrow \{\ast\}$ is certainly a $K^*( ; Z/p)$-equivalence (see [17, Theorem 2.7]).

Let $\mathcal{A}_G=\{A_H\}_{H \in C(G)}$ be an order preserving family of abelian groups and $N$ be a right $I$-module. For each $H \in C(G)$ put

$$L^K_{\mathcal{A}} N(G/H) = \begin{cases} 0 & \text{if } A_H \otimes Q = 0 \\ N1_{G/H} \otimes Q & \text{if } A_H \otimes Q \neq 0 \end{cases}$$

Then $L^K_{\mathcal{A}} N=\bigoplus_{H \in C(G)} L^K_{\mathcal{A}} N(G/H)$ is a right $I$-module. And the canonical map $l': N=\bigoplus_H N1_{G/H} \rightarrow L^K_{\mathcal{A}} N=\bigoplus_H L^K_{\mathcal{A}} N(G/H)$ is a homomorphism of right $I$-modules, which induces a $G$-map

$$\eta_N' = l'_*: K(N, n) \rightarrow K(L^K_{\mathcal{A}} N, n).$$

**Theorem 4.5.** Assume that a family $\mathcal{A}_G=\{A_H\}_{H \in C(G)}$ of abelian groups is order preserving. Let $N$ be a right $I$-module and $T$ its torsion subgroup. Then the following maps are all ($K^*$, $\mathcal{A}_G$)-localizations:
i) the $(H_\ast, \mathcal{A}_G)$-localization $\eta^G: K(N, 1) \to L_\Delta K(N, 1)$,

ii) the composite map $K(N, 2) \to K(N/T, 2) \xrightarrow{\eta^G_T} K(E_\Delta(N/T), 2) \to L_\Delta K(N/T, 2)$,

iii) the induced map $\eta^G_i: K(N, n) \to K(\Delta^n G, n)$ for $n \geq 3$.

Proof. Putting Propositions 4.1 and 4.4 together we can check that all the $H$-fixed point maps in the theorem are $K_\ast(\ ; A_H)$-localizations for any $H \in C(G)$.

Appendix. Proof of the existence theorem of the localization

Let $\sigma$ be a fixed infinite cardinal number such that $\text{Car} \oplus_{H \in C(G)} h^H(\ast; A_H) \leq \sigma$ where the abelian groups $A_H$ belong to the family $\mathcal{A}_G$. For a based $G$-CW complex $X$, let $\#X$ denote the number of $G$-cells in $X$.

Lemma A.1. Let $(X, Y)$ be a pair of based $G$-CW complexes such that $h^H(X^H, Y^H; A_H) = 0$ for each $H \in C(G)$, and $W_0$ be a $G$-CW subcomplex of $X$ with $\#W_0 \leq \sigma$. Then there exists a $G$-CW subcomplex $W$ of $X$ such that $\#W \leq \sigma$, $W_0 \subset W \subset Y$ and $h^H(W^H, W^H \cap Y^H; A_H) = 0$ for each $H \in C(G)$. (Cf., [3, Lemma 11.2]).

Proof. We construct a sequence of $G$-CW subcomplexes of $X$

$$W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_r \subset \cdots$$

such that $\#W_n \leq \sigma$, $W_n \subset Y$ and the map $h^H(W_n^H, W_n^H \cap Y^H; A_H) \to h^H(W_{n+1}^H, W_{n+1}^H \cap Y^H; A_H)$ is zero for $n \geq 1$ and each $H \in C(G)$. First, choose $W_1 \subset X$ such that $\#W_1 \leq \sigma$ and $W_0 \subset W_1 \subset Y$, and construct inductively $W_n$. Choose properly a finite subcomplex $F_x$ of $X^H$ for each element $x \in h^H(W_n^H, W_n^H \cap Y^H; A_H)$ and take as $W_{n+1}$ the union of $W_n$ with all $G \cdot F_x$, then each $x$ goes to zero in $h^H(W_{n+1}^H, W_{n+1}^H \cap Y^H; A_H)$ and $\#W_{n+1} \leq \sigma$. Finally we put $W = \cup_{n \geq 1} W_n$ to obtain the desired one.

Lemma A.2. Let $X$ be a based $G$-CW complex. Assume that for any inclusion map $i_\alpha: Y_\alpha \to Z_\alpha$ with $\#Z_\alpha \leq \sigma$ such that it is an $(h^H, \mathcal{A}_G)$-equivalence, $i^H_\alpha: [Z_\alpha, X]_\alpha \to [Y_\alpha, X]_\alpha$ is onto. Then $X$ is $(h^H, \mathcal{A}_G)$-local. (Cf., [3, Lemmas 2.5 and 11.3]).

Proof. Let $f: Y \to Z$ be an $(h^H, \mathcal{A}_G)$-equivalence. We may regard $Y$ as a $G$-CW subcomplex of $Z$ and $f$ as the inclusion $Y \subset Z$. Let $\gamma$ be an infinite ordinal of cardinality greater than $\#Z - \#Y$. Using Lemma A.1 we can construct a transfinite sequence

$$Y = Y_0 \subset Y_1 \subset \cdots \subset Y_s \subset Y_{s+1} \subset \cdots$$

of $G$-CW subcomplexes of $Z$ such that i) if $\lambda$ is a limit ordinal then $Y_\lambda=$
\[ \bigcup_{i<\gamma} Y_i, \text{ ii) if } Y_i = Z \text{ then } Y_{i+1} = Z, \text{ and iii) if } Y_i + Z \text{ then } Y_{i+1} = Y_i \cup W \] for some \( W \subset X \) with \( #W \leq \sigma \), \( W \cap Y \) and \( h_W(W^n, W^n \cap Y^n; A_H) = 0 \) for each \( H \in C(G) \). Clearly \( Z = Y_\gamma \) and \( f^*: [Z, Z_0, X_G] \rightarrow [Y, X_G]_0 \) is onto. Take two based \( G \)-maps \( g, h: Z \rightarrow X \) such that \( f^*g = f^*h \in [Y, X_G] \), to show the injectivity of \( f^* \). By the \( (h_*, \mathcal{A}_G) \)-version of [3, Lemma 3.6] there exists a based \( G \)-CW complex \( \tilde{X} \) and an \( (h_*, \mathcal{A}_G) \)-equivalence \( j: X \rightarrow \tilde{X} \) such that \( j^*g = j^*h \in [Z, \tilde{X}_G] \). Since we can find a left inverse \( k: \tilde{X} \rightarrow X \) of \( j \), it follows immediately that \( f^* \) is in fact a bijection.

Proof of Theorem 2.1. Choose a set \( \{i_\alpha: Y_\alpha \rightarrow Z_\alpha\}_{\alpha \in I} \) of inclusion maps with \( #Z_\alpha \leq \sigma \) which are \( (h_*, \mathcal{A}_G) \)-equivalences, such that it contains up to isomorphism each inclusion maps with these properties. Let \( \gamma \) be the first infinite ordinal of cardinality greater than \( \sigma \). We inductively construct a transfinite sequence of based \( G \)-CW complexes

\[ X = X_0 \subset X_1 \subset \cdots \subset X_s \subset X_{s+1} \subset \cdots \]

where \( X_\lambda = \bigcup_{i<\lambda} X_i \) for each limit ordinal \( \lambda \) and where \( X_s \subset X_{s+1} \) is given by the push-out square

\[
\begin{array}{ccc}
\bigvee_{s} a / f: & Y_\alpha \rightarrow X_s & \\
\downarrow & & \downarrow \\
\bigvee_{i} a / f: & Y_\alpha \rightarrow X_i & \rightarrow X_{i+1}
\end{array}
\]

Putting \( LX = X_\gamma \), the inclusion \( \eta: X \rightarrow LX \) is an \( (h_*, \mathcal{A}_G) \)-equivalence. Since each based \( G \)-map \( f: Y_\alpha \rightarrow LX \) passes through \( X_i \) for some \( i < \gamma \), \( i^*: [Z_\alpha, LX]_0 \rightarrow [Y_\alpha, LX]_0 \) is onto for any \( \alpha \in I \). By means of Lemma A.2 we observe that \( LX \) is \( (h_*, \mathcal{A}_G) \)-local.

References


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