General Equilibrium Model and Set Theoretic Finiteness

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Abstract

The axiom of infinity in the ZF set theory is usually considered essential for Brouwer's fixed point theorem and the proof of the existence of economic equilibria. In this paper, the meanings of the infinity axiom are reconsidered from the view point of the minimal requirements for the general equilibrium theory. We provide several tools and conditions under which a set-theoretically finite setting is sufficient, at least for an elementary equilibrium existence argument on the static general equilibrium model.

In view of the social science, the purpose of general equilibrium theory is to describe the whole structure of the human society based on the price and the market mechanism. However, there exists a classical problematic feature relating to the *objectivity*, i.e., the *introspection* problem for us to describe the world including ourselves, which also brings about important questions about the true sense of our individual *rationality*.

In order to consider about the introspective feature of the social science and the human rationality under an economic model, there may exist two approaches. One is to attempt to describe our *true* rationality into the model, so that we may treat directly the whole society as the totality of such rational individuals. The other approach is giving up to define the true sense of rationality (a general intelligence of human) and, instead, to seek a *minimal requirement* for our rationality to construct and recognize the world described by a certain economic model. In this paper, we base our argument on the second approach, and show that the static general equilibrium theory such as Debreu (1959), could essentially be treated under a finitistic mathematical argument.

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1 Introduction

In the view of social science, the purpose of general equilibrium theory is to describe the whole structure of human society based on price and market mechanism. However, there exists a classical problematic feature relating to *objectivity* (e.g. Weber 1904), i.e. the *introspection* problem for us to describe the world including ourselves, which also brings about important questions about the true sense of our individual *rationality*.

To consider the introspective feature of social science and human rationality under an economic model, there exist the following two approaches. One is to attempt to describe our *true* rationality in the model, so that we may directly treat the whole society as the totality of such rational individuals. In such a case, we always have to point out the *incompleteness* of such a description of our *real* society and rationality.[1] The other approach is giving up defining the true sense of rationality (a general intelligence of humans) and, instead, to seek a *minimal requirement* for our rationality to construct and recognize the world described by a certain economic model. For the latter attempt, the implication, the soundness, the consistency, etc. of the model would be evaluated under our *rationality*, so the totality of the theory describing a certain model is important. Such a soundness is necessary for the theory to be plausible for agents described in the theory from an introspective viewpoint.

Under the second approach, we have to admit that an economic theory is always related to certain kinds of value judgments, especially epistemologic and/or methodological ones, which will construct the fundamental relation between *economics and ethics*, and also provide the meanings, roles, and limits of economics.

In this paper, we base our argument on the second approach, and show that the static general equilibrium theory as in Debreu (1959) could essentially be treated under a finitistic mathematical argument. From the ZF set-theoretic viewpoint, the theory without using the infinity axiom is complete (e.g. Urai 2010), so our argument gives an affirmative answer to the question of whether the model of static general equilibrium, even when we allow each agent of it to have introspective views of the world to which they belong, is possibly complete as a model of the society, at least from the ZF meta-theoretic viewpoint.

2 Sperner's Lemma

Begle (1950a, 1950b) gives an algebraic argument on abstract simplices and barycentric subdivision, as well as the Vietoris Mapping Theorem and the fixed-point theory. These arguments could be utilized to give a certain kind of purely algebraic development for the abstract reconstruction of the general equilibrium theory.

In this paper, we do not use such a Čech-homological approach since our main concern in the following is restricted on the *finiteness* in the sense that the infinity axiom of the ZF theory of sets is not necessary for the development of a theory. Our argument is an extension of the setting in Nikaido (1968) for simplices of order 1 to the cases with a general complex and simplices of order ν. The

[1] This is a similar result to the Gödel-type incompleteness theorem. We usually have to admit that such a description of individual rationality must be introspectively incomplete as long as it is consistent. See Urai (2010, Chapter 9).

purpose of this section is as follows.

> We show that the concept of barycentric subdivision and Sperner's lemma is possible to be proved without using any concepts based on the infinity axiom.

2.1 Abstract Simplex

Let $x^1, x^2, \ldots, x^{n+1}$ be $n+1$ points. For example, if we regard that $x^1, x^2, \ldots, x^{n+1}$ are the points of R^ℓ ($\ell \geqq n$), $\overline{x^0 x^1 \cdots x^n}$ will construct n*-dimensional simplex*. However, we do not take such a concrete way. In the abstract sense, we consider that these points are in general position and construct *abstract* n*-simplex*, which are named $x^1 x^2 \cdots x^{n+1}$. Since these points are associated with fixed numbers, they are oriented. When we emphasize the orientation, we express these points as $\langle x^1, x^2, \ldots, x^{n+1} \rangle$.

2.2 Simplex and Vertex

Let $\boldsymbol{K}$ be the oriented complex that consists of all subsimplices of abstract simplex $x^1 x^2 \cdots x^{n+1}$.

(Definition 1: Increasing Sequence)

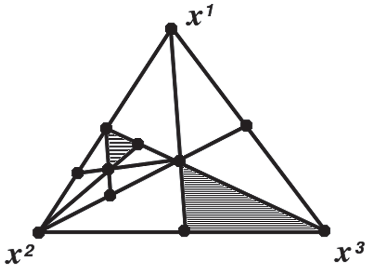
The m*-increasing sequence* ($1 \leqq m \leqq n+1$), which is made of $n+1$ points, $y^1, y^2, \ldots, y^{n+1}$, is the sequence, $V^1, \ldots, V^m$, of m non-empty subsets of $\{y^1, y^2, \ldots, y^{n+1}\}$, such that V^k is a proper subset of V^{k+1} ($1 \leqq k \leqq m-1$).

(Definition 2: Simplex and Vertices of order 0)

Abstract simplex $x^1 x^2 \cdots x^{n+1}$ is the n*-simplex of order* 0, and we call each of $x^1, x^2, \ldots, x^{n+1}$ as the *vertex of order* 0. In addition, we call an abstract simplex, $x^{i_1} x^{i_2} \cdots x^{i_{m+1}}$, which consists of $m+1$ vertices of order 0 ($0 \leqq m \leqq n$), $x^{i_1}, x^{i_2}, \ldots, x^{i_{m+1}}$, an m*-simplex of order* 0.

(Definition 3: Simplices and Vertices of order ν)

Let ν be greater than or equal to 1. We consider the $(n+1)$-increasing sequence $V_\nu^1, V_\nu^2, \ldots, V_\nu^{n+1}$, which consists of $n+1$ vertices of order $\nu - 1$, $y^0, y^1, \ldots, y^n$, constructing an n-simplex of order $\nu - 1$. We call $V_\nu^1, V_\nu^2, \ldots, V_\nu^{n+1}$ as *vertices of order* ν, and the abstract simplex $V_\nu^1 V_\nu^2 \cdots V_\nu^{n+1}$ as an n*-simplex of order* ν.



Lower-right shaded portion, 2-simplex of order 1, $V_1^1 V_1^2 V_1^3$
$= \{x^3\}\{x^2, x^3\}\{x^1, x^2, x^3\}$
Lower-left shaded portion, 2-simplex of order 2, $V_2^1 V_2^2 V_2^3$
$= \{\{x^1, x^2\}\}\{\{x^1, x^2\}, \{x^1, x^2, x^3\}\}\{\{x^1, x^2\}, \{x^1, x^2, x^3\}, \{x^2\}\}$

Figure 1: Simplices and Vertices of order ν

An m-simplex of order ν is an abstract *subsimplex* of an n-simplex of order ν. When we follow the above definitions, each number of elements of V_ν^1, V_ν^2, …, V_ν^{n+1} becomes $1,2,\ldots,n+1$. V_ν^m, the set of m vertices of order $\nu-1$, is not unique. If we fix one n-simplex of order $\nu-1$ and focus on the sets that are made from $n+1$ vertices of order $\nu-1$, the number of such increasing sequences is exactly the same with the number of permutations of $n+1$ vertices of order $\nu-1$. Then, the n -simplex of order $\nu-1$ could induce $(n+1)!$ kinds of n-simplex of order ν. Hence, with order ν, there exist $((n+1)!)^\nu$ kinds of n-simplices.

Consider a sufficiently large natural number $\bar{\nu}$, and the above definitions are made from 1 to $\nu \leqq \bar{\nu}$. In the following, we fix $\bar{\nu}$, and the above inductive definitions merely imply that we omit the descriptions of order 0, 1, …, ν. That is, we do not need the Peano axioms.

All vertices of order $0,\cdots,\bar{\nu}$ that appear in the above are only finite, and *as long as order ν and order ν' are different*, each of V_ν^m and $V_{\nu'}^m$ belongs to a different rank of sets (since they can become non-empty sets). Thus, V_ν^m and $V_{\nu'}^m$ are completely different, and we can easily make all of the sets totally-ordered. Especially for y^m, the vertices of order $\nu-1$, we could always define $\{y^m\}$, the vertex of order ν. Then, we could consider the mapping, $\iota_{\nu-1}$, from $\boldsymbol{V}_{\nu-1}$ (the set of vertices of order $\nu-1$) to $\boldsymbol{V}_\nu$ (the set of vertices of order ν).

(1) $$\iota_{\nu-1}\colon\ \boldsymbol{V}_{\nu-1} \ni y^m \mapsto \{y^m\} \in \boldsymbol{V}_\nu \quad (\nu = 1,2,\ldots,\bar{\nu}).$$

Based on this mapping, we could say that $\boldsymbol{V}_0 \subset \boldsymbol{V}_1 \subset \cdots \subset \boldsymbol{V}_{\bar{\nu}}$ and the above total ordering can be considered as typically given on $\boldsymbol{V}_{\bar{\nu}}$.

2.3 Oriented Complex

Let us define $\boldsymbol{K}^0=\boldsymbol{K}$. For each $\nu=1,2,\ldots,\bar{\nu}$, assume that the above total ordering is defined so that the identification $\boldsymbol{V}_\nu \subset \boldsymbol{V}_{\bar{\nu}}$ is possible. We define $\boldsymbol{K}^\nu$ as follows.

(2) $\boldsymbol{K}^\nu=\{\sigma_\nu^m \mid 0 \leqq m \leqq n+1,\ \sigma_\nu^m$ is an abstract m-subsimplex of an n-simplex of order ν $\}$.

It is easy to verify that $\boldsymbol{K}^\nu$ is an oriented complex with the set of vertices, $\boldsymbol{V}_\nu$.[2]

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It is easy to verify that $\boldsymbol{K}^\nu$ is an oriented complex with the set of vertices, $\boldsymbol{V}_\nu$.[3]

2.4 Carrier

Complex $\boldsymbol{K}^\nu$ represents *order ν barycentric subdivision of $\boldsymbol{K}=\boldsymbol{K}^0$*. The 0-simplex (vertices of order ν, $\sigma_\nu^0 \in \boldsymbol{V}_\nu$) is the set having ν-fold parentheses. We write the set of all subsets of $\boldsymbol{V}=\boldsymbol{V}_0$ $=\{x^1,\ldots,x^{n+1}\}$ as $\mathcal{P}(\boldsymbol{V})$ and the set of all subsets of $\mathcal{P}(\boldsymbol{V})$as $\mathcal{P}^2(\boldsymbol{V})$. Thereafter, we could make

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[2] See Urai and Murakami (2014).
[3] See Urai and Murakami (2014).

$\mathfrak{P}^3(\boldsymbol{V})$, $\mathfrak{P}^4(\boldsymbol{V})$, ..., and we have $\sigma_\nu^0 \in \mathfrak{P}^\nu(\boldsymbol{V})$. Let us define the symbol $\in^2$ as the relation "is an element of an element of" a set, and the symbol $\in^3$ as the relation "is an element of an element of an element of" a set, ... Then, we have

(4) $$\{x|\, x \in^\nu \sigma_\nu^0\} \subset \mathrm{V} = \mathrm{V}_0 = \{x^1, \ldots, x^{n+1}\}$$

We could identify $\in^1$ as $\in$ and $\in^0$ as $=$. The set $\{x|\, x \in^\nu \sigma_\nu^0\}$ is called the *carrier* of σ_ν^0 and is denoted by $C(\sigma_\nu^0)$. (For any $\nu \geqq 0$, $C(\sigma_\nu^0)$ is a subset of $\boldsymbol{V}=\boldsymbol{V}_0=\{x^1, \ldots, x^{n+1}\}$.) In general, for k -simplex of order ν, i.e. $\sigma_\nu^k = y_\nu^1 y_\nu^2 \cdots y_\nu^k$,

(5) $$C(\sigma_\nu^k) = \bigcup_{i=1}^{k} C(y_\nu^i)$$

is called the carrier of σ_ν^k. ($C(\sigma_\nu^k)$ is the union of carriers of vertices of 0-simplex of order ν constructing σ_ν^k.) Now, we can prove the next theorem, which is a complete finitistic algebraic version of Lemma 4.2 in Nikaido (1968, p.57). Note that in this theorem, we do not use any topological notion like the relative boundary of the original simplex.

Theorem 1: For σ_ν^{n-1}, $(n-1)$-simplex of $\boldsymbol{K}^\nu$ ($1 \leqq \nu \leqq \bar{\nu}$), we have either of the following conditions.

(i) When $C(\sigma_\nu^{n-1}) \neq \boldsymbol{V}$, σ_ν^{n-1} is a unique subsimplex of an n-dimensional simplex of order ν.

(ii) Otherwise, σ_ν^{n-1} is a common-subsimplex of exactly two n-dimensional simplices of order ν.

Proof:

(I) When $\nu = 1$, σ_1^{n-1} is a subsimplex of a certain σ_1^n, and σ_1^n is, by definition, an $(n+1)$ -increasing sequence of vertices of order 0. Hence, σ_1^{n-1} is an n-increasing sequence constructed by order 0vertices, $x^1, x^2, \ldots, x^{n+1}$.

Case (i): Since to construct an n-increasing sequence, n different vertices are necessary, the vertex that does not belong to $C(\sigma_1^{n-1})$ is unique. Hence, the method for constructing an $(n+1)$-increasing sequence from the n-increasing sequence is also unique, i.e. to add $\{x^1, x^2, \ldots, x^{n+1}\}$ at the end of the n-increasing sequence; so σ_1^{n-1} is a subsimplex of the unique n-simplex represented by the unique $(n+1)$-increasing sequence.

Case (ii): Since $C(\sigma_1^{n-1})$ contains all order 0 vertices, the last entry, $\boldsymbol{V}_0^n$, of the n-increasing sequence, $V_0^1 \cdots V_0^n$, representing σ_1^{n-1}, is necessarily equal to $\{x^1, x^2, \ldots, x^{n+1}\}$. Since this set has $n+1$ elements, there exists $k \in \{0, 1, \ldots, n\}$ such that the number of elements of V_0^{k+1} minus the number of elements of V_0^k, where $V_0^0 = \emptyset$, is 2. Then, for such an n-increasing sequence, the number of methods for constructing an $(n+1)$-increasing sequence from it is two, i.e. to insert $V_0^k \cup \{x^i\}$ or $V_0^k \cup \{x^j\}$ in the place between V_0^k and V_0^{k+1}, where $\{x^i, x^j\} = V_0^{k+1} \setminus V_0^k$. Therefore, σ_1^{n-1} is exactly a common subsimplex of these two n-simplices of order 1.

(II) Let us consider the case with $\nu \geqq 2$. We assume that the assertion of the theorem holds for $\nu - 1$, and we show for the case with ν. First, note that since every σ_ν^m can be identified with an $(m+1)$-increasing sequence, $V_{\nu-1}^1, \cdots, V_{\nu-1}^{m+1}$, of order $\nu - 1$ vertices constructing an n-simplex of order $\nu-1$, the last entry of such a sequence, $V_{\nu-1}^{m+1}$, must contain at least $m+1$ different order $\nu-1$ vertices constructing an n-simplex of order ν. Hence, there exists at least one abstract m-simplex of order $\nu-1$, $\sigma_{\nu-1}^{m}$, whose carrier is a subset of $C(\sigma_{\nu}^{m})$. Then again, since $\sigma_{\nu-1}^{m}$ can be identified with an $(m+1)$-increasing sequence, $V_{\nu-2}^{1}\cdots, V_{\nu-2}^{m+1}$, of order $\nu-2$ vertices constructing an n-simplex of order $\nu-2$, the last entry of such a sequence, $V_{\nu-2}^{m+1}$, must contain at least $m+1$ different order $\nu-2$ vertices constructing an n-simplex of order $\nu-1$. Thus, there exists at least one abstract m-simplex of order $\nu-2$, $\sigma_{\nu-2}^{m}$, whose carrier is a subset of $C(\sigma_{\nu}^{m})$. We can repeat such a process for ν times and obtain a sequence of m-simplices, $\sigma_{\nu-1}^{m}$, $\sigma_{\nu-2}^{m}$, …, σ_{0}^{m} such that $C(\sigma_{\nu}^{m})\supset C(\sigma_{\nu-1}^{m})\supset C(\sigma_{\nu-2}^{m})\supset\cdots\supset C(\sigma_{0}^{m})$. We call such a sequence a *resolution* of σ_{ν}^{m} to simplices (and vertices) of orders $\nu-1,\nu-2,\cdots,0$. The resolution of σ_{ν}^{m} may not be unique but always exists. It follows, in particular, that the number of elements of the carrier of σ_{ν}^{m} must be greater than or equal to the number of elements of $C(\sigma_{0}^{m})$, $m+1$.

Case (i): Consider a resolution of σ_{ν}^{n-1}, $\sigma_{\nu-1}^{n-1}=V_{\nu-1}^{1}\cdots V_{\nu-1}^{n}$, $\sigma_{\nu-2}^{n-1}=V_{\nu-2}^{1}\cdots V_{\nu-2}^{n}$, …, $\sigma_{1}^{n-1}=V_{1}^{1}\cdots V_{1}^{n}$, $\sigma_{0}^{n-1}=x^{i_1}\cdots x^{i_n}$. Since $C(\sigma_{\nu}^{n-1})\neq\{x^1,x^2,\ldots,x^{n+1}\}$ and the number of elements of $C(\sigma_{\nu}^{n-1})$ must be greater than or equal to n, the vertex of order 0 that does not belong to $C(\sigma_{\nu}^{n-1})$ is unique. Let x^i be such a unique vertex of order 0. Then, σ_{0}^{n-1} is $x^1\cdots x^{i-1}x^{i+1}\cdots x^{n+1}$, so the resolution of σ_{ν}^{n-1} of order 0 is uniquely determined. Moreover, for the n-increasing sequence of order 0 vertices, V_1^1, …, V_1^n, the possibility to extend it to the $(n+1)$-increasing sequence by adding V_1^{n+1} is unique since in that case, V_1^{n+1} must be equal to $\{x^1,\cdots,x^{i-1},x^i,x^{i+1},\cdots,x^{n+1}\}$. Thus, the resolution of σ_{ν}^{n-1} of order 1 is also uniquely determined as $V_1^1V_1^2,\ldots,V_1^nV_1^{n+1}$. We have shown that for any resolution of σ_{ν}^{n-1}, $\sigma_{\nu-1}^{n-1}=V_{\nu-1}^{1}\cdots V_{\nu-1}^{n}$, $\sigma_{\nu-2}^{n-1}=V_{\nu-2}^{1}\cdots V_{\nu-2}^{n}$, …, $\sigma_{1}^{n-1}=V_{1}^{1}\cdots V_{1}^{n}$, $\sigma_{0}^{n-1}=x^{i_1}\cdots x^{i_n}$, the possibility of the last two entries together with V_1^{n+1} is unique. Note that the resolution of σ_{ν}^{n-1} is constructed such that given an $(n-1)$-simplex of order k, $\sigma_{k}^{n-1}=V_k^1\cdots V_k^n$, the order $k-1$ resolution, σ_{k-1}^{n-1}, is an abstract $(n-1)$-subsimplex of a certain n-simplex of order $k-1$ having the form, $V_{k-1}^{1}\cdots V_{k-1}^{n}V_{k-1}^{n+1}$. Therefore, if the possibility of such V_{k-1}^{n+1} is unique, and if $C(\sigma_{k-1}^{n-1})\neq \mathrm{V}$, since the carrier of an n-simplex is always equal to $\boldsymbol{V}$, the possibility of $\sigma_{k}^{n-1}=V_k^1\cdots V_k^{n-1}$ together with $V_k^{n+1}=\{V_{k-1}^1,\cdots,V_{k-1}^n,V_{k-1}^{n+1}\}$ in a certain resolution of σ_{ν}^{n-1} is also unique. By repeating the same argument, it follows that the resolution of $\sigma_{\nu}^{n-1}=V_\nu^1\cdots V_\nu^n$ and the possibility of the vertex, $V_\nu^{n+1}=\{V_{\nu-1}^1,\cdots,V_{\nu-1}^n,V_{\nu-1}^{n+1}\}$, under which σ_{ν}^{n-1} is an abstract subsimplex of $V_\nu^1\cdots V_\nu^nV_\nu^{n+1}$, is unique.

Case (ii): We may represent σ_{ν}^{n-1} as an abstract n-subsimplex $V_\nu^1V_\nu^2\cdots V_\nu^n$, where $V_\nu^1\subset V_\nu^2\subset\cdots\subset V_\nu^n$ is an n-increasing sequence of order $\nu-1$ vertices, $y^1,y^2,\ldots,y^{n+1}$, which form a certain n-simplex of order $\nu-1$. Let us consider the next two cases.

(ii-1): When the last entry, V_ν^n, is equal to the set, $\{y^1,y^2,\ldots,y^{n+1}\}$. In this case, $y^1y^2\cdots y^{n+1}$ is the only n-simplex of order $\nu-1$ under which σ_{ν}^{n-1} is obtained as an abstract $(n-1)$-subsimplex of a certain n-simplex of order ν since this is the only n-simplex of order $\nu-1$ whose set of vertices includes all $V_\nu^1,\ldots,V_\nu^n$. Hence, the argument for case (ii) in (I) would be sufficient to confirm that there are exactly two n-simplices of order ν satisfying that σ_{ν}^{n-1} is an abstract $(n-1)$-subsimplex of it.

(ii-2): When the last entry, V_ν^n, is not equal to the set, $\{y^1, y^2, \ldots, y^{n+1}\}$. In this case, since $V_\nu^1 \subset \cdots \subset V_\nu^n$ is an n-increasing sequence, among the order $\nu - 1$ vertices, $y^1, y^2, \ldots, y^{n+1}$, there exists exactly one vertex that does not belong to V_ν^n. Without loss of generality, we can rename the unique vertex as y^{n+1}. Hence, we have $\sigma_\nu^{n-1} = V_\nu^1 V_\nu^2 \cdots V_\nu^n$ and $V_\nu^n = \{y^1, y^2, \ldots, y^n\}$. Then, $\sigma_{\nu-1}^{n-1} = y^1 y^2 \cdots y^n$ is an abstract n-subsimplex of order $\nu - 1$ such that $C(\sigma_{\nu-1}^{n-1}) = C(\sigma_\nu^{n-1})$, where the last equation follows from the fact that $V_\nu^n = \{y^1, y^2, \ldots, y^n\}$. Therefore, from our assumption that the assertion of the theorem holds for $\nu - 1$, $y^1 y^2 \cdots y^n$ is a common abstract subsimplex of exactly two n-simplices of order $\nu - 1$. Thus, one is, of course, $y^1 y^2 \cdots y^n y^{n+1}$, and there exists another n-simplex of order $\nu - 1$, $y^1 y^2 \cdots y^n y^{*n+1}$ together with another order $\nu - 1$ vertex, $y^{*n+1} \notin \{y^1, \ldots, y^n, y^{n+1}\}$. Except for these two simplices, there is no n-simplex of order $\nu - 1$ satisfying that $y^1 y^2 \cdots y^n$ is an abstract subsimplex of it. It follows that for $\sigma_\nu^{n-1} = V_\nu^1 V_\nu^2 \cdots V_\nu^n$, where $V_\nu^n = \{y^1, y^2, \ldots, y^n\}$, the possibility of V_ν^{n+1}, such that $V_\nu^1 \subset \cdots \subset V_\nu^n \subset V_\nu^{n+1}$ forms an $(n+1)$-increasing sequence of vertices of order $\nu - 1$, which in turn form a certain n-simplex of order $\nu - 1$, is restricted to the two cases where $V_\nu^{n+1} = \{y^1, y^2, \ldots, y^n, y^{n+1}\}$ or $V_\nu^{*n+1} = \{y^1, y^2, \ldots, y^n, y^{*n+1}\}$. Of course, this means that the possibility for σ_ν^{n-1} to be an abstract subsimplex of a certain n-simplex, σ_ν^n, of order ν is restricted to the two cases where $\sigma_\nu^n = V_\nu^1 V_\nu^2 \cdots V_\nu^n V_\nu^{n+1}$ or $\sigma_\nu^n = V_\nu^1 V_\nu^2 \cdots V_\nu^n V_\nu^{*n+1}$.

(III) By (I) and (II), we have shown that the assertion of the theorem holds for all $\nu \leqq \bar{\nu}$. ■

By using the above theorem, we formulate and prove Sperner's Lemma about $\boldsymbol{K}^\nu$ for all $\nu \leqq \bar{\nu}$. *Vertex assignment* of order ν is a function, ω: $\boldsymbol{V}_\nu \to \boldsymbol{V}_0 = \boldsymbol{V}$, which gives for each $y \in \boldsymbol{V}_\nu$ a certain $x \in C(\{y\})$, i.e. to assign a vertex x of order 0 belonging to the carrier for each vertex y of order ν.

Then, for each k-simplex of order ν $(k = 0, 1, \ldots, n)$, $\langle \sigma_\nu^n \rangle = \langle y_\nu^1 y_\nu^2 \cdots y_\nu^{n+1} \rangle$, we define as $\epsilon(\langle \sigma_\nu^n \rangle) = 1$ when $\langle \omega(y_\nu^1)\omega(y_\nu^2) \cdots \omega(y_\nu^{n+1}) \rangle$ is equal to $\langle x^1 x^2 \cdots x^{k+1} \rangle$, as $\epsilon(\langle \sigma_\nu^n \rangle) = -1$ when $\langle \omega(y_\nu^1)\omega(y_\nu^2) \cdots \omega(y_\nu^{n+1}) \rangle$ is equal to $-\langle x^1 x^2 \cdots x^{n+1} \rangle$, and as $\epsilon(\langle \sigma_\nu^n \rangle) = 0$ otherwise. When $\epsilon(\langle \sigma_\nu^k \rangle) \neq 0$, k-simplex, $\langle \sigma_\nu^k \rangle$ is called *regular*.

Theorem 2: (Sperner's Lemma) $\boldsymbol{K}^\nu$ $(0 \leqq \nu \leqq \bar{\nu})$ has an odd number of regular n-simplices (hence, at least one n-simplex).

Proof: Fix the order, ν, and the complex, $\boldsymbol{K}^\nu$. Let us denote by α the number of regular n-simplices. Denote by β the number of $(n-1)$-simplices whose carrier is equal to $\{x^1, x^2, \ldots, x^n\}$. Moreover, for each n-simplex, σ_ν^n, denote by $\beta'(\sigma_\nu^n)$ the number of regular subsimplices of σ_ν^n. Then, exactly the same argument as with Nikaido (1968, p.61, Lemma 4.3) for the same proof is applicable. Thus, we will omit the proof in this paper.[4] ■

4 Also for details of this proof, see Urai and Murakami (2014)

3 Economic Equilibrium

In this section, we provide a certain condition under which the economic equilibrium arguments need not necessarily be constructed on the commodity spaces of the real field.

Assumption 1: There is a positive number $\delta > 0$ such that any amount less than δ does not have essential meaning for our equilibrium concept (as well as all concepts necessary to grasp and/or measure the amounts in defining the equilibrium).

Such an assumption would enable us to define the economic equilibrium as the status that market excess demand is not greater than δ. Moreover, assume that there is a continuous excess demand function (as an instrument in the meta-theory), and we can take $\gamma > 0$ uniformly so that any price change less than γ would not cause the change of excess demands greater than δ.

Then, if $\bar{p}$ is not a price that leads to an equilibrium as we explained in the above recognizable sense, it would be possible that there exists no full-labeled simplex in a certain γ-neighborhood ($\gamma > 0$) of such a price, under the standard labeling for points in the price domain based on the excess demand function in the meta-theory.[5] Therefore, by considering the above Sperner's Lemma, Theorem 2, the full-labeled simplex when the diameter of the simplex is sufficiently smaller than γ, every point in the neighborhood represents the economic equilibrium price in the above meaning.

Although we have only discussed the existence of equilibria, it would be possible to consider the first and second fundamental theorems of welfare economics in a similar way. These problems require further research.

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[5] See, for example, the standard proof of an equilibrium existence in Nikaido (1970, chapter 10).

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