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THE REDUCIBILITY OF THE BOUNDARY CONDITIONS IN THE ONE-PARAMETER FAMILY OF ELLIPTIC LINEAR BOUNDARY VALUE PROBLEMS I

RYUICHI ASHINO

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1. Introduction

In this paper, we shall deal with the behaviour of boundary values of a family of singularly perturbed equations as the parameter tends to zero. Let $n \geqq 3$. Let P_1 and P_2 be elliptic operators on $\boldsymbol{R}^n$ with constant coefficients of order 2μ and 2ν with $\mu > \nu$, respectively and every $b_j(D)$, $j=0, \cdots, 2\mu-1$, be a normal boundary operator of order j. Let $j_1, \cdots, j_\mu$ be a series of integers with

$$0 \leqq j_1 < \cdots < j_\mu \leqq 2\mu - 1 . \tag{1.1}$$

We shall introduce the notion of "reducibility" for the following one-parameter family of boundary value problems:

$$\begin{cases} (\varepsilon \cdot P_1(D) + P_2(D))u = 0 , & \text{in } \boldsymbol{R}^n_+, \ 0 < \varepsilon < 1 ; \\ b_{j_k}(D)u|_{x_1 \downarrow 0} = \phi_k , & k = 1, \cdots, \mu . \end{cases} \tag{1.2}$$

Here $\boldsymbol{R}^n_+ = \{x \in \boldsymbol{R}^n ; x_1 > 0\}$ and ϕ_k, $k=1, \cdots, \mu$ belong to $\mathcal{S}(\boldsymbol{R}^{n-1})$. We shall deal with a distributional solution u_ε, which is prolongable to $x_1 \leqq 0$ as a distribution. Then we define a canonical extension $[u_\varepsilon]^+$ of a solution u_ε with support in $\overline{\boldsymbol{R}^n_+}$. We know that the canonical extension is unique, the boundary values

$$\lim_{\delta \downarrow 0} b_j(D)u_\varepsilon|_{x_1=\delta} , \qquad j = 0, \cdots, 2\mu-1, \text{ in } \mathcal{D}'(\boldsymbol{R}^{n-1})$$

are uniquely determined, and that

$$\lim_{\delta \downarrow 0} b_{j_k}(D)u_\varepsilon|_{x_1=\delta} = \phi_k , \qquad k = 1, \cdots, \nu, \text{ in } \mathcal{D}'(\boldsymbol{R}^{n-1}) .$$

Assume that there exists a prolongable distribution u_0 in $\mathcal{D}'(\boldsymbol{R}^n_+)$ such that $\lim_{\varepsilon \downarrow 0} [u_\varepsilon]^+ = u_0$ in $\mathcal{D}'(\boldsymbol{R}^n_+)$. Since P_1 and P_2 act continuously on $\mathcal{D}'(\boldsymbol{R}^n_+)$, we have

$$\lim_{\varepsilon \downarrow 0} (\varepsilon P_1(D) + P_2(D))u_\varepsilon = (\lim_{\varepsilon \downarrow 0} \varepsilon) \cdot P_1(D) \cdot (\lim_{\varepsilon \downarrow 0} u_\varepsilon) + P_2(D) (\lim_{\varepsilon \downarrow 0} u_\varepsilon) = P_2(D)u_0 .$$

Therefore u_0 satisfies the reduced equation

$$P_2(D)u_0 = 0 \quad \text{in} \quad \boldsymbol{R}^n_+ , \tag{1.3}$$

and the boundary values of u_0 in $\mathcal{D}'(\boldsymbol{R}^{n-1})$

$$\lim_{\delta \downarrow 0} b_j(D)u_0|_{x_1=\delta}, \qquad j = 0, \cdots, 2\nu-1$$

are uniquely determined. See [7].

In the previous works [1]–[5] and [8], they studied only the case when

$$u_\varepsilon \to u_0 \quad \text{in} \quad H^{j_\nu+1}(\boldsymbol{R}^n_+) .$$

In this case, the continuity of the trace operator implies that for $0 \leqq j \leqq j_\nu$

$$\lim_{\varepsilon \downarrow 0} (\lim_{\delta \downarrow 0} b_j(D)u_\varepsilon|_{x_1=\delta}) = \lim_{\delta \downarrow 0} b_j(D)(\lim_{\varepsilon \downarrow 0} u_\varepsilon)|_{x_1=\delta} .$$

Then u_0 satisfies the following boundary value problem:

$$\begin{cases} P_2(D)v = 0 , & \text{in} \quad \boldsymbol{R}^n_+ ; \\ b_{j_k}(D)v|_{x_1 \downarrow 0} = \phi_k , & k = 1, \cdots, \nu . \end{cases} \tag{1.4}$$

Here we assume that (1.4) has a unique solution v. But this does not necessarily hold if the convergence $u_\varepsilon \to u_0$ does not take place in $H^{j_\nu+1}(\boldsymbol{R}^n_+)$.

Why does u_0 happen to satisfy the first ν boundary conditions? Does u_0 satisfy a different set of ν boundary conditions in a different topology which does not ensure the continuity of the trace operator? We are going to give an affirmative example to this question and a detailed analysis of a framework, called *reducibility* of (1.2), by use of methods different from those of [1]–[5] and [8].

Definition 1.1. A one-parameter family of the boundary value problems (1.2) is said to be *reducible* if (1.2) satisfies the following four conditions:
(1) Every boundary value problem (1.2) has a prolongable solution u_ε in $\mathcal{D}'(\boldsymbol{R}^n_+)$ such that the canonical extension $[u_\varepsilon]^+$ bleongs to $\mathcal{S}'(\boldsymbol{R}^n)$.
(2) There exists a prolongable solution u_0 of (1.3) such that

$$\lim_{\varepsilon \downarrow 0} u_\varepsilon = u_0 \quad \text{in} \quad \mathcal{D}'(\boldsymbol{R}^n_+) .$$

(3) There exists a series $(k_1, \cdots, k_\nu)$ such that

$$1 \leqq k_1 < \cdots < k_\nu \leqq \mu ; \; 0 \leqq j_{k_1} < \cdots < j_{k_\nu} \leqq 2\nu - 1 ;$$

and u_0 satisfies the following boundary conditions:

$$b_{j_{k_l}}(D)u_0|_{x_1 \downarrow 0} = \phi_{k_l} , \qquad l = 1, \cdots, \nu . \tag{1.5}$$

(4) The reduced boundary value problem (1.3) with (1.5) is uniquely solvable.

In particular, if $k_l=l$, $l=1, \cdots, \nu$ then the family (1.2) is said to be *normally reducible*. The family (1.2) is said to be *abnormally reducible* if the family (1.2) is reducible but not normally reducible.

Our main theme is to look for conditions for reducibility and study an example of the abnormally reducible family such that the limit u_0 of the solution u_ε of (1.2) in $L^2(\boldsymbol{R}^n_+)$ satisfies the boundary conditions:

$$b_{j_k}(D)u_0|_{x_1\downarrow 0} = \phi_k\,, \qquad k=1, \cdots, \nu-1,\ \nu+1\,. \tag{1.6}$$

The main results are given in § 4. As a preliminary, we shall study in § 2 asymptotic behaviour of determinants appearing in the expression of solutions of the boundary value problems. In § 3 we shall examine necessary properties of the characteristic roots of the perturbed equations which were assumed in § 2.

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2. The order calculus of determinants

Let $j_1, \cdots, j_\mu$ be a series of integers with

$$0 \leqq j_1 < \cdots < j_\mu \leqq 2\mu-1\,. \tag{2.1}$$

Let $b_j(\tau, \xi')$, $j=j_1, \cdots, j_\mu$ be polynomials of (τ, ξ') as

$$b_j(\tau, \xi') = \tau^j + \textstyle\sum_{k=1}^{j} b_{j,k}(\xi')\tau^{j-k}, \qquad j=j_1, \cdots, j_\mu\,, \tag{2.2}$$

which are denoted by $b_j(\tau)$ when regarded as polynomials of τ with polynomial coefficients.

NOTATION 2.1.

For polynomials $b_j(\tau)$, $j=1, \cdots, \mu$ and for complex numbers or functions τ_j and ϕ_j, $j=1, \cdots, \mu$,

$$\mathrm{Mat}\, D_0 = \mathrm{Mat}\, D_0(\tau_1, \cdots, \tau_\mu; b_1, \cdots, b_\mu) = \begin{bmatrix} b_1(\tau_1) \cdots b_1(\tau_\mu) \\ \vdots \qquad \vdots \\ b_\mu(\tau_1) \cdots b_\mu(\tau_\mu) \end{bmatrix},$$

$$\begin{aligned} \mathrm{Mat}\, D_k &= \mathrm{Mat}\, D_k(\tau_1, \cdots, \tau_\mu; b_1, \cdots, b_\mu; \phi_1, \cdots, \phi_\mu) \\ &= \begin{bmatrix} b_1(\tau_1) \cdots b_1(\tau_{k-1}) & \phi_1 & b_1(\tau_{k+1}) \cdots b_1(\tau_\mu) \\ \vdots \qquad \vdots & \vdots & \vdots \qquad \vdots \\ b_\mu(\tau_1) \cdots b_\mu(\tau_{k-1}) & \phi_\mu & b_\mu(\tau_{k+1}) \cdots b_\mu(\tau_\mu) \end{bmatrix}, \end{aligned}$$

where $k=1, \cdots, \mu$.

For $J=j_{\nu+1}+\cdots+j_\mu$, $A=\{j_{\nu+1}, \cdots, j_\mu\}$, $\zeta=\exp\dfrac{2\pi i}{2\mu-2\nu}$, and $\Theta=\exp\dfrac{\theta i}{2\mu-2\nu}$, where $0\leqq\theta<2\pi$,

$$\operatorname{Mat} V_n(\zeta; j_1, \cdots, j_n) = \begin{pmatrix} 1 & \cdots & 1 \\ \zeta^{j_1} & & \zeta^{j_n} \\ \vdots & & \vdots \\ (\zeta^{j_1})^{n-1} & \cdots & (\zeta^{j_n})^{n-1} \end{pmatrix},$$

$$\det \operatorname{Mat} V_n(\zeta; j_1, \cdots, j_n) = V_n(\zeta; j_1, \cdots, j_n),$$

$$\operatorname{Mat} V_{\mu-\nu,0} = \operatorname{Mat} V_{\mu-\nu}(\zeta; j_{\nu+1}, \cdots, j_\mu).$$

For $1 \leqq k \leqq \mu-\nu$,

$$j'_k = j_{\nu+k} - 1$$

$$\operatorname{Mat} V_{\mu-\nu,,k} = \operatorname{Mat} V_{\mu-\nu}(\zeta; j_{\nu+1}, \cdots, j_{\nu+k-1}, j'_k, j_{\nu+k+1}, \cdots, j_\mu)$$

$$T_k = (j_{\nu+1} \cdot (\zeta^{k-1})^{j'_1} \cdot \tau_{\nu+k,1}(\xi'), \cdots, j_\mu \cdot (\zeta^{k-1})^{j'_{\mu-\nu}} \cdot \tau_{\nu+k,1}(\xi'))$$

$$\operatorname{Mat} \partial D_k = \operatorname{Mat} D_k(1, \zeta, \cdots, \zeta^{\mu-\nu-1}; \tau^{j_{\nu+1}}, \cdots, \tau^{j_\mu}; T_k).$$

For $F = (\tau^{j_{\nu+2}}, \cdots, \tau^{j_\mu})$,

$$\operatorname{Mat} D_{(\nu+1)} = \operatorname{Mat} D_0(\zeta\Theta, \cdots, \zeta^{\mu-\nu-1}\Theta; F).$$

For $\nu+2 \leqq k \leqq \mu$,

$$\operatorname{Mat} D_{(k)} = \operatorname{Mat} D_0(\Theta, \zeta\Theta, \cdots, \zeta^{k-\nu-2}\Theta, \zeta^{k-\nu}\Theta, \cdots, \zeta^{\mu-\nu-1}\Theta; F).$$

$$B_{\mu-\nu} = \textstyle\sum_{k=1}^{\mu-\nu} (b_{j_{\nu+k},1}(\xi') \cdot V_{\mu-\nu,k} + \partial D_k).$$

We shall abbreviate the determinant of Mat D as D, where Mat D is any of the matrices abbrevitated as above.

Our purpose of this section is to calculate

$$\lim_{\lambda \uparrow \infty} \frac{D_j}{D_0}, \qquad j = 1, \cdots, \mu.$$

Lemma 2.2.

$$D_0(\Theta, \Theta\zeta, \cdots, \Theta\zeta^{\mu-\nu-1}; \tau^{j_{\nu+1}}, \cdots, \tau^{j_\mu}) = \Theta^J \cdot V_{\mu-\nu,0}.$$

$V_{\mu-\nu,0}=0$ if and only if there exist two integers k and l in $A = \{j_{\nu+1}, \cdots, j_\mu\}$ such that

$$k \equiv l \pmod{2\mu-2\nu}, \qquad k \neq l.$$

Proof. Put

$$a_i = (1, (\zeta)^{j_{\nu+i}}, \cdots, (\zeta^{\mu-\nu-1})^{j_{\nu+i}}), \qquad 1 \leqq i \leqq \mu-\nu.$$

Then

$$\det {}^t(\Theta^{j_{\nu+1}} a_1, \cdots, \Theta^{j_\mu} a_{\mu-\nu}) = \Theta^J \det {}^t(a_1, \cdots, a_{\mu-\nu}).$$

Since we can rewrite a_i as

$$a_i = (1, (\zeta^{j_{\nu+i}}), \cdots, (\zeta^{j_{\nu+i}})^{\mu-\nu-1}), \qquad 1 \leqq i \leqq \mu-\nu,$$

we have

$$\det {}^t(a_1, \cdots, a_{\mu-\nu}) = \det({}^t a_1, \cdots, {}^t a_{\mu-\nu}) = V_{\mu-\nu}(\zeta; j_{\nu+1}, \cdots, j_\mu) = V_{\mu-\nu,0}.$$

Since ζ is a primitive root of 1 of order $2\mu-2\nu$, $\zeta^k=\zeta^l$ holds if and only if $k\equiv l \pmod{2\mu-2\nu}$. Recalling that $V_{\mu-\nu}$ is the difference product of $\zeta^{j_{\nu+1}}, \cdots, \zeta^{j_\mu}$, we have the conclusion. [Q.E.D.]

REMARK. rank Mat $V_{\mu-\nu,0}=\mu-\nu$, if and only if every pair (l_1, l_2) with $l_1<l_2$ and $l_1, l_2 \in A$ satisfies $l_1 \not\equiv l_2 \pmod{2\mu-2\nu}$.

rank Mat $V_{\mu-\nu,0}=\mu-\nu-1$, if and only if there exists only one pair (l_1, l_2) with $l_1<l_2$ and $l_1, l_2\in A$ such that $l_1\equiv l_2 \pmod{2\mu-2\nu}$.

rank Mat $V_{\mu-\nu,0}\leqq\mu-\nu-2$, if and only if there exists two different pairs (l_1, l_2) and (l_1', l_2') with $l_1<l_2$ and $l_1'<l_2'$ and $l_1, l_2, l_1', l_2' \in A$ such that $l_1\equiv l_2$ and $l_1'\equiv l_2' \pmod{2\mu-2\nu}$.

Assume that there exists only one pair (l_1, l_2) with $l_1<l_2$ and $l_1, l_2\in A$ such that $l_1\equiv l_2 \pmod{2\mu-2\nu}$, and put $l_1=j_{\nu+k_1}$ and $l_2=j_{\nu+k_2}$. Then

$$B_{\mu-\nu} = b_{l_1,1}\cdot V_{\mu-\nu,k_1}+b_{l_2,1}\cdot V_{\mu-\nu,k_2}+\partial D_{k_1}+\partial D_{k_2}.$$

ASSUMPTION 2.3.

$\tau_j(\lambda, \xi')$, $j=1, \cdots, \mu$ are continuous functions of (λ, ξ') in $\{\lambda>1\}\times \boldsymbol{R}^{n-1}_{\xi'}$ satisfying the following asymptotic properties: there exist continuous functions $\sigma_j(\xi')$, $j=1, \cdots, \nu$ of ξ' in $\boldsymbol{R}^{n-1}$ such that

$$\lim_{\lambda\uparrow\infty} \tau_j(\lambda, \xi') = \sigma_j(\xi'), \qquad 1\leqq j\leqq \nu, \tag{2.3}$$

$$\lim_{\lambda\uparrow\infty} \tau_j(\lambda, \xi')/\lambda = \zeta^{j-\nu-1}\Theta, \qquad \nu+1\leqq j\leqq \mu, \tag{2.4}$$

uniformly on every compact subset K of $\boldsymbol{R}^{n-1}_{\xi'}$.

We shall calculate the coefficients of the leading terms with respect to λ of the asymptotic expansions of D_j, $j=0, \cdots, \mu$.

Lemma 2.4. *Let Assumption* 2.3 *be satisfied. Then*

$$\begin{aligned}\lim_{\lambda\uparrow\infty} & D_0(\tau_1, \cdots, \tau_\mu; b_{j_1}, \cdots, b_{j_\mu})/\lambda^J \\ &= D_0(\sigma_1, \cdots, \sigma_\nu; b_{j_1}, \cdots, b_{j_\nu})\cdot\Theta^J\cdot V_{\mu-\nu,0}.\end{aligned} \tag{2.5}$$

For $k=1, \cdots, \nu$

$$\begin{aligned}\lim_{\lambda\uparrow\infty} & D_k(\tau_1, \cdots, \tau_\mu; b_{j_1}, \cdots, b_{j_\mu}; \hat\phi_1, \cdots, \hat\phi_\mu)/\lambda^J \\ &= D_k(\sigma_1, \cdots, \sigma_\nu; b_{j_1}, \cdots, b_{j_\nu}; \hat\phi_1, \cdots, \hat\phi_\nu)\cdot\Theta^J\cdot V_{\mu-\nu,0},\end{aligned} \tag{2.6}$$

and for $k=\nu+1, \cdots, \mu$

$$\lim_{\lambda\uparrow\infty} D_k(\tau_1, \cdots, \tau_\mu; b_{j_1}, \cdots, b_{j_\mu}; \hat{\phi}_1, \cdots, \hat{\phi}_\mu)/\lambda^J = 0 . \tag{2.7}$$

Here the convergences are uniform on every compact subset K *of* $\boldsymbol{R}^{n-1}$.

Proof. Let K be a compact subset of $\boldsymbol{R}^{n-1}$ and $\xi'\in K$. With a new variable $\tilde{\lambda}$, we replace $\tau_j(\lambda, \xi')$, $j=\nu+1, \cdots, \mu$ by $\tilde{\lambda}\cdot\tau_j(\lambda, \xi')/\lambda$ in D_k, $k=0, \cdots, \mu$ and denote the result by $\tilde{D}_k$. If we put $\lambda=\tilde{\lambda}$, then we have $D_k=\tilde{D}_k$. Since $\tilde{D}_k$ is a polynomial in $\tilde{\lambda}$, we can rewrite $\tilde{D}_k$ as

$$\tilde{D}_k = \textstyle\sum_{j=0}^{J} \tilde{d}_{k,j}(\lambda, \xi')\cdot\tilde{\lambda}^{J-j} . \tag{2.8}$$

First we prove (2.5). Recalling the definition of the determinant, the terms of $\tilde{d}_{0,0}\tilde{\lambda}^J$ are contained in the sum of the products of

$$(\operatorname{sgn}\rho)b_{j_{\rho(1)}}(\tau_1(\lambda, \xi')) \cdots b_{j_{\rho(\nu)}}(\tau_\nu(\lambda, \xi')) \tag{2.9}$$

and

$$b_{j_{\rho(\nu+1)}}(\tilde{\lambda}\cdot\tau_{\nu+1}(\lambda, \xi')/\lambda) \cdots b_{j_{\rho(\mu)}}(\tilde{\lambda}\cdot\tau_\mu(\lambda, \xi')/\lambda) , \tag{2.10}$$

where ρ runs over all the permutations of μ letters satisfying

$$1\leqq\rho(j)\leqq\nu , \qquad \text{for} \quad 1\leqq j\leqq\nu \tag{2.11}$$

and

$$\nu+1\leqq\rho(j)\leqq\mu , \qquad \text{for} \quad \nu+1\leqq j\leqq\mu . \tag{2.12}$$

Denote the symmetric group of order m by $\mathcal{S}_m$. Then there exist $\rho'\in\mathcal{S}_\nu$ and $\rho''\in\mathcal{S}_{\mu-\nu}$ such that $\rho(j)=\rho'(j)$ in (2.11) and $\rho(j)-\nu=\rho''(j-\nu)$ in (2.12). Since $(\operatorname{sgn}\rho)=(\operatorname{sgn}\rho')(\operatorname{sgn}\rho'')$, (2.9)$\times$(2.10) can be represented as

$$\begin{aligned}&(\operatorname{sgn}\rho')b_{j_{\rho'(1)}}(\tau_1, (\lambda, \xi')) \cdots b_{j_{\rho'(\nu)}}(\tau_\nu(\lambda, \xi'))\\ &\quad\times(\operatorname{sgn}\rho'')b_{j_{\nu+\rho''(1)}}(\tilde{\lambda}\cdot\tau_{\nu+1}(\lambda, \xi')/\lambda) \cdots b_{j_{\nu+\rho''(\mu-\nu)}}(\tilde{\lambda}\cdot\tau_\mu(\lambda, \xi')/\lambda) .\end{aligned}$$

Therefore $\tilde{d}_{0,0}(\lambda, \xi')$ is the product of

$$\sum_{\rho'\in\mathcal{S}_\nu} (\operatorname{sgn}\rho')b_{j_{\rho'(1)}}(\tau_1(\lambda, \xi')) \cdots b_{j_{\rho'(\nu)}}(\tau_\nu(\lambda, \xi')) \tag{2.13}$$

and

$$\sum_{\rho''\in\mathcal{S}_{\mu-\nu}} (\operatorname{sgn}\rho'')(\tau_{\nu+1}(\lambda, \xi')/\lambda)_1^{j_{\nu+\rho''(1)}} \cdots (\tau_\mu(\lambda, \xi')/\lambda)^{j_{\nu+\rho''(\mu-\nu)}} . \tag{2.14}$$

When $\lambda\uparrow\infty$, we have (2.13)$\to D_0(\sigma_1, \cdots, \sigma_\nu; b_{j_1}, \cdots, b_{j_\nu})$ and (2.14)$\to D_0(\Theta, \Theta\zeta, \cdots, \Theta\zeta^{\mu-\nu-1}; \tau^{j_{\nu+1}}, \cdots, \tau^{j_\mu})$. Then Lemma 2.2 shows (2.5).

Next we prove (2.6). By the definition of the determinant, we have

$$(2.15)\qquad D_k(\tau_1, \cdots, \tau_\mu; b_{j_1}, \cdots, b_{j_\mu}; \hat{\phi}_1, \cdots, \hat{\phi}_\mu) = \sum_{\rho\in\mathcal{S}_\mu} (\operatorname{sgn}\rho) b_{j_{\rho(1)}}(\tau_1)\cdots b_{j_{\rho(k-1)}}(\tau_{k-1})\cdot\hat{\phi}_{\rho(k)}\cdot b_{j_{\rho(k+1)}}(\tau_{k+1})\cdots b_{j_{\rho(\mu)}}(\tau_\mu)\,.$$

Since $\hat{\phi}_1, \cdots, \hat{\phi}_\mu$ are bounded on K, we can use the same argument as in the proof of (2.5). If we replace (2.9) by

$$(2.16)\qquad (\operatorname{sgn}\rho) b_{j_{\rho(1)}}(\tau_1)\cdots b_{j_{\rho(k-1)}}(\tau_{k-1})\cdot\hat{\phi}_{\rho(k)}\cdot b_{j_{\rho(k+1)}}(\tau_{k+1})\cdots b_{j_{\rho(\nu)}}(\tau_\nu)\,,$$

then we have (2.6).

Finally we prove (2.7). (2.15) shows that the formal highest term of $\tilde{D}_k$ with respect to $\tilde{\lambda}$ is contained in the sum of

$$(2.17)\qquad \begin{aligned}&(\operatorname{sgn}\rho) b_{j_{\rho(1)}}(\tau_1(\lambda, \xi'))\cdots b_{j_{\rho(\nu)}}(\tau_\nu(\lambda, \xi'))\\ &\times b_{j_{\rho(\nu+1)}}(\tilde{\lambda}\cdot\tau_{\nu+1}(\lambda, \xi')/\lambda)\cdots b_{j_{\rho(k-1)}}(\tilde{\lambda}\cdot\tau_{k-1}(\lambda, \xi')/\lambda)\\ &\times\hat{\phi}_{\rho(k)}\cdot b_{j_{\rho(k+1)}}(\tilde{\lambda}\cdot\tau_{k+1}(\lambda, \xi')/\lambda)\cdots b_{j_{\rho(\mu)}}(\tilde{\lambda}\cdot\tau_\mu(\lambda, \xi')/\lambda)\,.\end{aligned}$$

Since the formal highest order of (2.17) with respect to $\tilde{\lambda}$ is $J-j_{\nu+1}$, which is obtained by putting $\rho(k)=\nu+1$, we have

$$(2.18)\qquad \tilde{d}_{k,l}(\lambda, \xi') = 0\,,$$

for $0\leqq l\leqq j_{\nu+1}-1$. Since $j_{\nu+1}>j_1\geqq 0$, we have $\widetilde{d_{k,0}}=0$. This impilies (2.7).

[Q.E.D.]

When $V_{\mu-\nu,0}=0$, we need more assumptions on $\tau_j(\lambda, \xi')$, $j=1, \cdots, \mu$ to calculate the leading terms of D_j, $j=0, \cdots, \mu$.

ASSUMPTION 2.5.

Let δ be a positive number and N be the greatest integer satisfying $N\delta\leqq 1$. $\tau_j(\lambda, \xi')$, $j=1, \cdots, \mu$ satisfy the following asymptotic properties: for $j=1, \cdots, \nu$

$$(2.19)\qquad \tau_j(\lambda, \xi') = \sigma_j(\xi')+\textstyle\sum_{k=1}^N \lambda^{-k\delta}\cdot\tau_{j,k\delta}(\xi')+\lambda^{-(N+1)\delta}\cdot\tau_{j,(N+1)\delta}(\lambda, \xi')\,,$$

and for $j=\nu+1, \cdots, \mu$

$$(2.20)\qquad \tau_j(\lambda, \xi')/\lambda = \zeta^{j-\nu-1}\Theta+\lambda^{-1}\cdot\tau_{j,1}(\xi')+\lambda^{-2}\cdot\tau_{j,2}(\lambda, \xi')\,.$$

Here $\tau_{j,k\delta}(\xi')$ and $\tau_{j,1}(\xi')$ are continuous in $\boldsymbol{R}^{n-1}_{\xi'}$ and $\tau_{j,(N+1)\delta}(\lambda, \xi')$ and $\tau_{j,2}(\lambda, \xi')$ remain bounded on K when $\lambda\uparrow\infty$, where K is an arbitrary compact subset of $\boldsymbol{R}^{n-1}$.

With a new variable $\tilde{\lambda}$, we put for $j=1, \cdots, \nu$

$$\tilde{\tau}_j = \sigma_j(\xi')+\sum_{k=1}^{N} \tilde{\lambda}^{-k\delta}\cdot\tau_{j,\delta}(\xi')+\tilde{\lambda}^{-(N+1)\delta}\tau_{,(N+1)\delta}(\lambda, \xi')\,, \tag{2.21}$$

and for $j=\nu+1, \cdots, \mu$

$$\tilde{\tau}_j = \tilde{\lambda}\cdot\zeta^{j-\nu-1}\Theta+\tau_{j,1}(\xi')+\tilde{\lambda}^{-1}\cdot\tau_{j,2}(\lambda, \xi')\,. \tag{2.22}$$

If we substitute (2.21) and (2.22) for $\tau_j(\lambda, \xi')$, $j=1, \cdots, \mu$ in D_k, $k=0, \cdots, \mu$ and denote the result by $\tilde{D}_k$, we have asymptotic expansions of D_k with respect to $\tilde{\lambda}$ as

$$\tilde{D}_k = d_{k,0}(\xi')\cdot\tilde{\lambda}^J+\sum_{j=1}^{N} d_{k,j\delta}(\xi')\cdot\tilde{\lambda}^{J-j\delta}+\tilde{d}_{k,1}(\lambda, \xi')\cdot\tilde{\lambda}^{J-1}+o(\tilde{\lambda}^{J-1})\,. \tag{2.23}$$

Put

$$d_{k,1}(\xi') = \lim_{\lambda\uparrow\infty} \tilde{d}_{k,1}(\lambda, \xi')\,,$$

and $\lambda=\tilde{\lambda}$. Then asymptotic expansions of D_k are

$$D_k = d_{k,0}(\xi')\cdot\lambda^J+\sum_{j=1}^{N} d_{k,j\delta}(\xi')\cdot\lambda^{J-j\delta}+d_{k,1}(\xi')\cdot\lambda^{J-1}+o(\lambda^{J-1})\,. \tag{2.24}$$

Here even when $N\delta=1$, we deal with $d_{k,N\delta}$ and $d_{k,1}$ separately.

We have already calculated the leading terms $d_{k,0}(\xi')$ in Lemma 2.4, that is, $d_{0,0}=(2.5)$, $d_{k,0}=(2.6)$ for $k=1, \cdots, \nu$, and $d_{k,0}=(2.7)$ for $k=\nu+1, \cdots, \mu$. We shall calculate the leading terms of (2.24) when $V_{\mu-\nu,0}=0$.

Lemma 2.6. *Let Assumption* 2.5 *be satisfied. Assume that* $V_{\mu-\nu,0}=0$. *Then,* $\sum_{j=1}^{N} d_{k,j\delta}(\xi')\cdot\lambda^{J-j\delta}=0$, $k=0, \cdots, \mu$.

Furthermore, when $j_{\nu+1}-j_\nu\geqq 2$,

$$d_{0,1} = D_0(\sigma_1, \cdots, \sigma_\nu; b_{j_1}, \cdots, b_{j_\nu})\cdot\Theta^{J-1}\cdot B_{\mu-\nu}\,. \tag{2.25}$$

For $k=1, \cdots, \nu$

$$d_{k,1} = D_k(\sigma_1, \cdots, \sigma_\nu; b_{j_1}, \cdots, b_{j_\nu}; \hat{\phi}_1, \cdots, \hat{\phi}_\nu)\cdot\Theta^{J-1}\cdot B_{\mu-\nu}\,. \tag{2.26}$$

For $k=\nu+1, \cdots, \mu$

$$d_{k,1} = 0\,. \tag{2.27}$$

When $j_{\nu+1}-j_\nu=1$,

$$\begin{aligned} d_{0,1} = {} & D_0(\sigma_1, \cdots, \sigma_\nu; b_{j_1}, \cdots, b_{j_\nu})\cdot\Theta^{J-1}\cdot B_{\mu-\nu} \\ & +D_0(\sigma_1, \cdots, \sigma_\nu; b_{j_1}, \cdots, b_{j_{\nu-1}}, b_{j_{\nu+1}})\cdot\Theta^{J-1}\cdot V_{\mu-\nu,1}\,. \end{aligned} \tag{2.28}$$

For $k=1, \cdots, \nu$

$$\begin{aligned} d_{k,1} = {} & D_k(\sigma_1, \cdots, \sigma_\nu; b_{j_1}, \cdots, b_{j_\nu}; \hat{\phi}_1, \cdots, \hat{\phi}_\nu)\cdot\Theta^{J-1}\cdot B_{\mu-\nu} \\ & +D_0(\sigma_1, \cdots, \sigma_\nu; b_{j_1}, \cdots, b_{j_{\nu-1}}, b_{j_{\nu+1}}; \hat{\phi}_1, \cdots, \hat{\phi}_{\nu-1}, \hat{\phi}_{\nu+1})\cdot\Theta^{J-1}\cdot V_{\mu-\nu,1}\,. \end{aligned} \tag{2.29}$$

For $k=\nu+1, \cdots, \mu$ *there are two cases as follows.*

When $\nu=1$ *and* $j_\nu=0$, *it may be assumed that* $b_{j_1}=b_0=1$ *and* $b_{j_2}=b_1=\xi_1+b_{1,1}(\xi')$. *Then*

$$d_{k,1}=(-1)^{k-2}(\hat{\phi}_2-(\sigma_1+b_{1,1}(\xi'))\hat{\phi}_1)\cdot D_{(k)}\,. \tag{2.30}$$

When $\nu\geqq 2$ or $j_\nu\geqq 1$,

$$d_{k,1}=0\,. \tag{2.31}$$

Proof. By the same argument as in Lemma 2.4, $d_{0,0}\tilde{\lambda}^J$ and $d_{0,j\delta}\tilde{\lambda}^{J-j\delta}$, $j=1, \cdots, N$ are contained in the sum of the products of

$$(\operatorname{sgn}\rho')b_{j_{\rho'(1)}}(\tilde{\tau}_1)\cdots b_{j_{\rho'(\nu)}}(\tilde{\tau}_\nu) \tag{2.32}$$

and

$$(\operatorname{sgn}\rho'')(\tilde{\tau}_{\nu+1})^{j_{\nu+\rho''(1)}}\cdots(\tilde{\tau}_\mu)^{j_{\nu+\rho''(\mu-\nu)}}\,. \tag{2.33}$$

Put $\tau'_j=\zeta^{j-\nu-1}\Theta$, $j=\nu+1, \cdots, \mu$. Then, by the binomial theorem, for $k=j_{\nu+1}, \cdots, j_\mu$ and $j=\nu+1, \cdots, \mu$

$$(\tilde{\tau}_j/\tilde{\lambda})^k=(\tau'_j+\tilde{\lambda}^{-1}\tau_{j,1}(\xi')+O(\tilde{\lambda}^{-2}))^k=\tau'^k_j+\tilde{\lambda}^{-1}\cdot k\cdot\tau'^{k-1}_j\cdot\tau_{j,1}(\xi')+O(\tilde{\lambda}^{-2})\,. \tag{2.34}$$

Hence every term of (2.33) can be expanded as (2.35)+(2.36):

$$(\operatorname{sgn}\rho'')(\tau'_{\nu+1})^{j_{\nu+\rho''(1)}}\cdots(\tau'_\mu)^{j_{\nu+\rho''(\mu-\nu)}} \tag{2.35}$$

$$\begin{aligned}(\operatorname{sgn}\rho'')\cdot\textstyle\sum_{k=1}^{\mu-\nu}&(\tau'^{\,j_{\nu+\rho''(1)}}_{\nu+1})\cdots(\tau'_{\nu+k-1})^{j_{\nu+\rho''(k-1)}}\\&\times\tilde{\lambda}^{-1}\cdot j_{\nu+\rho''(k)}\cdot(\tau'_{\nu+k})^{j'_{\rho''(k)}}\tau_{\nu+k,1}(\xi')\\&\times(\tau'_{\nu+k+1})^{j_{\nu+\rho''(k+1)}}\cdots(\tau'_\mu)^{j_{\nu+\rho''(\mu-\nu)}}+O(\tilde{\lambda}^{-2})\,.\end{aligned} \tag{2.36}$$

Here $j'_k=j_k-1$. Since $V_{\mu-\nu,0}=0$ implies that (2.35)$=0$, we have (2.33)$=O(\tilde{\lambda}^{-1})$. Hence (2.33)$\times\tilde{\lambda}^{-j\delta}\tau_{l,j\delta}=O(\tilde{\lambda}^{-1-j\delta})$. Put $\tilde{\lambda}=\lambda$, then

$$\textstyle\sum_{j=1}^N d_{k,j\delta}(\xi')\cdot\lambda^{J-j\delta}=0\,,\qquad k=0, \cdots, \mu\,.$$

This implies that when we calculate $d_{k,1}(\xi')$, $k=0, \cdots, \mu$, we can ignore the terms $\tau_{l,j\delta}(\xi')$, $l=1, \cdots, \nu$, $j=1, \cdots, N$ under the condition that $V_{\mu-\nu,0}=0$. Put $\tau_{l,j\delta}(\xi')=0$, $l=1, \cdots, \nu$, $j=1, \cdots, N$, then for $k=0, \cdots, \mu$

$$\begin{aligned}d_{k,0}(\xi')&=\lim_{\lambda\uparrow\infty}\tilde{d}_{k,0}(\lambda, \xi')\,,\\ d_{k,1}(\xi')&=\lim_{\lambda\uparrow\infty}(\lambda(\tilde{d}_{k,0}(\lambda, \xi')-d_{k,0}(\xi'))+\tilde{d}_{k,1}(\lambda, \xi'))\,.\end{aligned} \tag{2.37}$$

When $j_{\nu+1}-j_\nu\geqq 2$, we prove (2.25) first. Since the product of (2.13) and the sum of (2.35) is $d_{0,0}(\xi')$, we have

$$(2.38)\qquad \lim_{\lambda\uparrow\infty}\lambda(\tilde{d}_{0,0}(\lambda,\xi')-d_{0,0}(\xi'))=D_0(\sigma_1,\cdots,\sigma_\nu;b_{j_1},\cdots,b_{j_\nu})\cdot\Theta^{J-1}\cdot\textstyle\sum_{k=1}^{\mu-\nu}\partial D_k\,.$$

Since every $b_j(\tau)$ is represented as (2.2), the coefficient of $\tilde{\lambda}^{J-1}$ in (2.10) is

$$(2.39)\qquad \textstyle\sum_{k=1}^{\mu-\nu}(\tau_{\nu+1})^{j_{\rho(\nu+1)}}\cdots(\tau_{\nu+k-1})^{j_{\rho(\nu+k-1)}}$$
$$\times b_{j_{\rho(k)},1}(\xi')(\tau_{\nu+k})^{j'_{\rho(k)}}(\tau_{\nu+k+1})^{j_{\rho(\nu+k+1)}}\cdots(\tau_\mu)^{j_{\rho(\mu)}}\,.$$

Hence

$$(2.40)\qquad \lim_{\lambda\uparrow\infty}\tilde{d}_{0,1}(\lambda,\xi')=D_0(\sigma_1,\cdots,\sigma_\nu;b_{j_1},\cdots,b_{j_\nu})\cdot\Theta^{J-1}\cdot\textstyle\sum_{k=1}^{\mu-\nu}b_{j_{\nu+k},1}(\xi')\cdot V_{\mu-\nu,k}\,.$$

Recalling (2.37), we have (2.25) by (2.38) and (2.40).

Now we prove (2.26). By a similar method substituting (2.16) for (2.9) in Lemma 2.4, and $D_k(\sigma_1,\cdots,\sigma_\nu;b_{j_1},\cdots,b_{j_\nu};\hat{\phi}_1,\cdots,\hat{\phi}_\nu)$ for $D_0(\sigma_1,\cdots,\sigma_\nu;b_{j_1},\cdots,b_{j_\nu})$, we have (2.26). Since $j_{\nu+1}-j_\nu\geqq 2$, (2.18) implies (2.27).

When $j_{\nu+1}-j_\nu=1$, we must consider the following extra terms:

$$(2.41)\qquad (\operatorname{sgn}\rho)b_{j_{\rho(1)}}(\tau_1(\lambda,\xi'))\cdots b_{j_{\rho(\nu-1)}}(\tau_{\nu-1}(\lambda,\xi'))$$
$$\times b_{j_{\rho(\nu+1)}}(\tau_\nu(\lambda,\xi'))\cdot b_{j_{\rho(\nu)}}(\tilde{\lambda}\cdot\tau_{\nu+1}(\lambda,\xi')/\lambda)$$
$$\times b_{j_{\rho(\nu+2)}}(\tilde{\lambda}\cdot\tau_{\nu+2}(\lambda,\xi')/\lambda)\cdots b_{j_{\rho(\mu)}}(\tilde{\lambda}\cdot\tau_\mu(\lambda,\xi')/\lambda)\,,$$

where ρ satisfies (2.11) and (2.12). By substituting $b_{j_{\nu+1}}$ for b_{j_ν} in Lemma 2.4, we can apply Lemma 2.4 to (2.41). Then the sum of (2.41) is expanded as

$$(2.42)\qquad \tilde{\lambda}^{J-1}\textstyle\sum_{\rho',\rho''}(\operatorname{sgn}\rho')$$
$$\times b_{j_{\rho'(1)}}(\tau_1(\lambda,\xi'))\cdots b_{j_{\rho'(\nu-1)}}(\tau_{\nu-1}(\lambda,\xi'))b_{j_{\rho'(\nu+1)}}(\tau_\nu(\lambda,\xi'))$$
$$\times(\operatorname{sgn}\rho'')(\tau_{\nu+1}(\lambda,\xi')/\lambda)^{j_{\lambda+\rho''(0)}}(\tau_{\nu+2}(\lambda,\xi')/\lambda)^{j_{\nu+\rho''(2)}}$$
$$\times\cdots(\tau_\mu(\lambda,\xi')/\lambda)^{j_{\nu+\rho''(\mu-\nu)}}+O(\tilde{\lambda}^{J-2})\,,$$

where ρ' is a permutation of $1,\cdots,\nu-1,\nu+1$ and ρ'' is of $0,2,\cdots,\mu-\nu$. Putting $\tilde{\lambda}=\lambda$ and dividing (2.42) by λ^{J-1}, we have, when $\lambda\uparrow\infty$,

$$(2.42)\to D_0(\sigma_1,\cdots,\sigma_\nu;b_{j_1},\cdots,b_{j_{\nu-1}},b_{j_{\nu+1}})\cdot\Theta^{J-1}\cdot V_{\mu-\nu,1}\,.$$

Adding this extra term to (2.25), we have (2.28).

If we substitute $b_{j_{\nu+1}}$ for b_{j_ν} and $\phi_{\nu+1}$ for ϕ_ν in Lemma 2.4, we can prove (2.29) similarly.

Now we prove (2.30). Denote the transposition of i and j by (i,j) and put $\tilde{\rho}=\rho$, for $k=2$ and

$$\tilde{\rho}=(2,3)\cdots(k-2,k-1)(k-1,k)\rho\,,\qquad \text{for}\quad k\geqq 3\,.$$

Then $\nu=1$ and $j_\nu=0$ implies that (2.17) becomes

$$(2.43)\qquad (-1)^{k-2}(\operatorname{sgn}\tilde{\rho})b_{j_{\tilde{\rho}(1)}}(\tau_1(\lambda,\xi'))\hat{\phi}_{\tilde{\rho}(2)} \times b_{j_{\tilde{\rho}(3)}}(\tilde{\lambda}\cdot\tau_2(\lambda,\xi')/\lambda)\cdots b_{j_{\tilde{\rho}(k)}}(\tilde{\lambda}\cdot\tau_{k-1}(\lambda,\xi')/\lambda) \times b_{j_{\tilde{\rho}(k+1)}}(\tilde{\lambda}\cdot\tau_{k+1}(\lambda,\xi')/\lambda)\cdots b_{j_{\tilde{\rho}(\mu)}}(\tilde{\lambda}\cdot\tau_\mu(\lambda,\xi')/\lambda)\,.$$

Therefore (2.30) can be calculated by the same method as in Lemma 2.4 with $\nu=k=2$. The order of $\tilde{\lambda}$ of the leading term of (2.43) is $J-j_2=J-1$. The term of order $J-1$ is the sum of

$$(2.44)\qquad (-1)^{k-2}(\operatorname{sgn}\tilde{\rho})b_{j_{\tilde{\rho}(1)}}(\sigma_1(\xi'))\hat{\phi}_{\tilde{\rho}(2)} \times(\tau_2')^{j_{\tilde{\rho}(3)}}\cdots(\tau_{k-1}')^{j_{\tilde{\rho}(k)}}(\tau_{k+1}')^{j_{\tilde{\rho}(k+1)}}\cdots(\tau_\mu')^{j_{\tilde{\rho}(\mu)}}\,,$$

where $\tilde{\rho}$ satisfies (2.11) and (2.12) with $\nu=2$. Hence

$$(2.45)\qquad d_{k,1}=(-1)^{k-2}D_2(\sigma_1(\xi'),\sigma;1,b_{j_2};\hat{\phi}_1,\hat{\phi}_2)\cdot D_{(k)}\,.$$

By the definition, we have

$$(2.46)\qquad D_2(\sigma_1,\sigma;1,b_{j_2};\hat{\phi}_1,\hat{\phi}_2)=\hat{\phi}_2-(\sigma_1+b_{1,1}(\xi'))\hat{\phi}_1\,.$$

Therefore (2.45) and (2.46) imply (2.30).

Since $j_{\nu+1}-1=j_\nu\geqq 1$, (2.18) implies (2.31). [Q.E.D.]

Lemma 2.7. *If* rank Mat $V_{\mu-\nu,0}\leqq\mu-\nu-2$, *then*

$$(2.47)\qquad d_{k,0}(\xi')=d_{k,1}(\xi')=0\,,\qquad \textit{for}\quad k=0,\cdots,\mu\,.$$

Proof. Put $l_1=j_{\nu+k_1}$, $l_2=j_{\nu+k_2}$, $l_1'=j_{\nu+k_1'}$ and $l_2'=j_{\nu+k_2'}$. Since Mat $V_{\mu-\nu,k}$ and Mat ∂D_k, $k=1,\cdots,\mu-\nu$ are different from Mat $V_{\mu-\nu,0}$ by the kth row, either the k_1th row and the k_2th row, or the k_1'th row and the k_2'th row remains equal. This implies that $V_{\mu-\nu,k}=0$ and $\partial D_k=0$, $k=1,\cdots,\mu-\nu$. Hence $B_{\mu-\nu}=0$ by the definition. Recalling that $D_{(k)}$ is one of the minors of order $\mu-\nu-1$ of Mat $D_0(\Theta,\cdots,\Theta\zeta^{\mu-\nu-1};\tau^{j_\nu},\cdots,{}^{j_\mu})$, which is zero by Lemma 2.2, we have $D_{(k)}=0$. Therefore Lemma 2.6 implies that $d_{k,1}(\xi')=0$, $k=0,\cdots,\mu$. [Q.E.D.]

REMARK. This lemma shows us that we need to calculate $d_{k,l}$ for $k=0,\cdots,\mu$ and $l\geqq 2$, when rank Mat $V_{\mu-\nu,0}\leqq\mu-\nu-2$.

3. The properties of the characteristic roots

Let $n\geqq 3$. Assume that $P_1(D)$ and $P_2(D)$ are elliptic linear operators with constant coefficients of order 2μ and 2ν with $\mu>\nu$, respectively, such that

$$P_1(\xi) = \xi_1^{2\mu}+\textstyle\sum_{j=1}^{2\mu} p_{1,j}(\xi')\xi_1^{2\mu-j}\,, \tag{3.1}$$

$$P_2(\xi) = -(\exp i\theta)\cdot\xi_1^{2\nu}+\textstyle\sum_{j=1}^{2\nu} p_{2,j}(\xi')\xi_1^{2\nu-j}\,. \tag{3.2}$$

Here $p_{1,j}(\xi')$ and $p_{2,j}(\xi')$ are polynomials of ξ' with their orders not higher than j, and θ satisfies $0\leqq\theta<2\pi$.

We shall deal with the following polynomial with a large positive parameter λ:

$$P_1(\xi)+\lambda^{2\mu-2\nu}P_2(\xi) = 0\,. \tag{3.3}$$

Denote the characteristic roots of (3.3) with respect to ξ_1 by $\tau_j(\lambda, \xi')$, $j=1, \cdots, 2\mu$ and those of

$$P_2(\xi) = 0 \tag{3.4}$$

with respect to ξ_1 by $\sigma_j(\xi')$, $j=1, \cdots, 2\nu$, respectively.

Assumption 3.1.

The characteristic roots of (3.4) are simple for all ξ'.

There exists a positive number C such that

$$P_1(\xi)+\lambda^{2\mu-2\nu}P_2(\xi)\neq 0\,, \qquad \text{for} \quad \lambda\geqq C \text{ and } |\xi|\geqq C\,.$$

Since P_2 is elliptic, we may assume that $\sigma_j(\xi')$, $j=1, \cdots, \nu$ have positive imaginary parts for $|\xi'|\geqq C$.

Lemma 3.2. *Let Assumption* 3.1 *be satisfied. If the suffixes* $\{j\}$ *of the characteristic roots* $\tau_j(\lambda, \xi')$, $j=1, \cdots, 2\mu$ *are properly chosen, then there exists a positive number* λ_R *for every positive number* R *with* $R>C$ *such that if* $\lambda>\lambda_R$, *then* $\tau_j(\lambda, \xi')$, $j=1, \cdots, \nu$ *have positive imaginary parts for* $|\xi'|>C$ *and satisfy* (2.19) *for* $\delta=1$ *and* $\tau_j(\lambda, \xi')$, $j=\nu+1, \cdots, \mu$ *have positive imaginary parts for all* ξ' *and satisfy* (2.20).

Proof. First we deal with

$$t^{2\mu}+\textstyle\sum_{j=1}^{2\mu} a_j t^{2\mu-j}-\exp(i\theta)t^{2\nu}+\sum_{j=1}^{2\nu} b_j t^{2\nu-j} = 0\,. \tag{3.5}$$

Denote the roots of (3.5) by $t_j=t_j(a, b)$, where $j=1, \cdots, 2\mu$, $a=(a_1, \cdots, a_{2\mu})$, and $b=(b_1, \cdots, b_{2\nu})$. They every $t_j(0, 0)$ satisfies

$$t^{2\mu}-\exp(i\theta)t^{2\nu} = 0\,. \tag{3.6}$$

The roots of (3.6) are 0 and $\Theta\zeta^{j-1}$, $j=1, \cdots, 2\mu-2\nu$ and the roots of (3.6) with positive imaginary parts are $\Theta\zeta^{j-1}, j=1, \cdots, \mu-\nu$. Since the roots of $t^{2\mu-2\nu}=\exp(i\theta)$ are simple, there exists a positive number η such that $t_j(a, b), j=2\nu+1, \cdots, 2\mu$ are simple for $|a|<\eta$ and $|b|<\eta$. Then every $t_j(a, b), j=2\nu+1, \cdots, 2\mu$ is analytic in $|a|<\eta$ and $|b|<\eta$. Hence every $t_j(a, b), j=2\nu+1, \cdots, 2\mu$ has its power series representation with centre 0:

$$t_j(a, b) = \sum_\alpha t_{j,\alpha}\cdot(a, b)^\alpha , \tag{3.7}$$

where $(a, b)^\alpha=(a_1)^{\alpha_1}\cdots(a_{2\mu})^{\alpha_{2\mu}}(b_1)^{\alpha_{2\mu+1}}\cdots(b_{2\nu})^{\alpha_{2\mu+2\nu}}$.

We may assume that

$$t_j(0, 0) = 0\ , \quad j = 1, \cdots, 2\nu; \quad t_j(0, 0) = \Theta\zeta^{j-2\nu-1}\ , \quad j = 2\nu+1, \cdots, 2\mu\ .$$

Then for $j=2\nu+1, \cdots, 2\mu$

$$t_j(a, b) = \Theta\zeta^{j-2\nu-1}+\sum_{\alpha\neq 0} t_{j,\alpha}\cdot(a, b)^\alpha. \tag{3.8}$$

If we divide (3.3) by $\lambda^{2\mu}$ and put $\xi_1/\lambda=t$ and

$$(\lambda^{-1}\cdot p_{1,1}, \cdots, \lambda^{-2\mu}\cdot p_{1,2\mu}, \lambda^{-1}\cdot p_{2,1}, \cdots, \lambda^{-2\nu}\cdot p_{2,2\nu}) = (a, b)\ , \tag{3.9}$$

then we have (3.5). For η and R, there exists a positive number λ_R'' with $\lambda_R''>C$ such that $\lambda>\lambda_R''$ implies that $|a|<\eta$ and $|b|<\eta$. Substitute (3.9) for (a, b) and put $\tau_j(\lambda, \xi')=\lambda\cdot t_j(a, b)$ in (3.8). Thus we have representations as (2.20).

Next we deal with

$$s^{2\nu}+\sum_{j=1}^{2\nu} b_j s^{2\nu-j} = 0\ . \tag{3.10}$$

Denote the roots of (3.10) by $s_j=s_j(b)$, where $j=1, \cdots, 2\nu$ and $b=(b_1, \cdots, b_{2\nu})$. Put $b_0=(-\exp(-i\theta)\cdot p_{2,1}(\xi'), \cdots, -\exp(-i\theta)\cdot p_{2,2\nu}(\xi'))$. Then Assumption 3.1 implies that $s_j(b_0)=\sigma_j(\xi'), j=1, \cdots, 2\nu$ are simple. Hence there exists a positive continuous function $\eta(\xi')$ such that $s_j(b), j=1, \cdots, 2\nu$ are simple and analytic of b in $|b-b_0(\xi')|<\eta(\xi')$ and $s_j(b), j=1, \cdots, \nu$ have positive imaginary parts in $|b-b_0(\xi')|<\eta(\xi')$ for $|\xi'|>C$. Denote

$$\prod_{j=1}^{2\nu} (\tau-\tau_j(\lambda, \xi')) = Q_1(\tau, \lambda, \xi')\ ,$$
$$\sum_{j=2\nu+1}^{2\mu} ((\tau/\lambda)-(\tau_j(\lambda, \xi')/\lambda)) = Q_2(\tau, \lambda, \xi')\ ,$$
$$A_1 = \{(\tau, \lambda^{-1}, \xi');\ |\tau|<\lambda_R'/2,\ \lambda^{-1}<\lambda_R'^{-1},\ |\xi'|<R\},$$

and

$$A_2 = \{(\lambda^{-1}, \xi');\ \lambda^{-1}<\lambda_R'^{-1},\ |\xi'|<R\}\ .$$

Then we can write

$$\lambda^{-(2\mu-2\nu)}P_1(\tau, \xi')+P_2(\tau, \xi') = Q_1\times Q_2\ . \tag{3.11}$$

Since $\lim_{\lambda\uparrow\infty} |\tau_j(\lambda, \xi')/\lambda| = 1$, $j = 2\nu+1, \cdots, 2\mu$, there exists a positive number λ'_R with $\lambda'_R > \lambda''_R$ such that $Q_2 \neq 0$ in A_1. Then Q_2 is a polynomial of τ with analytic coefficients of (λ^{-1}, ξ') in A_2 and $1/Q_2$ is analytic in A_1. Hence $(\lambda^{-(2\mu-2\nu)}P_1 + P_2)/Q_2$ has a power series representation of $(\tau, \lambda^{-1}, \xi')$ in A_1 and Q_1 is a polynomial of τ with analytic coefficients $b_j(\lambda^{-1}, \xi')$ in A_2 as

$$Q_1(\tau, \lambda, \xi') = \tau^{2\nu} + \textstyle\sum_{j=1}^{2\nu} b_j(\lambda^{-1}, \xi')\tau^{2\nu-j} .$$

Since $\lim_{\lambda\uparrow\infty} Q_1 = -\exp(-i\theta)\cdot P_2$, there exists a positive number λ_R with $\lambda_R > \lambda'_R$ such that if $\lambda > \lambda_R$, then $|b(\lambda^{-1}, \xi') - b_0| < \eta(\xi')$. Hence the characteristic roots $\tau_j(\lambda, \xi')$, $j = 1, \cdots, 2\nu$ of Q_1 are analytic of (λ^{-1}, ξ') in $\{(\lambda^{-1}, \xi');\ \lambda^{-1} < \lambda_R^{-1}, |\xi'| < R\}$. Thus $\tau_j(\lambda, \xi')$, $j=1, \cdots, 2\nu$ can be expanded as (2.19).

By renumbering the suffixes $\{j\}$ properly, we have the conclusion.

[Q.E.D.]

4. The reducibility of the one-parameter family

If we divide the equation of (1.2) by $\varepsilon = \lambda^{-2\mu+2\nu}$ then we have

$$\text{(4.1)} \qquad \begin{cases} (P_1(D) + \lambda^{2\mu-2\nu}P_2(D))u(x) = 0 & \text{in } \boldsymbol{R}^n_+ ; \\ b_{j_k}(D)u(x)|_{x_1\downarrow 0} = \phi_k(x') , & k = 1, \cdots, \mu . \end{cases}$$

Here we require Assumption 3.1 and that every symbol of $b_{j_k}(D)$ is represented as (2.2). We shall consider the one-parameter family (4.1) with $\lambda > 1$ instead of (1.2) and study the behaviour when $\lambda \uparrow \infty$. We denote by u_∞ the limit of the canonical extension $[u_\lambda]^+$ of a solution u_λ of (4.1). We know that $[u_\lambda]^+$ is uniquely determined when the boundary conditions of (4.1) are coercive.

We can define the reducibility of (4.1) by replacing u_ε, u_0, and $\lim_{\varepsilon\downarrow 0}$ by u_λ, u_∞, and $\lim_{\lambda\uparrow\infty}$, respectively, in the definition of the reducibility of (1.2).

We shall consider the solutions of (4.1) solved by the partial Fourier transformation with respect to x'. We denote by $\wedge$ the partial Fourier transformation with respect to x' and by $\mathcal{F}^{-1}$ the inverse partial Fourier transformation with respect to ξ'. The partial Fourier transform of (4.1) is

$$\text{(4.2)} \qquad \begin{cases} (P_1(D_1, \xi') + \lambda^{2\mu-2\nu}P_2(D_1, \xi'))\hat{u}(x_1, \xi') = 0 ; \\ b_{j_k}(D_1, \xi')\hat{u}_1(x_1, \xi')|_{x_1\downarrow 0} = \hat{\phi}_k(\xi') , & k = 1, \cdots, \mu . \end{cases}$$

This is an ordinary differential equation subjected to parameters (λ, ξ'). We shall consider only the solution $\hat{u}(x_1, \xi')$ of (4.2) represented by

$$\text{(4.3)} \qquad \hat{u}(x_1, \xi') = Y(x_1)\cdot \textstyle\sum_{k=1}^{\mu} C_k(\lambda, \xi')(\exp i\tau_k(\lambda, \xi')x_1) .$$

Here $Y(x_1)$ is the Heaviside function. The partial Fourier transforms of the boundary conditions in (4.2) are represented as

$$\sum_{k=1}^{\mu} b_{j_l}(\tau_k(\lambda, \xi'), \xi')C_k(\lambda, \xi') = \hat{\phi}_l(\xi'), \qquad l = 1, \cdots, \mu. \tag{4.4}$$

If $D_0 \neq 0$, then C_k can be obtained uniquely by Cramer's formula as

$$C_k(\lambda, \xi') = D_k/D_0, \qquad k = 1, \cdots, \mu. \tag{4.5}$$

For the definitions of D_0 and D_k see Notation 2.1.

ASSUMPTION 4.1.

There exist positive numbers J, C, and M independent of $\lambda > 1$ and ξ' in $\boldsymbol{R}^{n-1}$ such that

$$|(D_0/\lambda^J)^{-1}| \leqq C\langle\xi'\rangle^M, \tag{4.6}$$

and every cofactor $D_{0,k,l}$ of D_0, k, $l=1, \cdots, \mu$ satisfies

$$|D_{0,k,l}| \leqq C\lambda^J\langle\xi'\rangle^M. \tag{4.7}$$

Here $\langle\xi'\rangle = (1+|\xi'|^2)^{1/2}$.

REMARK. Since Assumption 4.1 assures the commutation of the limit $\lambda \uparrow \infty$ and the inverse Fourier transformation, we have only to calculate the pointwise limit of (4.3). It is difficult to check (4.6) when the data of (4.1) belong to $\mathcal{S}(\boldsymbol{R}^{n-1})$. But if we restrict the data of (4.1) from $\mathcal{S}(\boldsymbol{R}^{n-1})$ to $\mathcal{F}^{-1}(C_0^\infty(K))$, where K is a compact set of $\boldsymbol{R}_{\xi'}^{n-1}$, then (4.6) follows the estimate of $\lim_{\lambda \uparrow \infty} D_0$.

NOTATION 4.2.

$$\begin{aligned}
D_0(\tau) &= D_0(\tau_1, \cdots, \tau_\mu; b_{j_1}, \cdots, b_{j_\mu}) \\
D_k(\tau) &= D_k(\tau_1, \cdots, \tau_\mu; b_{j_1}, \cdots, b_{j_\mu}; \hat{\phi}_1, \cdots, \hat{\phi}_\mu) \\
D_0(\sigma) &= D_0(\sigma_1, \cdots, \sigma_\nu; b_{j_1}, \cdots, b_{j_\nu}) \\
D_k(\sigma) &= D_k(\sigma_1, \cdots, \sigma_\nu; b_{j_1}, \cdots, b_{j_\nu}; \hat{\phi}_1, \cdots, \hat{\phi}_\nu) \\
D_0(\sigma; \nu) &= D_0(\sigma_1, \cdots, \sigma_\nu; b_{j_1}, \cdots, b_{j_{\nu-1}}, b_{j_{\nu+1}}) \\
D_k(\sigma; \nu) &= D_k(\sigma_1, \cdots, \sigma_\nu; b_{j_1}, \cdots, b_{j_{\nu-1}}, b_{j_{\nu+1}}; \hat{\phi}_1, \cdots, \hat{\phi}_{\nu-1}, \hat{\phi}_{\nu+1})
\end{aligned}$$

ASSUMPTION 4.3. For all ξ' in $\boldsymbol{R}^{n-1}$,

(1) $D_0(\sigma) \neq 0$.
(2) $D_0(\sigma; \nu) \neq 0$.
(3) $D_0(\sigma) \cdot B_{\mu-\nu} + D_0(\sigma; \nu) \cdot V_{\mu-\nu,1} \neq 0$.

REMARK. Assumption 4.3 implies the unique solvability of the reduced problem. This requirement is natural from the viewpoint of singular perturbations. Recall that $V_{\mu-\nu,1}$ is independent of ξ'.

Theorem 4.4. *Let Assumption* 3.1, 4.1 *and* 4.3 *be satisfied. By restricting the data* ϕ_k, $k=1, \cdots, \mu$ *from* $\mathcal{S}(\boldsymbol{R}^{n-1})$ *to* $\mathcal{F}^{-1}(C_0^\infty(\boldsymbol{R}^{n-1}))$, *Assumption* 3.1 *can be removed.*

(1) *If* rank Mat $V_{\mu-\nu,0}=\mu-\nu$, *then the family* (4.1) *is normally reducible. In particular, if the boundary conditions are Dirichlet's*

$$b_{j_k}(D) = D_1^{k-1}\,, \qquad k = 1, \cdots, \mu\,, \tag{4.8}$$

then the family (4.1) *is normally reducible.*

(2) *Assume that* rank Mat $V_{\mu-\nu,0}=\mu-\nu-1$.

(2–1) *If* $j_{\nu+1}-j_\nu \geqq 2$ *and* $B_{\mu-\nu} \not\equiv 0$, *then the family* (4.1) *is normally reducible.*

(2–2) *If* $j_{\nu+1}-j_\nu=1$, *then there are three cases as follows.*

(2–2–a) *If* $B_{\mu-\nu} \not\equiv 0$ *and* $V_{\mu-\nu,1}=0$, *then the family* (4.1) *is normally reducible.*

(2–2–b) *If* $B_{\mu-\nu} \equiv 0$ *and* $V_{\mu-\nu,1} \not\equiv 0$, *then* u_∞ *satisfies the following boundary conditions:*

$$b_{j_k}(D)u(x)\,|_{x_1\downarrow 0} = \phi_k(x')\,, \qquad k = 1, \cdots, \nu-1,\ \nu+1\,. \tag{4.10}$$

In particular, the family (4.1) *is abnormally reducible.*

(2–2–c) *If* $B_{\mu-\nu} \not\equiv 0$ *and* $V_{\mu-\nu,1} \not\equiv 0$, *then the family* (4.1) *is not reducible.*

Proof. Assumption 4.1 implies that there exists a unique solution u_λ of (4.1) having representation (4.3).

Case (1). By Lemma 2.2, 2.4, Assumption 4.3-(1) and the condition $V_{\mu-\nu,0} \not\equiv 0$ we have for $k=1, \cdots, \nu$

$$\lim_{\lambda\uparrow\infty} C_k(\lambda, \xi') = \lim_{\lambda\uparrow\infty} \frac{D_k(\tau)/\lambda^J}{D_0(\tau)/\lambda^J} = D_k(\sigma)/D_0(\sigma)\,, \tag{4.11}$$

and for $k=\nu+1, \cdots, \mu$

$$\lim_{\lambda\uparrow\infty} C_k(\lambda, \xi') = 0\,. \tag{4.12}$$

Here the convergence is uniform in ξ' on every compact subset of $\boldsymbol{R}^{n-1}$. Since every $\tau_j(\lambda, \xi')$, $j=1, \cdots, \nu$ has a positive imaginary part, it follows that for every rapidly decreasing function $\psi(\xi')$ and for fixed x_1,

$$\lim_{\lambda\uparrow\infty} \langle \hat{u}_\lambda(x_1, \xi'), \psi\rangle_{\xi'} = \textstyle\sum_{k=1}^{\nu} Y(x_1)\cdot\langle(\exp i\sigma_k(\xi')x_1)D_k(\sigma)/D_0(\sigma), \psi\rangle_{\xi'}\,. \tag{4.13}$$

Thus u_λ converges to u_∞ satisfying (1.4). Therefore the family (4.1) is normally reducible.

If the boundary conditions are Dirichlet's conditions, then $A=\{\nu, \cdots, \mu-1\}$. For every pair (l, l') of two different integers in A with $l<l'$, we have $l'-l\leqq \mu-1-\nu<2\mu-2\nu$. Hence it is impossible that $l\equiv l'$ (mod $2\mu-2\nu$). This implies that the family (4.1) is normally reducible.

Case (2). We shall use the representation (2.24). Since $V_{\mu-\nu,0}=0$, we have for $k=0, \cdots, \mu$, $d_{k,0}(\xi')=0$ by Lemma 2.4 and $\sum_{j=1}^{N} d_{k,j\delta}(\xi')\lambda^{J-j\delta}=0$ by Lemma 2.6. Hence

$$\lim_{\lambda\uparrow\infty} C_k(\lambda, \xi') = \lim_{\lambda\uparrow\infty} \frac{D_k(\tau)/\lambda^{J-1}}{D_0(\tau)/\lambda^{J-1}} = \lim_{\lambda\uparrow\infty} \frac{d_{k,1}(\xi')+o(1)}{d_{0,1}(\xi')+o(1)} = \frac{d_{k,1}(\xi')}{d_{0,1}(\xi')}\,. \tag{4.14}$$

Case (2–1). Lemma 2.6 implies that for $k=1, \cdots, \nu$

$$\lim_{\lambda\uparrow\infty} C_k(\lambda, \xi') = D_k(\sigma)/D_0(\sigma)\,, \tag{4.15}$$

and that for $k=\nu+1, \cdots, \mu$

$$\lim_{\lambda\uparrow\infty} C_k(\lambda, \xi') = 0\,. \tag{4.16}$$

By the same argument as in Case (1), (4.1) proves to be normally reducible.

Case (2–2). Since the imaginary part of every $\tau_j(\lambda, \xi')$ is positive, even if $\nu=1$ and $j_\nu=0$, we have for $k=\nu+1, \cdots, \mu$

$$\lim_{\lambda\uparrow\infty} C_k(\lambda, \xi')(\exp i\tau_k(\lambda, \xi')x_1)\cdot Y(x_1) = 0\,. \tag{4.17}$$

By Lemma 2.6 we have for $k=1, \cdots, \nu$

$$\lim_{\lambda\uparrow\infty} C_k(\lambda, \xi') = \frac{D_k(\sigma)\cdot B_{\mu-\nu}+D_k(\sigma;\, \nu)\cdot V_{\mu-\nu,1}}{D_0(\sigma)\cdot B_{\mu-\nu}+D_0(\sigma;\, \nu)\cdot V_{\mu-\nu,1}} \tag{4.18}$$

where the denominator is not equal to zero by Assumption 4.3–(3).

Case (2–2–a).

$$\lim_{\lambda\uparrow\infty} C_k(\lambda, \xi') = \frac{D_k(\sigma)\cdot B_{\mu-\nu}}{D_0(\sigma)\cdot B_{\mu-\nu}} = \frac{D_k(\sigma)}{D_0(\sigma)}\,. \tag{4.19}$$

This implies that u_∞ satisfies the following boundary conditions:

$$b_{j_k}(D)u(x)\,|_{x_1\downarrow 0} = \phi_k(x')\,, \qquad k=1, \cdots, \nu\,. \tag{4.20}$$

Therefore the family (4.1) is normally reducible.

Case (2–2–b).

$$\lim_{\lambda\uparrow\infty} C_k(\lambda, \xi') = \frac{D_k(\sigma;\, \nu)\cdot V_{\mu-\nu,1}}{D_0(\sigma;\, \nu)\cdot V_{\mu-\nu,1}} = \frac{D_k(\sigma;\, \nu)}{D_0(\sigma;\, \nu)}\,. \tag{4.21}$$

This implies that u_∞ satisfies (4.10) and that the family (4.1) is abnormally reducible.

Case (2–2–c). Since the right-hand side of (4.18) is different from that of (4.19), the unique solvability of the reduced problem in Case (2–2–a) implies that u_∞ can not satisfy every of the boundary conditions of (4.20). As the right-hand side of (4.18) does not contain ϕ_k, $k=\nu+2, \cdots, \mu$, u_∞ can not satisfy any of the following boundary conditions:

$$b_{j_k}(D)u_\infty|_{x_1\downarrow 0} = \phi_k\,, \qquad k = \nu+2, \cdots, \mu\,. \tag{4.22}$$

If u_∞ satisfies the boundary condition

$$b_{j_{\nu+1}}(D)u_\infty|_{x_1\downarrow 0} = \phi_{\nu+1}\,,$$

then the unique solvability of the reduced problem in Case (2–2–b) implies that the right-hand side of (4.18) is equal to the right-hand side of (4.21), which is a contradiction. Therefore the family (4.1) is not reducible. Since the convergence is locally uniform, Assumption 4.3 assures (4.8) for large λ.

[Q.E.D.]

We give and study an example of the abnormally reducible family.

Example 4.5. In (4.1) consider the following symbols:

$$P_1(\xi) = (\xi_1^4+\langle\xi'\rangle^4)(\xi_1^2+\langle\xi'\rangle^2)\,, \tag{4.23}$$

$$P_2(\xi) = (\xi_1^2+\langle\xi'\rangle^2)\,, \tag{4.24}$$

and

$$b_{j_1}(\xi) = 1\,, \quad b_{j_2}(\xi) = \xi_1\,, \quad \text{and} \quad b_{j_3}(\xi) = \xi_1^5\,, \tag{4.25}$$

where $\langle\xi'\rangle=(1+|\xi'|^2)^{1/2}$. Denote the boundary conditions with symbols (4.25) by

$$u|_{x_1\downarrow 0} = \phi_0\,, \quad D_1u|_{x_1\downarrow 0} = \phi_1\,, \quad \text{and} \quad D_1^5u|_{x_1\downarrow 0} = \phi_5\,. \tag{4.26}$$

Then $\mu=3$, $\nu=1$, and $A=\{1, 5\}$. Obviously Assumption 3.1 is satisfied. Let

$$\Theta = \exp \pi i/4, \quad \zeta = \exp \pi i/2 = i\,, \quad a = (\langle\xi'\rangle^4+\lambda^4)^{1/4}, \quad \text{and} \quad b = \langle\xi'\rangle/a\,.$$

Then the characteristic roots of $P_1+\lambda^4\cdot P_2=0$ are

$$\pm i\langle\xi'\rangle,\ \Theta a,\ \Theta ia,\ -\Theta a, \text{ and } -\Theta ia\,.$$

Let

$$\tau_1(\lambda, \xi') = \sigma_1(\xi') = i\langle\xi'\rangle\,, \quad \tau_2(\lambda, \xi') = \Theta a\,, \quad \text{and} \quad \tau_3(\lambda, \xi') = \Theta ia\,.$$

When $\langle\xi'\rangle/\lambda<1$, the binomial expansion of a/λ is

$$a/\lambda = 1+\frac{1}{4}\cdot(\langle\xi'\rangle/\lambda)^4+\textstyle\sum_{k=2}^{\infty}\binom{1/4}{k}(\langle\xi'\rangle/\lambda)^{4k}.$$

This implies that $\tau_{2,1}(\xi')=\tau_{3,1}(\xi')=0$ and therefore $\partial D_1=\partial D_2=0$. Recalling that $b_{j_2,1}(\xi')=b_{j_3,1}(\xi')=0$, we have $B_{3-1}=0$. A routine calculation gives

$$\begin{aligned}
&V_{3-1,0} = 0, \quad V_{3-1,1} = i-1, \quad D_0(\sigma) = 1, \quad D_0(\sigma;1) = i\langle\xi'\rangle,\\
&D_0(\sigma)\cdot B_{3-1}+D_0(\sigma;1)\cdot V_{3-1,1} = -(1+i)\langle\xi'\rangle,\\
&D_0 = -\Theta i(1-i)a^5\langle\xi'\rangle(1+b^4), \quad D_1 = -\Theta(1-i)a^5(\hat\phi_1+(\hat\phi_5/a^4)),\\
&D_2 = -ia^5(-\Theta i\langle\xi'\rangle(1+b^4)\hat\phi_0+(\Theta+b^5)\hat\phi_1+(\Theta-b)(\hat\phi_5/a^4)),
\end{aligned}$$

and

$$D_3 = -\Theta a^5(i\langle\xi'\rangle(1+b^4)\hat\phi_0-(1+\Theta b^5)\hat\phi_1-(1-\Theta b)(\hat\phi_5/a^4)).$$

Since $|\zeta|=1$, $0<\lambda<a$, and $0<b<1$, it follows that

$$\begin{aligned}
&|D_0| = \sqrt{2}\,a^5\langle\xi'\rangle(1+b^4)\geqq\sqrt{2}\,a^5\langle\xi'\rangle,\\
&|D_1| \leqq \sqrt{2}\,a^5(|\hat\phi_1|+\lambda^{-4}|\hat\phi_5|),
\end{aligned}$$

and

$$|D_2|,\ |D_3|\leqq 2a^5(\langle\xi'\rangle|\hat\phi_0|+|\hat\phi_1|+\lambda^{-4}|\hat\phi_5|).$$

Since $C_1=D_1/D_0$, $C_2=D_2/D_0$, and $C_3=D_3/D_0$, Assumption 4.1 and 4.3 are satisfied. Therefore we can apply Theorem 4.4 to this example.

We show that the convergence of $[u_\lambda]^+$ is in $L^2(\boldsymbol{R}^n_+)$. Let $|\cdot|_s$ be the norm of the Sobolev space $H^s(\boldsymbol{R}^{n-1})$. Then

$$\begin{aligned}
&|C_1\exp(-\langle\xi'\rangle x_1)|\leqq\langle\xi'\rangle^{-1}(|\hat\phi_1|+\lambda^{-4}|\hat\phi_5|)\cdot\exp(-x_1),\\
&|C_2\exp i\Theta ax_1|,\ |C_3\exp(-\Theta ax_1)|\\
&\quad\leqq\sqrt{2}\cdot\langle\xi'\rangle^{-1}(\langle\xi'\rangle|\hat\phi_0|+|\hat\phi_1|+\lambda^{-4}|\hat\phi_5|)\cdot\exp(-\sqrt{2}\lambda x_1/2).
\end{aligned}$$

Integrate $|u_\lambda|^2$ over $\boldsymbol{R}^n_+$. Then the partial Fourier transformation and the formula

$$\int_0^\infty \exp(-ct)\,dt = \frac{1}{c}, \qquad \text{for} \quad c>0,$$

imply that

$$|u_\lambda|^2_{L^2(\boldsymbol{R}^n_+)}\leqq 3(|\phi_1|^2_{-1}+\lambda^{-8}|\phi_5|^2_{-1})+18\sqrt{2}\cdot\lambda^{-1}(|\phi_0|^2_0+|\phi_1|^2_{-1}+\lambda^{-8}|\phi_5|^2_{-1})$$

This estimate shows that every u_λ belongs to $L^2(\boldsymbol{R}^n_+)$,

By the same argument as above we have

$$\hat u_\infty = -i\cdot\langle\xi'\rangle^{-1}\cdot\hat\phi_1\cdot\exp(-\langle\xi'\rangle x_1)$$

and

$$|u_\infty|^2_{L^2(\boldsymbol{R}^n_+)} \leqq \frac{1}{2} |\phi_1|^2_{-1}\,.$$

This estimate implies that u_∞ belongs to $L^2(\boldsymbol{R}^n_+)$. Let $\tilde{D}_0 = i\langle\xi'\rangle D_0$ and

$$\tilde{D}_1 = i\langle\xi'\rangle D_1 - D_0\cdot\hat{\phi}_1 = -\Theta i(1-i)a\langle\xi'\rangle(-\langle\xi'\rangle^4\hat{\phi}_1 + \hat{\phi}_5/a)\,.$$

Replacing u_λ by $u_\lambda - u_\infty$ and C_1 by $\tilde{C}_1 = \tilde{D}_1/\tilde{D}_0$ in the above estimates and using

$$|D_0|/a \geqq \sqrt{2}\,a^4\langle\xi'\rangle \geqq \sqrt{2}\,\lambda^4\langle\xi'\rangle\,,$$

we have

$$|\tilde{C}_1 \exp(-\langle\xi'\rangle x_1)| \leqq \lambda^{-4}\langle\xi'\rangle^{-1}(\langle\xi'\rangle|\hat{\phi}_1| + |\hat{\phi}_5|)$$

and

$$|u_\lambda - u_\infty|^2_{L^2(\boldsymbol{R}^n_+)} \leqq 3\cdot\lambda^{-8}(|\phi_1|^2_0 + |\phi_5|^2_{-1}) + 18\sqrt{2}\cdot\lambda^{-1}(|\phi_0|^2_0 + |\phi_1|^2_{-1} + \lambda^{-8}|\phi_5|^2_{-1})\,.$$

This estimate shows that $[u_\lambda]^+ \to [u_\infty]^+$ in $L^2(\boldsymbol{R}^n_+)$.

If $[u_\lambda]^+ \to [u_\infty]^+$ in $H^1(\boldsymbol{R}^n_+)$ then the continuity of the trace operator implies that u_∞ must satisfy

$$u|_{x_1\downarrow 0} = \phi_0\,. \tag{4.27}$$

But this is a contradiction. Therefore the convergence in $L^2(\boldsymbol{R}^n_+)$ is the strongest among the Sobolev topologies of $H^s(\boldsymbol{R}^n_+)$, where s runs over all integers.

Replace the boundary conditions (4.26) by the Dirichlet's conditions:

$$u|_{x_1\downarrow 0} = \phi_0\,, \quad D_1 u|_{x_1\downarrow 0} = \phi_1\,, \quad \text{and} \quad D_1^2 u|_{x_1\downarrow 0} = \phi_2\,. \tag{4.28}$$

Then the limit u_∞ satisfies (4.27). Therefore the family with (4.28) becomes normally reducible.

The family not with (4.26) but with (4.28) can be regarded as the perturbation of (1.4) with (4.24) and (4.27). Thus the situation proves to be delicate in perturbing the boundary conditions.

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Department of Mathematics
Osaka City University
Sugimoto, Sumiyoshi-ku
Osaka 558, Japan