Kuwako, K.
Osaka J. Math.
43 (2006), 401–412

DEHN SURGERY CREATING KLEIN BOTTLES

Dedicated to Professor Yukio Matsumoto for his 60th birthday

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(Received February 28, 2005, revised April 15, 2005)

Abstract

We consider Dehn surgery creating Klein bottles. It is shown that if a non-cabled knot, which is not the figure eight knot, admits two Dehn surgeries creating Klein bottles, then the geometric intersection number of these slopes is bounded by four.

1. Introduction

In this paper, we treat Dehn surgery creating Klein bottles. There are results concerning Dehn surgeries which create closed non-orientable surfaces. Ichihara and Teragaito [7, 8] showed that if a nontrivial knot in S^3 admits Dehn surgery creating Klein bottles, then the geometric intersection number of this slope and the longitude class is bounded by $4g+4$. Here g is the genus of a knot. Matignon and Sayari [11] also examined Dehn surgery on a nontrivial knot in S^3 which produce a closed non-orientable surface with higher genus. In this paper, we examine the geometric intersection number of slopes creating Klein bottles. We remark that Lee [9] and Matignon and Sayari [12] have announced some extensions to our results independently.

To state our theorem, let us introduce some definitions needed in this paper. Let K be a nontrivial knot in S^3. The knot K is called a *cabled knot* if there is an embedded (maybe knotted) torus in S^3 such that K is isotoped on this torus and that K runs longitudinally at least twice. For example, any torus knot is a cabled knot. If K is not a cabled knot, then K is called a *non-cabled* knot. In this paper, we assume that K is non-cabled.

Let $N(K)$ be a regular neighborhood of K and M the exterior of K, $S^3 - \operatorname{int} N(K)$. The unoriented isotopy class of a nontrivial simple closed curve on ∂M is called a *slope*. We parametrize slopes as in [13], that is, if a simple closed curve c on ∂M runs meridionally a times and longitudinally b times, then the slope of c is a/b. For example, the meridian class μ has a slope $1/0$ and the (preferred) longitude class λ has a slope $0/1$. Under this parametrization, if a slope r is equal to $a/1$ ($a \in \mathbf{Z}$), then r is called an *integral slope*.

2000 Mathematics Subject Classification. Primary 57M25.

In this paper, $M(r)$ means the closed manifold obtained by r-Dehn filling, that is, $M(r)$ is a manifold $M \cup_f J$. Here, J is a solid torus and f is a homeomorphism from ∂J to ∂M which sends the boundary of a meridian disk of J to an essential simple closed curve representing the slope r.

Theorem 1.1. *Let K be a non-cabled knot in S^3. Assume that K is a nontrivial knot, which is not the figure eight knot. If there are two slopes r and s such that Dehn surgered manifolds $M(r)$ and $M(s)$ both contain Klein bottles, then the geometric intersection number $\Delta(r, s)$ is bounded by four.*

EXAMPLE 1.2 ([6]). Let K be the $(-2, 3, 7)$-pretzel knot. Then 16-surgery and 20-surgery yield Klein bottles.

This paper is organized as follows. We prove Theorem 1.1 by using combinatorial methods. In Section 2, we recall properties of Klein bottles created after Dehn surgery. The material of this section is proved in [14, 15]. In Section 3, we examine graphs of intersection. Roughly speaking, we prove that there cannot be too many parallel edges. Section 4 is devoted to a proof of Theorem 1.1. We divide the proof into two cases according to whether one of the created Klein bottles is once-punctured in the knot complement or not. Most of this section is devoted to the case where both of created Klein bottles are not once-punctured in the knot complement. In this case, we use an inequality obtained by [11] which tells us an upper bound of geometric intersection number.

2. Klein bottles created after surgery

In this section, we recall fundamental facts about Klein bottles created after Dehn surgery. If a surgered manifold $M(r)$ contains Klein bottles, then this slope r has the following property.

Lemma 2.1 ([4, Theorem 1.3], [14, Lemma 2.4]). *Slope r is integral, and is a multiple of four.*

REMARK 2.2. Combining this lemma and our theorem, we find that there are at most two surgeries creating Klein bottles in our situation.

We use a combinatorial method to prove Theorem 1.1. Let us recall basic notions and fix our notation. Let $\widehat{P}$ (resp. $\widehat{Q}$) be a created Klein bottle in $M(r)$ (resp. $M(s)$). Assume that $P := \widehat{P} \cap M$ (resp. $Q := \widehat{Q} \cap M$) is chosen to minimize the number of boundary components. Let p (resp. q) be the number of the boundary components of P (resp. Q). By a small perturbation, we can assume that P and Q intersect transversely and that each component of ∂P meets each component of ∂Q exactly $\Delta(r, s)$ times. Then the following holds. For a detailed proof, consult [14].

Lemma 2.3. (1) *Both p and q are odd.*
(2) *Both P and Q are incompressible and boundary incompressible in M.*
(3) *No arc component of $P \cap Q$ is boundary-parallel in P and in Q.*

Proof. (1) If p is even, then we can construct a closed non-orientable surface in $M \subset S^3$. Since S^3 cannot contain a closed non-orientable surface, this leads to a contradiction.

(2) See [14, Lemmas 2.1, 2.2].

(3) If there exists such an arc, then it contradicts (2) (See [14, Lemma 3.1]). $\square$

Let $G_P \subset \widehat{P}$ be a graph of intersection with Q. That is, vertices of G_P are disks $\widehat{P} - \operatorname{int} P$, edges of G_P are arc components of $P \cap Q$ in P. We often call such vertices as *fat vertices*. We define a graph $G_Q \subset \widehat{Q}$ similarly.

We give an orientation on all components of ∂P so that they are all homologous on ∂M. Let e be an edge of G_P. We may assume that a regular neighborhood $N(e)$ of e is a disk in P. Let a and b are segments of $\partial N(e) \cap \partial P$. We can give an induced orientations on a and b from the orientation of ∂P. If orientations of a and b give an orientation of $\partial N(e)$, then this edge e is called a *positive edge*. If orientations of a and b give inconsistent orientations on $\partial N(e)$, then this edge e is called a *negative edge*. Similar definitions are applied for edges in G_Q. Then the following *parity rule* holds.

Lemma 2.4 ([15, p.872 (parity rule)]). *An edge e is a positive edge in G_P if and only if e is a negative edge in G_Q.*

3. Graphs of intersection

In this section, we assume that $\sharp\partial P = p \geq 3$. Namely, at least one of the created Klein bottles is not once-punctured in the knot complement.

Let us recall some definitions needed in our discussion. A subgraph σ of G_Q is called a *cycle* if it becomes a closed loop when we regard fat vertices of G_Q as points. The *length* of a cycle σ is the number of edges of G_Q constructing σ. If a length one cycle σ bounds a disk in Q, then such a cycle σ is called a *trivial loop*. Lemma 2.3 (3) shows that there exist no trivial loops in G_Q.

We number components of ∂P $1, 2, \ldots, p$ in the order in which they appear on ∂M. Then each edge of G_Q has a label on each endpoint. If an edge e of G_Q has a label i on both endpoints, then e is called a *level i-edge*. If we do not have to specify this number i, then we call e a *level edge*.

Around a vertex of G_Q, these labels occur in the order $1, 2, \ldots, p, 1, 2, \ldots, p, \ldots$ repeated $\Delta(r, s)$ times. A cycle σ of G_Q is called an *i-cycle* if all edges of σ are positive edges and if we can give an orientation on σ so that every tail of each edge has label i. An i-cycle σ of G_Q is called a *Scharlemann cycle* if σ bounds a disk face of $\widehat{Q} - G_Q$ (see Fig. 1 left). Edges e_1, e_2 in G_Q are called *parallel* if there is a disk

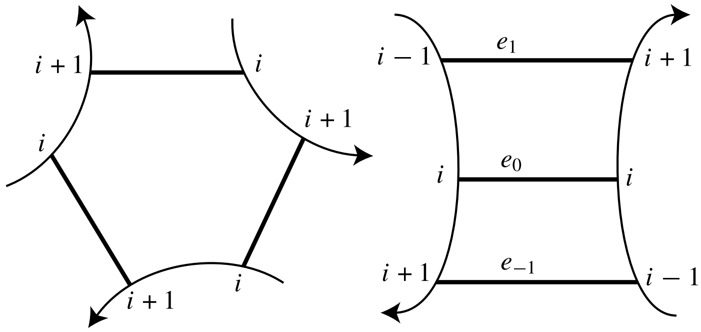

Fig. 1. Scharlemann cycle, generalized S-cycle

D in Q such that ∂D are edges e_1, e_2 and corners (subarcs of fat vertices) and that $\operatorname{int} D \cap G_Q$ is empty. A triple of successive parallel positive edges $e_{-1}, e_0, e_1 \subset G_Q$ is called a *generalized S-cycle* if e_0 is a level i-edge, and e_{-1}, e_1 have labels $\{i-1, i+1\}$ respectively (see Fig. 1 right).

Let us recall some basic facts about Scharlemann cycles and generalized S-cycles in a Klein bottle.

Lemma 3.1. (1) *There exist neither Scharlemann cycles nor generalized S-cycles in G_Q.*
(2) *Let $\mathcal{P}$ be the family of positive parallel edges in G_Q. If $\sharp\mathcal{P} \geq (p+3)/2$, then there exists a Scharlemann cycle or a generalized S-cycle in G_P.*

Proof. See [14, Lemmas 3.2, 3.3] for proof of (1), and see [14, Lemma 3.4] for (2). Remark that in the paper [14], Q is assumed to be a minimal genus Seifert surface. But examining the proof of these lemmas, we find that the statements still hold in our situation. □

Next we consider parallel negative edges. In this case, we will see that we cannot find too many edges in a negative parallel family.

Lemma 3.2. *Let $\mathcal{N}$ be a family of negative parallel edges in G_Q. If $\sharp\mathcal{N} \geq p+1$, then K is cabled.*

Proof. Let $e_1, e_2, \dots, e_k$ $(k > p)$ be the successive parallel negative edges of $\mathcal{N}$. By changing labels if necessary, we can assume that labels of e_i are i and $i+l$ (mod p).

CASE 3.3. The number l is equal to zero.

In this case, $e_1, \dots, e_{p+1}$ are all level edges (see Fig. 2 left). Since $\mathcal{N}$ is a family of negative edges in G_Q, the edges $e_1, \dots, e_{p+1}$ are all positive in G_P. Remark that each of these edges becomes an essential loop in $\widehat{P}$ if we regard fat vertices as points

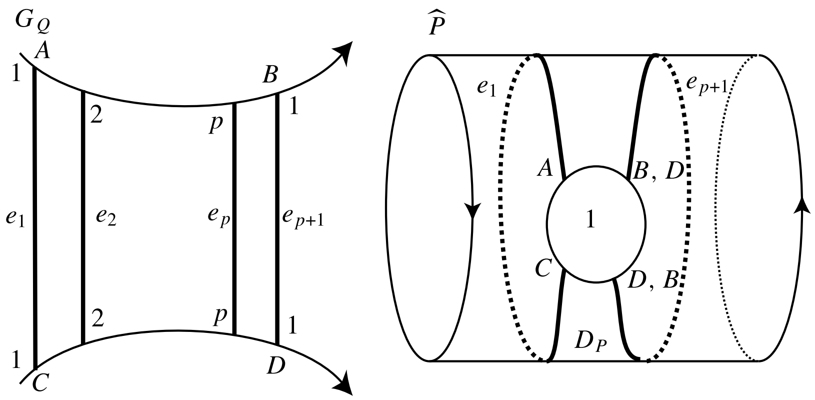

Fig. 2. Both e_1 and e_{p+1} join the same fat vertex in G_P.

because there exists no trivial loop in G_P. Since there exists only one family of positive loops in $\widehat{P}$, the edges e_1 and e_{p+1} are parallel in G_P. We name endpoints of e_1 as A, C and e_{p+1} as B, D so that points A, B lie in the same fat vertex of G_Q and C, D lie in the same fat vertex of G_Q. Remark that these points A, B, C, D do not need to lie in the same fat vertex of G_Q. If these four points appear on the same component of ∂Q, we may assume that they appear in the order A, B, C, D.

We first consider the case where the four points A, B, C, D are all contained in the same fat vertex of G_Q. Then since the surgery slopes are integral, this ordering must be respected in G_P. But this is impossible because e_1 and e_{p+1} are also parallel in G_P (see Fig. 2 right).

Next we consider the case where the fat vertex containing A, B and the fat vertex containing C, D are different. In this case, let us consider the subgraph Γ of G_P consisting of all fat vertices of G_P and the edges $e_1, \ldots, e_{p+1}$. In this subgraph, e_1 and e_{p+1} are parallel. Let D_P be the disk representing parallelism of e_1 and e_{p+1} in Γ. Remark that this disk D_P has no fat vertices in $\operatorname{int} D_P$ because there exist no trivial loops in G_P. Let D_{Q_i} be the disk representing parallelism of e_i and e_{i+1} in G_Q, and D_Q the disk consisting of $\bigcup_{i=1}^{p} D_{Q_i}$ in G_Q. Topologically $A := D_P \cup D_Q$ is an annulus or a Möbius band according to the orientation of the edges of e_1 and e_{p+1} in G_P. By taking small perturbation if necessary, we can assume that A is properly embedded in M.

If A is a Möbius band, then since slopes of ∂P and ∂Q are both integral, it follows that ∂A runs longitudinally only once. Thus, considering the boundary of the regular neighborhood $N(A)$ in M, we find a cabling torus, which shows that K is cabled.

If A is an annulus, then let γ_1, γ_2 be the boundary components of A. We orient γ_1 and γ_2 from the orientation of ∂Q. Exchanging γ_1 and γ_2 if necessary, we may assume that the orientation induced on γ_1 is consistent with the orientation of ∂P while the orientation induced on γ_2 is inconsistent. Then examining the intersection number

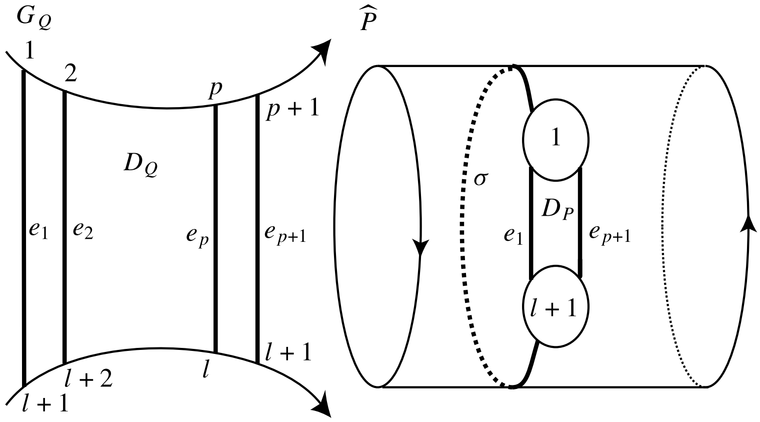

Fig. 3. $A = D_P \cup D_Q$ is an annulus in M.

between γ_i and μ, the meridian class of ∂M, we find that γ_1 and γ_2 have different slopes. Namely, they represent different slopes. Since A is a properly embedded annulus, γ_1 and γ_2 must represent the same slope. Hence we find a contradiction.

CASE 3.4. The number l is not equal to zero.

In this case, $e_1, e_2, \dots, e_p$ make cycles (may be one cycle) in G_P. Since we assume $l \not\equiv 0 \pmod{p}$, each of these cycles has the same length greater than one. Remark that $e_1, \dots, e_{p+1}$ are all positive edges in G_P so that each regular neighborhood of these cycles is an annulus in $\widehat{P}$. If one of these cycles bounds a disk in P, then the argument [3, Section 5] shows that K is cabled. Thus we can assume that none of these cycles bound a disk.

Let Γ be the subgraph of G_P consisting of all fat vertices G_P and edges $e_1, \dots, e_{p+1}$. Let σ be the cycle of Γ containing e_1. If e_1 and e_{p+1} are not parallel in Γ, then the cycle $(\sigma - e_1) \cup e_{p+1}$ bounds a disk. Again, using the argument [3, Section 5], we find that K is cabled. Thus we can assume that e_1 and e_{p+1} are parallel in Γ.

Let D_P be the disk representing parallelism of e_1 and e_{p+1} in Γ. Also let D_{Q_i} be the disk representing parallelism of e_i, and e_{i+1} in G_Q and D_Q the disk consisting of $\bigcup_{i=1}^{p} D_{Q_i}$ in G_Q. By taking small perturbation if necessary, we can assume that $A := D_P \cup D_Q$ is a properly embedded annulus in M. See Fig. 3. Arguing as in the last paragraph of Case 3.3, we find a contradiction. $\square$

4. Main argument

In this section, we prove Theorem 1.1. First we consider the case $\sharp\partial Q = q = 1$. The case $\sharp\partial P = p = 1$ is already examined in the paper [6]. Recall that the *cross-cap number* is the minimal number of the first Betti numbers of non-orientable Seifert surfaces for the knot.

Lemma 4.1 ([6, Theorem 2], [10, Theorem 1.2]). *For a crosscap number two knot K, the boundary slope of its minimal genus non-orientable Seifert surface F is a multiple of four, and there are at most two slopes which can be a boundary slope of F. If there are two, α and β, then $|\alpha - \beta| = 4$ or 8. Furthermore, if $|\alpha - \beta| = 8$, then K is the figure eight knot and $\{\alpha, \beta\} = \{-4, 4\}$.*

REMARK 4.2. To use Lemma 4.1 for the proof of Theorem 1.1 in the case $p = q = 1$, we need to check that K is a crosscap number two knot. This is shown as follows.

Because K bounds a once-punctured Klein bottle, it follows that the crosscap number of K is less than or equal to two. If the crosscap number of K is one, then by definition the of a crosscap number, we find a Möbius band $B \subset M$ whose boundary is K. Taking a regular neighborhood of B, we find that $K \subset \partial N(B)$. This shows that K is a cabled knot, which is a contradiction.

Thus, we only have to consider the case $p \geq 3$.

Proposition 4.3. *If $q = 1$ and $p \geq 3$, then $\Delta(r, s) \leq 4$.*

Proof. All edges in G_Q are partitioned into one of the following families: the one consisting of parallel positive edges $\mathcal{P}$, and the one consisting of parallel negatives edges $\mathcal{N}_1, \mathcal{N}_2$ (possibly $\mathcal{P}, \mathcal{N}_i$ is empty). By Lemma 3.1, we get $\sharp\mathcal{P} \leq (p+1)/2$. By Lemma 3.2, $\sharp\mathcal{N}_i \leq p$. Therefore the total number of endpoints around the fat vertex of G_Q is less than or equal to $2[(p+1)/2 + 2 \cdot p] = 5p + 1$. Hence we obtain an inequality $p\Delta(r, s) \leq 5p+1$. Since we know that $\Delta(r, s)$ is a multiple of four, we find that $\Delta(r, s) \leq 4$. $\square$

In the remainder of this section, we treat the case where both p and q are larger than one. Since p and q are odd numbers, it follows that $p, q \geq 3$. Before analyzing G_P and G_Q, we examine their "double coverings."

Let $\widehat{R}$ be the surface $\partial N(\widehat{P})$ in $M(r)$ and $\widehat{S}$ be the surface $\partial N(\widehat{Q})$ in $M(s)$. Topologically $\widehat{R}$ and $\widehat{S}$ are tori, the double coverings of $\widehat{P}$ and $\widehat{Q}$. As usual, let R be the surface $\widehat{R} \cap M$. We can assume that R and S intersect transversely and that each component of ∂R meets each component of ∂S in exactly $\Delta(r, s)$ times. Taking a thin regular neighborhood if necessary, we can think of $N(\widehat{P})$ as a twisted I-bundle over $\widehat{P}$. Thus each boundary component of P corresponds to the two boundary components of R. Let us think of $R \cap Q$. Each arc component of $P \cap Q$ (edge component of G_P) corresponds to the two arc components of $R \cap Q$. Thus if we take a graph of intersection $G_{R\cap Q} \subset \widehat{R}$ with Q, we can think $G_{R\cap Q}$ as a double covering of G_P. Since $\widehat{S} = \partial N(\widehat{Q})$, each arc component of $R \cap Q$ corresponds to the two arc components of $R \cap S$. Thus each arc component of $P \cap Q$ corresponds to the four arc components of $R \cap S$. Now let us take a graph of intersection $G_R \subset \widehat{R}$ with S. Similar construction gives a graph G_S. For a detailed construction, see [1, p.139–141]. We use graphs G_R and G_S to examine positive level edges in G_P and in G_Q.

Lemma 4.4. *There is no trivial loop in G_R and in G_S.*

Proof. Assume that there is a trivial loop e in G_R. Using the projection $\mathrm{pr}\colon \partial N(\widehat{P}) \to \widehat{P}$, we find that the edge corresponding to $\mathrm{pr}(e)$ in G_P must be a trivial loop. Since there is no trivial loop in G_P, this is a contradiction. $\square$

Since $\widehat{S}$ is a separating surface, we can color $\operatorname{int} N(\widehat{Q})$ black and $M(s) - N(\widehat{Q})$ white. Using this coloring, we can divide all faces of G_R into two families: white faces and black faces. Remark that the boundary of any black face is a length two cycle because black face corresponds to an edge in G_P. We can introduce an orientation on each component of ∂R from an orientation on ∂P. Using this orientation on each boundary component, we can define a Scharlemann cycle as in the previous section. If a Scharlemann cycle σ bounds a white disk in G_R, then the cycle σ is called a *white Scharlemann cycle*. Similarly we can define a *black Scharlemann cycle*. Remark that this coloring can be applied to faces of G_S.

We can also give a label on each endpoint of an edge. We number components of ∂R $1^-, 1^+, 2^-, \dots, p^-, p^+$ in the order in which they appear on ∂M. We assume that this numbering has the following property. If we cut an attaching solid torus J along meridian disks corresponding to 1^- and 1^+ of G_R, then two 1-handles J_1 and J_2 are obtained. Let J_1 be the 1-handle containing no meridian disks corresponding to other fat vertices of G_R. We give a numbering of ∂G_R so that J_1 contains the meridian disk corresponding to 1 of G_P.

Under this numbering, we note that an edge of a white Scharlemann cycle in G_R has labels $\{a^+, (a+1)^-\}$ and that an edge of a black Scharlemann cycle has labels $\{a^-, a^+\}$. By assumptions on G_P and G_Q, we can show the following.

Lemma 4.5. *There exists no white Scharlemann cycle in G_R and in G_S.*

Proof. Assume that there exists a white Scharlemann cycle σ in G_R. Let a^+ and $(a+1)^-$ be labels of each edge constructing σ. By construction of G_R and G_S, if we take an edge e of σ, then there exists a black face next to e. Remark that these black faces consist of two edges. One of the edge has labels a^+ and $(a+1)^-$, the other has labels a^- and $(a+1)^+$. Since black faces correspond to edges in G_P, existence of σ shows that there exists a cycle σ' in G_P. Remark that each edge of σ' has labels a and $a+1$ in G_P. Since σ bounds a disk in G_R, σ' also bounds a disk in G_P. By construction of G_R, each edge of σ' is a positive edge. Thus we find that this σ' is in fact a Scharlemann cycle in G_P. See Fig. 4. Since we are assuming $q \geq 3$, we can apply Lemma 3.1 (1). Therefore we find a contradiction. $\square$

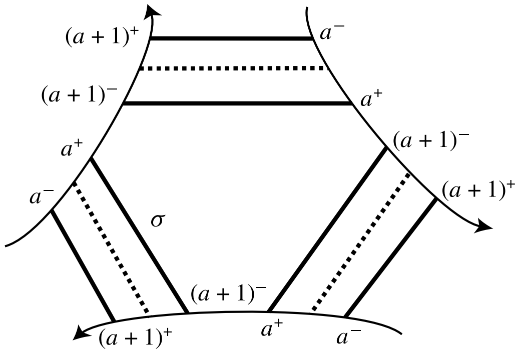

Fig. 4. A white Scharlemann cycle in G_R yields a Scharlemann cycle in G_P.

Proposition 4.6. *If* $p, q \geq 3$, *then* $\Delta(r, s) \leq 4$.

Proof. Assume that $\Delta(r, s) \geq 8$ seeking a contradiction. Applying Lemma 4.4 and the inequality [11, Lemma 2.1], we get

$$\begin{aligned}\Delta(r, s) &\leq 2 + \frac{\left(0 - \chi\left(\widehat{R}\right)\right)}{2p} + \frac{\left(0 - \chi\left(\widehat{S}\right)\right)}{2q} + \frac{m_R}{2p} + \frac{m_S}{2q} \\ &= 2 + \frac{m_R}{2p} + \frac{m_S}{2q}.\end{aligned}$$

Here m_R stands for the number of Scharlemann cycles in G_R, and m_S the number of Scharlemann cycles in G_S. Since there are no white Scharlemann cycles by Lemma 4.5, all Scharlemann cycles in G_R and in G_S are black Scharlemann cycles. Thus we can assume that there are at least $3 \cdot 2p$ black Scharlemann cycles in G_R. Since a black Scharlemann cycle corresponds to a positive level edge, there are at least $3p$ positive level edges in G_P. By the parity rule, this shows that there are at least $3p$ negative loops in G_Q.

Since $\widehat{Q}$ has at most two parallel families of negative loops, we find that one of those families has at least $3p/2 + 1/2$ (p is odd) negative loops joining the same fat vertex v. Let $e_1, \ldots, e_k (k \geq 3p/2 + 1/2)$ be such edges in G_Q and Γ the subgraph of G_Q consisting these edges and the fat vertex i. In this subgraph, edges e_i and e_{i+1} are parallel by construction. Let D_i be the disk representing the parallelism of e_i and e_{i+1} in Γ.

If none of the disks $D_1, \ldots, D_{k-1}$ contain fat vertices of G_Q, then edges $e_1, \ldots, e_k$ are parallel in G_Q. By applying Lemma 3.2, we find that this contradicts our assumption on K.

Thus we have to consider the case where these edges $e_1, \ldots, e_k$ are not parallel in G_Q. Let $D := D_i$ be the disk containing at least one fat vertex of G_Q. By examining the Euler characteristic of D, we will find a contradiction.

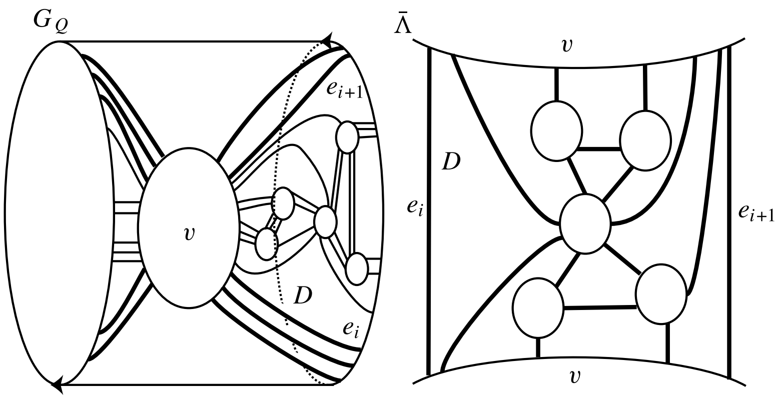

Fig. 5. Reduced graph $\overline{\Lambda}$

Let $v_1, \ldots, v_V$ be the fat vertices of G_Q contained in $\operatorname{int} D$. Let us define the subgraph Λ of G_Q on D as follows: vertices of Λ are two corners of v facing at D and the fat vertices $v_1, \ldots, v_V$, edges of Λ are edges of G_Q in $\operatorname{int} D$ and e_i, e_{i+1}. Instead of Λ, we consider its reduced graph $\overline{\Lambda}$, that is, parallel edges of Λ are amalgamated (see Fig. 5). Remark that $\overline{\Lambda}^{(0)}$, the 0-skeleton of $\overline{\Lambda}$, is the same as $\Lambda^{(0)}$ by definition so that we get $\sharp\overline{\Lambda}^{(0)} = \sharp\Lambda^{(0)} = V + 2$. Let E be the number of the edges of $\overline{\Lambda}$ such that at least one of their endpoints belongs to some v_m $(1 \le m \le V)$. Because the endpoints of e_i and e_{i+1} lie in the fat vertex v, we find that $\overline{\Lambda}$ has $E + 2$ edges. Let $\mathcal{F}$ be the faces of D separated by $\overline{\Lambda}$ and F the number of components of $\mathcal{F}$. We remark that since $\overline{\Lambda}$ is a reduced graph and G_Q has no trivial loops, the boundary of each face of $\mathcal{F}$ is a cycle having length at least three. Thus we have $2E + 2 \ge 3F$. Calculating the Euler characteristic $\chi(D)$, we find that

$$1 = \chi(D) = (V + 2) - (E + 2) + \sum_{f \in \mathcal{F}} \chi(f),$$

$$1 - V + E = \sum_{f \in \mathcal{F}} \chi(f) \le F \le \frac{2E + 2}{3}.$$

Then we get the inequality

$$(\flat) \qquad E \le 3V - 1.$$

Now let us evaluate the number E. By Lemmas 3.1, 3.2, we find that each v_m has at least $\Delta(r, s)p/p = \Delta(r, s)$ endpoints of $\overline{\Lambda}^{(1)}$. Let l be the number of edges of $\overline{\Lambda}$ which connect v and some v_m. Then E is greater than or equal to $(\Delta(r, s)V + l)/2$. Substituting this inequality into the previous inequality $(\flat)$, we get

$$(\Delta(r, s) - 6)V + l + 2 \le 0.$$

Since we are assuming $\Delta(r, s) \geq 8$, this is a contradiction.

Therefore we obtain the desired inequality $\Delta(r, s) \leq 4$. $\square$

Now Theorem 1.1 follows from Lemma 4.1, Propositions 4.3, 4.6.

ACKNOWLEDGEMENT. The author would like to thank Professor Kazuhiro Ichihara and Professor Masakazu Teragaito for encouraging advice and pointing out errors in the old version. He would like to thank Professor Ken'ichi Ohshika, Professor Makoto Sakuma and Professor Takashi Tsuboi for helpful comments. He would like to express his gratitude to the referee for detailed comments.

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