



Title	Dehn surgery creating Klein bottles
Author(s)	Kuwako, Kazuyuki
Citation	Osaka Journal of Mathematics. 2006, 43(2), p. 401-412
Version Type	VoR
URL	<a href="https://doi.org/10.18910/5682">https://doi.org/10.18910/5682</a>
rights	
Note	

*The University of Osaka Institutional Knowledge Archive : OUKA*

<https://ir.library.osaka-u.ac.jp/>

The University of Osaka

## DEHN SURGERY CREATING KLEIN BOTTLES

Dedicated to Professor Yukio Matsumoto for his 60th birthday

KAZUYUKI KUWAKO

(Received February 28, 2005, revised April 15, 2005)

### Abstract

We consider Dehn surgery creating Klein bottles. It is shown that if a non-cabled knot, which is not the figure eight knot, admits two Dehn surgeries creating Klein bottles, then the geometric intersection number of these slopes is bounded by four.

### 1. Introduction

In this paper, we treat Dehn surgery creating Klein bottles. There are results concerning Dehn surgeries which create closed non-orientable surfaces. Ichihara and Teragaito [7, 8] showed that if a nontrivial knot in  $S^3$  admits Dehn surgery creating Klein bottles, then the geometric intersection number of this slope and the longitude class is bounded by  $4g + 4$ . Here  $g$  is the genus of a knot. Matignon and Sayari [11] also examined Dehn surgery on a nontrivial knot in  $S^3$  which produce a closed non-orientable surface with higher genus. In this paper, we examine the geometric intersection number of slopes creating Klein bottles. We remark that Lee [9] and Matignon and Sayari [12] have announced some extensions to our results independently.

To state our theorem, let us introduce some definitions needed in this paper. Let  $K$  be a nontrivial knot in  $S^3$ . The knot  $K$  is called a *cabled knot* if there is an embedded (maybe knotted) torus in  $S^3$  such that  $K$  is isotoped on this torus and that  $K$  runs longitudinally at least twice. For example, any torus knot is a cabled knot. If  $K$  is not a cabled knot, then  $K$  is called a *non-cabled* knot. In this paper, we assume that  $K$  is non-cabled.

Let  $N(K)$  be a regular neighborhood of  $K$  and  $M$  the exterior of  $K$ ,  $S^3 - \text{int } N(K)$ . The unoriented isotopy class of a nontrivial simple closed curve on  $\partial M$  is called a *slope*. We parametrize slopes as in [13], that is, if a simple closed curve  $c$  on  $\partial M$  runs meridionally  $a$  times and longitudinally  $b$  times, then the slope of  $c$  is  $a/b$ . For example, the meridian class  $\mu$  has a slope  $1/0$  and the (preferred) longitude class  $\lambda$  has a slope  $0/1$ . Under this parametrization, if a slope  $r$  is equal to  $a/1$  ( $a \in \mathbf{Z}$ ), then  $r$  is called an *integral slope*.

In this paper,  $M(r)$  means the closed manifold obtained by  $r$ -Dehn filling, that is,  $M(r)$  is a manifold  $M \cup_f J$ . Here,  $J$  is a solid torus and  $f$  is a homeomorphism from  $\partial J$  to  $\partial M$  which sends the boundary of a meridian disk of  $J$  to an essential simple closed curve representing the slope  $r$ .

**Theorem 1.1.** *Let  $K$  be a non-cabled knot in  $S^3$ . Assume that  $K$  is a nontrivial knot, which is not the figure eight knot. If there are two slopes  $r$  and  $s$  such that Dehn surgered manifolds  $M(r)$  and  $M(s)$  both contain Klein bottles, then the geometric intersection number  $\Delta(r, s)$  is bounded by four.*

EXAMPLE 1.2 ([6]). Let  $K$  be the  $(-2, 3, 7)$ -pretzel knot. Then 16-surgery and 20-surgery yield Klein bottles.

This paper is organized as follows. We prove Theorem 1.1 by using combinatorial methods. In Section 2, we recall properties of Klein bottles created after Dehn surgery. The material of this section is proved in [14, 15]. In Section 3, we examine graphs of intersection. Roughly speaking, we prove that there cannot be too many parallel edges. Section 4 is devoted to a proof of Theorem 1.1. We divide the proof into two cases according to whether one of the created Klein bottles is once-punctured in the knot complement or not. Most of this section is devoted to the case where both of created Klein bottles are not once-punctured in the knot complement. In this case, we use an inequality obtained by [11] which tells us an upper bound of geometric intersection number.

## 2. Klein bottles created after surgery

In this section, we recall fundamental facts about Klein bottles created after Dehn surgery. If a surgered manifold  $M(r)$  contains Klein bottles, then this slope  $r$  has the following property.

**Lemma 2.1** ([4, Theorem 1.3], [14, Lemma 2.4]). *Slope  $r$  is integral, and is a multiple of four.*

REMARK 2.2. Combining this lemma and our theorem, we find that there are at most two surgeries creating Klein bottles in our situation.

We use a combinatorial method to prove Theorem 1.1. Let us recall basic notions and fix our notation. Let  $\widehat{P}$  (resp.  $\widehat{Q}$ ) be a created Klein bottle in  $M(r)$  (resp.  $M(s)$ ). Assume that  $P := \widehat{P} \cap M$  (resp.  $Q := \widehat{Q} \cap M$ ) is chosen to minimize the number of boundary components. Let  $p$  (resp.  $q$ ) be the number of the boundary components of  $P$  (resp.  $Q$ ). By a small perturbation, we can assume that  $P$  and  $Q$  intersect transversely and that each component of  $\partial P$  meets each component of  $\partial Q$  exactly  $\Delta(r, s)$  times. Then the following holds. For a detailed proof, consult [14].

- Lemma 2.3.** (1) *Both  $p$  and  $q$  are odd.*  
 (2) *Both  $P$  and  $Q$  are incompressible and boundary incompressible in  $M$ .*  
 (3) *No arc component of  $P \cap Q$  is boundary-parallel in  $P$  and in  $Q$ .*

Proof. (1) If  $p$  is even, then we can construct a closed non-orientable surface in  $M \subset S^3$ . Since  $S^3$  cannot contain a closed non-orientable surface, this leads to a contradiction.

(2) See [14, Lemmas 2.1, 2.2].

(3) If there exists such an arc, then it contradicts (2) (See [14, Lemma 3.1]).  $\square$

Let  $G_P \subset \widehat{P}$  be a graph of intersection with  $Q$ . That is, vertices of  $G_P$  are disks  $\widehat{P} - \text{int } P$ , edges of  $G_P$  are arc components of  $P \cap Q$  in  $P$ . We often call such vertices as *fat vertices*. We define a graph  $G_Q \subset \widehat{Q}$  similarly.

We give an orientation on all components of  $\partial P$  so that they are all homologous on  $\partial M$ . Let  $e$  be an edge of  $G_P$ . We may assume that a regular neighborhood  $N(e)$  of  $e$  is a disk in  $P$ . Let  $a$  and  $b$  be segments of  $\partial N(e) \cap \partial P$ . We can give an induced orientations on  $a$  and  $b$  from the orientation of  $\partial P$ . If orientations of  $a$  and  $b$  give an orientation of  $\partial N(e)$ , then this edge  $e$  is called a *positive edge*. If orientations of  $a$  and  $b$  give inconsistent orientations on  $\partial N(e)$ , then this edge  $e$  is called a *negative edge*. Similar definitions are applied for edges in  $G_Q$ . Then the following *parity rule* holds.

**Lemma 2.4** ([15, p.872 (parity rule)]). *An edge  $e$  is a positive edge in  $G_P$  if and only if  $e$  is a negative edge in  $G_Q$ .*

### 3. Graphs of intersection

In this section, we assume that  $\sharp \partial P = p \geq 3$ . Namely, at least one of the created Klein bottles is not once-punctured in the knot complement.

Let us recall some definitions needed in our discussion. A subgraph  $\sigma$  of  $G_Q$  is called a *cycle* if it becomes a closed loop when we regard fat vertices of  $G_Q$  as points. The *length* of a cycle  $\sigma$  is the number of edges of  $G_Q$  constructing  $\sigma$ . If a length one cycle  $\sigma$  bounds a disk in  $Q$ , then such a cycle  $\sigma$  is called a *trivial loop*. Lemma 2.3 (3) shows that there exist no trivial loops in  $G_Q$ .

We number components of  $\partial P$   $1, 2, \dots, p$  in the order in which they appear on  $\partial M$ . Then each edge of  $G_Q$  has a label on each endpoint. If an edge  $e$  of  $G_Q$  has a label  $i$  on both endpoints, then  $e$  is called a *level  $i$ -edge*. If we do not have to specify this number  $i$ , then we call  $e$  a *level edge*.

Around a vertex of  $G_Q$ , these labels occur in the order  $1, 2, \dots, p, 1, 2, \dots, p, \dots$  repeated  $\Delta(r, s)$  times. A cycle  $\sigma$  of  $G_Q$  is called an  *$i$ -cycle* if all edges of  $\sigma$  are positive edges and if we can give an orientation on  $\sigma$  so that every tail of each edge has label  $i$ . An  $i$ -cycle  $\sigma$  of  $G_Q$  is called a *Scharlemann cycle* if  $\sigma$  bounds a disk face of  $\widehat{Q} - G_Q$  (see Fig. 1 left). Edges  $e_1, e_2$  in  $G_Q$  are called *parallel* if there is a disk

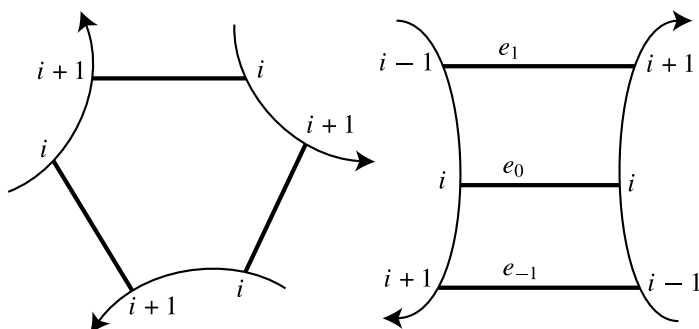


Fig. 1. Scharlemann cycle, generalized S-cycle

$D$  in  $Q$  such that  $\partial D$  are edges  $e_1, e_2$  and corners (subarcs of fat vertices) and that  $\text{int } D \cap G_Q$  is empty. A triple of successive parallel positive edges  $e_{-1}, e_0, e_1 \subset G_Q$  is called a *generalized S-cycle* if  $e_0$  is a level  $i$ -edge, and  $e_{-1}, e_1$  have labels  $\{i-1, i+1\}$  respectively (see Fig. 1 right).

Let us recall some basic facts about Scharlemann cycles and generalized S-cycles in a Klein bottle.

**Lemma 3.1.** (1) *There exist neither Scharlemann cycles nor generalized S-cycles in  $G_Q$ .*

(2) *Let  $\mathcal{P}$  be the family of positive parallel edges in  $G_Q$ . If  $\sharp \mathcal{P} \geq (p+3)/2$ , then there exists a Scharlemann cycle or a generalized S-cycle in  $G_P$ .*

Proof. See [14, Lemmas 3.2, 3.3] for proof of (1), and see [14, Lemma 3.4] for (2). Remark that in the paper [14],  $Q$  is assumed to be a minimal genus Seifert surface. But examining the proof of these lemmas, we find that the statements still hold in our situation.  $\square$

Next we consider parallel negative edges. In this case, we will see that we cannot find too many edges in a negative parallel family.

**Lemma 3.2.** *Let  $\mathcal{N}$  be a family of negative parallel edges in  $G_Q$ . If  $\sharp \mathcal{N} \geq p+1$ , then  $K$  is cabled.*

Proof. Let  $e_1, e_2, \dots, e_k$  ( $k > p$ ) be the successive parallel negative edges of  $\mathcal{N}$ . By changing labels if necessary, we can assume that labels of  $e_i$  are  $i$  and  $i+l \pmod{p}$ .

CASE 3.3. The number  $l$  is equal to zero.

In this case,  $e_1, \dots, e_{p+1}$  are all level edges (see Fig. 2 left). Since  $\mathcal{N}$  is a family of negative edges in  $G_Q$ , the edges  $e_1, \dots, e_{p+1}$  are all positive in  $G_P$ . Remark that each of these edges becomes an essential loop in  $\widehat{P}$  if we regard fat vertices as points

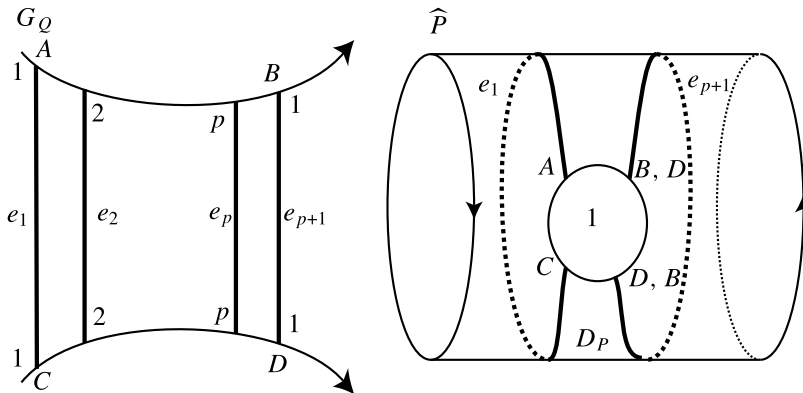


Fig. 2. Both  $e_1$  and  $e_{p+1}$  join the same fat vertex in  $G_P$ .

because there exists no trivial loop in  $G_P$ . Since there exists only one family of positive loops in  $\widehat{P}$ , the edges  $e_1$  and  $e_{p+1}$  are parallel in  $G_P$ . We name endpoints of  $e_1$  as  $A, C$  and  $e_{p+1}$  as  $B, D$  so that points  $A, B$  lie in the same fat vertex of  $G_Q$  and  $C, D$  lie in the same fat vertex of  $G_Q$ . Remark that these points  $A, B, C, D$  do not need to lie in the same fat vertex of  $G_Q$ . If these four points appear on the same component of  $\partial Q$ , we may assume that they appear in the order  $A, B, C, D$ .

We first consider the case where the four points  $A, B, C, D$  are all contained in the same fat vertex of  $G_Q$ . Then since the surgery slopes are integral, this ordering must be respected in  $G_P$ . But this is impossible because  $e_1$  and  $e_{p+1}$  are also parallel in  $G_P$  (see Fig. 2 right).

Next we consider the case where the fat vertex containing  $A, B$  and the fat vertex containing  $C, D$  are different. In this case, let us consider the subgraph  $\Gamma$  of  $G_P$  consisting of all fat vertices of  $G_P$  and the edges  $e_1, \dots, e_{p+1}$ . In this subgraph,  $e_1$  and  $e_{p+1}$  are parallel. Let  $D_P$  be the disk representing parallelism of  $e_1$  and  $e_{p+1}$  in  $\Gamma$ . Remark that this disk  $D_P$  has no fat vertices in  $\text{int } D_P$  because there exist no trivial loops in  $G_P$ . Let  $D_{Q_i}$  be the disk representing parallelism of  $e_i$  and  $e_{i+1}$  in  $G_Q$ , and  $D_Q$  the disk consisting of  $\bigcup_{i=1}^p D_{Q_i}$  in  $G_Q$ . Topologically  $A := D_P \cup D_Q$  is an annulus or a Möbius band according to the orientation of the edges of  $e_1$  and  $e_{p+1}$  in  $G_P$ . By taking small perturbation if necessary, we can assume that  $A$  is properly embedded in  $M$ .

If  $A$  is a Möbius band, then since slopes of  $\partial P$  and  $\partial Q$  are both integral, it follows that  $\partial A$  runs longitudinally only once. Thus, considering the boundary of the regular neighborhood  $N(A)$  in  $M$ , we find a cabling torus, which shows that  $K$  is cabled.

If  $A$  is an annulus, then let  $\gamma_1, \gamma_2$  be the boundary components of  $A$ . We orient  $\gamma_1$  and  $\gamma_2$  from the orientation of  $\partial Q$ . Exchanging  $\gamma_1$  and  $\gamma_2$  if necessary, we may assume that the orientation induced on  $\gamma_1$  is consistent with the orientation of  $\partial P$  while the orientation induced on  $\gamma_2$  is inconsistent. Then examining the intersection number

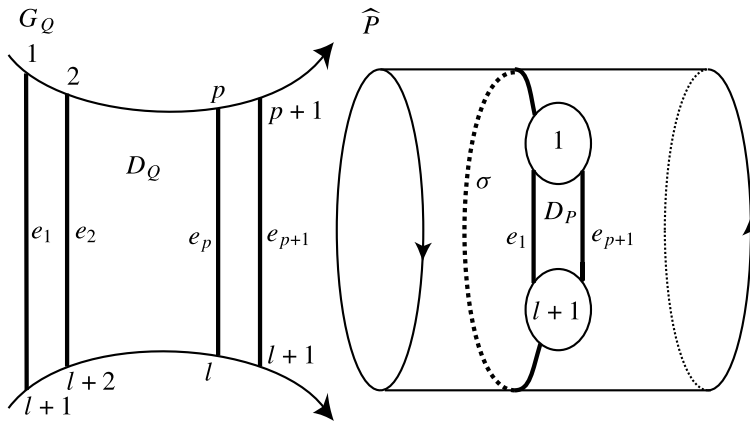


Fig. 3.  $A = D_P \cup D_Q$  is an annulus in  $M$ .

between  $\gamma_i$  and  $\mu$ , the meridian class of  $\partial M$ , we find that  $\gamma_1$  and  $\gamma_2$  have different slopes. Namely, they represent different slopes. Since  $A$  is a properly embedded annulus,  $\gamma_1$  and  $\gamma_2$  must represent the same slope. Hence we find a contradiction.

CASE 3.4. The number  $l$  is not equal to zero.

In this case,  $e_1, e_2, \dots, e_p$  make cycles (may be one cycle) in  $G_P$ . Since we assume  $l \not\equiv 0 \pmod{p}$ , each of these cycles has the same length greater than one. Remark that  $e_1, \dots, e_{p+1}$  are all positive edges in  $G_P$  so that each regular neighborhood of these cycles is an annulus in  $\widehat{P}$ . If one of these cycles bounds a disk in  $P$ , then the argument [3, Section 5] shows that  $K$  is cabled. Thus we can assume that none of these cycles bound a disk.

Let  $\Gamma$  be the subgraph of  $G_P$  consisting of all fat vertices  $G_P$  and edges  $e_1, \dots, e_{p+1}$ . Let  $\sigma$  be the cycle of  $\Gamma$  containing  $e_1$ . If  $e_1$  and  $e_{p+1}$  are not parallel in  $\Gamma$ , then the cycle  $(\sigma - e_1) \cup e_{p+1}$  bounds a disk. Again, using the argument [3, Section 5], we find that  $K$  is cabled. Thus we can assume that  $e_1$  and  $e_{p+1}$  are parallel in  $\Gamma$ .

Let  $D_P$  be the disk representing parallelism of  $e_1$  and  $e_{p+1}$  in  $\Gamma$ . Also let  $D_{Q_i}$  be the disk representing parallelism of  $e_i$ , and  $e_{i+1}$  in  $G_Q$  and  $D_Q$  the disk consisting of  $\bigcup_{i=1}^p D_{Q_i}$  in  $G_Q$ . By taking small perturbation if necessary, we can assume that  $A := D_P \cup D_Q$  is a properly embedded annulus in  $M$ . See Fig. 3. Arguing as in the last paragraph of Case 3.3, we find a contradiction.  $\square$

#### 4. Main argument

In this section, we prove Theorem 1.1. First we consider the case  $\sharp \partial Q = q = 1$ . The case  $\sharp \partial P = p = 1$  is already examined in the paper [6]. Recall that the *cross-cap number* is the minimal number of the first Betti numbers of non-orientable Seifert surfaces for the knot.

**Lemma 4.1** ([6, Theorem 2], [10, Theorem 1.2]). *For a crosscap number two knot  $K$ , the boundary slope of its minimal genus non-orientable Seifert surface  $F$  is a multiple of four, and there are at most two slopes which can be a boundary slope of  $F$ . If there are two,  $\alpha$  and  $\beta$ , then  $|\alpha - \beta| = 4$  or  $8$ . Furthermore, if  $|\alpha - \beta| = 8$ , then  $K$  is the figure eight knot and  $\{\alpha, \beta\} = \{-4, 4\}$ .*

**REMARK 4.2.** To use Lemma 4.1 for the proof of Theorem 1.1 in the case  $p = q = 1$ , we need to check that  $K$  is a crosscap number two knot. This is shown as follows.

Because  $K$  bounds a once-punctured Klein bottle, it follows that the crosscap number of  $K$  is less than or equal to two. If the crosscap number of  $K$  is one, then by definition the of a crosscap number, we find a Möbius band  $B \subset M$  whose boundary is  $K$ . Taking a regular neighborhood of  $B$ , we find that  $K \subset \partial N(B)$ . This shows that  $K$  is a cabled knot, which is a contradiction.

Thus, we only have to consider the case  $p \geq 3$ .

**Proposition 4.3.** *If  $q = 1$  and  $p \geq 3$ , then  $\Delta(r, s) \leq 4$ .*

**Proof.** All edges in  $G_Q$  are partitioned into one of the following families: the one consisting of parallel positive edges  $\mathcal{P}$ , and the one consisting of parallel negatives edges  $\mathcal{N}_1, \mathcal{N}_2$  (possibly  $\mathcal{P}, \mathcal{N}_i$  is empty). By Lemma 3.1, we get  $\sharp \mathcal{P} \leq (p+1)/2$ . By Lemma 3.2,  $\sharp \mathcal{N}_i \leq p$ . Therefore the total number of endpoints around the fat vertex of  $G_Q$  is less than or equal to  $2[(p+1)/2 + 2 \cdot p] = 5p+1$ . Hence we obtain an inequality  $p\Delta(r, s) \leq 5p+1$ . Since we know that  $\Delta(r, s)$  is a multiple of four, we find that  $\Delta(r, s) \leq 4$ .  $\square$

In the remainder of this section, we treat the case where both  $p$  and  $q$  are larger than one. Since  $p$  and  $q$  are odd numbers, it follows that  $p, q \geq 3$ . Before analyzing  $G_P$  and  $G_Q$ , we examine their “double coverings.”

Let  $\widehat{R}$  be the surface  $\partial N(\widehat{P})$  in  $M(r)$  and  $\widehat{S}$  be the surface  $\partial N(\widehat{Q})$  in  $M(s)$ . Topologically  $\widehat{R}$  and  $\widehat{S}$  are tori, the double coverings of  $\widehat{P}$  and  $\widehat{Q}$ . As usual, let  $R$  be the surface  $\widehat{R} \cap M$ . We can assume that  $R$  and  $S$  intersect transversely and that each component of  $\partial R$  meets each component of  $\partial S$  in exactly  $\Delta(r, s)$  times. Taking a thin regular neighborhood if necessary, we can think of  $N(\widehat{P})$  as a twisted  $I$ -bundle over  $\widehat{P}$ . Thus each boundary component of  $P$  corresponds to the two boundary components of  $R$ . Let us think of  $R \cap Q$ . Each arc component of  $P \cap Q$  (edge component of  $G_P$ ) corresponds to the two arc components of  $R \cap Q$ . Thus if we take a graph of intersection  $G_{R \cap Q} \subset \widehat{R}$  with  $Q$ , we can think  $G_{R \cap Q}$  as a double covering of  $G_P$ . Since  $\widehat{S} = \partial N(\widehat{Q})$ , each arc component of  $R \cap Q$  corresponds to the two arc components of  $R \cap S$ . Thus each arc component of  $P \cap Q$  corresponds to the four arc components of  $R \cap S$ . Now let us take a graph of intersection  $G_R \subset \widehat{R}$  with  $S$ . Similar construction



gives a graph  $G_S$ . For a detailed construction, see [1, p.139–141]. We use graphs  $G_R$  and  $G_S$  to examine positive level edges in  $G_P$  and in  $G_Q$ .

**Lemma 4.4.** *There is no trivial loop in  $G_R$  and in  $G_S$ .*

*Proof.* Assume that there is a trivial loop  $e$  in  $G_R$ . Using the projection  $\text{pr}: \partial N(\widehat{P}) \rightarrow \widehat{P}$ , we find that the edge corresponding to  $\text{pr}(e)$  in  $G_P$  must be a trivial loop. Since there is no trivial loop in  $G_P$ , this is a contradiction.  $\square$

Since  $\widehat{S}$  is a separating surface, we can color  $\text{int} N(\widehat{Q})$  black and  $M(s) - N(\widehat{Q})$  white. Using this coloring, we can divide all faces of  $G_R$  into two families: white faces and black faces. Remark that the boundary of any black face is a length two cycle because black face corresponds to an edge in  $G_P$ . We can introduce an orientation on each component of  $\partial R$  from an orientation on  $\partial P$ . Using this orientation on each boundary component, we can define a Scharlemann cycle as in the previous section. If a Scharlemann cycle  $\sigma$  bounds a white disk in  $G_R$ , then the cycle  $\sigma$  is called a *white Scharlemann cycle*. Similarly we can define a *black Scharlemann cycle*. Remark that this coloring can be applied to faces of  $G_S$ .

We can also give a label on each endpoint of an edge. We number components of  $\partial R$   $1^-, 1^+, 2^-, \dots, p^-, p^+$  in the order in which they appear on  $\partial M$ . We assume that this numbering has the following property. If we cut an attaching solid torus  $J$  along meridian disks corresponding to  $1^-$  and  $1^+$  of  $G_R$ , then two 1-handles  $J_1$  and  $J_2$  are obtained. Let  $J_1$  be the 1-handle containing no meridian disks corresponding to other fat vertices of  $G_R$ . We give a numbering of  $\partial G_R$  so that  $J_1$  contains the meridian disk corresponding to 1 of  $G_P$ .

Under this numbering, we note that an edge of a white Scharlemann cycle in  $G_R$  has labels  $\{a^+, (a+1)^-\}$  and that an edge of a black Scharlemann cycle has labels  $\{a^-, a^+\}$ . By assumptions on  $G_P$  and  $G_Q$ , we can show the following.

**Lemma 4.5.** *There exists no white Scharlemann cycle in  $G_R$  and in  $G_S$ .*

*Proof.* Assume that there exists a white Scharlemann cycle  $\sigma$  in  $G_R$ . Let  $a^+$  and  $(a+1)^-$  be labels of each edge constructing  $\sigma$ . By construction of  $G_R$  and  $G_S$ , if we take an edge  $e$  of  $\sigma$ , then there exists a black face next to  $e$ . Remark that these black faces consist of two edges. One of the edge has labels  $a^+$  and  $(a+1)^-$ , the other has labels  $a^-$  and  $(a+1)^+$ . Since black faces correspond to edges in  $G_P$ , existence of  $\sigma$  shows that there exists a cycle  $\sigma'$  in  $G_P$ . Remark that each edge of  $\sigma'$  has labels  $a$  and  $a+1$  in  $G_P$ . Since  $\sigma$  bounds a disk in  $G_R$ ,  $\sigma'$  also bounds a disk in  $G_P$ . By construction of  $G_R$ , each edge of  $\sigma'$  is a positive edge. Thus we find that this  $\sigma'$  is in fact a Scharlemann cycle in  $G_P$ . See Fig. 4. Since we are assuming  $q \geq 3$ , we can apply Lemma 3.1 (1). Therefore we find a contradiction.  $\square$

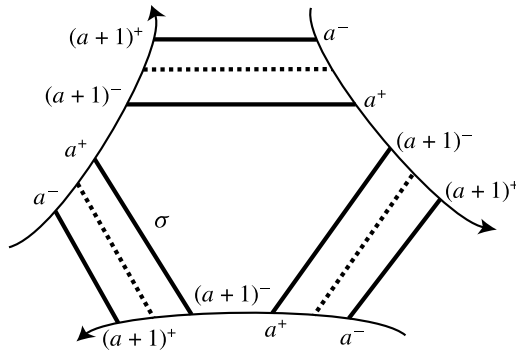


Fig. 4. A white Scharlemann cycle in  $G_R$  yields a Scharlemann cycle in  $G_P$ .

**Proposition 4.6.** *If  $p, q \geq 3$ , then  $\Delta(r, s) \leq 4$ .*

*Proof.* Assume that  $\Delta(r, s) \geq 8$  seeking a contradiction. Applying Lemma 4.4 and the inequality [11, Lemma 2.1], we get

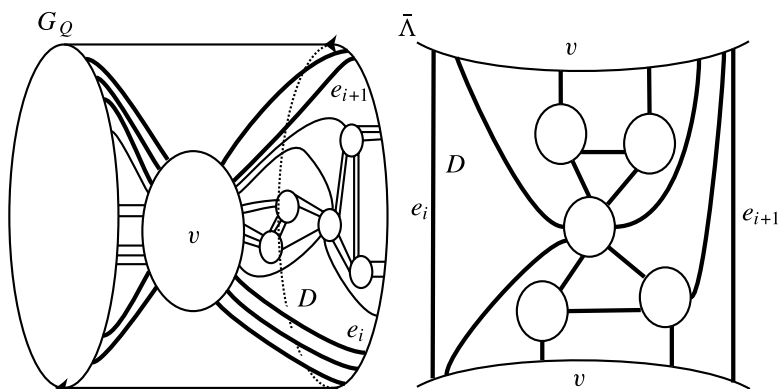
$$\begin{aligned} \Delta(r, s) &\leq 2 + \frac{(0 - \chi(\widehat{R}))}{2p} + \frac{(0 - \chi(\widehat{S}))}{2q} + \frac{m_R}{2p} + \frac{m_S}{2q} \\ &= 2 + \frac{m_R}{2p} + \frac{m_S}{2q}. \end{aligned}$$

Here  $m_R$  stands for the number of Scharlemann cycles in  $G_R$ , and  $m_S$  the number of Scharlemann cycles in  $G_S$ . Since there are no white Scharlemann cycles by Lemma 4.5, all Scharlemann cycles in  $G_R$  and in  $G_S$  are black Scharlemann cycles. Thus we can assume that there are at least  $3 \cdot 2p$  black Scharlemann cycles in  $G_R$ . Since a black Scharlemann cycle corresponds to a positive level edge, there are at least  $3p$  positive level edges in  $G_P$ . By the parity rule, this shows that there are at least  $3p$  negative loops in  $G_Q$ .

Since  $\widehat{Q}$  has at most two parallel families of negative loops, we find that one of those families has at least  $3p/2 + 1/2$  ( $p$  is odd) negative loops joining the same fat vertex  $v$ . Let  $e_1, \dots, e_k$  ( $k \geq 3p/2 + 1/2$ ) be such edges in  $G_Q$  and  $\Gamma$  the subgraph of  $G_Q$  consisting these edges and the fat vertex  $i$ . In this subgraph, edges  $e_i$  and  $e_{i+1}$  are parallel by construction. Let  $D_i$  be the disk representing the parallelism of  $e_i$  and  $e_{i+1}$  in  $\Gamma$ .

If none of the disks  $D_1, \dots, D_{k-1}$  contain fat vertices of  $G_Q$ , then edges  $e_1, \dots, e_k$  are parallel in  $G_Q$ . By applying Lemma 3.2, we find that this contradicts our assumption on  $K$ .

Thus we have to consider the case where these edges  $e_1, \dots, e_k$  are not parallel in  $G_Q$ . Let  $D := D_i$  be the disk containing at least one fat vertex of  $G_Q$ . By examining the Euler characteristic of  $D$ , we will find a contradiction.

Fig. 5. Reduced graph  $\bar{\Lambda}$ 

Let  $v_1, \dots, v_V$  be the fat vertices of  $G_Q$  contained in  $\text{int } D$ . Let us define the subgraph  $\Lambda$  of  $G_Q$  on  $D$  as follows: vertices of  $\Lambda$  are two corners of  $v$  facing at  $D$  and the fat vertices  $v_1, \dots, v_V$ , edges of  $\Lambda$  are edges of  $G_Q$  in  $\text{int } D$  and  $e_i, e_{i+1}$ . Instead of  $\Lambda$ , we consider its reduced graph  $\bar{\Lambda}$ , that is, parallel edges of  $\Lambda$  are amalgamated (see Fig. 5). Remark that  $\bar{\Lambda}^{(0)}$ , the 0-skeleton of  $\bar{\Lambda}$ , is the same as  $\Lambda^{(0)}$  by definition so that we get  $\sharp \bar{\Lambda}^{(0)} = \sharp \Lambda^{(0)} = V + 2$ . Let  $E$  be the number of the edges of  $\bar{\Lambda}$  such that at least one of their endpoints belongs to some  $v_m$  ( $1 \leq m \leq V$ ). Because the endpoints of  $e_i$  and  $e_{i+1}$  lie in the fat vertex  $v$ , we find that  $\bar{\Lambda}$  has  $E + 2$  edges. Let  $\mathcal{F}$  be the faces of  $D$  separated by  $\bar{\Lambda}$  and  $F$  the number of components of  $\mathcal{F}$ . We remark that since  $\bar{\Lambda}$  is a reduced graph and  $G_Q$  has no trivial loops, the boundary of each face of  $\mathcal{F}$  is a cycle having length at least three. Thus we have  $2E + 2 \geq 3F$ . Calculating the Euler characteristic  $\chi(D)$ , we find that

$$1 = \chi(D) = (V + 2) - (E + 2) + \sum_{f \in \mathcal{F}} \chi(f),$$

$$1 - V + E = \sum_{f \in \mathcal{F}} \chi(f) \leq F \leq \frac{2E + 2}{3}.$$

Then we get the inequality

$$(b) \quad E \leq 3V - 1.$$

Now let us evaluate the number  $E$ . By Lemmas 3.1, 3.2, we find that each  $v_m$  has at least  $\Delta(r, s)p/p = \Delta(r, s)$  endpoints of  $\bar{\Lambda}^{(1)}$ . Let  $l$  be the number of edges of  $\bar{\Lambda}$  which connect  $v$  and some  $v_m$ . Then  $E$  is greater than or equal to  $(\Delta(r, s)V + l)/2$ . Substituting this inequality into the previous inequality (b), we get

$$(\Delta(r, s) - 6)V + l + 2 \leq 0.$$

Since we are assuming  $\Delta(r, s) \geq 8$ , this is a contradiction.

Therefore we obtain the desired inequality  $\Delta(r, s) \leq 4$ . □

Now Theorem 1.1 follows from Lemma 4.1, Propositions 4.3, 4.6.

ACKNOWLEDGEMENT. The author would like to thank Professor Kazuhiro Ichihara and Professor Masakazu Teragaito for encouraging advice and pointing out errors in the old version. He would like to thank Professor Ken'ichi Ohshika, Professor Makoto Sakuma and Professor Takashi Tsuboi for helpful comments. He would like to express his gratitude to the referee for detailed comments.

---

### References

- [1] M. Domerque and D. Matignon: *Dehn surgeries and  $\mathbb{P}^2$ -reducible 3-manifolds*, Topology Appl. **72** (1996), 135–148.
- [2] C.McA. Gordon: *Boundary slopes of punctured tori in 3-manifolds*, Trans. Amer. Math. Soc. **350** (1998), 1713–1790.
- [3] C.McA. Gordon and R.A. Litherland: *Incompressible planar surface in 3-manifolds*, Topology Appl. **18** (1984), 121–144.
- [4] C.McA. Gordon and J. Luecke: *Dehn surgeries on knots creating essential tori*. I, Comm. Anal. Geom. **3** (1995), 597–644.
- [5] C.McA. Gordon and J. Luecke: *Dehn surgeries on knots creating essential tori*. II, Comm. Anal. Geom. **8** (2000), 671–725.
- [6] K. Ichihara, M. Ohtouge and M. Teragaito: *Boundary slopes of non-orientable Seifert surfaces for knots*, Topology Appl. **122** (2002), 467–478.
- [7] K. Ichihara and M. Teragaito: *Klein bottle surgery and genera of knots*, Pacific J. Math. **210** (2003), 317–333.
- [8] K. Ichihara and M. Teragaito: *Klein bottle surgery and genera of knots II*, Topology Appl. **146/147** (2005), 195–199.
- [9] S. Lee: *Exceptional Dehn fillings on hyperbolic 3-manifolds with at least two boundary components*, preprint.
- [10] E. Ramírez-Losada and L.G. Valdez-Sánchez: *Once-punctured Klein bottles in knot complements*, Topology Appl. **146/147** (2005), 159–188.
- [11] D. Matignon and N. Sayari: *Non-orientable surfaces and Dehn surgeries*, Canad. J. Math. **56** (2004), 1022–1033.
- [12] D. Matignon and N. Sayari: *Dehn fillings producing Klein bottles*, in preparation.
- [13] D. Rolfsen: *Knots and Links*, Math. Lec. Ser. **7**, Publish or Perish, Berkeley, California, 1976.
- [14] M. Teragaito: *Creating Klein bottles by surgery on knots*, J. Knot Theory Ramifications **10** (2001), 781–794.
- [15] M. Teragaito: *Dehn surgery on crosscap number two knots and projective planes*, J. Knot Theory Ramifications **11** (2002), 869–886.

Graduate School of Mathematical Sciences  
The University of Tokyo  
3-8-1 Komaba, Meguro, Tokyo, 153-8914, Japan  
e-mail: kkuwa@ms.u-tokyo.ac.jp

Current address:  
Department of Mathematics  
Graduate School of Science  
Osaka University  
Machikaneyama 1-16, Toyonaka  
Osaka, 560-0043, Japan  
e-mail: kuwako@gaia.math.wani.osaka-u.ac.jp