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A CHARACTERIZATION OF FOUR-GENUS OF KNOTS

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Introduction

We shall work in piecewise linear category. All knots and links will be assumed to be oriented in a 3-sphere S^3.

The 4-*genus* $g^*(K)$ of a knot K in $S^3 = \partial B^4$ is the minimum genus of orientable surfaces in B^4 bounded by K [1]. The *nonorientable* 4-*genus* $\gamma^*(K)$ is the minimum first Betti number of nonorientable surfaces in B^4 bounded by K [3]. For a slice knot, it is defined to be zero instead of one. The first author [4] defined the 4-*dimensional clasp number* $c^*(K)$ to be the minimum number of the double points of transversely immersed 2-disks in B^4 bounded by K. He gave an inequality $g^*(K) \leq c^*(K)$ [4, Lemma 9] and asked whether an equality $g^*(K) = c^*(K)$ holds or not. For this question, H. Murakami and the second author [3] gave an negative answer, i.e., they proved that there is a knot K such that $g^*(K) < c^*(K)$. Thus $c^*(K)$ is not enough to characterize $g^*(K)$. In this paper we give characterizations of 4-genus and nonorientable 4-genus by using certain 4-dimensional numerical invariants.

The local move as illustrated in Fig. 1(a) (resp. 1(b)) is called an H-*move* (resp. H'-*move*) for some positive integer n. Both an H-move and an H'-move realize a crossing change when $n = 1$. Thus these moves are certain kinds of unknotting operations of knots. Let L_n (resp. L'_n) be a link as illustrated in Fig. 2(a) (resp. 2(b)). Then we easily see that an H-move (resp. H'-move) can be realized by a *fusion/fission* [2, p. 95] of L_n (resp. L'_n); see Fig. 3. Therefore, for a knot K in ∂B^4, there is a *singular* disk D in B^4 with $\partial D = K$ that satisfies the following:

(1) D is a locally flat except for points $p_1, p_2, \ldots, p_{m(D)}$ in the interior of D.

(2) For each p_i ($i = 1, 2, \ldots, m(D)$) there is a small neighborhood $N(p_i)$ of p_i in B^4 such that $(\partial N(p_i), \partial(N(p_i) \cap D))$ is a link L_{n_i} (resp. L'_{n_i}) for some integer n_i.

We call these points $p_1, p_2, \ldots, p_{m(D)}$ *singularities of type* H (resp. *type* H'). Among these disks satisfying the above, $c^*_H(K)$ (resp. $c^*_{H'}(K)$) is the minimum number of $m(D)$. Note that $c^*_H(K) \leq c^*(K)$ and $c^*_{H'}(K) \leq c^*(K)$.

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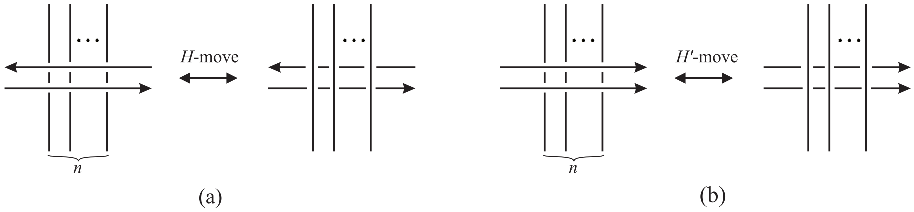


Fig. 1.

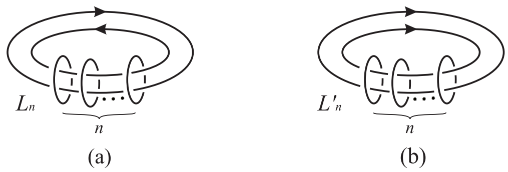


Fig. 2.

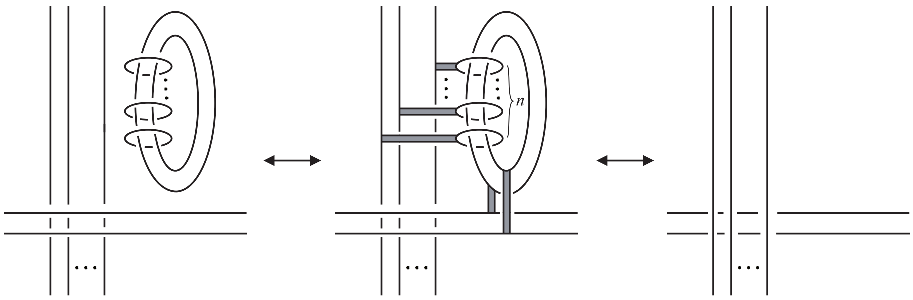


Fig. 3.

In this paper, we shall prove the following.

Theorem 1. *For any knot K, the following equalities hold.*
(1) $c^*_H(K) = g^*(K)$.
(2) $c^*_{H'}(K) = \left[(\gamma^*(K)+1)/2\right]$.
Here $[x]$ is the maximum integer that is not greater than x.

Since the inequality $\gamma^*(K) \leq 2g^*(K)+1$ holds for any knot K [3, Proposition 2.2], by Theorem 1, we have the following corollary.

Corollary 2. *For any knot K, $c^*_{H'}(K) \leq c^*_H(K)+1$.*

REMARK. Let K_n be a $(2, 2n+1)$-torus knot ($n = 1, 2, \ldots$). It is known that $g^*(K_n) = n$. On the other hand, we note that K_n bounds a Möbius band in a 4-ball and that K_n is not a slice knot. This implies $\gamma^*(K_n) = 1$. Therefore, by Theorem 1, we have $c^*_{H'}(K_n) = 1$ and $c^*_H(K_n) = n$.

Proof of Theorem 1

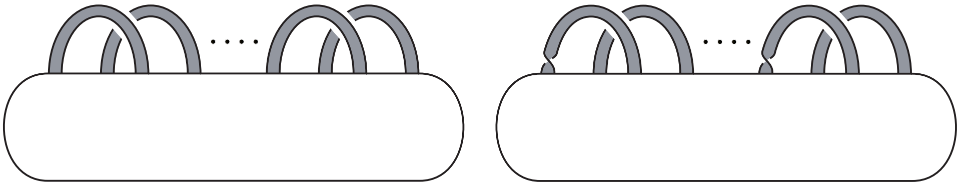
In order to prove Theorem 1, we shall show the following lemma.

Lemma 3. *Let K (resp. $-K'$) be a knot in $S^3 \times \{0\}$ (resp. $S^3 \times \{1\}$). Suppose that K and $-K'$ cobound a twice punctured surface F in $S^3 \times [0,1]$ such that F has neither maximal points nor minimal points. Then the following hold.*
(1) If F is orientable and oriented so that $\partial F = K \cup (-K')$, then K is obtained from K' by $g(F)$ H-moves.
(2) If F is nonorientable, then K is obtained from K' or $-K'$ by $[\beta_1(F)/2]$ H'-moves.
Here $-K'$ denotes the knot K' with reversed orientation, $g(F)$ the genus of F and $\beta_1(F)$ the first Betti number of F.

Proof. Suppose that F is orientable. Then $2g(F) = \beta_1(F) - 1$. We regard each saddle point as a saddle band in the sense of [2, p. 107]. We can deform F so that all saddle bands lie in $S^3 \times \{1/2\}$; see [2]. Note that $F \cap (S^3 \times \{1/2\})$ is a 2-complex that consists of K and $2g(F)$ bands $b_1, b_2, \ldots, b_{2g(F)}$, and that K' is obtained from K by *hyperbolic transformations* [2, Definition 1.1] along the bands $b_1, b_2, \ldots, b_{2g(F)}$. Moreover we may assume that $F \cap (S^3 \times \{1/2\})$ is homeomorphic to a 2-complex as illustrated in Fig. 4(a). Hence K, $F \cap (S^3 \times \{1/2\})$ and K' can be given as shown in Fig. 5(a). Then we can deform $F \cap (S^3 \times \{1/2\})$ into a 2-complex as illustrated in Fig. 6(a) by combining the three kinds of local moves; (1) changing a crossing of b_{2i} and b_j, (2) changing a crossing of b_{2i} and K, and (3) adding a ± 1-full twist to b_{2i}, where $i = 1, 2, \ldots, g(F)$ and $j = 1, 2, \ldots, 2g(F)$. We note that this deformation is realized by $g(F)$ local moves as illustrated in Fig. 7. Since the result of hyperbolic transformations along the bands in Fig. 6(a) is K, K is obtained from K' by $g(F)$ local moves as illustrated in Fig. 8. It is not hard to see that the local move as in Fig. 8 is realized by a single H-move; see Fig. 9 for example. Thus K is obtained from K' by $g(F)$ H-moves.

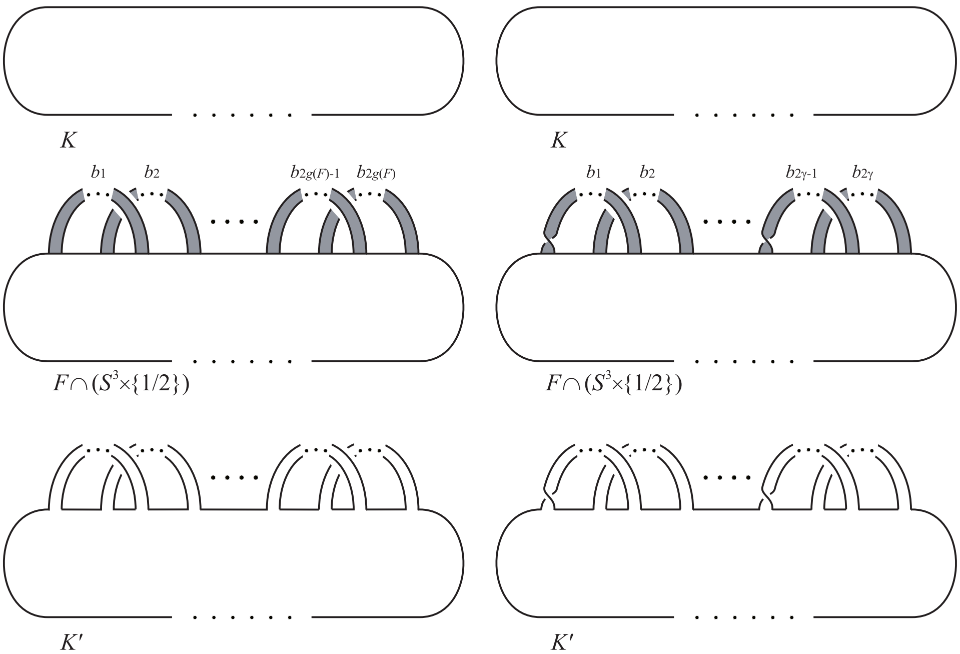
Suppose F is nonorientable and that $\beta_1(F) - 1$ is even. Set $\beta_1(F) - 1 = 2\gamma$. In the above arguments, by replacing $g(F)$, K', Fig. 4(a), 5(a), 6(a) and H-move with γ, $\pm K'$, Fig. 4(b), 5(b), 6(b) and H'-move respectively, we have the required result.

In the case that $\beta_1(F) - 1$ is odd, we have the conclusion by the following. By attaching a small half-twisted band to $F \cap (S^3 \times \{1/2\})$, we find a new surface F' in $S^3 \times [0,1]$ such that K and $-K'$ cobound F', $\beta_1(F') = \beta_1(F) + 1$ and F' has neither maximal points nor minimal points. $\square$



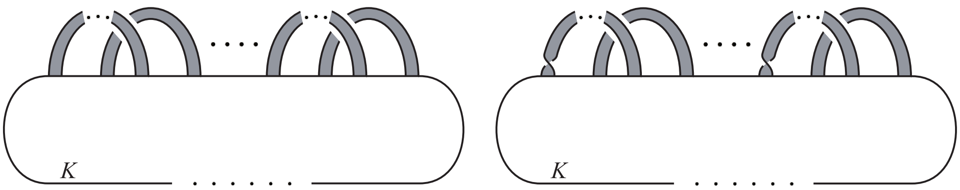
(a) (b)

Fig. 4.



(a) (b)

Fig. 5.



(a) (b)

Fig. 6.

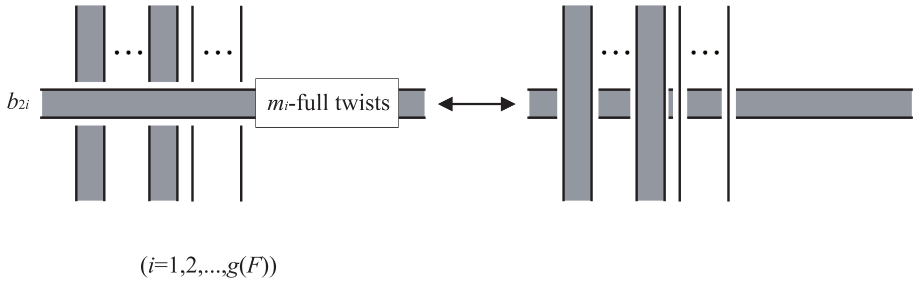


Fig. 7.

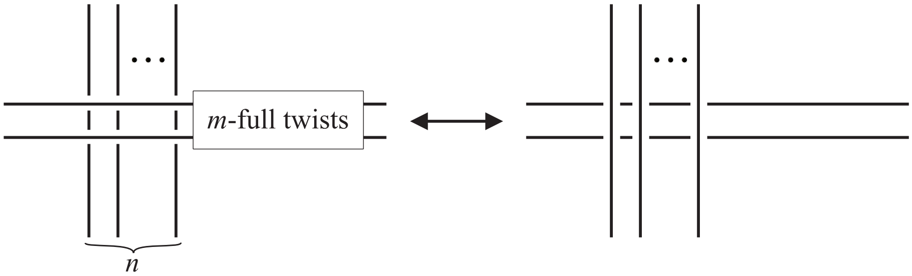


Fig. 8.

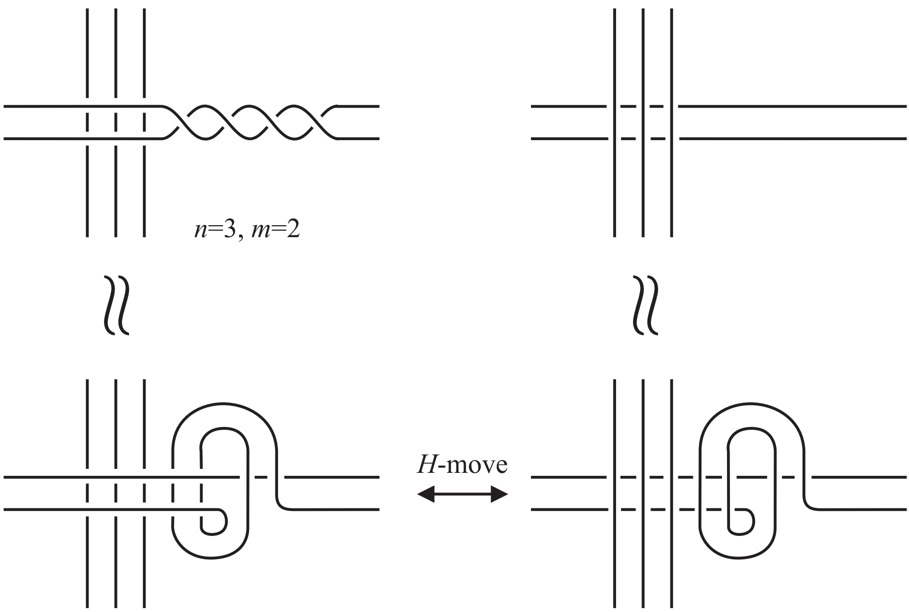


Fig. 9.

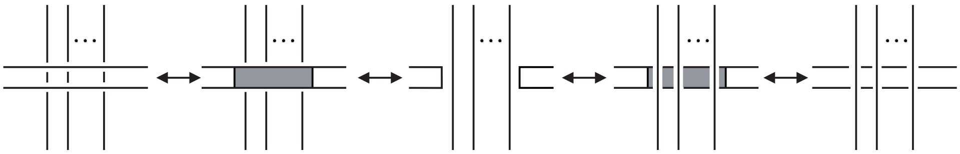


Fig. 10.

Proof of Theorem 1. An H-move (H'-move) is realized by twice hyperbolic transformations as illustrated in Fig. 10. Hence we have $c^*_H(K) \geq g^*(K)$ and $2c^*_{H'}(K) \geq \gamma^*(K)$. Note that $2c^*_{H'}(K) \geq \gamma^*(K)$ implies $c^*_{H'}(K) \geq [(\gamma^*(K)+1)/2]$.

Suppose that a knot K in ∂B^4 bounds a surface F in B^4. We assume that $B^4 = (S^3 \times [0, \infty)) \cup \{1\mathrm{pt}\}$. We can deform F so that the following conditions are satisfied [2]:

(1) $F \cap (S^3 \times [0, 1])$ is an annulus that does not have maximal points.
(2) $F \cap (S^3 \times [1, 2])$ is a surface that has neither maximal points nor minimal points.
(3) $F \cap (S^3 \times [2, \infty))$ is a disk that does not have minimal points, i.e., $F \cap (S^3 \times \{2\})$ is a ribbon knot.

Set $\partial(F \cap (S^3 \times [0, 1])) \setminus K = -K'$ and $\partial(F \cap (S^3 \times [2, \infty))) = K''$. If F is orientable (resp. nonorientable), then by Lemma 3, we have that the ribbon knot K'' is obtained from K' by $g(F)$ H-moves (resp. from K' or $-K'$ by $[(\beta_1(F)+1)/2]$ H'-moves). This implies that K' and $-K''$ (resp. $\pm K''$) cobound a singular annulus in $S^3 \times [1, 2]$ with $g(F)$ singularities of type H (resp. $[(\beta_1(F)+1)/2]$ singularities of type H'). Hence we have $c^*_H(K) \leq g^*(K)$ and $c^*_{H'}(K) \leq [(\gamma^*(K)+1)/2]$. This completes the proof. $\square$

Since both H-move and H'-move are unknotting operations, we can define 4-dimentional unknotting numbers, $u^*_H(K), u^*_{rH}(K), u^*_{H'}(K)$ and $u^*_{rH'}(K)$, of a knot K by the similar ways to those of $u^*(K)$ and $u^*_r(K)$ in [4]. Namely $u^*_H(K)$ (resp. $u^*_{rH}(K)$) is the minimum number of H-moves that is needed to transform K into a slice knot (resp. a ribbon knot), and $u^*_H(K')$ (resp. $u^*_{rH'}(K)$) is the minimum number of H'-moves that is needed to transform K into a slice knot (resp. a ribbon knot). The *ribbon* 4-*genus* $g^*_r(K)$ of a knot K in $S^3 = \partial B^4$ is the minimum genus of orientable surfaces in B^4 bounded by K that has no minimal points [4]. The *nonorientable ribbon* 4-*genus* $\gamma^*_r(K)$ is the minimum first Betti number of nonorientable surfaces in B^4 bounded by K that has no minimal points. For a ribbon knot, it is defined to be 0 instead of 1. We define c^*_{rH} (resp. $c^*_{rH'}$) to be the minimum number of type H (resp. type H') singular points of singular disks in B^4 bounded by K that has no minimal points and whose singularities are of type H (resp. type H'). From the proof of Theorem 1, we have the following theorem.

Theorem 4. *For any knot K, the following equalities hold.*
(1) $c^*_{rH}(K) = g^*_r(K) = u^*_{rH}(K)$.
(2) $c^*_{rH'}(K) = \left[(\gamma^*_r(K) + 1)/2\right] = u^*_{rH'}(K)$.

Since the trivial knot in ∂B^4 bounds a Möbius band in B^4 without minimal points, we have $\gamma^*_r(K) \leq 2g^*_r(K) + 1$ for any knot K. By Theorem 4, we have the following corollary.

Corollary 5. *For any knot K, $u^*_{rH'}(K) \leq u^*_{rH}(K) + 1$.*

REMARK. Let K_n be a $(2, 2n + 1)$-torus knot ($n = 1, 2, \ldots$). Since $g^*(K) \leq g^*_r(K) \leq g(K)$ [4, Lemma 2], we have $g^*_r(K_n) = n$, where $g(K)$ is the *genus* of K. On the other hand, since K_n is not a ribbon knot and K_n bounds a Möbius band in a 4-ball that has no minimal points, we have $\gamma^*_r(K_n) = 1$. Therefore, by Theorem 4, we have $c^*_{rH'}(K_n) = 1$ and $c^*_{rH}(K_n) = n$.

By the definitions of $c^*_H(K), c^*_{H'}(K), u^*_H(K)$ and $u^*_{H'}(K)$, we have $c^*_H(K) \leq u^*_H(K)$ and $c^*_{H'}(K) \leq u^*_{H'}(K)$.

Conjecture. *For any knot K, $c^*_H(K) = u^*_H(K)$ and $c^*_{H'}(K) = u^*_{H'}(K)$.*

REMARK. If $g^*(K) = g^*_r(K)$, then by Theorems 1 and 4, $c^*_H(K) = g^*(K) = g^*_r(K) = u^*_{rH}(K) \geq u^*_H(K)$. If $\gamma^*(K) = \gamma^*_r(K)$, then by Theorems 1 and 4, $c^*_{H'}(K) = [(\gamma^*(K) + 1)/2] = [(\gamma^*_r(K) + 1)/2] = u^*_{rH'}(K) \geq u^*_{H'}(K)$. Thus if $g^*(K) = g^*_r(K)$ and $\gamma^*(K) = \gamma^*_r(K)$ for any knot K, then the conjecture above is true.

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