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Author(s)	Jäger, Willi; Saitō, Yoshimi
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THE REDUCED WAVE EQUATION IN LAYERED MATERIALS

WILLI JÄGER and YOSHIMI SAITŌ[1]

(Recieved March 12, 1996)

1. Introduction

The mathematical theory of wave in layered media is still posing interesting mathematical problems even in the linear, stationary case.

In Jäger-Saitō [9] and [8], we studied the spectrum of the reduced wave operator

$$H_0 = -\mu_0(x)^{-1}\Delta, \tag{1.1}$$

where $\mu_0(x)$ is a simple function which takes a two positive values μ_{01} and μ_{02} on Ω_1 and Ω_2 respectively. Here Ω_ℓ, $\ell = 1, 2$, are open sets of $\mathbf{R}^N$ such that

$$\begin{cases} \Omega_1 \cap \Omega_2 = \emptyset, \\ \overline{\Omega_1} \cup \Omega_2 = \Omega_1 \cup \overline{\Omega_2} = \mathbf{R}^N, \end{cases} \tag{1.2}$$

$\overline{\Omega_\ell}$ being the closure of Ω_ℓ. Under a new condition on the separating surface $S = \partial\Omega_1 = \partial\Omega_2$, we have established the limiting absorption principle for H_0 which implies that H_0 is absolute continuous. Our condition is satisfied, for example, for the case where S is a cylinder.

In this work we are going to extend the results in [9] to the multimedia case, the case where $\mu_0(x)$ can take finitely or infinitely many values (see §2). The limiting absorption principle will be established and, again, the operator H_0 is absolute continuous. Also we shall consider short-range or long-range perturbation of H_0, that is, we shall study the operator

$$H = -\mu(x)^{-1}\Delta, \tag{1.3}$$

where

$$\mu(x) = \mu_0(x) + \mu_1(x) \tag{1.4}$$

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and $\mu_1(x)$ is short-range or long-range. In this case we shall prove that the point spectrum, if it exists, is discrete, and the limiting absorption principle holds on any interval which does not contain an eigenvalue.

As for the study of the reduced wave operators with discontinuous coefficients, many works have been done for the stratified media in which the coefficients of the operator are the functions of $x' \in \mathbf{R}^k \subset \mathbf{R}^N$, $k < N$. Some perturbed operators of the above type have been discussed, too. Here we refer Wilcox [16], Ben-Artzi-Dermanjian-Guillot [2], Weder [14], [15], DeBiévre-Pravica [4], [5], Boutet de Monvel-Berthier-Manda [3], and Zhang [17]. In [5] S. DeBiévre and D.W. Pravica proved there is no point spectrum for the stratified propagators without any additional conditions other than sufficient smoothness of the coefficients at infinity.

It seems that there are rather few results for the nonstratified case. Eidus [6] was the first to consider the reduced wave operators H_0 with a *cone-shape discontinuity*. He imposed the following assumptions on the separating surface S: *there exist positive constants* c_1 *and* c_2 *such that*

$$(1.5) \qquad \left|n_N^{(1)}(x)\right| \geq c_1 \qquad (x \in S),$$

and

$$(1.6) \qquad |x \cdot n^{(1)}(x)| \leq c_2 \qquad (x \in S),$$

where $n^{(\ell)}(x)$, $\ell = 1, 2$, *is the unit outward normal of* Ω_ℓ *at* x, *and* $x \cdot n^{(1)}(x)$ *is the inner product of* x *and* $n^{(1)}(x)$ *in* $\mathbf{R}^N$. Note that a cone having its vertex at the origin and the positive x_N-axis as its axis satisfies (1.5) and (1.6). Under the above assumptions, Eidus [6] proved the limiting absorption principle for H_0, that is, by denoting by $R_0(z)$ the resolvent of H_0, the limits

$$(1.7) \qquad \lim_{\eta \downarrow 0} R_0(\lambda \pm i\eta) = R_{0\pm}(\lambda) \qquad \text{in } \mathbf{B}(L_{2,1}(\mathbf{R}^N), L_{2,-1}(\mathbf{R}^N))$$

exist for $\lambda > 0$, where the weighted L_2 space $L_{2,t}(\mathbf{R}^N)$, $t \in \mathbf{R}$, is defined by

$$(1.8) \qquad L_{2,t}(\mathbf{R}^N) = \{f \ : \ (1 + |x|)^t f(x) \in L_2(\mathbf{R}^N)\}$$

with its inner product and norm

$$(1.9) \qquad \begin{cases} (f,\ g)_t = \displaystyle\int_{\mathbf{R}^N} f(x)\overline{g(x)}\, dx, \\ \|f\|_t = (f,\ f)_t^{1/2}, \end{cases}$$

and $\mathbf{B}(X, Y)$ is the Banach space of all bounded linear operators from X into Y. Then, Saitō [13] showed that $L_{2,1}(\mathbf{R}^N)$ and $L_{2,-1}(\mathbf{R}^N)$ in (1.7) can be replaced by

$L_{2,\delta}(\mathbf{R}^N)$ and $L_{2,-\delta}(\mathbf{R}^N)$ with $\delta > 1/2$, respectively. This means that the limiting absorption principle for H_0 holds on the same weighted L_2 spaces as are used for the Schrödinger operator (cf. Agmon [1], Ikebe-Saitō [7] and Saitō [8]). Recently Roach-Zhang [10] has shown that $u = R^{\pm}(\lambda)f$, where $\lambda > 0$ and $f \in L_{2,\delta}(\mathbf{R}^N)$ with $\delta > 1/2$, is characterized as a unique solution of the equation

$$(1.10) \qquad (-\mu_0(x)^{-1}\Delta - \lambda)u = f$$

with the radiation condition

$$(1.11) \qquad \lim_{R\to\infty} \frac{1}{R}\int_{B_R} |\nabla u \mp i\sqrt{\lambda\mu(x)}\widetilde{x}u|^2\,dx = 0 \qquad \left(\widetilde{x} = \frac{x}{|x|}\right),$$

B_R being the ball with radius R and center at the origin. The condition (1.11) is a natural extension of the radiation condition for the Schrödinger operators ([7], [8]). [10] also gave another proof of the limiting absorption principle for H_0.

In the recent work [9] and [8], we studied the reduced wave operators H_0 with a *cylindrical discontinuity* in which the separating surface is assumed to satisfy that

$$(1.12) \qquad (\mu_{02} - \mu_{01})(x \cdot n^{(1)}) = (\mu_{01} - \mu_{02})(x \cdot n^{(2)}) \geq 0 \quad (x \in S).$$

The condition (1.12) is satisfied if Ω_1 is an infinite cylindrical domain which contains the origin and $\mu_{02} > \mu_{01}$. Then it has been shown again that H_0 is absolutely continuous. So far it seems that the absence of the point spectrum can not be obtained without imposing some additional conditions such as (1.5)–(1.6) or (1.12).

In §2, we define the reduced wave operator H_0 with multimedia and we state our assumption on the separating surface S and the positive function μ_0, in which μ_0 can take countably infinite values although the condition is a natural extension of the condition (1.12). §3 is devoted to showing the limiting absorption principle for the unperturbed operator H_0. Here the arguments are quite parallel to the one in [8] or [9], and hence we shall omit some of the proof. In §4 we shall discuss the point spectrum of the perturbed operator (1.3). It will be shown that the point spectrum of H is discrete. Also some sufficient conditions for the nonexistence of the point spectrum of H will be given. We shall show in §5 that the limiting absorption principle for H holds on any closed interval which does not contain the point spectrum.

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2. The operators H_0 and H

In this section, we are going to define a reduced wave operator

$$H_0 = -\mu_0(x)^{-1}\Delta, \tag{2.1}$$

where $\mu_0(x)$ is a positive, simple function on $\mathbf{R}^N$ which will be specified below, and its perturbed operator

$$H = -\mu(x)^{-1}\Delta, \tag{2.2}$$

where

$$\mu(x) = \mu_0(x) + \mu_1(x) \tag{2.3}$$

such that $\mu(x)$ is a positive function on $\mathbf{R}^N$ and $\mu_1(x)$ decays to 0 at infinity.

Let us describe the conditions on $\mu_0(x)$. Let $\mathbf{N}$ be all positive integers and let $\mathbf{N}_-$ be all negative integers. Let L be a subset of integers satisfying one of the following:

$$\begin{array}{ll} \text{(I)} & L = \mathbf{N}_- \cup \{0\} \cup \mathbf{N}, \\ \text{(II)} & L = \{L_-, L_- + 1, \cdots, -1, 0\} \cup \mathbf{N}, \\ \text{(III)} & L = \mathbf{N}_- \cup \{0, 1, \cdots, L_+\}, \\ \text{(IV)} & L = \{L_-, L_- + 1, \cdots, -1\} \cup \{0\} \cup \{1, 2, \cdots, L_+\}, \end{array} \tag{2.4}$$

where $L_- \in \mathbf{N}_- \cup \{0\}$ and $L_+ \in \{0\} \cup \mathbf{N}$.

ASSUMPTION 2.1. Let N be a positive integer such that $N \geq 2$. Let L be as in (2.4). For each $\ell \in L$, let Ω_ℓ be an open set in $\mathbf{R}^N$. Let μ_0 be a positive function on $\mathbf{R}^N$. The family $\{\Omega_\ell\}_{\ell \in L}$ and the function μ_0 are assumed to satisfy the following (i) $\sim$ (iii):

(i) $\{\Omega_\ell\}_{\ell \in L}$ is a disjoint family of open sets of $\mathbf{R}^N$ such that

$$\mathbf{R}^N = \bigcup_{\ell \in L} \overline{\Omega_\ell}, \tag{2.5}$$

where $\overline{\Omega_\ell}$ is the closure of Ω_ℓ. For any $R > 0$, the open ball B_R with center at the origin and radius R is covered by a union of a finite number of $\overline{\Omega_\ell}$, i.e., for $R > 0$ there is a finite subset L_R of L such that

$$\Omega_\ell \cap B_R = \emptyset \qquad (\ell \in L - L_R). \tag{2.6}$$

(ii) For each $\ell \in L$, the boundary $\partial\Omega_\ell$ of Ω_ℓ is a disjoint union of two continuous surfaces $S_\ell^{(-)}$ and $S_\ell^{(+)}$, i.e.,

$$\begin{cases} \partial\Omega_\ell = S_\ell^{(-)} \cup S_\ell^{(+)}, \\ S_\ell^{(-)} \cap S_\ell^{(+)} = \emptyset \end{cases} \tag{2.7}$$

for $\ell \in L$, where $S_\ell^{(-)}$ and $S_\ell^{(+)}$ are unions of a finite number of smooth surfaces. Here we assume that $S_{L_-}^{(-)} = \emptyset$ when L has the smallest number L_- and $S_{L_+}^{(+)} = \emptyset$ when L has the largest number L_+. Further we assume that

$$S_\ell^{(+)} = S_{\ell+1}^{(-)} \qquad (\ell \in L), \tag{2.8}$$

where we set $S_{\ell+1}^{(-)} = \emptyset$ if $\ell + 1 \notin L$.

(iii) μ_0 is a simple function which takes the value ν_ℓ on each Ω_ℓ, where ν_ℓ is a positive number such that

$$0 < m_0 \equiv \inf_{\ell \in L} \nu_\ell \leq \sup_{\ell \in L} \nu_\ell \equiv M_0 < \infty. \tag{2.9}$$

Let

$$n^{(\ell)}(x) = (n_1^{(\ell)}(x), n_2^{(\ell)}(x), \cdots, n_N^{(\ell)}(x)) \quad (\ell \in L), \tag{2.10}$$

be the unit outward normal of Ω_ℓ at a.e. $x \in \partial\Omega_\ell$. Then we assume that

$$(\nu_\ell - \nu_{\ell+1})(n^{(\ell)}(x) \cdot x) \leq 0 \quad (x \in S_\ell^{(+)},\ \ell \in L) \tag{2.11}$$

although $\ell \neq L_+$ if L has the largest number L_+, where $n^{(\ell)}(x) \cdot x$ is the usual inner product of $n^{(\ell)}(x)$ and x in $\mathbf{R}^N$.

As for the function μ, we have

ASSUMPTION 2.2. Let μ be a measurable function on $\mathbf{R}^N$ satisfying the following (i) and (ii):

(i) We have

$$0 < \widetilde{m_0} \equiv \inf_{x \in \mathbf{R}^N} \mu(x) \leq \sup_{x \in \mathbf{R}^N} \mu(x) \equiv \widetilde{M_0} < \infty. \tag{2.12}$$

(ii) Let $\mu_1 = \mu - \mu_0$. Then either μ_1 is short-range, that is,

$$|\mu_1(x)| \leq c_1(1 + |x|)^{-1-\epsilon} \qquad (x \in \mathbf{R}^N), \tag{2.13}$$

or μ_1 is long-range, that is, μ_1 is differentiable such that

$$\begin{cases} |\mu_1(x)| \leq c_1(1 + |x|)^{-\epsilon} & (x \in \mathbf{R}^N), \\ |\nabla\mu_1(x)| \leq c_1(1 + |x|)^{-1-\epsilon} & (x \in \mathbf{R}^N), \end{cases} \tag{2.14}$$

with constants $c_1, \epsilon > 0$. Throughout this work we assume that $0 < \epsilon < 1/2$ with no loss of generality.

Let X_0 and X be Hilbert spaces given by

$$
\begin{cases} X_0 = L_2(\mathbf{R}^N;\ \mu_0(x)dx), \\ X = L_2(\mathbf{R}^N;\ \mu(x)dx). \end{cases} \tag{2.15}
$$

The inner product and norm of X_0 [or X] will be denoted by $(\ ,\ )_{X_0}$ and $\|\ \|_{X_0}$ [or $(\ ,\ )_X$ and $\|\ \|_X$], respectively. Then define the operator H_0 in X_0 by

$$
\begin{cases} D(H_0) = H^2(\mathbf{R}^N), \\ H_0 u = -\mu_0(x)^{-1}\Delta u, \end{cases} \tag{2.16}
$$

where $D(T)$ is the domain of T, $H^2(\mathbf{R}^N)$ is the second order Soblev space on $\mathbf{R}^N$. and Δu is defined in the sense of distributions. Similarly the operator H in X is given by

$$
\begin{cases} D(H) = H^2(\mathbf{R}^N), \\ Hu = -\mu(x)^{-1}\Delta u, \end{cases} \tag{2.17}
$$

Then it is easy to see that H_0 and H are selfadjoint operators in X_0 and X, respectively.

Now we are going to give some examples of $\{\Omega_\ell\}_{\ell \in L}$ and μ_0 which satisfy Assumption 2.1. In the following examples we take $N = 3$ although the N-dimensional versions of these examples can be easily obtained.

EXAMPLE 2.3. Let $L = \mathbf{N}_- \cup \{0\} \cup \mathbf{N}$. Let $\{b_\ell\}_{\ell \in \mathbf{N}_- \cup \mathbf{N}}$ be such that

$$
\begin{cases} \cdots < b_m < b_{m+1} < \cdots < b_{-1} < 0 < b_1 < \cdots < b_\ell \cdots, \\ b_\ell \to \infty \qquad (\ell \to \infty), \\ b_m \to -\infty \quad (m \to -\infty), \end{cases} \tag{2.18}
$$

and define $\{\Omega_\ell\}_{\ell \in L}$ by

$$
\begin{cases} \Omega_\ell = \{x = (x_1, x_2, x_3) \in \mathbf{R}^3 : b_\ell < x_3 < b_{\ell+1}\} \quad (\ell \in \mathbf{N}), \\ \Omega_0 = \{x = (x_1, x_2, x_3) \in \mathbf{R}^3 : b_{-1} < x_3 < b_1\}, \\ \Omega_\ell = \{x = (x_1, x_2, x_3) \in \mathbf{R}^3 : b_{\ell-1} < x_3 < b_\ell\} \quad (\ell \in \mathbf{N}_-), \end{cases} \tag{2.19}
$$

Then the separating surfaces $S_\ell^{(\pm)}$ are given by

$$
\begin{cases} S_\ell^{(+)} = \{x = (x_1, x_2, x_3) \in \mathbf{R}^3 : x_3 = b_{\ell+1}\}, \\ S_\ell^{(-)} = \{x = (x_1, x_2, x_3) \in \mathbf{R}^3 : x_3 = b_\ell\} \end{cases} \tag{2.20}
$$

for $\ell \in \mathbf{N}$,

$$\begin{cases} S_0^{(+)} = \{x = (x_1, x_2, x_3) \in \mathbf{R}^3 : x_3 = b_1\}, \\ S_0^{(-)} = \{x = (x_1, x_2, x_3) \in \mathbf{R}^3 : x_3 = b_{-1}\}, \end{cases} \tag{2.21}$$

and

$$\begin{cases} S_\ell^{(+)} = \{x = (x_1, x_2, x_3) \in \mathbf{R}^3 : x_3 = b_\ell\}, \\ S_\ell^{(-)} = \{x = (x_1, x_2, x_3) \in \mathbf{R}^3 : x_3 = b_{\ell-1}\} \end{cases} \tag{2.22}$$

for $\ell \in \mathbf{N}_-$. Define μ_0 by

$$\mu_0(x) = \nu_\ell \qquad (x \in \Omega_\ell) \tag{2.23}$$

such that

$$\begin{cases} \nu_0 < \nu_1 < \nu_2 < \cdots < \nu_\ell < \cdots, \\ \nu_0 < \nu_{-1} < \nu_{-2} < \cdots < \nu_{-\ell} < \cdots \end{cases} \tag{2.24}$$

with

$$\begin{cases} \nu_0 > 0, \\ M_0 \equiv \sup\limits_{\ell \in \mathbf{N} \cup \mathbf{N}_-} \nu_\ell < \infty. \end{cases} \tag{2.25}$$

Since we have

$$\begin{cases} n^{(\ell)}(x) \cdot x \geq 0 \quad (x \in S_\ell^{(+)},\ \ell \in \{0\} \cup \mathbf{N}) \\ n^{(\ell)}(x) \cdot x \leq 0 \quad (x \in S_\ell^{(+)},\ \ell \in \mathbf{N}_-), \end{cases} \tag{2.26}$$

we see that the condition (2.11) is satisfied. Although this is a reduced wave operator in stratified media studied by many authors (see, e.g., [16], [14], [4]), note that Ω_ℓ can be modified as far as the condition (2.26) holds good.

EXAMPLE 2.4. Let $L = \{0\} \cup \mathbf{N}$. Let $\{b_\ell\}_{\ell \in \mathbf{N}}$ be such that

$$\begin{cases} 0 < b_1 < b_2 < \cdots < b_\ell < \cdots, \\ b_\ell \to \infty \quad (\ell \to \infty), \end{cases} \tag{2.27}$$

and define $\{\Omega_\ell\}_{\ell \in L}$ by

$$\begin{cases} \Omega_0 = \{x = (x_1, x_2, x_3) \in \mathbf{R}^3 : x_1^2 + x_2^2 < b_1^2\}, \\ \Omega_\ell = \{x = (x_1, x_2, x_3) \in \mathbf{R}^3 : b_\ell^2 < x_1^2 + x_2^2 < b_{\ell+1}^2\} \quad (\ell \in \mathbf{N}). \end{cases} \tag{2.28}$$

The separating surfaces $S_\ell^{(\pm)}$ are given by

$$
(2.29)\qquad \begin{cases} S_\ell^{(+)} = \{x = (x_1, x_2, x_3) \in \mathbf{R}^3 : x_1^2 + x_2^2 = b_{\ell+1}^2\} \quad (\ell \in L), \\ S_\ell^{(-)} = \{x = (x_1, x_2, x_3) \in \mathbf{R}^3 : x_1^2 + x_2^2 = b_\ell^2\} \quad (\ell \in \mathbf{N}), \\ S_0^{(-)} = \emptyset. \end{cases}
$$

Define μ_0 by

$$
(2.30)\qquad \mu_0(x) = \nu_\ell \qquad (x \in \Omega_\ell)
$$

such that

$$
(2.31)\qquad \nu_0 < \nu_1 < \nu_2 < \cdots < \nu_\ell < \cdots,
$$

with

$$
(2.32)\qquad \begin{cases} \nu_0 > 0, \\ M_0 \equiv \sup_{\ell \in \mathbf{N}} \nu_\ell < \infty. \end{cases}
$$

Since

$$
(2.33)\qquad n^{(\ell)}(x) \cdot x \geq 0 \quad (x \in S_\ell^{(+)},\ \ell \in L),
$$

from (2.31) it is seen that the condition (2.11) is satisfied. Again Ω_ℓ are allowed to be deformed as far as (2.33) holds good.

3. The unperturbed operator H_0

In this section we are going to discuss the unperturbed operator H_0 given by (2.16). First we shall show the uniqueness theorem for the equation

$$
(3.1)\qquad (-\mu_0(x)^{-1}\Delta - \lambda)u = f
$$

with radiation condition. Then, after showing several a priori estimates of the solution u of the equation (3.1), the limiting absorption principle for H_0 will be proved. The arguments in this section are quite parallel to the ones in Jäger-Saitō [9], and hence we shall omit the proof or give a sketch of proof in most of the theorems given in this section.

We shall start with some notations.

NOTATION 3.1. Let $z \in \mathbf{C}$, $x = (x_1, x_2, \cdots, x_N)$, $r = |x|$, $\widetilde{x} = (\widetilde{x}_1, \widetilde{x}_2, \cdots, \widetilde{x}_N) = x/r$, $\partial_j = \partial/\partial x_j$ and $\nabla = (\partial/\partial x_1, \partial/\partial x_2, \cdots, \partial/\partial x_N)$. Then we set

(1) $k = k(x) = k(x,z) = [z\mu_0(x)]^{1/2}$, where the branch is taken so that $\operatorname{Im} k(x,z) \geq 0$;

(2) $a = a(x) = a(x,z) = \operatorname{Re} k(x,z)$;

(3) $b = b(x) = b(x,z) = \operatorname{Im} k(x,z)$;

(4) $\mathcal{D}u = \nabla u + \{(N-1)/(2r)\}\widetilde{x}u - ik(x)\widetilde{x}u$;

(5) $\mathcal{D}_r u = \mathcal{D}u \cdot \widetilde{x} = \partial u/\partial r + \{(N-1)/(2r)\}u - ik(x)u$;

Let $u \in H^2(\mathbf{R}^N)_{\text{loc}}$. Then the restrictions $u|_G$ and $\partial_j u|_G$, $j = 1, 2, \cdots, N$, of u and $\partial_j u = \partial u/\partial x_j$ onto a smooth surface G are defined as the traces of u and $\partial_j u$ on G, respectively. Thus $u|_G$ and $\partial_j u|_G$ are considered to belong to $L_2(G)_{\text{loc}}$.

Let $z \in \mathbf{C}$ and let $u \in H^2(\mathbf{R}^N)_{\text{loc}}$. Define f by

$$f = -\mu_0(x)^{-1}\Delta u - zu = \mu_0(x)^{-1}\big(-\Delta u - k^2 u\big) \tag{3.2}$$

with k given by (1) of Notation 3.1. Now we are going to show an identity which is an extension of Proposition 3.3 of [9] and will be used throughout this section.

Proposition 3.2. *Let $u \in H^2(\mathbf{R}^N)_{\text{loc}}$ and let f be given by* (3.2). *Let ξ be a real-valued, continuous function on $[0, \infty)$ such that ξ has piecewise continuous derivative. Set $\varphi(x) = \alpha(x)\xi(|x|)$, where α is a simple function which is constant on each Ω_ℓ. For $0 < r < R < \infty$, set*

$$B_{rR} = \{x \in \mathbf{R}^N \ : \ r < |x| < R\}, \tag{3.3}$$

Then we have

$$\begin{aligned}
&\int_{B_{rR}} \left(b\varphi + \frac{1}{2}\frac{\partial \varphi}{\partial r}\right) |\mathcal{D}u|^2 \, dx + \sum_{\ell \in L} \int_{\partial\Omega_\ell \cap B_{rR}} \varphi \operatorname{Im}\left\{\overline{k}\frac{\partial u}{\partial n}\overline{u}\right\} dS \\
&\qquad + \int_{B_{rR}} \left(\frac{\varphi}{r} - \frac{\partial \varphi}{\partial r}\right)(|\mathcal{D}u|^2 - |\mathcal{D}_r u|^2)\, dx \\
&\qquad + c_N \int_{B_{rR}} r^{-2}\left(\frac{\varphi}{r} - 2^{-1}\frac{\partial \varphi}{\partial r} + b\varphi\right)|u|^2 \, dx \\
&= \operatorname{Re}\int_{B_{rR}} \varphi\mu_0(x) f \overline{\mathcal{D}_r u}\, dx \\
&\quad + 2^{-1}\sum_{\ell \in L}\int_{\partial\Omega_\ell \cap B_{rR}} \varphi \left\{\frac{(N-1)b}{r} + |k|^2\right\}(\widetilde{x} \cdot n)|u|^2 \, dS \\
&\quad + 2^{-1}\int_{S_R} \varphi\left(2|\mathcal{D}_r u|^2 - |\mathcal{D}u|^2 - c_N r^{-2}|u|^2\right) dS \\
&\quad - 2^{-1}\int_{S_r} \varphi(2|\mathcal{D}_r u|^2 - |\mathcal{D}u|^2 - c_N r^{-2}|u|^2)\, dS,
\end{aligned} \tag{3.4}$$

where Ω_ℓ *satisfies* (i), (ii) *of* Assumption 2.1, $\partial/\partial n$ *in the integrand of the surface integral over* $\partial\Omega_\ell \cap B_{rR}$ *means the directional derivative in the direction of the outward normal* $n = n^{(\ell)}$ *of* $\partial\Omega_\ell$, *and*

$$c_N = (N-1)(N-3)/4. \tag{3.5}$$

The proof will be omitted since it is essentially the same as the proof of Proposition 3.3 of [9].

Theorem 3.3. *Assume* Assumption 2.1. *Let* $u \in H^2(\mathbf{R}^N)_{\text{loc}}$ *be a solution of the homogeneous equation*

$$-\mu_0(x)^{-1}\Delta u - \lambda u = 0 \qquad (\lambda > 0) \tag{3.6}$$

on $\mathbf{R}^N$ *such that*

$$\liminf_{R\to\infty} \int_{S_R} \left(\left| \frac{\partial u}{\partial r} \right|^2 + |u|^2 \right) dS = 0, \tag{3.7}$$

for $N \geq 3$, *or*

$$\liminf_{R\to\infty} R^\alpha \int_{S_R} \left(\left| \frac{\partial u}{\partial r} \right|^2 + |u|^2 \right) dS = 0 \tag{3.8}$$

with $\alpha > 0$ *for* $N = 2$, *where*

$$S_R = \{x \in \mathbf{R}^N \ : \ |x| = R\}. \tag{3.9}$$

Then u *is identically zero.*

Sketch of Proof. Theorme 3.3 can be proved by starting with Proposition 3.2 and proceeding as in the proof of Theorem 3.2 of [9] (for $N \geq 3$) or proof of Theorem 7.1 of [9] (for $N = 2$). Only difference here is that, instead of the last inequality of (3.20) in [9], we have to use

$$\sum_{\ell\in L} \int_{\partial\Omega_\ell \cap B_{rR}} \varphi |k|^2 (\tilde{x}\cdot n)|u|^2 \, dS \leq 0, \tag{3.10}$$

where n in the integrand is the unit outward normal $n^{(\ell)}(x)$ of Ω_ℓ at x. In fact, it follows from (i) and (ii) of Assumption 2.1 that

$$
(3.11)\qquad
\begin{aligned}
&\sum_{\ell\in L}\int_{\partial\Omega_\ell\cap B_R}\varphi|k|^2(\widetilde{x}\cdot n)|u|^2\,dS\\
&=\sum_{\ell\in L}\left(\int_{S_\ell^{(-)}\cap B_R}+\int_{S_\ell^{(+)}\cap B_R}\right)\varphi|k|^2(\widetilde{x}\cdot n)|u|^2\,dS\\
&=\lambda\sum_{\ell\in L}\int_{S_\ell^{(-)}\cap B_R}\varphi\left(\nu_\ell-\nu_{\ell+1}\right)(\widetilde{x}\cdot n^{(\ell)})|u|^2\,dS,
\end{aligned}
$$

where we should note that we are dealing with a finite sum because of (i) of Assumption 2.1. Then (3.10) is obtained from (iii) of Assumption 2.1. //

The following corollary guarantees the uniqueness of the inhomogeneous equation

$$
(3.12)\qquad -\mu_0(x)^{-1}\Delta u-\lambda u=f
$$

with one of the conditions

$$
(3.13)\qquad \|\mathcal{D}_r^{(\pm)}u\|_{\delta-1,\,E_1}<\infty,
$$

where $\delta>1/2$,

$$
(3.14)\qquad \mathcal{D}_r^{(\pm)}u=\partial u/\partial r+\{(N-1)/(2r)\}u\mp ik(x)u,
$$

$$
(3.15)\qquad E_R=\{x\in\mathbf{R}^N\ :\ |x|>R\},
$$

and, for a measurable set G in $\mathbf{R}^N$,

$$
(3.16)\qquad \|v\|_{\delta-1,\,G}^2=\int_G(1+|x|)^{2(\delta-1)}|v(x)|^2\,dx.
$$

Corollary 3.4. *Let $\lambda>0$ and let $f\in L_2(\mathbf{R}^N)_{\text{loc}}$. Then the solution $u\in H^2(\mathbf{R}^N)_{\text{loc}}$ of the equation* (3.12) *with one of the radiation conditions in* (3.13) *is unique.*

The proof is the same as the proof of Corollary 3.8 of [9].

Let $L_{2,t}(\mathbf{R}^N)$ be the weighted Hilbert space defined by (1.8). Let the resolvent $(H_0-z)^{-1}$ of the operator H_0 be denoted by $R_0(z)$. Now consider $u\in X_0$ defined by

$$
(3.17)\qquad
\begin{cases}
u=R_0(z)f,\\
z=\lambda+i\eta & (\lambda\ge 0,\eta\ne 0),\\
f\in L_{2,\delta}(\mathbf{R}^N).
\end{cases}
$$

For $0 < c < d < \infty$ a subset $J_{\pm}(c,\, d)$ of $\mathbf{C}$ are defined by

$$\left\{\begin{array}{l} J_{+}(c,\, d) = \{\ z = \lambda + i\eta\ :\ c \le \lambda \le d,\ 0 < \eta \le 1\ \}, \\ J_{-}(c,\, d) = \{\ z = \lambda + i\eta\ :\ c \le \lambda \le d,\ -1 \le \eta < 0\ \}. \end{array}\right. \tag{3.18}$$

In the next theorem, we are going to evaluate the radiation condition terms $\mathcal{D}u$. Here and in the sequel we agree that $C = C(A, B, \cdots)$ in an inequality means a positive constant depending on $A, B, \cdots$. Now we are evaluating the radiation condition term $\mathcal{D}u$.

Theorem 3.5. *Suppose that* Assumption 2.1 *holds. Let* $1/2 < \delta \le 1$. *Let* u *be given by* (3.17).

(i) *Let* $N \ge 3$. *Then there exists a constant* $C = C(\delta, m_0, M_0) > 0$ *such that*

$$\|\mathcal{D}u\|_{\delta-1} \le C\|f\|_{\delta}, \tag{3.19}$$

where $\mathcal{D}u$ *is as in* Notation 3.1, $\|\ \|_t$ *is the norm of* $L_{2,t}(\mathbf{R}^N)$, *and the constant* $C(\delta)$ *is independent of* f *and* z *satisfying* (3.17).

(ii) *Let* $N = 2$. *Let* $0 < c < d < \infty$ *and let* $J_{\pm}(c, d)$ *be as in* (3.18). *Let* u *be given by* (3.17) *with* $z \in J_{+}(c, d) \cup J_{-}(c, d)$. *Then there exists a positive constant* $C = C(\delta, c, d, m_0, M_0)$ *such that*

$$\|\mathcal{D}u\|_{\delta-1,*} \le C\big(\|f\|_{\delta} + \|u\|_{-\delta}\big) \tag{3.20}$$

where

$$\|v\|_{t,*}^2 = \int_{B_1} |x||v(x)|^2\, dx + \int_{E_1} (1 + |x|)^{2t}|v(x)|^2\, dx. \tag{3.21}$$

Sketch of Proof. We have only to proceed as in the proof of Theorems 4.1 and 7.2 of [9]. Set in (3.4) $\alpha(x) = 1/\sqrt{\mu_0}$,

$$\xi(r) = \left\{\begin{array}{ll} r & (0 \le r \le 1), \\ 2^{-(2\delta-1)}(1+r)^{2\delta-1} & (r \ge 1) \end{array}\right. \tag{3.22}$$

for $N \ge 3$, and

$$\xi(r) = \left\{\begin{array}{ll} \dfrac{1}{2}r^2 & (r \le 1/2), \\ \dfrac{1}{2^{2\delta}}(1+r)^{2\delta-1} & (r \ge 1). \end{array}\right. \tag{3.23}$$

for $N = 2$. Let the second term of the left-hand side of (3.4) be denoted by I_{L2}. Then it is easy to see that

$$I_{L2} = \sum_{\ell \in L} \int_{\partial\Omega_\ell \cap B_{rR}} \varphi \mathrm{Im} \left\{ \overline{k} \frac{\partial u}{\partial n} \overline{u} \right\} dS = 0 \tag{3.24}$$

(cf. (3.11)). Similarly we see that the second term of the right-hand side of (3.4) is nonpositive. All other terms of (3.4) can be evaluated exactly in the same manner as in the proof of Theorems 4.1 and 7.2 of [9], which completes the proof. //

Now that we have established the uniqueness of the solution of the equation (3.12) with the radiation condition (Corollary 3.4) and the estimate of the radiation condition term (Theorem 3.5), we can show the limiting absorption principle for H_0 by proceeding as in §5, §6, and §7 of [9]. Let $t \in \mathbf{R}$. The weighted Sobolev spaces $H_t^j(\mathbf{R}^N)$, $j = 1, 2$, are defined as the completion of $C_0^\infty(\mathbf{R}^N)$ by the norms

$$\|u\|_{1,t} = \left[\int_{\mathbf{R}^N} (1+r)^{2t} \big(|\nabla u|^2 + |u(x)|^2 \big)\, dx \right]^{1/2}, \tag{3.25}$$

and

$$\|u\|_{2,t} = \left[\int_{\mathbf{R}^N} (1+r)^{2t} \sum_{|\gamma| \le 2} |\partial^\gamma u|^2\, dx \right]^{1/2}, \tag{3.26}$$

respectively, where

$$\begin{cases} \gamma = (\gamma_1, \gamma_2, \cdots, \gamma_N), \\ |\gamma| = \gamma_1 + \gamma_2 + \cdots + \gamma_N, \\ \partial^\gamma u = (\partial_1)^{\gamma_1} \cdots (\partial_N)^{\gamma_N} u \qquad (\partial_j = \partial/\partial x_j). \end{cases} \tag{3.27}$$

The inner product and norm of $H_t^j(\mathbf{R}^N)$ will be denoted by $(\ ,\ )_{j,t}$ and $\|\ \|_{j,t}$. For an operator T, the operator norm in $\mathbf{B}(H_s^j(\mathbf{R}^N), H_t^\ell(\mathbf{R}^N))$ will be denoted by $\|T\|_{(j,s)}^{(\ell,t)}$, where $j, \ell = 0, 1, 2$, $s, t \in \mathbf{R}$, and we set

$$H_s^0(\mathbf{R}^N) = L_{2,s}(\mathbf{R}^N). \tag{3.28}$$

Let $D_\pm \subset \mathbf{C}$ be given by

$$\begin{cases} D_+ = \{ z = \lambda + i\eta \, : \, \lambda > 0,\ \eta \ge 0 \}, \\ D_- = \{ z = \lambda + i\eta \, : \, \lambda > 0,\ \eta \le 0 \}. \end{cases} \tag{3.29}$$

Also, for $0 < c < d < \infty$, let $J_\pm(c,d)$ be as in (3.18). The closures $\overline{J}_\pm(c,d)$ are given by

$$(3.30)\quad \begin{cases} \overline{J}_+(c,\, d) = \{\, z = \lambda + i\eta \;:\; c \le \lambda \le d,\; 0 \le \eta \le 1 \,\} \subset D_+, \\ \overline{J}_-(c,\, d) = \{\, z = \lambda + i\eta \;:\; c \le \lambda \le d,\; -1 \le \eta \le 0 \,\} \subset D_-. \end{cases}$$

For $\lambda > 0$, let

$$(3.31)\quad R_{0\pm}(\lambda) = \lim_{\eta \downarrow 0} R_0(\lambda \pm i\eta),$$

and extend the resolvent $R_0(z)$ on $D_\pm$ by

$$(3.32)\quad R_0(\lambda + i\eta) = \begin{cases} R_0(\lambda + i\eta) & (\lambda > 0,\ \eta > 0), \\ R_{0+}(\lambda) & (\lambda > 0,\ \eta = 0) \end{cases}$$

for $z \in D_+$ and

$$(3.33)\quad R_0(\lambda + i\eta) = \begin{cases} R_0(\lambda + i\eta) & (\lambda > 0,\ \eta < 0), \\ R_{0-}(\lambda) & (\lambda > 0,\ \eta = 0) \end{cases}$$

for $z \in D_-$. Then we have

Theorem 3.6. *Suppose that* Assumption 2.1 *holds. Let* $1/2 < \delta \le 1$.

(i) *Then the limits* (3.31) *is well-defined in* $\mathbf{B}(L_{2,\delta}(\mathbf{R}^N),\, H^2_{-\delta}(\mathbf{R}^N))$, *and the extended resolvent* $R_0(z)$ *is a* $\mathbf{B}(L_{2,\delta}(\mathbf{R}^N),\, H^2_{-\delta}(\mathbf{R}^N))$*-valued continuous function on each of* D_+ *and* D_-.

(ii) *For any* $z \in D_+$ [*or* D_-], $R_0(z)$ *is a compact operator from* $L_{2,\delta}(\mathbf{R}^N)$ *into* $H^1_{-\delta}(\mathbf{R}^N)$.

(iii) *The selfadjoint operator* H_0 *is absolutely continuous on the interval* $(0,\infty)$. *The operator* H_0 *has neither point spectrum nor singular continuous spectrum.*

(iv) *For* $0 < c < d < \infty$ *there exists a constant* $C = C(c,d,\delta,m_0,M_0) > 0$ *such that, for* $z \in \overline{J}_+(c,\, d) \cup \overline{J}_-(c,\, d)$,

$$(3.34)\quad \begin{cases} \displaystyle\int_{E_s} (1+r)^{-2\delta}\big(|\nabla R_0(z)f|^2 + |k|^2 |R_0(z)f|^2\big)\, dx \\ \qquad\qquad \le C^2(1+s)^{-(2\delta-1)}\|f\|_\delta^2 \qquad (s \ge 1,\ f \in L_{2,\delta}(\mathbf{R}^N)), \\ \|\mathcal{D}R_0(z)f\|_{\delta-1} \le C\|f\|_\delta \qquad (f \in L_{2,\delta}(\mathbf{R}^N)), \end{cases}$$

where, for $\lambda \in D_+ \cap (0,\, \infty)$ *or* $D_- \cap (0,\, \infty)$, $\mathcal{D}u$ *should be interpreted as*

$$(3.35)\quad \mathcal{D}u = \begin{cases} \mathcal{D}^{(+)}u = \nabla u + \{(N-1)/(2r)\}\widetilde{x}u - ik(x)\widetilde{x}u \\ \qquad\qquad\qquad\qquad\qquad (\lambda \in D_+ \cap (0,\, \infty)), \\ \mathcal{D}^{(-)}u = \nabla u + \{(N-1)/(2r)\}\widetilde{x}u + ik(x)\widetilde{x}u \\ \qquad\qquad\qquad\qquad\qquad (\lambda \in D_- \cap (0,\, \infty)). \end{cases}$$

(v) *Let $N \geq 3$. Then there exists a constant $C = C(\delta, m_0, M_0) > 0$ such that*

$$\tag{3.36} \begin{cases} \displaystyle\int_{E_s} (1+r)^{-2\delta}|R_0(z)f|^2\,dx \leq \frac{C^2}{|z|}(1+s)^{-2(2\delta-1)}\|f\|_\delta^2 \\ \qquad\qquad\qquad\qquad (s \geq 0,\ f \in L_{2,\delta}), \\ \|R_0(z)\|_{(0,\delta)}^{(0,-\delta)} \leq \dfrac{C}{\sqrt{|z|}} \quad (z \in D_+ \cup D_-), \\ \|\mathcal{D}R_0(z)f\|_{\delta-1} \leq C\|f\|_\delta \quad (z \in D_+ \cup D_-,\ f \in L_{2,\delta}). \end{cases}$$

Finally we are going to prove a modification of Theorem 3.5, where the range of δ is slightly wider. This modification will be useful in the next section.

Proposition 3.7. *Let* Assumption 2.1 *be satisfied. Let $1/2 < \delta < 3/2$. Let $f \in L_{2,\delta}(\mathbf{R}^N)$ and let $z \in D_+ \cup D_-$. Let $u = R_0(z)f$.*

(i) *Let $N \geq 3$. Then there exists a constant $C = C(\delta, m_0, M_0) > 0$ such that*

$$\tag{3.37} \|\mathcal{D}_r u\|_{\delta-1} \leq C\|f\|_\delta \quad (f \in L_{2,\delta}(\mathbf{R}^N),\ z \in D_+ \cup D_-)$$

where $\mathcal{D}_r u$ is given in Notation 3.1, (6), *and for $\lambda \in D_+ \cap (0, \infty)$ [or $D_- \cap (0, \infty)$], $\mathcal{D}_r u$ should be interpreted as $\mathcal{D}_r^{(+)}$ [or $\mathcal{D}_r^{(-)}$].*

(ii) *Let $N = 2$. Let $0 < c < d < \infty$. Then there exists $C = C(\delta, c, d, m_0, M_0) > 0$ such that*

$$\tag{3.38} \|\mathcal{D}_r u\|_{\delta-1,*} \leq C\|f\|_\delta \quad (f \in L_{2,\delta}(\mathbf{R}^N),\ z \in \overline{J}_+(c,d) \cup \overline{J}_-(c,d)),$$

where $\|\mathcal{D}_r u\|_{\delta-1,*}$ is given by (3.21).

Proof. In view of the continuity of the extended resolvent of $R_0(z)$, we only have to prove (3.37) and (3.38) for non real z. Then we should note that we have $u = R_0(z)f \in H_\delta^2(\mathbf{R}^N)$ (cf., e.g., [12], Lemma 2.1). As in the proof of Theorem 3.5, we start with (3.4) with ξ given by (3.22) or (3.23) and $\alpha(x) = 1/\sqrt{\mu_0}$. Let the j-th term of the left-hand side be denoted by I_{Lj}, where $j = 1, 2, 3, 4$. Then we have $I_{L2} = 0$, and, since

$$\tag{3.39} \frac{\varphi}{r} - 2^{-1}\frac{\partial\varphi}{\partial r} \geq 0$$

and $|\mathcal{D}u| \geq |\mathcal{D}_r u|$, we have

$$
\begin{aligned}
I_{L1} + I_{L3} &\geq \int_{B_{rR}} \frac{1}{2}\frac{\partial \varphi}{\partial r}|\mathcal{D}u|^2\,dx + \int_{B_{rR}} \left(\frac{\varphi}{r} - \frac{\partial \varphi}{\partial r}\right)(|\mathcal{D}u|^2 - |\mathcal{D}_r u|^2)\,dx \\
&= \int_{B_{rR}} \left(\frac{\varphi}{r} - \frac{1}{2}\frac{\partial \varphi}{\partial r}\right)|\mathcal{D}u|^2\,dx - \int_{B_{rR}} \left(\frac{\varphi}{r} - \frac{\partial \varphi}{\partial r}\right)|\mathcal{D}_r u|^2\,dx \\
&\geq \int_{B_{rR}} \left(\frac{\varphi}{r} - \frac{1}{2}\frac{\partial \varphi}{\partial r}\right)|\mathcal{D}_r u|^2\,dx - \int_{B_{rR}} \left(\frac{\varphi}{r} - \frac{\partial \varphi}{\partial r}\right)|\mathcal{D}_r u|^2\,dx \\
&= \int_{B_{rR}} \frac{1}{2}\frac{\partial \varphi}{\partial r}|\mathcal{D}_r u|^2\,dx.
\end{aligned}
\tag{3.40}
$$

As for the fourth term I_{L4}, we have $I_{L4} \geq 0$ for $N \geq 3$, and for $N = 2$ we have, as in (7.19) and (7.20) of [9],

$$
\begin{aligned}
-I_{L4} &\leq C_1\|u\|_{\delta-2}^2 + C_2\int_{\mathbf{R}^2} |\eta||u|^2\,dx \\
&\leq C_1\|u\|_{\delta-2}^2 + C_3(|f|, |u|)_0 \\
&\leq C_1\|u\|_{\delta-2}^2 + \frac{C_3}{2}(\|f\|_\delta^2 + \|u\|_{-\delta}^2)
\end{aligned}
\tag{3.41}
$$

with $C_1 = C_1(\delta, m_0)$, $C_2 = C_2(c, d)$, and $C_3 = C_3(\delta, c, d, m_0, M_0)$. Here we can evaluate $\|u\|_{\delta-2}$ as

$$
\begin{cases}
\delta \in (1/2, 1] \Longrightarrow \|u\|_{\delta-2} \leq \|u\|_{-\delta} \leq C'\|f\|_\delta, \\
\delta \in (1, 3/2) \Longrightarrow \|u\|_{\delta-2} \leq C''\|f\|_{2-\delta} \leq C''\|f\|_\delta,
\end{cases}
\tag{3.42}
$$

with $C' = C'(\delta, c, d, \mu_0)$ and $C'' = C''(\delta, c, d, \mu_0)$. We can evaluate the right-hand of (3.4) by proceeding as in the proof of Theorems 4.1 and 7.2 of [9] Therefore, letting $r \to 0$ and $R \to \infty$, where we have to use the fact that $u = R_0(z)f \in H_\delta^2(\mathbf{R}^N)$, we obtain (3.37) and (3.38), respectively. This completes the proof. □

4. The point spectrum for H

Throughout this and the following sections we shall assume that Assumptions 2.1 and 2.2 are satisfied. Let the operator H be as in §2. Let $\sigma_p(H)$ be the set of all eigenvalues of H, and let $V_p(H)$ be the set of all eigenvectors of H, i.e.,

$$
V_p(H) = \{u \in H^2(\mathbf{R}^N) : u \neq 0,\ (H - \lambda)u = 0 \quad \text{with} \quad \lambda \in \sigma_p(H)\}. \tag{4.1}
$$

Now we need to introduce a set $\widetilde{\sigma}_p^{(\pm)}(H)$ of the extended eigenvalues of H and a set $\widetilde{V}_p^{(\pm)}(H)$ of the extended eigenvectors of H.

DEFINITION 4.1. Let $1/2 < \delta < 1/2 + \epsilon$ if μ_1 is short-range, i.e., μ_1 satisfies (2.13), and let $1/2 < \delta < (1+\epsilon)/2$ if μ_1 is long-range, i.e., μ_1 satisfies (2.14), where ϵ is given in Assumption 2.2. Denote by $\widetilde{\sigma}_p^{(+)}(H)$ [or $\widetilde{\sigma}_p^{(-)}(H)$] the set of all $\lambda > 0$ such that there exists a function u satisfying

$$\text{(4.2)}\qquad \begin{aligned} &\text{(i)} && u \in H^2(\mathbf{R}^N)_{\text{loc}},\ u \neq 0,\\ &\text{(ii)} && u \in L_{2,-\delta}(\mathbf{R}^N),\\ &\text{(iii)} && \|\mathcal{D}^{(+)}u\|_{\delta-1,E_1} < \infty,\ [\text{or } \|\mathcal{D}^{(-)}u\|_{\delta-1,E_1} < \infty],\\ &\text{(iv)} && -\mu(x)^{-1}\Delta u - \lambda u = 0, \end{aligned}$$

where $\mathcal{D}^{(\pm)}u$ are given by (3.35), and $k = k(x) = [\lambda\mu_0(x)]^{1/2}$ is as in §3. Let $\widetilde{V}_p^{(\pm)}(H)$ be the set of all $u \in X$ which satisfy (4.2) with $\lambda \in \widetilde{\sigma}_p^{(\pm)}(H)$. We call $u \in \widetilde{V}_p^{(\pm)}(H)$ which satisfies (4.2) an extended eigenvector of H associated with the extended eigenvalue λ.

Since $0 \notin \sigma_p(H)$, we have $V_p(H) \subset \widetilde{V}_p^{(\pm)}(H)$ and $\sigma_p(H) \subset \widetilde{\sigma}_p^{(\pm)}(H)$ by definition. In this section we are going to prove that

$$\sigma_p(H) = \widetilde{\sigma}_p^{(+)}(H) = \widetilde{\sigma}_p^{(-)}(H), \tag{4.3}$$

and $\sigma_p(H)$ is a discrete set on $(0,\infty)$.

Proposition 4.2. *Assume that* Assumptions 2.1 *and* 2.2 *hold. Let* $u \in H^2(\mathbf{R}^N)_{\text{loc}}$ *be a solution of the equation* $-\mu(x)^{-1}\Delta u - \lambda u = 0$. *Let* $\varphi(x) = \xi(|x|)$ *and let* $\xi(r)$ *satisfy the following* (a) $\sim$ (c)$'$:

(a) ξ *is a nonnegative, continuous function on* $(0,\infty)$,

(b) ξ *has a piecewise continuous derivative* ξ' *such that*

$$\xi'(r) \geq 0, \tag{4.4}$$

and

$$\frac{\xi(r)}{r} - \frac{1}{2}\xi'(r) \geq 0 \tag{4.5}$$

for almost all $r > 0$.

(c) *If* $N \geq 3$,

$$\xi(r) = O(r) \qquad (r \to 0). \tag{4.6}$$

(c)$'$ *If* $N = 2$,

$$\begin{cases} \xi(r) = O(r^2) & (r \to 0),\\ \xi'(r) = O(r) & (r \to 0). \end{cases} \tag{4.7}$$

(i) *Suppose that* μ_1 *is short-range in the sense of* (2.13). *Then there exists a constant* $C = C(\lambda, \mu_0) > 0$ *such that, for* $R > 1$,

$$\int_{B_R} \frac{\partial \varphi}{\partial r} |u|^2 \, dx \le 2m_0^{-1} \int_{B_R} \varphi |\mu_1||u||\mathcal{D}_r^{(\pm)} u| \, dx + C\xi(R) \int_{S_R} |\mathcal{D}_r^{(\pm)} u|^2 \, dS \tag{4.8}$$

for $N \ge 3$, *or*

$$\begin{aligned}\int_{B_R} \frac{\partial \varphi}{\partial r} |u|^2 \, dx \le 2m_0^{-1} &\int_{B_R} \varphi |\mu_1||u||\mathcal{D}_r^{(\pm)} u| \, dx \\ &+ (2m_0\lambda)^{-1} \int_{B_R} r^{-2} \left(\frac{\varphi}{r} - 2^{-1} \frac{\partial \varphi}{\partial r} \right) |u|^2 \, dx \\ &+ C\xi(R) \int_{S_R} |\mathcal{D}_r^{(\pm)} u|^2 \, dS\end{aligned} \tag{4.9}$$

for $N = 2$.

(ii) *Suppose that* μ_1 *is long-range in the sense of* (2.14). *Then the relation* (4.8) *or* (4.9) *holds again with the first term* K_{R1} *of the right-hand side of* (4.8) *or* (4.9) *replaced by*

$$K'_{R1} = m_0^{-1} \int_{B_R} \left(\left| \frac{\partial \varphi}{\partial r} \mu_1 \right| + \left| \frac{\partial \mu_1}{\partial r} \varphi \right| \right) |u|^2 \, dx. \tag{4.10}$$

Proof. (I) We shall prove (4.8) and (4.9) for $\mathcal{D}_r^{(+)} u$. These formulas for $\mathcal{D}_r^{(-)} u$ can be proved in quite a similar way. For the sake of simplicity of notation we set $\mathcal{D}_r^{(+)} u = \mathcal{D}_r u$ and $\mathcal{D}^{(+)} u = \mathcal{D} u$. Since we have from (iv) of (4.2) $-\Delta u - \lambda \mu_0(x) u = \lambda \mu_1(x) u$, we can apply the formula (3.4) with f and z replaced by $\lambda \mu_0(x)^{-1} \mu_1(x) u$ and λ, respectively, to obtain, for $0 < r < 1 < R < \infty$,

$$\begin{aligned}&\int_{B_{rR}} \frac{1}{2} \frac{\partial \varphi}{\partial r} |\mathcal{D} u|^2 \, dx + \sum_{\ell \in L} \int_{\partial \Omega_\ell \cap B_{rR}} \varphi \mathrm{Im} \left\{ \overline{k} \frac{\partial u}{\partial n} \overline{u} \right\} dS \\ &\quad + \int_{B_{rR}} \left(\frac{\varphi}{r} - \frac{\partial \varphi}{\partial r} \right) (|\mathcal{D} u|^2 - |\mathcal{D}_r u|^2) \, dx \\ &\quad + c_N \int_{B_{rR}} r^{-2} \left(\frac{\varphi}{r} - 2^{-1} \frac{\partial \varphi}{\partial r} \right) |u|^2 \, dx \\ &= \mathrm{Re} \int_{B_{rR}} \lambda \varphi \mu_1(x) u \overline{\mathcal{D}_r u} \, dx \\ &\quad + 2^{-1} \sum_{\ell \in L} \int_{\partial \Omega_\ell \cap B_{rR}} \varphi k^2 (\tilde{x} \cdot n) |u|^2 \, dS\end{aligned} \tag{4.11}$$

$$+ 2^{-1} \int_{S_R} \varphi\big(2|\mathcal{D}_r u|^2 - |\mathcal{D}u|^2 - c_N r^{-2}|u|^2\big)\, dS$$
$$- 2^{-1} \int_{S_r} \varphi(2|\mathcal{D}_r u|^2 - |\mathcal{D}u|^2 - c_N r^{-2}|u|^2)\, dS.$$

(II) Suppose that μ_1 is short-range. Proceeding as in the proof of [9], Theorems 3.2 or 7.1, we have

$$\begin{aligned}
&\int_{B_R} \frac{1}{2}\frac{\partial \varphi}{\partial r}|\mathcal{D}u|^2\, dx + \sum_{\ell \in L} \int_{\partial\Omega_\ell \cap B_R} \varphi \mathrm{Im}\left\{ k \frac{\partial u}{\partial n}\overline{u} \right\} dS \\
&\qquad \geq \int_{B_R} \frac{1}{2}\frac{\partial \varphi}{\partial r} k^2 |u|^2\, dx + \int_{B_R} \frac{1}{2}\frac{\partial \varphi}{\partial r}\left(|\nabla u|^2 - \left|\frac{\partial u}{\partial r}\right|^2 \right) dx \qquad (4.12)\\
&\qquad\qquad - \xi(R) \int_{S_R} \mathrm{Im}\left\{ k \frac{\partial u}{\partial r}\overline{u} \right\} dS,
\end{aligned}$$

where we should note that all the integrals in (4.12) is well-defined even in the case of $N = 2$ because of (4.7). Therefore it follows from (4.11) and (4.12) that

$$\begin{aligned}
&\int_{B_R} \frac{1}{2} k^2 \frac{\partial \varphi}{\partial r}|u|^2\, dx \\
&\qquad + \int_{B_{rR}} \left(\frac{\varphi}{r} - 2^{-1}\frac{\partial \varphi}{\partial r} \right)\left(|\nabla u|^2 - \left|\frac{\partial u}{\partial r}\right|^2 \right) dx \\
&\qquad + c_N \int_{B_{rR}} r^{-2}\left(\frac{\varphi}{r} - 2^{-1}\frac{\partial \varphi}{\partial r} \right)|u|^2\, dx \\
&\leq \mathrm{Re} \int_{B_{rR}} \lambda \varphi \mu_1(x) u \overline{\mathcal{D}_r u}\, dx \\
&\qquad + 2^{-1} \sum_{\ell \in L} \int_{\partial\Omega_\ell \cap B_{rR}} \varphi |k|^2 (\widetilde{x} \cdot n)|u|^2\, dS \qquad (4.13)\\
&\qquad + 2^{-1}\xi(R) \int_{S_R} (2|\mathcal{D}_r u|^2 - |\mathcal{D}u|^2 - c_N r^{-2}|u|^2)\, dS \\
&\qquad - 2^{-1}\xi(r) \int_{S_r} (2|\mathcal{D}_r u|^2 - |\mathcal{D}u|^2 - c_N r^{-2}|u|^2)\, dS, \\
&\qquad + \int_{B_r} \frac{1}{2}\frac{\partial \varphi}{\partial r}|\mathcal{D}u|^2\, dx + \sum_{\ell \in L} \int_{\partial\Omega_\ell \cap B_r} \varphi \mathrm{Im}\left(k \frac{\partial u}{\partial n}\overline{u} \right) dS \\
&\qquad + \xi(R) \int_{S_R} \mathrm{Im}\left(k \frac{\partial u}{\partial r}\overline{u} \right) dS.
\end{aligned}$$

It follows from (2.11) that the second term of the right-hand side of (4.13) is nonpositive. Therefore we can drop it from the right-hand side of (4.13). Further, we see from (4.5) in (b) that the second term of the left-hand side of (4.13) is nonnegative, and it can be dropped, too. Then, letting $r \to 0$ along an appropriate sequence $\{r_n\}$, we obtain

$$
\begin{aligned}
\lambda \int_{B_R} \frac{1}{2}\mu_0(x)\frac{\partial\varphi}{\partial r}|u|^2\,dx + c_N \int_{B_R} r^{-2}\left(\frac{\varphi}{r} - 2^{-1}\frac{\partial\varphi}{\partial r}\right)|u|^2\,dx \\
\le \lambda \int_{B_R} \varphi|\mu_1(x)||u||\mathcal{D}_r u|\,dx \\
+ 2^{-1}\xi(R)\int_{S_R}(2|\mathcal{D}_r u|^2 - |\mathcal{D}u|^2 - c_N r^{-2}|u|^2)\,dS \\
+ \xi(R)\int_{S_R} \operatorname{Im}\left(k\frac{\partial u}{\partial r}\overline{u}\right)dS. \\
\le \lambda \int_{B_R} \varphi|\mu_1(x)||u||\mathcal{D}_r u|\,dx \\
+ C\xi(R)\int_{S_R}\left(\left|\frac{\partial u}{\partial r}\right|^2 + |u|^2\right)dS.
\end{aligned}
\tag{4.14}
$$

Here we noticed from (c) and (c)$'$ that

$$
\begin{cases}
\liminf_{r\to 0}\ \xi(r)\int_{S_r}(2|\mathcal{D}_r u|^2 - |\mathcal{D}u|^2 - c_N r^{-2}|u|^2)\,dS = 0, \\
2^{-1}\xi(R)\int_{S_R}(2|\mathcal{D}_r u|^2 - |\mathcal{D}u|^2 - c_N r^{-2}|u|^2)\,dS \\
\qquad + \xi(R)\int_{S_R}\operatorname{Im}\left(k\frac{\partial u}{\partial r}\overline{u}\right)dS. \\
\qquad \le C\xi(R)\int_{S_R}\left(\left|\frac{\partial u}{\partial r}\right|^2 + |u|^2\right)dS
\end{cases}
\tag{4.15}
$$

with a constant C. Since $c_N \ge 0$ for $N \ge 3$ and $c_N = -1/4$ for $N = 2$, (4.8) and (4.9) follows from (4.14) if we can prove

$$
\int_{S_R}\left(\left|\frac{\partial u}{\partial r}\right|^2 + |u|^2\right)dS \le C'\int_{S_R}|\mathcal{D}_r^{(\pm)}u|^2\,dS
\tag{4.16}
$$

with a constant C'.

(III) (Proof of (4.16).) Multiply both sides of the equation $-\Delta u - \mu(x)\lambda u = 0$ by $\overline{u}$, integrating over B_R, use partial integration and take the imaginary part to obtain

$$
\int_{S_R}\operatorname{Im}\left\{\frac{\partial u}{\partial r}\overline{u}\right\}dS = 0.
\tag{4.17}
$$

Now we can proceed as in the proof of [9], Theorem 3.7 to obtain (4.16).

(IV) Suppose that μ_1 is long-range. We have only to evaluate the first term K''_{R1} of the right-hand side of (4.11). In fact, we have by partial integration

$$
\begin{aligned}
K''_{R1} &= \lambda \operatorname{Re} \int_{B_{rR}} \varphi\mu_1(x) u \overline{\mathcal{D}_r u}\, dx \\
&= \frac{\lambda}{2} \int_{B_{rR}} \varphi\mu_1(x) \left(\frac{\partial |u|^2}{\partial r} + (N-1) r^{-1} |u|^2 \right) dx \\
&= -\frac{\lambda}{2} \int_{B_{rR}} \frac{\partial(\varphi\mu_1)}{\partial r} |u|^2\, dx + \frac{\lambda}{2} \int_{S_R} \varphi\mu_1 |u|^2\, dS \\
&\quad - \frac{\lambda}{2} \int_{S_r} \varphi\mu_1 |u|^2\, dS
\end{aligned}
\tag{4.18}
$$

Since the third term of the right-hand side of (4.18) converges to 0 as $r \to 0$, we see that K'_{R1} in (4.10) can replace the first term K_{R1} of the right-hand side of (4.8) or (4.9), which completes the proof. □

Proposition 4.3. *Assume that* Assumptions 2.1 *and* 2.2 *hold. Suppose that* μ_1 *is short-range. Suppose that* $u \in \widetilde{V}_p^{(+)}(H)$ [*or* $u \in \widetilde{V}_p^{(-)}(H)$] *with an extended eigenvalue* $\lambda \in \widetilde{\sigma}_p^{(+)}(H)$ [*or* $\lambda \in \widetilde{\sigma}_p^{(-)}(H)$] *such that*

$$
\begin{cases} u \in L_{2,-\delta+j\epsilon}(\mathbf{R}^N), \\ -\delta + j\epsilon \le 0 \end{cases}
\tag{4.19}
$$

with a nonnegative integer j. *Then we have*

$$
\begin{cases} u \in L_{2,-\delta+(j+1)\epsilon}(\mathbf{R}^N), \\ \mathcal{D}_r^{(+)} u \in L_{2,-\delta+(j+1)\epsilon}(\mathbf{R}^N) \quad [\text{or } \mathcal{D}_r^{(-)} u \in L_{2,-\delta+(j+1)\epsilon}(\mathbf{R}^N)\,], \end{cases}
\tag{4.20}
$$

and

$$
\begin{cases} \|u\|_{-\delta+(j+1)\epsilon} \le \sqrt{\lambda} c_1 C_j^{(N)} \|u\|_{-\delta+j\epsilon} & (N \ge 3), \\ \|u\|_{-\delta+(j+1)\epsilon} \le c_1 C_j^{(2)} \|u\|_{-\delta+j\epsilon} & (N = 2), \end{cases}
\tag{4.21}
$$

where $C_j^{(N)} = C_j^{(N)}(\delta, \epsilon, m_0, M_0)$ *for* $N \ge 3$ *and* $C_j^{(2)} = C_j^{(2)}(\lambda, \delta, \epsilon, m_0, M_0)$.

Proof. (I) We shall prove (4.20) for $\mathcal{D}_r^{(+)} u$, and set $\mathcal{D}_r^{(+)} u = \mathcal{D}_r u$. Then u satisfies the equation $(-\Delta - \lambda\mu_0) u = \lambda\mu_1 u$ and the radiation condition $\|\mathcal{D}_r^{(+)} u\|_{\delta-1,E_1} < \infty$, i.e.,

$$
u = \lambda R_0(\lambda)\big(\mu_0^{-1}\mu_1 u\big),
\tag{4.22}
$$

where we set $R_{0+}(\lambda) = R_0(\lambda)$. Since we have

$$|\mu_0^{-1}\mu_1 u| \le c_1 m_0^{-1}(1+|x|)^{-1-\epsilon+\delta-j\epsilon}\big[(1+|x|)^{-\delta+j\epsilon}|u|\big], \tag{4.23}$$

it follows that

$$\mu_0^{-1}\mu_1 u \in L_{2,1-\delta+(j+1)\epsilon}(\mathbf{R}^N). \tag{4.24}$$

Noting that (4.19), $1/2 < \delta < 1/2 + \epsilon$ and $0 < \epsilon < 1/2$, we see that

$$\begin{cases} \delta < 1/2+\epsilon \Longrightarrow 1-\delta+(j+1)\epsilon \ge 1-\delta+\epsilon > 1/2, \\ -\delta + j\epsilon \le 0 \text{ and } 0 < \epsilon < 1/2 \\ \qquad\qquad \Longrightarrow 1-\delta+(j+1)\epsilon \le 1+\epsilon < 3/2, \end{cases} \tag{4.25}$$

and hence we can apply Proposition 3.7 with δ replaced by $1-\delta+(j+1)\epsilon$ to obtain

$$\|\mathcal{D}_r u\|_{-\delta+(j+1)\epsilon} \le \lambda m_0^{-1} c_1 C_j' \|u\|_{-\delta+j\epsilon} \tag{4.26}$$

with $C_j' = C_j'(\delta, \epsilon, m_0, M_0)$ for $N \ge 3$, and

$$\|\mathcal{D}_r u\|_{-\delta+(j+1)\epsilon,*} \le \lambda m_0^{-1} c_1 C_j'' \|u\|_{-\delta+j\epsilon} \tag{4.27}$$

with $C_j'' = C_j''(\delta, \epsilon, \lambda, m_0, M_0)$ for $N = 2$.

(II) Let $R_0 > 1$. Set $\beta = 2(-\delta + (j+1)\epsilon) + 1$,

$$\xi(r) = \begin{cases} r & (0 < r \le 1), \\ 2^{-\beta}(1+r)^\beta & (1 < r \le R_0), \\ 2^{-\beta}(1+R_0)^\beta & (r > R_0) \end{cases} \tag{4.28}$$

if $N \ge 3$, and

$$\xi(r) = \begin{cases} 2^{-1}r^2 & (0 < r \le 1), \\ 2^{-\beta-1}(1+r)^\beta & (1 < r \le R_0), \\ 2^{-\beta-1}(1+R_0)^\beta & (r > R_0) \end{cases} \tag{4.29}$$

for $N = 2$. It follows from (4.25) that $0 < \beta < 2$, and hence ξ satisfies (a), (b), (c) or (c)′ in Proposition 4.2. Therefore we can apply (i) of Proposition 4.2 to obtain

$$\begin{aligned} &\int_{B_1} |u|^2\,dx + \int_{B_{1R_0}} \beta 2^{-\beta}(1+r)^{\beta-1}|u|^2\,dx \\ &\quad \le 2m_0^{-1}c_1 \int_{B_1} r(1+r)^{-1-\epsilon}|u||\mathcal{D}_r u|\,dx \\ &\qquad + 2^{-\beta+1} m_0^{-1} c_1 \int_{B_{1R}} (1+r)^{\beta-1-\epsilon}|u||\mathcal{D}_r u|\,dx \\ &\qquad + C2^{-\beta}(1+R_0)^\beta \int_{S_R} |\mathcal{D}_r u|^2\,dS \end{aligned} \tag{4.30}$$

for $R > R_0$ and $N \geq 3$, or

$$\begin{aligned}\int_{B_1} r|u|^2\,dx &+ \int_{B_{1R_0}} \beta 2^{-\beta-1}(1+r)^{\beta-1}|u|^2\,dx \\ &\leq m_0^{-1}c_1 \int_{B_1} r^2(1+r)^{-1-\epsilon}|u||\mathcal{D}_r u|\,dx \\ &\quad + 2^{-\beta}m_0^{-1}c_1 \int_{B_{1R}} (1+r)^{\beta-1-\epsilon}|u||\mathcal{D}_r u|\,dx \\ &\quad + 2^{-\beta+1}m_0^{-1}\lambda^{-1} \int_{B_{1R}} (1+r)^{2(-\delta+j\epsilon)}|u|^2\,dx \\ &\quad + C2^{-\beta-1}(1+R_0)^\beta \int_{S_R} |\mathcal{D}_r u|^2\,dS\end{aligned} \tag{4.31}$$

for $R > R_0$ and $N = 2$, where we have used the facts that

$$\begin{cases} (1+R_0)^\beta \leq (1+R)^\beta & (R \geq R_0), \\ \dfrac{\varphi}{r} - 2^{-1}\dfrac{\partial\varphi}{\partial r} = 0 & (0 < r < 1), \end{cases} \tag{4.32}$$

and, for $r > 1$,

$$\begin{aligned} r^{-2}\left(\frac{\varphi}{r} - 2^{-1}\frac{\partial\varphi}{\partial r}\right) &\leq 2^{-\beta-2}2^3\left(1-\frac{\beta}{2}\right)(1+r)^{\beta-3} \\ &\leq 2^{-\beta+1}(1+r)^{2(-\delta+j\epsilon)}. \end{aligned} \tag{4.33}$$

Since

$$\beta - 1 - \epsilon = (-\delta + j\epsilon) + (-\delta + (j+1)\epsilon), \tag{4.34}$$

we have

$$\int_{B_{1R}} (1+r)^{\beta-1-\epsilon}|u||\mathcal{D}_r u|\,dx \leq \|u\|_{-\delta+j\epsilon}\|\mathcal{D}_r u\|_{-\delta+(j+1)\epsilon}. \tag{4.35}$$

Therefore we have for $N \geq 3$

$$\begin{aligned}&\|u\|^2_{-\delta+(j+1)\epsilon,B_{R_0}} \\ &\quad\leq C_0 c_1\|u\|_{-\delta+j\epsilon}\|\mathcal{D}_r u\|_{-\delta+(j+1)\epsilon} + \widetilde{C}(1+R_0)^\beta \int_{S_R} |\mathcal{D}_r u|^2\,dS,\end{aligned} \tag{4.36}$$

and for $N = 2$

$$\begin{aligned}&\|u\|^2_{-\delta+(j+1)\epsilon,B_{R_0}} \\ &\quad\leq C_0'\left(c_1\|u\|_{-\delta+j\epsilon}\|\mathcal{D}_r u\|_{-\delta+(j+1)\epsilon,*} + \lambda^{-1}\|u\|^2_{-\delta+j\epsilon}\right) \\ &\qquad + \widetilde{C}'(1+R_0)^{-\delta+j\epsilon} \int_{S_R} |\mathcal{D}_r u|^2\,dS,\end{aligned} \tag{4.37}$$

where C_0, C_0', $\widetilde{C}$, and $\widetilde{C}'$ depend on j, δ, ϵ, m_0, and M_0. Note that we have from (iii) of (4.2)

$$\liminf_{R\to\infty} \int_{S_R} |\mathcal{D}_r u|^2\, dS = 0. \tag{4.38}$$

Let $R \to \infty$ along an appropriate sequence so that the last terms of the right-hand sides of (4.36) and (4.37) tends to 0. Therefore, noting that $R_0 > 1$ is arbitrary, and using (4.26) and (4.27), too, we obtain (4.21), which completes the proof. □

Proposition 4.4. *Assume that* Assumptions 2.1 *and* 2.2 *hold. Suppose that* μ_1 *is long-range. Suppose that* $u \in \widetilde{V}_p^{(+)}(H)$ [*or* $u \in \widetilde{V}_p^{(-)}(H)$] *with an extended eigenvalue* $\lambda \in \widetilde{\sigma}_p^{(+)}(H)$ [*or* $\lambda \in \widetilde{\sigma}_p^{(-)}(H)$]. *Let* $\epsilon' = \epsilon/2$. *Suppose that*

$$\begin{cases} u \in L_{2,-\delta+j\epsilon'}(\mathbf{R}^N), \\ -\delta + j\epsilon' \le 0 \end{cases} \tag{4.39}$$

with a nonnegative integer j. *Then we have*

$$u \in L_{2,-\delta+(j+1)\epsilon'}(\mathbf{R}^N), \tag{4.40}$$

and

$$\|u\|_{-\delta+(j+1)\epsilon'} \le \sqrt{c_1}\, C_j^{(N)} \|u\|_{-\delta+j\epsilon'}, \tag{4.41}$$

where $C_j^{(N)} = C_j^{(N)}(\delta, \epsilon, m_0, M_0)$ *for* $N \ge 3$ *and* $C_j^{(2)} = C_j^{(2)}(\lambda, \delta, \epsilon, m_0, M_0)$.

Proof. We shall prove (4.40) and (4.41) in the case that $\lambda \in \widetilde{V}_p^{(+)}(H)$. We set $\mathcal{D}_r^{(+)} u = \mathcal{D}_r u$. Let $R_0 > 1$. Set $\beta = 2(-\delta + (j+1)\epsilon') + 1$, and let $\xi(r)$ be given by (4.28) (with ϵ replaced by ϵ'). Here we should note that $0 < \beta < 3/2$, and hence our $\xi(r)$ satisfies the conditions (a), (b) and (c) of Proposition 4.2. Then we have from (ii) of Proposition 4.2

$$
\begin{aligned}
&\int_{B_1} |u|^2\,dx + \int_{B_{1R_0}} \beta 2^{-\beta}(1+r)^{\beta-1}|u|^2\,dx \\
&\quad \le m_0^{-1}c_1 \int_{B_1} \{(1+r)^{-\epsilon} + r(1+r)^{-1-\epsilon}\}|u|^2\,dx \\
&\qquad + m_0^{-1}c_1(\beta+1)2^{-\beta} \int_{B_{1R_0}} (1+r)^{\beta-1-\epsilon}|u|^2\,dx \\
&\qquad + m_0^{-1}c_1 2^{-\beta} \int_{B_{1R_0}} (1+r)^{\beta}(1+r)^{-1-\epsilon}|u|^2\,dx \\
&\qquad + C2^{-\beta}(1+R_0)^{\beta} \int_{S_R} |\mathcal{D}_r u|^2\,dS, \\
&\quad \le \widetilde{C} \int_{B_R} (1+r)^{2(-\delta+j\epsilon')}|u|^2\,dx \\
&\qquad + C2^{-\beta}(1+R_0)^{\beta} \int_{S_R} |\mathcal{D}_r u|^2\,dS,
\end{aligned} \tag{4.42}
$$

where $R_0 < R$, $\widetilde{C} = \widetilde{C}(j,\delta,\epsilon,m_0)$, and we should note that $\beta - 1 - \epsilon = 2(-\delta + j\epsilon')$. The inequality (4.41) follows from (4.42). The case that $N = 2$ can be treated in quite a similar way, which completes the proof. □

Now we are in a position to show that $V_p(H) = \widetilde{V}_p^{(\pm)}(H)$. Let

$$
\begin{cases} j_0 = \min\{j \in \mathbf{N} : -\delta + j\epsilon > 0\}, \\ \delta_0 = -\delta + j_0\epsilon \end{cases} \tag{4.43}
$$

if μ_1 is short-range, and let

$$
\begin{cases} j_0 = \min\{j \in \mathbf{N} : -\delta + j\epsilon' > 0\}, \\ \delta_0 = -\delta + j_0\epsilon' \end{cases} \tag{4.44}
$$

if μ_1 is long-range.

Theorem 4.5. *Let* Assumptions 2.1 *and* 2.2 *be satisfied.*
(i) *Then we have*

$$
\widetilde{V}_p^{(\pm)}(H) \subset H^2_{\delta_0}(\mathbf{R}^N), \tag{4.45}
$$

where δ_0 *is given by* (4.43) *or* (4.44), *and hence*

$$
\begin{cases} V_p(H) = \widetilde{V}_p^{(+)}(H) = \widetilde{V}_p^{(-)}(H), \\ \sigma_p(H) = \widetilde{\sigma}_p^{(+)}(H) = \widetilde{\sigma}_p^{(-)}(H). \end{cases} \tag{4.46}
$$

(ii) *Let μ_1 be short-range. Let $u \in V_p(H)$ associated with $\lambda \in \sigma_p(H)$. Then, for each $N \geq 2$, there exists a positive constant $C^{(N)}$ such that*

$$\begin{cases} \|u\|_{\delta_0} \leq C^{(N)}\big(c_1\sqrt{\lambda}\big)^{j_0}\|u\|_{-\delta} & (N \geq 3), \\ \|u\|_{\delta_0} \leq C^{(2)} c_1^{j_0}\|u\|_{-\delta} & (N = 2), \end{cases} \tag{4.47}$$

where j_0 is given by (4.43), *and $C^{(N)} = C^{(N)}(\delta, \epsilon, m_0, M_0)$ for $N \geq 3$ and $C^{(2)} = C^{(2)}(\lambda, \delta, \epsilon, m_0, M_0)$. Further, for $N \geq 3$, we have*

$$\sigma_p(H) \subset [c_1^{-2}(C^{(N)})^{-2/j_0},\ \infty). \tag{4.48}$$

(iii) *Let μ_1 be long-range. Then, for each $N \geq 2$, there exists a positive constant $C^{(N)} = C^{(N)}(\delta, \epsilon, m_0, M_0)$ $(N \geq 3), = C^{(2)}(\lambda, \delta, \epsilon, m_0, M_0)$ $(N = 2)$ such that*

$$\|u\|_{\delta_0} \leq C^{(N)} c_1^{j_0/2}\|u\|_{-\delta}, \tag{4.49}$$

where $u \in V_p(H)$, and j_0 and δ_0 are given by (4.44).

Proof. Using Propositions 4.3 and 4.4 repeatedly, we obtain

$$\widetilde{V}_p^{(\pm)}(H) \subset L_{2,\delta_0}(\mathbf{R}^N), \tag{4.50}$$

and the inequalities (4.47) and (4.49), where $C^{(N)} = C_0^{(N)}C_1^{(N)} \cdots C_{j_0}^{(N)}$ and j_0 and δ_0 are in (4.43) or (4.44). Let $u \in \widetilde{V}_p^{(\pm)}(H)$ associated with $\lambda \in \widetilde{\sigma}_p^{(\pm)}(H)$. Then it follows from the equation $-\Delta u - \lambda\mu u = 0$ that $u,\ \Delta u \in L_{2,\delta_0}(\mathbf{R}^N)$ which implies that $u \in H^2_{\delta_0}(\mathbf{R}^N)$. Thus we have proved (4.45). Let $N \geq 3$ and let μ_1 is short-range. Since we have from the first inequality of (4.47)

$$\|u\|_{-\delta} \leq \|u\|_{\delta_0} \leq C^{(N)}\big(c_1\sqrt{\lambda}\big)^{j_0}\|u\|_{-\delta}, \tag{4.51}$$

or

$$(1 - C^{(N)}\big(c_1\sqrt{\lambda}\big)^{j_0})\|u\|_{-\delta} \leq 0, \tag{4.52}$$

whence (4.48) follows. This completes the proof. □

Theorem 4.6. *Let* Assumptions 2.1 *and* 2.2 *be satisfied. Let $\sigma_p(H)$ be as above.*

(i) *Then the multiplicity of each $\lambda \in \sigma_p(H)$ is finite.*

(ii) *$\sigma_p(H)$ does not have any accumulation point except $\lambda = 0$ and $\lambda = \infty$. If $N \geq 3$, then the only possible accumulation point of $\sigma_p(H)$ is $\lambda = \infty$.*

Proof. Suppose that $\sigma_p(H)$ has an accumulation point $\lambda_0 \in (0, \infty)$. Then there exist infinite sequences $\{\lambda_n\} \subset \sigma_p(H)$ and $\{u_n\} \subset V_p(H)$ such that

$$\begin{cases} \lambda_n \to \lambda_0 \quad (n \to \infty), \\ (u_m, u_n)_X = \delta_{mn} \quad (m, n \in \mathbf{N}), \\ -\mu(x)^{-1}\Delta u_n - \lambda_n u_n = 0 \quad (n \in \mathbf{N}), \end{cases} \tag{4.53}$$

where δ_{mn} is Kronecker's delta. Since

$$\begin{aligned} \|\nabla u_n\|_0^2 &= \lambda_n (u_n, u_n)_X = \lambda_n \|u_n\|_X^2 \\ &= \lambda_n \le \sup_n \lambda_n < \infty, \end{aligned} \tag{4.54}$$

we can apply the Rellich selection theorem to choose a subsequence $\{u_{n_m}\}$ which converges in $L_2(\mathbf{R}^N)_{\rm loc}$ as $m \to \infty$. Let $u_0 \in L_2(\mathbf{R}^N)_{\rm loc}$ be the limit function. On the other hand, in view of Theorem 4.5, there exists a positive constant C such that, for any $s > 0$,

$$\begin{aligned} \|u_{n_m}\|_{0,E_s} &\le (1+s)^{-\delta_0} \|u_{n_m}\|_{\delta_0, E_s} \le (1+s)^{-\delta_0}\|u_{n_m}\|_{\delta_0} \\ &\le C(1+s)^{-\delta_0}\|u_{n_m}\|_{-\delta} \le C(1+s)^{-\delta_0}\|u_{n_m}\|_0, \end{aligned} \tag{4.55}$$

and hence

$$\|u_{n_m}\|_{0,E_s} \le \frac{C}{\sqrt{m_0}}(1+s)^{-\delta_0}\|u_{n_m}\|_X \le \frac{C}{\sqrt{m_0}}(1+s)^{-\delta_0}, \tag{4.56}$$

where δ_0 is given by (4.43) or (4.44). Therefore u_{n_m} is small at infinity uniformly for $m \in \mathbf{N}$. Thus it follows that u_{n_m} converges to u_0 in X and $\|u_0\|_X = 1$. Noting that $\{u_{n_m}\}$ is an orthonormal system in X, we have

$$0 = \lim_{n\to\infty} (u_{n_m}, u_{n_{m+1}})_X = \|u_0\|_X^2 = 1, \tag{4.57}$$

which is a contradiction. Therefore $\sigma_p(H)$ is discrete in $(0, \infty)$. If $N \ge 3$, (ii) of Theorem 4.5 implies that $\lambda = 0$ cannot be an accumulation point of $\sigma_p(H)$. This completes the proof. □

Consider the following additional condition on $\mu_1(x)$:

ASSUMPTION 4.7. (i) The function μ_1 is measurable such that

$$\mu(x) \ge N\mu_1(x) + \lambda_0(|x||\mu_1(x)|)^2 \qquad (\text{a.e. } x \in \mathbf{R}^N) \tag{4.58}$$

with $\lambda_0 > 0$.

(ii) The function μ_1 is differentiable and μ_1 satisfies

$$\mu(x) + |x|\frac{\partial \mu_1}{\partial |x|} \geq 0 \qquad (\text{a.e. } x \in \mathbf{R}^N). \tag{4.59}$$

The following theorem gives sufficient conditions that the absence of $\sigma_p(H) = 0$ on some interval or whole positive half line (cf. Roach-Zhang [10], the proof of Theorem 3.1).

Theorem 4.8. *Suppose that* $\mu(x) = \mu_0(x)+\mu_1(x)$, μ_0 *satisfies* Assumptions 2.1, *and* μ *satisfies* (2.12) *of* Assumptions 2.2. *Suppose that* (i) *or* (ii) *of* Assumption 4.7 *hold. Then we have*

$$\begin{cases} \sigma_p(H) \cap [0, \lambda_0] = \emptyset & (\text{if (i) holds}), \\ \sigma_p(H) = \emptyset & (\text{if (ii) holds}). \end{cases} \tag{4.60}$$

Proof. (I) Let $u \in H^2(\mathbf{R}^N)$ satisfy the homogeneous equation $-\Delta u - \lambda\mu(x)u = 0$ with $\lambda > 0$. We have only to show that $u \equiv 0$. We are going to multiply both sides of the equation by $2r(\partial_r\overline{u}) + (N-1)\overline{u}$, integrate over B_R, $R > 0$, and take the real part.

(II) Using the identity

$$2\text{Re}\,[(\Delta u)r(\partial_r\overline{u})] = \text{div}\,[2\text{Re}\{r(\partial_r\overline{u})\nabla u\} - |\nabla u|^2 x] + (N-2)|\nabla u|^2 \tag{4.61}$$

(Roach-Zhang [10], (3.4) with $h(r) \equiv 1$) and the divergence theorem, we have

$$\begin{aligned} &2\text{Re}\int_{B_R} (-\Delta u)r(\partial_r\overline{u})\,dx \\ &\qquad = -\int_{B_R} (N-2)|\nabla u|^2\,dx - R\int_{S_R} (2|\partial_r\overline{u}|^2 - |\nabla u|^2)\,dS, \end{aligned} \tag{4.62}$$

where $\partial_r v = \partial v/\partial r$, $r = |x|$. Since it is easy to see that

$$\begin{aligned} &\text{Re}\int_{B_R} (-\Delta u)(N-1)\overline{u}\,dx \\ &\qquad = \int_{B_R} (N-1)|\nabla u|^2\,dx - (N-1)\int_{S_R} \text{Re}\,[(\partial_r u)\overline{u}]\,dS, \end{aligned} \tag{4.63}$$

it follows that

$$\begin{aligned} &\text{Re}\int_{B_R} (-\Delta u)\{2r(\partial_r\overline{u}) + (N-1)\overline{u}\}\,dx \\ &\qquad = \int_{B_R} |\nabla u|^2\,dx - R\int_{S_R} \left(2|\partial_r\overline{u}|^2 - |\nabla u|^2 + \frac{N-1}{R}\text{Re}\,[(\partial_r u)\overline{u}]\right)dS. \end{aligned} \tag{4.64}$$

(III) Suppose that (ii) of Assumption 4.7 holds. By the use of the integration by parts, we have

$$
\begin{aligned}
&2\mathrm{Re} \int_{B_R} (-\lambda\mu u) r(\partial_r \overline{u})\, dx \\
&\quad = -\int_{B_R} (\lambda\mu) r(\partial_r |u|^2)\, dx \\
&\quad = \lambda \int_{B_R} (N\mu + r(\partial_r \mu_1))|u|^2\, dx \\
&\qquad\qquad - \lambda \sum_{\ell\in L} \int_{\partial\Omega_\ell \cap B_R} \mu_0 (x\cdot n^{(\ell)})|u|^2\, dS - \lambda R \int_{S_R} \mu |u|^2\, dS,
\end{aligned}
\tag{4.65}
$$

where we should note that μ_0 does not appear in the first term of the right-hand side since it is constant on each Ω_ℓ, and μ_1 does not appear in the second term of the right-hand side since it is continuous on $\mathbf{R}^N$. Also we should note that

$$
\begin{aligned}
&-\lambda \sum_{\ell\in L} \int_{\partial\Omega_\ell \cap B_R} \mu_0 (x\cdot n^{(\ell)})|u|^2\, dS \\
&\quad = \lambda \sum_{\ell\in L} \int_{S_\ell^{(+)} \cap B_R} (\nu_{\ell+1} - \nu_\ell)(x\cdot n^{(\ell)})|u|^2\, dS \geq 0
\end{aligned}
\tag{4.66}
$$

since the integrand is nonnegative by (2.11). Thus,

$$
\begin{aligned}
&\mathrm{Re} \int_{B_R} (-\lambda\mu u)[2r(\partial_r \overline{u}) + (N-1)\overline{u}]\, dx \\
&\quad = \lambda \int_{B_R} (\mu + r(\partial_r \mu_1))|u|^2\, dx \\
&\qquad + \lambda \sum_{\ell\in L} \int_{S_\ell^{(+)} \cap B_R} (\nu_{\ell+1} - \nu_\ell)(x\cdot n^{(\ell)})|u|^2\, dS - \lambda R \int_{S_R} \mu |u|^2\, dS
\end{aligned}
\tag{4.67}
$$

(IV) It follows from (4.64) and (4.67) that

$$
\begin{aligned}
0 &= \mathrm{Re} \int_{B_R} (-\Delta u - \lambda\mu u)\{2r(\partial_r \overline{u}) + (N-1)\overline{u}\}\, dx \\
&= \int_{B_R} \left(|\nabla u|^2 + \lambda(\mu + r(\partial_r \mu_1))|u|^2\right) dx \\
&\quad + \lambda \sum_{\ell\in L} \int_{S_\ell^{(+)} \cap B_R} (\nu_{\ell+1} - \nu_\ell)(x\cdot n^{(\ell)})|u|^2\, dS \\
&\quad + R \int_{S_R} \left(|\nabla u|^2 - 2|\partial_r \overline{u}|^2 - \frac{N-1}{R} \mathrm{Re}\,[(\partial_r u)\overline{u}] - \lambda\mu |u|^2\right) dS.
\end{aligned}
\tag{4.68}
$$

Since $2|\partial_r \overline{u}|^2 + |\nabla u|^2 + \lambda\mu|u|^2 + (N-1)|\partial_r u||u|$ is integrable on $\mathbf{R}^N$, we see that the third term of the right-hand side goes to 0 as $R \to \infty$ along an appropriate sequence. Therefore it follows from (4.68) that

$$\begin{aligned} 0 = &\int_{\mathbf{R}^N} \left(|\nabla u|^2 + \lambda(\mu + r(\partial_r \mu_1))|u|^2\right) dx \\ &+ \lambda \sum_{\ell \in L} \int_{S_\ell^{(+)}} (\nu_{\ell+1} - \nu_\ell)(x \cdot n^{(\ell)})|u|^2 \, dS. \end{aligned} \tag{4.69}$$

Noting that all the integrands in the right-hand side are nonnegative, we have $\nabla u = 0$ a.e., and hence $u \equiv 0$ since $u \in H^2(\mathbf{R}^N)$.

(V) Suppose that (i) of Assumption 4.7 holds. By using partial integration only for the term containing μ_0, we obtain

$$\begin{aligned} \mathrm{Re} &\int_{B_R} (-\lambda\mu u)\left[2r(\partial_r \overline{u}) + (N-1)\overline{u}\right] dx \\ &= -\lambda \int_{B_R} \left[\mu_0 r(\partial_r |u|^2) + (N-1)\mu_0 |u|^2\right] dx \\ &\qquad - \lambda \int_{B_R} \mu_1 \left[2r\mathrm{Re}(u(\partial_r \overline{u})) + (N-1)|u|^2\right] dx \\ &= \lambda \int_{B_R} \mu_0 |u|^2 \, dx \\ &\qquad - \lambda \sum_{\ell \in L} \int_{\partial\Omega_\ell \cap B_R} \mu_0 (x \cdot n^{(\ell)})|u|^2 \, dS - \lambda R \int_{S_R} \mu_0 |u|^2 \, dS, \\ &\qquad - \lambda \int_{B_R} \mu_1 \left[2r\mathrm{Re}(u(\partial_r \overline{u})) + (N-1)|u|^2\right] dx \end{aligned} \tag{4.70}$$

Let $h(x)$ be a positive function to be specified later. Since we have

$$2|r\mu_1 u(\partial_r \overline{u})| \le |r\mu_1| \left(h|u|^2 + \frac{|\nabla u|^2}{h} \right), \tag{4.71}$$

it follows from (4.70) that

$$\begin{aligned} \mathrm{Re} &\int_{B_R} (-\lambda\mu u)\left[2r(\partial_r \overline{u}) + (N-1)\overline{u}\right] dx \\ &\ge -\lambda \int_{B_R} \frac{|r\mu_1|}{h} |\nabla u|^2 \, dx \\ &\qquad + \lambda \int_{B_R} \left[\mu_0 - (N-1)\mu_1 - h|r\mu_1|\right] |u|^2 \, dx \\ &\qquad - \lambda \sum_{\ell \in L} \int_{\partial\Omega_\ell \cap B_R} \mu_0 (x \cdot n^{(\ell)})|u|^2 \, dS - \lambda R \int_{S_R} \mu_0 |u|^2 \, dS. \end{aligned} \tag{4.72}$$

Thus (4.64) and (4.72) are combined to give

$$
\begin{aligned}
0 &= \mathrm{Re} \int_{B_R} (-\Delta u - \lambda\mu u)\big[2r(\partial_r \overline{u}) + (N-1)\overline{u}\big]\, dx \\
&\geq \int_{B_R} \Big(1 - \frac{\lambda|r\mu_1|}{h}\Big)|\nabla u|^2\, dx \\
&\quad + \lambda \int_{B_R} \big[\mu - N\mu_1 - h|r\mu_1|\big]|u|^2\, dx \\
&\quad - \lambda \sum_{\ell\in L} \int_{\partial\Omega_\ell \cap B_R} \mu_0 (x\cdot n^{(\ell)})|u|^2\, dS - \lambda R \int_{S_R} \mu_0 |u|^2\, dS \\
&\quad - R \int_{S_R} (2|\partial_r \overline{u}|^2 - |\nabla u|^2 + \frac{N-1}{R}\mathrm{Re}\,[(\partial_r u)\overline{u}])\, dS.
\end{aligned} \tag{4.73}
$$

Then, letting $R \to \infty$ along an appropriate sequence in (4.73), we obtain

$$
\begin{aligned}
0 &\geq \int_{\mathbf{R}^N} \Big(1 - \frac{\lambda|r\mu_1|}{h}\Big)|\nabla u|^2\, dx \\
&\quad + \lambda \int_{\mathbf{R}^N} \big[\mu - N\mu_1 - h|r\mu_1|\big]|u|^2\, dx,
\end{aligned} \tag{4.74}
$$

where we have used (4.66), too. Let $\lambda \in [0, \lambda_0)$ be an eigenvalue of H with its eigenfunction u. Set $\eta = \lambda/\lambda_0 \in [0, 1)$ and

$$
h(x) = \begin{cases} 1 & (\text{if } \mu_1(x) = 0), \\ \lambda_0 |r\mu_1(x)| & (\text{if } \mu_1(x) \neq 0). \end{cases} \tag{4.75}
$$

Then we have

$$
1 - \frac{\lambda|r\mu_1(x)|}{h(x)} = \begin{cases} 1 & (\text{if } \mu_1(x) = 0), \\ 1 - \eta & (\text{if } \mu_1(x) \neq 0), \end{cases} \tag{4.76}
$$

and hence, by using (4.58)

$$
\mu(x) - N\mu_1(x) - h|r\mu_1(x)| = \begin{cases} \mu_0 > 0 & (\text{if } \mu_1(x) = 0), \\ \mu(x) - N\mu_1(x) - \lambda_0 (r\mu_1(x))^2 \geq 0 & \\ & (\text{if } \mu_1(x) \neq 0). \end{cases} \tag{4.77}
$$

Therefore, we have from (4.74)

$$
0 \geq (1-\eta) \int_{\mathbf{R}^N} |\nabla u|^2\, dx, \tag{4.78}
$$

i.e., $\nabla u \equiv 0$ or u is identically zero almost everywhere. This completes the proof. □

5. The limiting absorption principle for H

Throughout this section we assume that δ satisfies

$$\frac{1}{2} < \delta \leq \frac{1}{2} + \frac{\epsilon}{4}, \tag{5.1}$$

where ϵ is as in (2.13) or (2.14). Let $u \in X$ be given by

$$\begin{cases} u = R(z)f, \\ z = \lambda + i\eta \qquad\qquad (\lambda \geq 0,\ \eta \neq 0), \\ f \in L_{2,\delta}(\mathbf{R}^N), \end{cases} \tag{5.2}$$

where $R(z) = (H - z)^{-1}$. Then u satisfies the inhomogeneous equation $(-\mu^{-1}\Delta - z)u = f$ which is equivalent to

$$(-\mu_0^{-1}\Delta - z)u = g \qquad (g = \mu_0^{-1}(\mu f + z\mu_1 u)) \tag{5.3}$$

with $k = \sqrt{z\mu_0}$. Let μ_1 be short-range. Then, since

$$u \in L_{2,-\delta}(\mathbf{R}^N) \Longrightarrow \begin{cases} \mu_1 u \in L_{2,\delta}(\mathbf{R}^N), \\ \|\mu_1 u\|_\delta \leq c_1\|u\|_{-\delta}, \end{cases} \tag{5.4}$$

we see that $g \in L_{2,-\delta}(\mathbf{R}^N)$. In the case that μ_1 is long-range, the inequality

$$|\eta|\|u\|_{1,0}^2 \leq C(|f|, |u|)_0 \tag{5.5}$$

will be useful, where $C = C(\mu)$, $\|\ \|_{1,0}$ is the norm of $H^1(\mathbf{R}^N)$, and $(\ ,\ )_0$ is the inner product of $L_2(\mathbf{R}^N)$. For the proof of (5.5), see, e.g., Eidus [6], [13], Lemma 2.1. Then, by a direct application of Theorem 3.5 to our case, we can evaluate the radiation condition term $\mathcal{D}u$.

Theorem 5.1. *Suppose that* Assumptions 2.1 *and* 2.2 *hold. Let* δ *be as in* (5.1). *Let* $0 < c < d < \infty$ *and let* $J_\pm(c,d)$ *be as in* (3.18). *Let* u *be given by* (5.2) *with* $z \in J_+(c,d) \cup J_-(c,d)$. *Then there exists a positive constant* $C = C(\delta, c, d, m_0, M_0)$ *such that*

$$\|\mathcal{D}u\|_{\delta-1} \leq C\big(\|f\|_\delta + \|u\|_{-\delta}\big) \tag{5.6}$$

for $N \geq 3$, *and*

$$\|\mathcal{D}u\|_{\delta-1,*} \leq C\big(\|f\|_\delta + \|u\|_{-\delta}\big) \tag{5.7}$$

for $N = 2$, *where* $\|\ \|_{t,*}$ *is as in* (3.21).

Proof. We can proceed as in the proof of Theorem 3.5. Since f in the proof of Theorem 3.5 should be replaced by $g = \mu_0^{-1}(\mu f + z\mu_1 u)$ (see (5.3)), our additional task is to prove, for any $e > 0$,

$$I = \mathrm{Re}\int_{B_{rR}} z\varphi\mu_1(x)u\overline{\mathcal{D}_r u}\,dx \le C(\|f\|_\delta^2 + \|u\|_{-\delta}^2 + e\|\mathcal{D}u\|_{\delta-1}^2) \tag{5.8}$$

with $C = C(e, \delta, c, d, m_0, M_0)$, where φ is as in the proof of Theorem 3.5, and $\|\mathcal{D}u\|_{\delta-1}$ in (5.8) should read $\|\mathcal{D}u\|_{\delta-1,*}$ if $N = 2$. Suppose that μ_1 is short-range. Then (5.8) follows directly from (5.4). Suppose that μ_1 is long-range. Then we have from the definition of $\mathcal{D}_r u$ ((6) of Notation 3.1) and partial integration

$$\begin{aligned} I &= \lambda\mathrm{Re}\int_{B_{rR}} \varphi\mu_1(x)u\overline{\mathcal{D}_r u}\,dx - \eta\mathrm{Im}\int_{B_{rR}} \varphi\mu_1(x)u\overline{\mathcal{D}_r u}\,dx \\ &= -\frac{\lambda}{2}\int_{B_{rR}} \partial_r(\varphi\mu_1)|u|^2\,dx + \lambda\int_{B_{rR}} b\varphi\mu_1|u|^2\,dx \\ &\qquad - \eta\mathrm{Im}\int_{B_{rR}} \varphi\mu_1(x)u\overline{\mathcal{D}_r u}\,dx + I_4(r) + I_5(R) \\ &= I_1 + I_2 + I_3 + I_4(r) + I_5(R), \end{aligned} \tag{5.9}$$

where the terms $I_4(r)$ and $I_5(R)$ tend to zero as $r \to 0$ and $R \to \infty$ along appropriate sequences, respectively. It follows from (5.1) and the definition of φ ((3.22) or (3.23)) that $\varphi\mu_1$ is bounded on $\mathbf{R}^N$, and hence I_2 and I_3 can be evaluated by using (5.5). On the other hand, since

$$\partial_r(\varphi\mu_1) = O((1+r)^{2\delta-2-\epsilon}) = O((1+r)^{-2\delta}), \tag{5.10}$$

the term I_1 is evaluated by $\|u\|_{-\delta}^2$, which completes the proof. □

As in §4, let $\sigma_p(H)$ be the set of all eigenvalues of H which is a discrete set in $(0, \infty)$ (Theorem 4.6). Let $\lambda > 0$ such that $\lambda \notin \sigma_p(H)$. Let $u \in H^2(\mathbf{R}^N)_{\mathrm{loc}} \cap L_{2,-\delta}(\mathbf{R}^N)$ be a solution of the homogeneous equation $-\mu(x)^{-1}\Delta u - \lambda u = 0$ with the radiation condition $\|\mathcal{D}^{(+)}u\|_{\delta-1,E_1} < \infty$ or $\|\mathcal{D}^{(-)}u\|_{\delta-1,E_1} < \infty$. Then it follows from Theorem 4.5 that $u \in L_{2,\delta_0}(\mathbf{R}^N)$ where δ_0 is given by (4.43) or (4.44). Since λ is supposed not to be an eigenvalue, we have $u \equiv 0$. Therefore we can prove the limiting absorption principle for $\lambda \in (0,\infty)\backslash\sigma_p(H)$ by starting with Theorem 5.1, proceeding as in §5 ~ §7 of [9]. Let $D_\pm \subset \mathbf{C}$ be given by (3.29). For $\lambda > 0$, let

$$R_\pm(\lambda) = \lim_{\eta\downarrow 0} R(\lambda \pm i\eta), \tag{5.11}$$

and extend the resolvent $R(z)$ on $D_\pm$ by

$$R(\lambda + i\eta) = \begin{cases} R(\lambda + i\eta) & (\lambda > 0,\ \eta > 0), \\ R_+(\lambda) & (\lambda > 0,\ \eta = 0) \end{cases} \tag{5.12}$$

for $z \in D_+$ and

$$R(\lambda + i\eta) = \begin{cases} R(\lambda + i\eta) & (\lambda > 0,\ \eta < 0), \\ R_-(\lambda) & (\lambda > 0,\ \eta = 0) \end{cases} \tag{5.13}$$

for $z \in D_-$. Then we have

Theorem 5.2. *Suppose that* Assumptions 2.1 *and* 2.2 *holds. Let* δ *satisfy* (5.1).

(i) *Then the limits* (5.11) *is well-defined in* $\mathbf{B}(L_{2,\delta}(\mathbf{R}^N),\ H^2_{-\delta}(\mathbf{R}^N))$ *for* $\lambda \in (0,\infty)\backslash\sigma_p(H)$, *and the extended resolvent* $R(z)$ *is a* $\mathbf{B}(L_{2,\delta}(\mathbf{R}^N),\ H^2_{-\delta}(\mathbf{R}^N))$*-valued continuous function on each of* $D_+\backslash\sigma_p(H)$ *and* $D_-\backslash\sigma_p(H)$.

(ii) *For any* $z \in D_+\backslash\sigma_p(H)$ [*or* $D_-\backslash\sigma_p(H)$], $R(z)$ *is a compact operator from* $L_{2,\delta}(\mathbf{R}^N)$ *into* $H^1_{-\delta}(\mathbf{R}^N)$.

(iii) *The selfadjoint operator* H *is absolutely continuous on the interval* $[c, d]$ *such that* $0 < c < d < \infty$ *and*

$$[c,\ d] \cap \sigma_p(H) = \emptyset. \tag{5.14}$$

The operator H *has no singular continuous spectrum.*

(iv) *For* $0 < c < d < \infty$ *satisfying* (5.14) *there exists* $C = C(c, d, \delta, m_0, M_0) > 0$ *such that, for* $z \in \overline{J}_+(c,\ d) \cup \overline{J}_-(c,\ d)$,

$$\begin{cases} \displaystyle\int_{E_s} (1+r)^{-2\delta}\left(|\nabla R(z)f|^2 + |k|^2|R(z)f|^2\right) dx \\ \qquad\qquad \le C^2(1+s)^{-(2\delta-1)}\|f\|^2_\delta (s \ge 1,\ f \in L_{2,\delta}(\mathbf{R}^N)), \\ \|\mathcal{D}R(z)f\|_{\delta-1} \le C\|f\|_\delta \quad (f \in L_{2,\delta}(\mathbf{R}^N)), \end{cases} \tag{5.15}$$

where, for $\lambda \in D_+ \cap (0,\ \infty)$ [*or* $D_- \cap (0,\ \infty)$], $\mathcal{D}u$ *should be interpreted as* $\mathcal{D}^{(+)}$ [*or* $\mathcal{D}^{(-)}$], *and* $\overline{J}_\pm(c,\ d)$ *are given by* (3.30).

References

[1] S. Agmon: *Spectral properties of Schrödinger operators and scattering theory*, Ann. Scuola Sup. Pisa **2** (1975), 151–218.

[2] M. Ben-Artzi, Y. Dermanjian and J.-C. Guillot: *Acoustic waves in perturbed stratified fluids: a spectral theory*, Commun. Partial Differential Equations **14** (1989), 479–517.

[3] A. Boutet de Monvel-Berthier and D, Manda: *Spectral and scattering theory for wave propagation in perturbed stratified media*, Universität Bielefeld, BiBos. Preprint, Nr. 606/11/93.

[4] S. DeBiévre and D. W. Pravica: *Spectral analysis for optical fibres and stratified fluids* I: *The limiting absorption principle*, J. Functional Analysis **98** (1991), 406–436.

[5] S. DeBiévre and D. W. Pravica: *Spectral analysis for optical fibres and stratified fluids* II: *Absence of eigenvalues*, Commun. Partial Differential Equations **17** (1992), 69–97.

[6] D. Eidus: *The limiting absorption and amplitude problems for the diffraction problem with two unbounded media*, Comm. Math. Phys. **107** (1986), 29–38.

[7] T. Ikebe and Y. Saitō: *Limiting absorption method and absolute continuity for the Schrödinger operator*, J. Math. Kyoto Univ. **12** (1972), 513–542.

[8] W. Jäger and Y. Saitō: *The limiting absorption principle for the reduced wave operator with cylindrical discontinuity*. Seminar Notes. IWR (SFB359) 94–74, University of Heidelberg. 1994.

[9] W. Jäger and Y. Saitō: *On the Spectrum of the reduced wave operator with cylindrical discontinuity*, Forum Mathematicum **9** (1997), 29–60.

[10] G. Roach and B. Zhang: *On Sommerfeld radiation conditions for the diffraction problem with two unbounded media*, Proc. Royal Soc. Edinburgh **121A** (1992), 149–161.

[11] Y. Saitō: *The principle of limiting absorption for second-order differential equations with operator-valued coefficients*, Publ. RIMS, Kyoto Univ. **7** (1972), 581–619.

[12] Y. Saitō: *The principle of limiting absorption for the non-selfadjoint Schrödinger operators in* $\mathbf{R}^N$ $(N \neq 2)$, Publ. RIMS, Kyoto Univ. **9** (1974), 397–428.

[13] Y. Saitō: *A remark on the limiting absorption principle for the reduced wave equation with two unbounded media,* Pacific J. Math. **136** (1989), 183–208.

[14] R. Weder: *Absence of eigenvalues of the acoustic propagators in deformed waveguides*, Rocky Mountain J. Math. **18** (1988), 495–503.

[15] R. Weder, Spectral and Scattering Theory for Wave Propagation in Perturbed Stratified Media, Springer-Verlaga, Berlin, 1991.

[16] C. Wilcox, Sound Propagation in Stratified Fluids, Springer-Verlag, New York, 1984.

[17] B. Zhang: *On radiation conditions for acoustic propagators in perturbed stratified fluids.* Preprint. 1994.

W. Jäger
Department of Mathematics
Heidelberg University, Garmany

Y. Saitō
Department of Mathematics
University of Alabama at Birmingham
Birmingham, Alabama 35294, U.S.A.
e-mail: saito@math.uab.edu