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LOCALIZATION OF EILENBERG-MACLANE G-SPACES WITH RESPECT TO HOMOLOGY THEORY

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In [11] and [12] May and others have constructed and have characterized equivariant localizations and completions of G-nilpotent G-spaces when G is a compact Lie group. Let J be a set of primes and X be a based G-nilpotent G-space. Then the equivariant localization $\lambda: X\to X_J$ is characterized by the universal property that the H-fixed point space X_J^H is J-local for each closed subgroup H of G and $\lambda^*: [X_J, Y]_G\to[X, Y]_G$ is a bijection for any based G-nilpotent G-space Y with Y^H J-local, and it is constructed as a based G-map whose restriction to H-fixed point spaces is a J-localization in the non-equivariant sense. The equivariant completion $\gamma: X\to\hat{X}_J$ is similarly characterized and constructed.

According to Bousfield [3], each non-equivariant homology theory h_* determines h_*-localizations of based CW-complexes. Special cases of the h_*-localization $\eta: X\to L_{h_*}X$ are familiar if a based CW-complex X is nilpotent. Taking $H_*(\ ;Z[J^{-1}])$ as h_*, then the $H_*(\ ;Z[J^{-1}])$-localization is the usual J^c-localization where J^c denotes the complement of the set J. Taking $H_*(\ ;\oplus_{p\in J}Z/p)$, then the $H_*(\ ;\oplus_{p\in J}Z/p)$-localization is the usual J-completion.

In this paper we study a localization $\eta: X\to L_{(h_*,G)}X=LX$ of a based G-CW complex X such that its H-fixed point map $\eta^H: X^H\to(LX)^H$ is an h^*-localization for any closed subgroup H of G. This localization is characterized by the universal property that $\eta^H: h_*(X^H)\to h_*((LX)^H)$ is an isomorphism for each H, and for any based G-map $f: X\to Y$ inducing isomorphisms $f_*^H: h_*(X^H)\to h_*(Y^H)$ there is a unique based G-map $r: Y\to LX$ with $r\cdot f=\eta\in[X, LX]_G$.

First we investigate some relations between Bredon homology and cohomology theories and ordinary homology and cohomology theories in § 1, following Wilson [16]. Given a homology theory h_* and a family $\mathcal{A}_G=\{A_H\}$ of abelian groups we define our localization in a general form in § 2, and prove the existence theorem of our localizations in Appendix using the technique developed by Bousfield. In § 3 we treat of the special case that h_* is the ordinary homology theory H_*. We proceed cocellularly the construction of the localizations of G-nilpotent G-CW complexes so that we obtain our main result (Theorem 3.3), which may be a slight generalization of main results in [11] and [12]. Finally we compute the localizations of Eilenberg-MacLane G-spaces $K(N, n)$ with respect to the complex homology K-theory K_* (Theorem 4.5) in § 4, as Mislin [14] did in the non-equivariant case.

1. Bredon homology and cohomology

Let G be a compact Lie group and $\mathcal{C}_G$ be the category of pairs (X, Y) of G-CW complexes, where $Y\subset X$, and G-maps $f: (X, Y)\to(X', Y')$. A *covariant coefficient system* M for G is a covariant functor from the category of left homogeneous spaces G/H by closed subgroups H and G-homotopy classes of G-maps to the category of abelian groups. A *contravariant coefficient system* N for G is similarly defined.

Bredon homology and cohomology theories written as ${}_GH_*(X, Y; M)$ and ${}_GH^*(X, Y; N)$ are Z-graded equivariant homology and cohomology theories with coefficients in the covariant coefficient system M and the contravariant one N respectively defined on the category $\mathcal{C}_G$, both of which satisfy the dimension axiom. For any coefficient system M or N for G and each closed subgroup K of G, denote by i_K^*M or i_K^*N the induced coefficient system for K which assigns to each K/H the abelian group $M(G/H)$ or $N(G/H)$ respectively. The composites

$$ {}_KH_*(X, Y; i_K^*M) \to {}_KH_*(G\overset{\times}{_K}X, G\overset{\times}{_K}Y; i_K^*M) \to {}_GH_*(G\overset{\times}{_K}X, G\overset{\times}{_K}Y; M) $$

and

$$ {}_GH^*(G\overset{\times}{_K}X, G\overset{\times}{_K}Y; N) \to {}_KH^*(G\overset{\times}{_K}X, G\overset{\times}{_K}Y; i_K^*N) \to {}_KH^*(X, Y; i_K^*N) $$

are isomorphisms for every closed subgroup K of G and $(X, Y)\in\mathcal{C}_K$.

We now recall the useful notion in interpreting coefficient systems for G, introduced by Wilson [16, § 4]. Let $C(G)$ be a collection of closed subgroups of G which contains precisely one subgroup from every conjugacy class of closed subgroups of G and fix it. We have a partial ordering on $C(G)$, namely $H\leqq K$ if and only if H is subconjugate to K. The *isotropy ring* I_G is defined to be the free abelian group on the set of G-homotopy classes of G-maps from G/H to G/K for all pairs $H\leqq K$ in $C(G)$, whose ring structure is imposed by compositions of G-maps. When G is a finite group, the ring I_G has the multiplicative unit $1=\Sigma_{H\in C(G)}1_{G/H}$ where $1_{G/H}$ denotes the identity map on G/H. However we notice that the ring I_G has in general no multiplicative unit.

We call a *left I_G-module* an abelian group M together with a structure map $\phi: I_G\times M\to M$ written as $\phi(\lambda, x)=\lambda x$ satisfying the condition that $M\cong\oplus_{H\in C(G)}1_{G/H}\,M$ in place of the unitary property in the usual case. A *right I_G-module* is similarly treated. According to [16, Theorem 5.1] there is a one to one correspondence between covariant coefficient systems and left I_G-modules and analogously between contravariant coefficient systems and right I_G-modules.

Write $I=I_G$ for short. Given any abelian group A and each $H\in C(G)$, the abelian group $I1_{G/H}\otimes A=\oplus_{K\in C(G)}1_{G/K}I1_{G/H}\otimes A$ is a left I-module, and the abelian group $\oplus_{K\in C(G)}\mathrm{Hom}(1_{G/K}I1_{G/H}, A)$ denoted by $\mathrm{Hom}(I1_{G/H}, A)_I$ is a right I-module. The following identifications of Bredon homology and cohomology with ordinary homology and cohomology were given implicitly in [16, Theorem 7.3].

Lemma 1.1. *Let G be a compact Lie group, H a closed subgroup of G contained in $C(G)$ and A be any abelian group. Then for any pair (X, Y) of G-CW complexes we have natural isomorphisms*
i) ${}_GH_*(X, Y; I1_{G/H}\otimes A)\cong H_*(X^H/W(H)_0, Y^H/W(H)_0; A)$
ii) ${}_GH^*(X, Y; \mathrm{Hom}(I1_{G/H}, A)_I)\cong H^*(X^H/W(H)_0, Y^H/W(H)_0; A)$
where $W(H)_0$ denotes the identity component of the Weyl group $N(H)/H$ and $X^H/W(H)_0$ denotes the orbit space of the H-fixed point space X^H by $W(H)_0$.

Let M be a left I-module. Denote by $I(M)$ the left I-module defined to be

$$I(M)=\oplus_{H\in C(G)}I1_{G/H}\otimes 1_{G/H}M\,.$$

The map π: $I(M)=\oplus_H I1_{G/H}\otimes 1_{G/H}M\to M$ given by $\pi(\lambda 1_{G/H}\otimes 1_{G/H}x)=\lambda 1_{G/H}x$ is a homomorphism of left I-modules, which is obviously epic. Let N be a right I-module. Denote by $I(N)$ the right I-module defined to be

$$I(N)=\oplus_{K\in C(G)}\Pi_{H\in C(G)}\mathrm{Hom}(1_{G/K}I1_{G/H}, N1_{G/H})\,.$$

The map i: $N=\oplus_K N1_{G/K}\to I(N)=\oplus_K\Pi_H\mathrm{Hom}(1_{G/K}I1_{G/H}, N1_{G/H})$ given by $i(x1_{G/K})(1_{G/K}\lambda 1_{G/H})=x1_{G/K}\lambda 1_{G/H}$ is a homomorphism of right I-modules and it is monic.

Corollary 1.2. *Let G be a compact Lie group, M be a left I-module and N a right I-module. Then for any pair (X, Y) of G-CW complexes we have natural isomorphisms*
i) ${}_GH_*(X, Y; I(M))\cong\oplus_{H\in C(G)}H_*(X^H/W(H)_0, Y^H/W(H)_0; 1_{G/H}M)$
ii) ${}_GH^*(X, Y; I(N))\cong\Pi_{H\in C(G)}H^*(X^H/W(H)_0, Y^H/W(H)_0; N1_{G/H})$.

By means of Corollary 1.2 we show

Proposition 1.3. *Let G be a compact Lie group, f: $(X, Y)\to(X', Y')$ be a G-map of pairs of G-CW complexes and $n\geqq 0$.*
i) *Let M be a left I-module. If f^*: $H_i(X^K/W(K)_0, Y^K/W(K)_0; 1_{G/H}M)\to H_i(X'^K/W(K)_0, Y'^K/W(K)_0; 1_{G/H}M)$ is an isomorphism for each $i\leqq n$ and every pair $H\leqq K$ in $C(G)$, then f_*; ${}_GH_i(X, Y; M)\to{}_GH_i(X', Y'; M)$ is an isomorphism for each $i\leqq n$.*
ii) *Let N be a right I-module. If f^*: $H^i(X'^K/W(K)_0, Y'^K/W(K)_0; N1_{G/H})\to$*

$H^i(X^K/W(K)_0, Y^K/W(K)_0; N1_{G/H})$ is an isomorphism for each $i\leqq n$ and every pair $H\leqq K$ in $C(G)$, then f^*: ${}_GH^i(X', Y'; N)\to{}_GH^i(X, Y; N)$ is an isomorphism for each $i\leqq n$.

Proof. i) Consider the exact sequence $0\to M_1\to I(M)\xrightarrow{\pi} M\to 0$ of left I-modules. Since the exact sequence

$$0\to 1_{G/H}M_1\to 1_{G/H}I(M)=\oplus_L 1_{G/H}I1_{G/L}\otimes 1_{G/L}M\to 1_{G/H}M\to 0$$

is split as abelian groups, our assumption is maintained for M_1 as well as M. By induction on i, $0\leqq i\leqq n$, we will show that f_*: ${}_GH_i(X, Y; M)\to{}_GH_i(X', Y'; M)$ is an isomorphism. By Corollary 1.2 our assumption implies that f_*: ${}_GH_i(X, Y; I(M))\to{}_GH_i(X', Y'; I(M))$ is an isomorphism for each $i\leqq n$. Using induction hypothesis and the weak four lemma we first verify that f_*: ${}_GH_i(X, Y; M)\to{}_GH_i(X', Y'; M)$ is epic and hence f_*: ${}_GH_i(X, Y; M_1)\to{}_GH_i(X', Y'; M_1)$ is epic, too. Using again induction hypothesis and the weak four lemma we next see that f_*: ${}_GH_i(X, Y; M)\to{}_GH_i(X', Y'; M)$ is monic.

The case ii) is analogously shown, considering the exact sequence $0\to N\xrightarrow{i} I(N)\to N_1\to 0$ of right I-modules.

Let ${}_Gh_*$ and ${}_Gh^*$ be $RO(G)$-graded (or Z-graded) equivariant homology and cohomology theories defined on the category $\mathcal{C}_{\mathcal{G}}$. By definition the composites

$${}_Hh_{\alpha|H}(X, Y)\to{}_Hh_{\alpha|H}(G\overset{\times}{_H}X, G\overset{\times}{_H}Y)\to{}_Gh_\alpha(G\overset{\times}{_H}X, G\overset{\times}{_H}Y)$$

and

$${}_Gh^\alpha(G\overset{\times}{_H}X, G\overset{\times}{_H}Y)\to{}_Hh^{\alpha|H}(G\overset{\times}{_H}X, G\overset{\times}{_H}Y)\to{}_Hh^{\alpha|H}(X, Y)$$

are isomorphisms for each degree $\alpha\in RO(G)$ (or $\in Z$) and each pair $(X, Y)\in\mathcal{C}_{\mathcal{H}}$, taking every closed subgroup H of G (see Kosniowski [10]). Applying entirely the same method adopted in [11] we obtain the following proposition regarded as a generalization of the result [11, Proposition 2].

Proposition 1.4. *Let G be a compact Lie group, (X, Y) be a pair of G-CW complexes and $\alpha\in RO(G)$ (or $\in Z$).*
i) Let ${}_Gh_*$ be an $RO(G)$-graded (or Z-graded) equivariant homology theory. If ${}_Hh_{\alpha|H-i}(X^K, Y^K)=0$ for each $i\geqq 0$ and any pair $H\subset K$ of closed subgroups of G, then ${}_Gh_{\alpha-i}(X, Y)=0$ for each $i\geqq 0$.
ii) Let ${}_Gh^*$ be an $RO(G)$-graded (or Z-graded) equivariant cohomology theory. If ${}_Hh^{\alpha|H-i}(X^K, Y^K)=0$ for each $i\geqq 0$ and any pair $H\subset K$ of closed subgroups of G, then ${}_Gh^{\alpha-i}(X, Y)=0$ for each $i\geqq 0$.

Proof. We first prove the cohomology case ii). We may assume that ${}_Hh^{\alpha|H-i}(X, Y)=0$ for each $i\geqq 0$ and all H in the family F of proper closed subgroups of G and that ${}_Hh^{\alpha|H-i}(X^G, Y^G)=0$ for each $i\geqq 0$ and all H in the family

F_∞ of all closed subgroups of G. There is a commutative diagram with exact rows

$$\begin{array}{ccccccc}
{}_Gh^{\alpha-i-1}[F](X, Y) & \longrightarrow & {}_Gh^{\alpha-i}[F_\infty, F](X, Y) & \longrightarrow & {}_Gh^{\alpha-i}(X, Y) & \longrightarrow & {}_Gh^{\alpha-i}[F](X, Y) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
{}_Gh^{\alpha-i-1}[F](X^G, Y^G) & \rightarrow & {}_Gh^{\alpha-i}[F_\infty, F](X^G, Y^G) & \rightarrow & {}_Gh^{\alpha-i}(X^G, Y^G) & \rightarrow & {}_Gh^{\alpha-i}[F](X^G, Y^G)
\end{array}$$

where all vertical arrows are induced by the inclusion $(X^G, Y^G)\rightarrow(X, Y)$ (see [7]). Observing exactly Segal's spectral sequence in the proof of Jackowski [9, Proposition 1.4], it is easy to check under the above assumptions that ${}_Gh^{\alpha-i}[F](X, Y)=0={}_Gh^{\alpha-i}[F]\ (X^G, Y^G)$ for each $i\geqq 0$. On the other hand, the inclusion $(X^G, Y^G)\rightarrow(X, Y)$ induces an isomorphism ${}_Gh^*[F_\infty, F](X, Y)\xrightarrow{\cong}{}_Gh^*[F_\infty, F](X^G, Y^G)$ as investigated by tom Dieck (see [7, Proposition 7.4.2]). Consequently we obtain that ${}_Gh^{\alpha-i}(X, Y)\cong{}_Gh^{\alpha-i}(X^G, Y^G)=0$ for each $i\geqq 0$.

We next prove the homology case i) by coming back to the cohomology case ii) by duality. For any divisible abelian group A, consider the equivariant cohomology theory ${}_Gh(A)^*$ given by setting ${}_Gh(A)^*(X, Y)=\mathrm{Hom}({}_Gh_*(X, Y), A)$. Applying the cohomology case ii) it follows at once that ${}_Gh(A)^{\alpha-i}(X, Y)=0$ for each $i\geqq 0$ and any divisible abelian group A. Taking the injective resolution $0\rightarrow Z\rightarrow Q\rightarrow Q/Z\rightarrow 0$ of the integers Z, we have that $\mathrm{Hom}({}_Gh_{\alpha-i}(X, Y), Z)=0=\mathrm{Ext}({}_Gh_{\alpha-i}(X, Y), Z)$. This means that ${}_Gh_{\alpha-i}(X, Y)=0$ for each $i\geqq 0$.

Let H be a closed subgroup of G and M be a covariant coefficient system for G and N a contravariant coefficient system for G. If (X, Y) is a pair of trivial H-CW complexes, then we have natural isomorphisms

$${}_HH_*(X, Y; i_H^*M)\cong H_*(X, Y; M(G/H))$$

and

$${}_HH^*(X, Y; i_H^*N_*)\cong H^*(X, Y; N(G/H))$$

where i_H^*M and i_H^*N are the induced coefficient systems for H. Taking Bredon homology and cohomology as ${}_Gh_*$ and ${}_Gh^*$ in the above proposition we have

Corollary 1.5. *Let G be a compact Lie group and (X, Y) be a pair of G-CW complexes and $n\geqq 0$.*
i) *Let M be a left I-module. If $H_i(X^K, Y^K; 1_{G/H}M)=0$ for each $i\leqq n$ and every pair $H\leqq K$ in $C(G)$, then ${}_GH_i(X, Y; M)=0$ for each $i\leqq n$.*
ii) *Let N be a right I-module. If $H^i(X^K, Y^K; N1_{G/H})=0$ for each $i\leqq n$ and every pair $H\leqq K$ in $C(G)$, then ${}_GH^i(X, Y; N)=0$ for each $i\leqq n$.* (Cf., [11, Proposition 4]).

Let h_* and h^* be (non-equivariant) homology and cohomology theories defined on the category C of pairs of CW-complexes. Putting ${}_Gh_*(X)=h_*(X/G)$

and ${}_Gh^*(X)=h^*(X/G)$, ${}_Gh_*$ and ${}_Gh^*$ are equivariant homology and cohomology theories respectively defined on the category $\mathcal{C}_G$.

Corollary 1.6. *Let G be a compact Lie group, H its closed subgroup and $N(H)$ the normalizer of H in G. Let (X, Y) be a pair of G-CW complexes and $n\geqq 0$.*

i) *If $h_i(X^K, Y^K)=0$ for each $i\leqq n$ and each K, $H\subset K\subset N(H)$, then $h_i(X^H/W(H)_0, Y^H/W(H)_0)=0$ for each $i\leqq n$.*

ii) *If $h^i(X^K, Y^K)=0$ for each $i\leqq n$ and each K, $H\subset K\subset N(H)$, then $h^i(X^H/W(H)_0, Y^H/W(H)_0)=0$ for each $i\leqq n$.*

2. h_*-localization of G-CW complexes

For a (non-equivariant) homology theory defined on the category $\mathcal{C}$, let $h_*(\ ;A)$ denote the associated homology theory with coefficients in the abelian group A. Let $\mathcal{A}_G=\{A_H\}$ be a family of abelian groups indexed by $H\in C(G)$.

A based G-map $f\colon X\to Y$ of based G-CW complexes is said to be an *$(h_*, \mathcal{A}_G)$-equivalence* if its H-fixed point map f^H induces an isomorphism $f^H_*\colon h_*(X^H; A_H)\to h_*(Y^H; A_H)$ for every $H\in C(G)$. A based G-CW complex X is said to be *$(h_*, \mathcal{A}_G)$-local* if any $(h_*, \mathcal{A}_G)$-equivalence $f\colon X'\to Y'$ induces a bijection $f^*\colon [Y', X]_G\to[X', X]_G$ where $[\ ,\ ]_G$ denotes the set of G-homotopy classes of based G-maps.

Using the technique developed by Bousfield [3, 5] we can show the existence of $(h_*, \mathcal{A}_G)$-localizations of based G-CW complexes, although we avoid to rely on the simplicial homotopy theory.

Theorem 2.1. *Let G be a compact Lie group, h_* be a homology theory defined on the category $\mathcal{C}$ and $\mathcal{A}_G=\{A_H\}_{H\in C(G)}$ be a family of abelian groups. Given any based G-CW complex X there is a based G-map*

$$\eta_X\colon X\to LX$$

of based G-CW complexes such that η_X is an $(h_*, \mathcal{A}_G)$-equivalence and LX is $(h_*, \mathcal{A}_G)$-local.

The proof is deferred to Appendix. Remark that our construction given in Appendix is not necessarily functorial.

Let A be an abelian group and J_A denote the set of primes p such that A is uniquely p-divisible. Associate with A the special abelian groups

$$S_A=\begin{cases}\oplus_{p\notin J_A}Z/p & \text{if } A\otimes Q=0\\ Z[J_A^{-1}] & \text{if } A\otimes Q\neq 0\end{cases}$$

and

$$S_A^* = \begin{cases} \Pi_{p \notin J_A} Z/p & \text{if } \operatorname{Hom}(Q, A) = 0 = \operatorname{Ext}(Q, A) \\ Z[J_A^{-1}] & \text{if not} \end{cases}$$

Since each homology theory h_* holds a universal coefficient sequence

$$0 \to h_*(X, Y) \otimes A \to h_*(X, Y; A) \to \operatorname{Tor}(h_{*-1}(X, Y), A) \to 0,$$

from [2, Proposition 2.3] it follows easily that

(2.1) $\quad h_*(X, Y; A) = 0 \quad$ *if and only if* $\quad h_*(X, Y; S_A) = 0$

(see [13, Proposition 1.9]). Let h^* be a cohomology theory of finite type, thus its coefficient group $h^i(*)$ is finitely generated for every degree i. Then there exists a homology theory ∇h_* related with the universal coefficient sequence [6, 18]

$$0 \to \operatorname{Ext}(\nabla h_{*-1}(X, Y), A) \to h^*(X, Y; A) \to \operatorname{Hom}(\nabla h_*(X, Y), A) \to 0.$$

As a similar result we have

(2.2) $\quad h^*(X, Y; A) = 0 \quad$ *if and only if* $\quad h^*(X, Y; S_A^*) = 0$.

Moreover, when either $S = \oplus_{p \in J} Z/p$ and $\nabla S = \Pi_{p \in J} Z/p$ or $S = \nabla S = Z[J^{-1}]$ for some set J of primes, we get

(2.3) $\quad h^*(X, Y; \nabla S) = 0 \quad$ *if and only if* $\quad \nabla h_*(X, Y; S) = 0$.

For a family $\mathcal{A}_G = \{A_H\}_{H \in C(G)}$ of abelian groups we define the family $\mathcal{S}_{\mathcal{A}} = \{S_H\}_{H \in C(G)}$ of special abelian groups given by

$$S_H = \begin{cases} \oplus_{p \notin J_H} Z/p & \text{if } A_H \otimes Q = 0 \\ Z[J_H^{-1}] & \text{if } A_H \otimes Q \neq 0 \end{cases}$$

where J_H denotes the set of primes such that A_H is uniquely p-divisible. Clearly (2.1) implies

(2.4) *A based G-CW complex X is $(h_*, \mathcal{A}_G)$-local if and only if it is $(h_*, \mathcal{S}_{\mathcal{A}})$-local.*

Given abelian groups A and B, write $\langle A \rangle \leqq \langle B \rangle$ if $J_A \supset J_B$ and $B \otimes Q = 0$ implies $A \otimes Q = 0$, and $\langle A \rangle^* \leqq \langle B \rangle^*$ if $J_A \supset J_B$ and $\operatorname{Hom}(Q, B) = 0 = \operatorname{Ext}(Q, B)$ implies $\operatorname{Hom}(Q, A) = 0 = \operatorname{Ext}(Q, A)$. Obviously it follows from (2.1) that

(2.5) $\quad h_*(X, Y; B) = 0 \quad$ *implies* $\quad h_*(X, Y; A) = 0$

when $\langle B \rangle \geqq \langle A \rangle$, and under the restriction that h^* is of finite type it follows from (2.2) that

(2.6) $\quad h^*(X, Y; B) = 0 \quad$ *implies* $\quad h^*(X, Y; A) = 0$

when $\langle B\rangle^* \geqq \langle A\rangle^*$.

A family $\mathcal{A}_G=\{A_H\}_{H\in C(G)}$ of abelian groups is said to be (homologically) *order preserving* if $\langle A_H\rangle \leqq \langle A_K\rangle$ for every pair $H\leqq K$ in the partially ordered set $C(G)$.

Lemma 2.2. *Assume that a family $\mathcal{A}_G=\{A_H\}$ is order preserving. If a based G-CW complex X is $(h_*, \mathcal{A}_G)$-local, then its H-fixed point space X^H is $h_*(\ ; A_H)$-local for every $H\in C(G)$.*

Proof. Let f: $U\to V$ be a based map which induces an isomorphism f_*: $h_*(U; A_H)\to h_*(V; A_H)$ for a fixed $H\in C(G)$. Then $1_\wedge f$: $(G/H)_+\wedge U\to (G/H)_+\wedge V$ becomes an $(h_*, \mathcal{A}_G)$-equivalence under the hypothesis on $\mathcal{A}_G=\{A_H\}$. Therefore it induces a bijection $(1_\wedge f)^*$: $[(G/H)_+\wedge V, X]_G\to[(G/H)_+\wedge U, X]_G$. This means that f^*: $[V, X^H]\to[U, X^H]$ is certainly a bijection, too.

Owing to the existence theorem of $(h_*, \mathcal{A}_G)$-localizations we show that the converse of Lemma 2.2 is also valid.

Proposition 2.3. *Assume that a family $\mathcal{A}_G=\{A_H\}$ is order preserving. Then a based G-CW complex X is $(h_*, \mathcal{A}_G)$-local if and only if its H-fixed point spaces X^H are $h_*(\ ; A_H)$-local for all $H\in C(G)$.*

Proof. We have to show only the "if" part. Let η: $X\to LX$ be an $(h_*, \mathcal{A}_G)$-localization of X. By the "only if" part, Lemma 2.2, $(LX)^H$ is $h_*(\ ; A_H)$-local for each $H\in C(G)$. Since the H-fixed point map η^H: $X^H\to (LX)^H$ is an $h_*(\ ; A_H)$-equivalence, we see easily that η^H: $X^H\to(LX)^H$ is a homotopy equivalence for each $H\in C(G)$. Thus η: $X\to LX$ itself is a homotopy equivalence, and hence X is $(h_*, \mathcal{A}_G)$-local as desired.

An Eilenberg-MacLane G-space $K(N, n)$ is a based G-CW complex by which the n-th (reduced) Bredon cohomology group ${}_G\tilde{H}^n(X; N)$ with coefficients in N is represented as ${}_G\tilde{H}^n(X; N)\cong[X, K(N, n)]_G$. For Eilenberg-MacLane G-spaces $K(N, n)$ we can give another proof of Proposition 2.3 without use of the existence theorem of $(h_*, \mathcal{A}_G)$-localizations.

Proposition 2.4 (as a special case of Proposition 2.3). *Assume that a family $\mathcal{A}_G=\{A_H\}$ is order preserving. Let N be a right I-module. Then an Eilenberg-MacLane G-space $K(N, n)$ is $(h_*, \mathcal{A}_G)$-local if and only if Eilenberg-MacLane spaces $K(N1_{G/H}, n)$ are $h_*(\ ; A_H)$-local for all $H\in C(G)$.*

Proof. The "if" part: Let f: $X\to Y$ be an $(h_*, \mathcal{A}_G)$-equivalence. Then by Corollary 1.6 f_*: $h_*(X^K/W(K)_0; A_H)\to h_*(Y^K/W(K)_0; A_H)$ is an isomorphism for each pair $H\leqq K$ in $C(G)$. Since the Eilenberg-MacLane space $K(N1_{G/H}, m)$ is $h_*(\ ; A_H)$-local for any $m\leqq n$, f^*: $H^m(Y^K/W(K)_0; N1_{G/H})\to H^m(X^K/W(K)_0; N1_{G/H})$ is an isomorphism for any $m\leqq n$ and every pair $H\leqq K$ in $C(G)$. Using

Proposition 1.3 we observe that $f^*\colon {}_GH^n(Y; N)\to {}_GH^n(X; N)$ is an isomorphism, thus the Eilenberg-MacLane G-space $K(N, n)$ is $(h_*, \mathcal{A}_G)$-local.

3. H_*-localization of G-nilpotent G-CW complexes

Let $\mathcal{A}_G=\{A_H\}_{H\in C(G)}$ be a family of abelian groups which is order preserving and N be a right I-module. For each $H\in C(G)$ put

$$E_{\mathcal{A}}N(G/H) = \begin{cases} \Pi_{p\notin J_H} \mathrm{Ext}(Z_{p^\infty}, N1_{G/H}) & \text{if } A_H\otimes Q=0 \\ N1_{G/H}\otimes Z[J_H^{-1}] & \text{if } A_H\otimes Q\neq 0 \end{cases}$$

and

$$H_{\mathcal{A}}N(G/H)= \begin{cases} \Pi_{p\notin J_H} \mathrm{Hom}(Z_{p^\infty}, N1_{G/H}) & \text{if } A_H\otimes Q=0 \\ 0 & \text{if } A_H\otimes Q\neq 0 \end{cases}$$

where J_H denotes the set of primes p such that A_H is uniquely p-divisible and $Z_{p^\infty}=\varinjlim Z/p^n$. As is easily seen, setting $E_{\mathcal{A}}N=\oplus_{H\in C(G)}E_{\mathcal{A}}N(G/H)$ it is a right I-module and the canonical map $l\colon N=\oplus_H N1_{G/H}\to E_{\mathcal{A}}N=\oplus_H E_{\mathcal{A}}N(G/H)$ is a homomorphism of right I-modules. Similarly for $H_{\mathcal{A}}N$. Note that $H_{\mathcal{A}}N=0$ if N is torsion free as an abelian group.

Lemma 3.1. *Assume that a family $\mathcal{A}_G=\{A_H\}$ is order preserving. Let N be a right I-module such that $H_{\mathcal{A}}N=0$. Then the induced G-map*

$$\eta_N = l_*\colon K(N, n)\to K(E_{\mathcal{A}}N, n)\,, \qquad n\geqq 1\,,$$

is an $(H_*, \mathcal{A}_G)$-localization.

Proof. According to Bousfield [3, Proposition 4.3] the H-fixed point map $\eta_N^H\colon K(N1_{G/H}, n)\to K(E_{\mathcal{A}}N(G/H), n)$ is an $H_*(\ ; A_H)$-localization. Now the result follows from Proposition 2.4 (or Proposition 2.3).

Let M be a left I-module. We say a based G-map $f\colon X\to Y$ of based G-CW complexes an *${}_GH_*(\ ; M)$-equivalence* if $f_*\colon {}_GH_*(X; M)\to {}_GH_*(Y; M)$ is an isomorphism, and a based G-CW complex X *${}_GH_*(\ ; M)$-local* if any ${}_GH_*(\ ; M)$-equivalence $f\colon X'\to Y'$ induces a bijection $f^*\colon [Y', X]_G\to [X', X]_G$. Similarly for *${}_GH^*(\ ; N)$-equivalence* and *${}_GH^*(\ ; N)$-local space* when N is a right I-module.

For the order preserving family $\mathcal{A}_G=\{A_H\}_{H\in C(G)}$ we put $I(\mathcal{A})=\oplus_{H\in C(G)}I1_{G/H}\otimes A_H$, which is a left I-module.

Lemma 3.2. *Assume that $\mathcal{A}_G=\{A_H\}$ is order preserving. Let N be a right I-module such that $H_{\mathcal{A}}N=0$. Then the Eilenberg-MacLane G-space $K(E_{\mathcal{A}}N, n)$ is ${}_GH_*(\ ; I(\mathcal{A}))$-local for every $n\geqq 1$.*

Proof. By use of Lemma 1.1, (2.1) and (2.3) we can see that the following four conditions are all equivalent:
i) a based G-map $f: X\to Y$ is an ${}_GH_*(\ ;I(\mathcal{A}))$-equivalence,
ii) each $f^H: X^H/W(H)_0\to Y^H/W(H)_0$ is an $H_*(\ ;A_H)$-equivalence,
iii) each f^H is an $H_*(\ ;S_H)$-equivalence,
iv) each f^H is an $H^*(\ ;\nabla S_H)$-equivalence
where $S_H=\oplus_{p\notin J_H}Z/p$ and $\nabla S_H=\Pi_{p\notin J_H}Z/p$ if $A_H\otimes Q=0$, or $S_H=\nabla S_H=Z[J_H^{-1}]$ if $A_H\otimes Q\neq 0$. Obviously $\langle\nabla S_H\rangle^*\geqq\langle E_{\mathcal{A}}N(G/H)\rangle^*$, hence (2.5) says that the condition iv) implies
v) each f^H is an $H^*(\ ;E_{\mathcal{A}}N(G/H))$-equivalence.
Moreover it follows from Proposition 1.3 that the condition v) implies
vi) $f: X\to Y$ is an ${}_GH^*(\ ;E_{\mathcal{A}}N)$-equivalence.
Therefore the Eilenberg-MacLane G-space $K(E_{\mathcal{A}}N, n)$ is ${}_GH_*(\ ;I(\mathcal{A}))$-local as desired.

Following [11] we say a based G-CW complex X *G-nilpotent* if each X^H is connected and nilpotent and if for every $n\geqq 1$ the orders of nilpotency of the $\pi_1(X^H)$-groups $\pi_n(X^H)$ have a common bound for varying H. According to [11, Proposition 8] we have the following analogous result to the non-equivariant case.

(3.1) If a based G-CW complex X is G-nilpotent, then there is a (nilpotent) G-tower $\mathcal{X}=\{X_n\}$ such that
i) X is weakly G-homotopy equivalent to the inverse limit of X_n.
ii) $X_0=\{*\}$ and X_{n+1} is the fiber of a based G-map $k_n: X_n\to K(N_n, q_n)$ where $q_n\geqq 2$, and
iii) $q_{n+1}\geqq q_n$ and only finitely many $q_n=r$ for each r.

Theorem 3.3. *Let $\mathcal{A}_G=\{A_H\}_{H\in C(G)}$ be a family of abelian groups which is order preserving and denote $I(\mathcal{A})=\oplus_{H\in C(G)}I1_{G/H}\otimes A_H$. Given any G-nilpotent G-CW complex X there exists a based G-map*

$$\eta_X: X\to L_{\mathcal{A}}X$$

of G-nilpotent G-CW complexes such that
i) *its H-fixed point map $\eta_X^H: X^H\to(L_{\mathcal{A}}X)^H$ is an $H_*(\ ;A_H)$-localization for each $H\in C(G)$, and*
ii) *$L_{\mathcal{A}}X$ is ${}_GH_*(\ ;I(\mathcal{A}))$-local.*

Proof. First take a right I-module N and an exact sequence $0\to F_2\to F_1\to N\to 0$ of right I-modules such that F_1 is projective. Note that both F_1 and F_2 are free as abelian groups. Denote by $L_{\mathcal{A}}K(N, n)$, $n\geqq 1$, the fiber of the G-map $K(E_{\mathcal{A}}F_2, n+1)\to K(E_{\mathcal{A}}F_1, n+1)$. It is a 2-stage G-CW complex with homotopy groups only in dimensions n and $n+1$. We have a dotted arrow

η_N: $K(N, n) \to L_{\mathcal{A}}\, K(N, n)$ making the diagram below G-homotopy commutative

$$\begin{array}{ccccccc} K(F_1, n) & \to & K(N, n) & \to & K(F_2, n+1) & \to & K(F_1, n+1) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K(E_{\mathcal{A}}F_1, n) & \to & L_{\mathcal{A}}K(N, n) & \to & K(E_{\mathcal{A}}F_2, n+1) & \to & K(E_{\mathcal{A}}F_1, n+1)\,. \end{array}$$

By use of the Serre spectral sequences and Lemma 3.1 we see easily that η_N is an $(H_*, \mathcal{A}_G)$-equivalence and by Lemma 3.2 that $L_{\mathcal{A}}K(N, n)$ is ${}_GH_*(\ ;\ I(\mathcal{A}))$-local.

For a G-nilpotent G-CW complex X we may regard that it is given as the inverse limit of a nilpotent G-tower $\mathcal{X} = \{X_n\}$. Inductively we can construct ${}_GH_*(\ ;\ I(\mathcal{A}))$-local spaces $L_{\mathcal{A}}X_{n+1}$ being the fiber of $L_{\mathcal{A}}k_n$: $L_{\mathcal{A}}X_n \to L_{\mathcal{A}}K(N_n, q_n)$, and also $(H_*, \mathcal{A}_G)$-equivalences η_{n+1}: $X_{n+1} \to L_{\mathcal{A}}X_{n+1}$ such that the following diagram is G-homotopy commutative

$$\begin{array}{ccccccc} K(N_n, q_n-1) & \to & X_{n+1} & \to & X_n & \to & K(N_n, q_n) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ L_{\mathcal{A}}K(N_n, q_n-1) & \to & L_{\mathcal{A}}X_{n+1} & \to & L_{\mathcal{A}}X_n & \to & L_{\mathcal{A}}K(N_n, q_n)\,. \end{array}$$

Put as $L_{\mathcal{A}}X$ a G-CW approximation of the inverse limit space of $L_{\mathcal{A}}X_n$. Then the above construction shows that $L_{\mathcal{A}}X_n$ is G-nilpotent for each n and that $L_{\mathcal{A}}X$ is also G-nilpotent (see [8]). So we obtain a desired localization map η_X: $X \to L_{\mathcal{A}}X$.

Because of the localization theorem 3.3 we have the following characterization, although it is trivial if G is a finite group.

Proposition 3.4. *Assume that a family $\mathcal{A}_G = \{A_H\}$ is order preserving. On a based G-map f: $X \to Y$ the following three conditions are equivalent*:
i) f^H_*: $H_*(X^H; A_H) \to H_*(Y^H; A_H)$ *is an isomorphism for each* $H \in C(G)$,
ii) f^H_*: $H_*(X^H/W(H)_0; A_H) \to H_*(Y^H/W(H)_0; A_H)$ *is an isomorphism for each* $H \in C(G)$, *and*
iii) f_*: ${}_GH_*(X;\ I(\mathcal{A})) \to {}_GH_*(Y;\ I(\mathcal{A}))$ *is an isomorphism.*
(Cf., [12, Proposition 6]).

Proof. Lemma 1.1 asserts that the conditions ii) and iii) are equivalent, and Corollary 1.5 says that the condition i) implies iii). It remains to show only the implication iii)$\to$i). Consider the G-homotopy commutative square

$$\begin{array}{ccc} S^2X & \xrightarrow{S^2f} & S^2Y \\ {\scriptstyle \eta_{S^2X}}\downarrow & & \downarrow{\scriptstyle \eta_{S^2Y}} \\ L_{\mathcal{A}}S^2X & \longrightarrow & L_{\mathcal{A}}S^2Y. \end{array}$$

Then the condition iii) implies that $L_{\mathcal{A}}S^2f$: $L_{\mathcal{A}}S^2X \to L_{\mathcal{A}}S^2Y$ is an ${}_GH_*(\ ;\ I(\mathcal{A}))$-equivalence, and hence it becomes a G-homotopy equivalence since $L_{\mathcal{A}}S^2X$

and $L_{\mathcal{A}}S^2Y$ are both ${}_GH_*(\ ;I(\mathcal{A}))$-local. Hence $f\colon X\to Y$ is certainly an $(H_*, \mathcal{A}_G)$-equivalence.

As a dual of Proposition 3.4 we have

Corollary 3.5. *Let $\mathcal{A}_G=\{A_H\}_{H\in C(G)}$ be a family of abelian groups such that $\langle A_H\rangle^*\leqq\langle A_K\rangle^*$ for every pair $H\leqq K$ in $C(G)$. On a based G-map $f\colon X\to Y$ the following three conditions are equivalent:*
i) *$f^{H*}\colon H^*(Y^H;A_H)\to H^*(X^H;A_H)$ is an isomorphism for each $H\in C(G)$,*
ii) *$f^{H*}\colon H^*(Y^H/W(H)_0;A_H)\to H^*(X^H/W(H)_0;A_H)$ is an isomorphism for each $H\in C(G)$, and*
iii) *$f^*\colon {}_GH^*(Y;I(\mathcal{A})^*)\to {}_GH^*(X;I(\mathcal{A})^*)$ is an isomorphism. Here $I(\mathcal{A})^*=\oplus_K\Pi_H \operatorname{Hom}(1_{G/K}I1_{G/H}, A_H)$.*

Proof. Use Lemma 1.1, Corollary 1.5, (2.2), (2.3) and Proposition 3.4.

4. K_*-localization of Eilenberg-MacLane G-spaces

Denote by $K_*(\ ;A)$ and $K^*(\ ;A)$ respectively the complex homology and cohomology K-theories with coefficients in A. Let BUA be the connected component of the base point of the CW-complex which represents $K^0(\ ;A)$. As a consequence of [14, Theorem 1.11] we notice that

(4.1) *$BU(A\otimes R)$ is $K_*(\ ;R)$-local when $R=Z[J^{-1}]$ is a subring of the rationals Q.*

Moreover, Mislin [14, Lemma 2.1] showed that

(4.2) *the Eilenberg-MacLane space $K(A, 2)$ is a factor of BUA if the abelian group A is torsion free.*

Combining (4.1) with (4.2) we obtain examples of $K_*(\ ;R)$-local spaces. If R is a subring of Q, then

(4.3) *the Eilenberg-MacLane spaces $K(A\otimes R, 1)$ and $K(A/T\otimes R, 2)$ are $K_*(\ ;R)$-local where T denotes the torsion subgroup of A.*

By aid of the computation of Mislin [14, Theorem 2.2] we get immediately

Proposition 4.1. *Let A be an abelian group, T its torsion subgroup and R be a subring of Q. Then the following induced maps are respectively $K_*(\ ;R)$-localizations:*
i) $K(A, 1)\to K(A\otimes R, 1)$
ii) $K(A, 2)\to K((A/T)\otimes R, 2)$
iii) $K(A, n)\to K(A\otimes Q, n)$ *for $n\geqq 3$.*

When $R=\oplus_{p\in J}Z/p$ for some set J of primes, we consider the cofibering

$\bigvee_\alpha S^1 \xrightarrow{j} \bigvee_\beta S^1 \to M_R$ associated with a free resolution $0 \to \bigoplus_\alpha Z \to \bigoplus_\beta Z \to \bigoplus_{p \in J} Z_{p^\infty} \to 0$. M_R is a Moore space of type $(\bigoplus_{p \in J} Z_{p^\infty}, 1)$.

Lemma 4.2. *Let A be a torsion free abelian group and $R = \bigoplus_{p \in J} Z/p$. Putting $E_R A = \Pi_{p \in J} \operatorname{Ext}(Z_{p^\infty}, A)$, then $BUE_R A$ is homotopy equivalent to the connected component of the constant map of the based mapping space $F(M_R, BUA)$.*

Proof. It is sufficient to show that there is a natural isomorphism between $\tilde{K}^0(X; E_R A)$ and $\tilde{K}^0(X_\wedge M_R; A)$ for any based CW-complex X. We work in the category of CW-spectra. Let MA be a Moore spectrum of type A and $ME_R A$ of type $E_R A$. Consider the pairing u_α: $(\prod_\alpha MA)_\wedge(\bigvee_\alpha S^2) \to S^2 MA$ induced by the projections p_α: $(\prod_\alpha MA)_\wedge S^2 \to S^2 MA$. The canonical morphism $K_\wedge \prod_\alpha MA \to \prod_\alpha K_\wedge MA$ is a homotopy equivalence since $\prod_\alpha MA$ is a Moore spectrum of type $\prod_\alpha A$ (see [18, Lemma 4] or [1]). Thus the map

$$T(u_\alpha)_K\colon \{X, K_\wedge \prod_\alpha MA\} \to \{X_\wedge(\bigvee_\alpha S^2), K_\wedge S^2 MA\}$$

defined by $T(f) = (1_\wedge u_\alpha)(f_\wedge 1)$ for any CW-spectrum X, is an isomorphism. Similarly for u_β. Consider the homotopy commutative square

$$\begin{array}{ccc} (\prod_\beta MA)_\wedge(\bigvee_\alpha S^2) & \xrightarrow{k_\wedge 1} & (\prod_\alpha MA)_\wedge(\bigvee_\alpha S^2) \\ {\scriptstyle 1_\wedge j}\big\downarrow & & \big\downarrow{\scriptstyle u_\alpha} \\ (\prod_\beta MA)_\wedge(\bigvee_\beta S^2) & \xrightarrow[u_\beta]{} & S^2 MA \end{array}$$

where the map k: $\prod_\beta MA \to \prod_\alpha MA$ is one induced by the map j: $\bigvee_\alpha S^1 \to \bigvee_\beta S^1$. Then, by [18, Lemma 1] (or [15, Theorem 6.10]) there exists a nice map

$$w\colon ME_R A_\wedge M_R \to S^2 MA$$

which induces an isomorphism

$$T(w)_K\colon \{X, K_\wedge ME_R A\} \xrightarrow{\cong} \{X_\wedge M_R, K_\wedge S^2 MA\}$$

for any CW-spectrum X. Composing the Bott isomorphism with the above map we obtain a natural isomorphism

$$\tilde{K}^*(X; E_R A) \xrightarrow{\cong} \tilde{K}^*(X_\wedge M_R; A)$$

for any CW-spectrum X, and in particular for any based CW-complex X.

Applying Mislin's method [14, Corollary 2.5] with Lemma 4.2 we have

Lemma 4.3. *Let A be a torsion free abelian group and $R = \bigoplus_{p \in J} Z/p$. Then* *the Eilenberg-MacLane space $K(E_R A, 2)$ is $K_*(\ ; R)$-local.*

Proof. Let $f: X \to Y$ be a $K_*(\ ; R)$-equivalence. Then $f_\wedge 1: X_\wedge M_R \to Y_\wedge M_R$ is clearly a K_*-equivalence. Therefore the based mapping space $F(M_R, BUA)$ is $K_*(\ ; R)$-local because BUA is K_*-local by (4.1). On the other hand, by (4.2) the Eilenberg-MacLane space $K(E_R A, 2)$ is a factor of $BUE_R A$ since the abelian group $E_R A$ is torsion free. We use Lemma 4.2 to obtain that $K(E_R A, 2)$ is $K_*(\ ; R)$-local.

By use of Lemma 4.3 we obtain

Proposition 4.4. *Let A be an abelian group, T its torsion subgroup and $R = \oplus_{p \in J} Z/p$ for some set J of primes. Then the following canonical maps are respectively $K_*(\ ; R)$-localizations:*

i) $K(A, 1) \to L_R K(A, 1)$

ii) $K(A, 2) \to K(E_R(A/T), 2) = L_R K(A/T, 2)$

iii) $K(A, n) \to \{*\}$ *for* $n \geqq 3$

where $L_R X$ denotes the $H_*(\ ; R)$-localization of X and $E_R(A/T) = \Pi_{p \in J} \mathrm{Ext}(Z_{p^\infty}, A/T)$. (Cf., [14, Corollaries 2.3 and 2.5]).

Proof. i) Take a free resolution $0 \to F_2 \to F_1 \to A \to 0$. Since $L_R K(A, 1)$ is the fiber of the map $K(E_R F_2, 2) \to K(E_R F_1, 2)$, it is $K_*(\ ; R)$-local by Lemma 4.3. The $H_*(\ ; R)$-localization map $\eta_A: K(A, 1) \to L_R K(A, 1)$ is obviously a $K_*(\ ; R)$-equivalence.

ii) In the composite map $K(A, 2) \to K(A/T, 2) \to K(E_R(A/T), 2)$ the former is a K_*-equivalence and the latter is an $H_*(\ ; R)$-equivalence, and hence the composite map is a $K_*(\ ; R)$-equivalence.

iii) For $n \geqq 3$ the constant map $K(A, n) \to \{*\}$ is certainly a $K_*(\ ; Z/p)$-equivalence (see [17, Theorem 2.7]).

Let $\mathcal{A}_G = \{A_H\}_{H \in C(G)}$ be an order preserving family of abelian groups and N be a right I-module. For each $H \in C(G)$ put

$$L^K_{\mathcal{A}} N(G/H) = \begin{cases} 0 & \text{if } A_H \otimes Q = 0 \\ N1_{G/H} \otimes Q & \text{if } A_H \otimes Q \neq 0 . \end{cases}$$

Then $L^K_{\mathcal{A}} N = \oplus_{H \in C(G)} L^K_{\mathcal{A}} N(G/H)$ is a right I-module. And the canonical map $l': N = \oplus_H N1_{G/H} \to L^K_{\mathcal{A}} N = \oplus_H L^K_{\mathcal{A}} N(G/H)$ is a homomorphism of right I-modules, which induces a G-map

$$\eta'_N = l'_*: K(N, n) \to K(L^K_{\mathcal{A}} N, n) .$$

Theorem 4.5. *Assume that a family $\mathcal{A}_G = \{A_H\}_{H \in C(G)}$ of abelian groups is order preserving. Let N be a right I-module and T its torsion subgroup. Then the following maps are all $(K_*, \mathcal{A}_G)$-localizations:*

i) *the* $(H_*, \mathcal{A}_G)$*-localization* $\eta_N\colon K(N, 1)\to L_{\mathcal{A}}K(N, 1)$,

ii) *the composite map* $K(N, 2)\to K(N/T, 2)\xrightarrow{\eta_{N/T}} K(E_{\mathcal{A}}(N/T), 2)=L_{\mathcal{A}}K(N/T, 2)$,

iii) *the induced map* $\eta'_N\colon K(N, n)\to K(L^K_{\mathcal{A}}N, n)$ *for* $n\geqq 3$.

Proof. Putting Propositions 4.1 and 4.4 together we can check that all the H-fixed point maps in the theorem are $K_*(\ ; A_H)$-localizations for any $H\in C(G)$.

Appendix. Proof of the existence theorem of the localization

Let σ be a fixed infinite cardinal number such that Car $\bigoplus_{H\in C(G)}h_*(*; A_H)\leqq\sigma$ where the abelian groups A_H belong to the family $\mathcal{A}_G$. For a based G-CW complex X, let $\sharp X$ denote the number of G-cells in X.

Lemma A.1. *Let* (X, Y) *be a pair of based G-CW complexes such that* $h_*(X^H, Y^H; A_H)=0$ *for each* $H\in C(G)$, *and* W_0 *be a G-CW subcomplex of* X *with* $\sharp W_0\leqq\sigma$. *Then there exists a G-CW subcomplex W of X such that* $\sharp W\leqq\sigma$, $W_0\subset W\not\subset Y$ *and* $h_*(W^H, W^H\cap Y^H; A_H)=0$ *for each* $H\in C(G)$. (Cf., [3, Lemma 11.2]).

Proof. We construct a sequence of G-CW subcomplexes of X

$$W_0\subset W_1\subset W_2\subset\cdots\subset W_n\subset\cdots$$

such that $\sharp W_n\leqq\sigma$, $W_n\not\subset Y$ and the map $h_*(W^H_n, W^H_n\cap Y^H; A_H)\to h_*(W^H_{n+1}, W^H_{n+1}\cap Y^H; A_H)$ is zero for $n\geqq 1$ and each $H\in C(G)$. First, choose $W_1\subset X$ such that $\sharp W_1\leqq\sigma$ and $W_0\subset W_1\not\subset Y$, and construct inductively W_n. Choose properly a finite subcomplex F_x of X^H for each element $x\in h_*(W^H_n, W^H_n\cap Y^H; A_H)$ and take as W_{n+1} the union of W_n with all $G\cdot F_x$, then each x goes to zero in $h_*(W^H_{n+1}, W^H_{n+1}\cap Y^H; A_H)$ and $\sharp W_{n+1}\leqq\sigma$. Finally we put $W=\bigcup_{n\geqq 1}W_n$ to obtain the desired one.

Lemma A.2. *Let X be a based G-CW complex. Assume that for any inclusion map* $i_\alpha\colon Y_\alpha\to Z_\alpha$ *with* $\sharp Z_\alpha\leqq\sigma$ *such that it is an* $(h_*, \mathcal{A}_G)$*-equivalence,* $i^*_\alpha\colon[Z_\alpha, X]_G\to[Y_\alpha, X]_G$ *is onto. Then X is* $(h_*, \mathcal{A}_G)$*-local.* (Cf., [3, Lemmas 2.5 and 11.3]).

Proof. Let $f\colon Y\to Z$ be an $(h_*, \mathcal{A}_G)$-equivalence. We may regard Y as a G-CW subcomplex of Z and f as the inclusion $Y\subset Z$. Let γ be an infinite ordinal of cardinality greater than $\sharp Z-\sharp Y$. Using Lemma A.1 we can construct a transfinite sequence

$$Y=Y_0\subset Y_1\subset\cdots\subset Y_s\subset Y_{s+1}\subset\cdots$$

of G-CW subcomplexes of Z such that i) if λ is a limit ordinal then $Y_\lambda=$

$\bigcup_{s<\lambda} Y_s$, ii) if $Y_s=Z$ then $Y_{s+1}=Z$, and iii) if $Y_s \neq Z$ then $Y_{s+1}=Y_s \cup W$ for some $W \subset X$ with $\# W \leqq \sigma$, $W \not\subset Y$ and $h_*(W^H, W^H \cap Y_s^H; A_H)=0$ for each $H \in C(G)$. Clearly $Z=Y_\gamma$, and $f^*\colon [Z, X]_G \to [Y, X]_G$ is onto. Take two based G-maps $g, h\colon Z \to X$ such that $f^*g=f^*h \in [Y, X]_G$, to show the injectivity of f^*. By the $(h_*, \mathcal{A}_G)$-version of [3, Lemma 3.6] there exists a based G-CW complex $\tilde{X}$ and an $(h_*, \mathcal{A}_G)$-equivalence $j\colon X \to \tilde{X}$ such that $j_*g=j_*h \in [Z, \tilde{X}]_G$. Since we can find a left inverse $k\colon \tilde{X} \to X$ of j, it follows immediately that f^* is in fact a bijection.

Proof of Theorem 2.1. Choose a set $\{i_\alpha\colon Y_\alpha \to Z_\alpha\}_{\alpha \in I}$ of inclusion maps with $\# Z_\alpha \leqq \sigma$ which are $(h_*, \mathcal{A}_G)$-equivalences, such that it contains up to isomorphism each inclusion maps with these properties. Let γ be the first infinite ordinal of cardinality greater than σ. We inductively construct a transfinite sequence of based G-CW complexes

$$X=X_0 \subset X_1 \subset \cdots \subset X_s \subset X_{s+1} \subset \cdots$$

where $X_\lambda=\bigcup_{s<\lambda} X_s$ for each limit ordinal λ and where $X_s \subset X_{s+1}$ is given by the push-out square

$$\begin{array}{ccc} \bigvee_\alpha \bigvee_{f\colon Y_\alpha \to X_s} Y_\alpha & \to & X_s \\ \downarrow & & \downarrow \\ \bigvee_\alpha \bigvee_{f\colon Y_\alpha \to X_s} Z_\alpha & \to & X_{s+1} \end{array}$$

Putting $LX=X_\gamma$, the inclusion $\eta\colon X \to LX$ is an $(h_*, \mathcal{A}_G)$-equivalence. Since each based G-map $f\colon Y_\alpha \to LX$ passes through X_s for some $s<\gamma$, $i_\alpha^*\colon [Z_\alpha, LX]_G \to [Y_\alpha, LX]_G$ is onto for any $\alpha \in I$. By means of Lemma A.2 we observe that LX is $(h_*, \mathcal{A}_G)$-local.

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