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<th><strong>Title</strong></th>
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Osaka University
1. Introduction

Let \( \{\xi_i\}_{i \geq 1} \) be a sequence of independent identically distributed random variables, and set \( S_0 = 0, S_n = \sum_{i=1}^{n} \xi_i, n \geq 1 \). Let \( f(x), h(x) \) be two functions on \([0, \infty)\). For \( n \geq 1 \), set

\[
P_n = E \left[ f(e^{S_n})h \left( \sum_{i=1}^{n} e^{S_i} \right) \right].
\]

One object of this paper is to obtain an estimate of \( P_n \) under certain conditions of \( \xi_1, f(x) \) and \( h(x) \).

We can consider a similar problem in case of Brownian motion. Let \( \{B(t) : t \geq 0\} \) be a Brownian motion with \( B(0) = 0 \). We assume that \( f(x) \) and \( h(x) \) satisfy the following: for some \( 0 < \alpha < \beta < \infty \),

\[
\sup_{x \geq 0} x^{-\alpha}|f(x)| < \infty, \quad \sup_{x \geq 0} x^{\beta}|h(x)| < \infty.
\]

Set

\[
P(t) = E \left[ f(e^{B(t)})h \left( \int_{0}^{t} e^{B(s)} ds \right) \right].
\]

Applying a very useful formula of Yor ([9],(6.e)), we see as in [5] that

\[
P(t) \sim ct^{-3/2} \quad \text{as } t \to \infty,
\]

where

\[
c = 2^{3/2} \pi^{-1/2} \int_{0}^{\infty} \int_{0}^{\infty} f(y^2)h(4/z)e^{-\lambda z} u \sinh udydzdu, \quad \lambda = (1 + y^2)/2 + y \cosh u.
\]

Kawazu-Tanaka [5] used this fact to obtain the rate of decay of the tail probability of the maximum of a diffusion process in a drifted Brownian environment. This
result in case of a Brownian motion suggests that an analogous result in a random
walk case also holds. In fact we obtain Theorem 1 below. But to our regret, we can
not find the precise value corresponding to $c$.

The original motivation of this problem is to clarify and simplify the proof of
Afanas'ev's result [1] on a random walk in a random medium. In the last section
we get Theorem 2 which is a slight extention of the theorem in [1] by using our
Theorem 1.

Now we state the conditions on $\xi = \xi_1$ and Theorem 1.

Conditions (a)-(d).
(a) $E\xi = 0$, $0 < E\xi^2 = \sigma^2 < \infty$.
(b) The distribution of $\xi$ is continuous and $E|\xi|^3 < \infty$.
(c) $Ee^{\theta \xi}$ converges for some $\theta > 0$.
The condition (b) can be replaced by the following condition (d).
(d) The distribution of $\xi$ is concentrated on the set $\{id : i \in \mathbb{Z}\}$ with some $d > 0$.

**Theorem 1.** Let functions $f$, $h$ and $W$ satisfy the following conditions:
(i) $f$ and $h$ are continuous on $[0, \infty)$.
(ii) $W$ is continuous on $\mathbb{R}$ and non-negative.
(iii) There exist positive numbers $\alpha$, $\beta$, $\varepsilon$, $\eta$ such that

\[
\sup_{x \geq 0} x^{-\alpha}|f(x)| < \infty, \quad \sup_{x \geq 0} x^\beta|h(x)| < \infty,
\]

\[
\limsup_{x \to -\infty} W(x)e^{-\varepsilon x} < \infty, \quad \liminf_{x \to +\infty} W(x)e^{-\eta x} > 0,
\]

with $\beta \eta > \alpha > 0$. If the conditions (a), (b) and (c) are satisfied, then

\[
E \left[ f(e^{S_n})h \left( \sum_{i=1}^{n} W(S_i) \right) \right] \sim cn^{-3/2},
\]

as $n \to \infty$ with a constant $c$. The same conclusion holds under the conditions
(a), (d) and (c).

**Remark.** A sufficient condition for the positivity of $c$ is given in Corollary
10.

2. Preliminaries

We introduce the following quantities together with their corresponding Laplace
transforms.

\[
u_n(x) = P(S_1 > 0, \cdots, S_{n-1} > 0, 0 < S_n \leq x),
\]
\[ v_n(x) = P(S_1 \leq 0, \ldots, S_{n-1} \leq 0, -x \leq S_n \leq 0), \]

\[ u_0(x) = v_0(x) = 1_{[0, \infty)}(x), \]

\[ \varphi_n(\theta) = \int_0^\infty e^{-\theta x} du_n(x), \quad \psi_n(\theta) = \int_0^\infty e^{-\theta x} dv_n(x). \]

The three lemmas below will be used later. We state them without proof.

**Lemma 1** (Spitzer-Baxter identity. See [6], p. 49). For any \( \theta > 0, |t| < 1, \)

\[ 1 + \sum_{n=1}^{\infty} \varphi_n(\theta)t^n = \exp \left\{ \sum_{n=1}^{\infty} \frac{t^n}{n} E(e^{-\theta S_n}; S_n > 0) \right\}. \]

**Lemma 2** (See [2]). If the conditions (a) and (b) hold, then for any \( \theta > 0, \)

\[ E(e^{-\theta S_n}; S_n > 0) \sim (\sqrt{2\pi\sigma\theta})^{-\frac{1}{2}} n^{-\frac{3}{2}} \text{ as } n \to \infty. \]

**Lemma 3** (See [4] and [8]).

1. Let \( \sum_{n=0}^{\infty} a_n t^n = \exp(\sum_{n=1}^{\infty} b_n t^n) \) for \( |t| < 1. \) If \( b_n \sim bn^{-3/2}, \) then \( a_n \sim \Theta \exp(B)n^{-3/2} \) with \( B = \sum_{n=1}^{\infty} b_n. \)

2. Let \( c_n \geq 0, d_n \geq 0, c_n \sim cn^{-3/2} \) and \( d_n \sim dn^{-3/2}. \) If \( a_n = \sum_{j=0}^{n} c_{n-j}d_j, \) then \( a_n \sim (cD+dC)n^{-3/2} \) with \( C = \sum_{n=0}^{\infty} c_n \) and \( D = \sum_{n=0}^{\infty} d_n. \)

Set

\[ u(x) = \sum_{n=0}^{\infty} u_n(x), \quad v(x) = \sum_{n=0}^{\infty} v_n(x), \]

and

\[ U(x) = \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{x} u(y)dy, \quad \varphi(\theta) = \int_{0}^{\infty} e^{-\theta x} dU(x), \]

\[ V(x) = \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{x} v(y)dy, \quad \psi(\theta) = \int_{0}^{\infty} e^{-\theta x} dV(x). \]

Then we can prove the following lemma.

**Lemma 4.** If the conditions (a) and (b) hold, then

1. \( \lim_{n \to \infty} n^{3/2} \varphi_n(\theta) = \varphi(\theta), \quad \lim_{n \to \infty} n^{3/2} \psi_n(\theta) = \psi(\theta) \) for each \( \theta > 0. \)

2. \( \lim_{n \to \infty} n^{3/2} u_n(x) = U(x), \quad \lim_{n \to \infty} n^{3/2} v_n(x) = V(x) \) compact uniformly on \([0, \infty).\)

3. \( \lim_{n \to \infty} n^{3/2} P(M_n \leq x, M_n - S_n \leq y) = U(x)v(y) + u(x)V(y) \) where \( M_n = \max_{0 \leq j \leq n} S_j. \)
Proof. Combining Lemmas 1, 2 with (1) of Lemma 3, for each $\theta > 0$, we have
\[
\lim_{n \to \infty} n^{3/2} \varphi_n(\theta) = \frac{1}{\sqrt{2 \pi \sigma \theta}} \left( 1 + \sum_{n=1}^{\infty} \varphi_n(\theta) \right).
\]
It is easy to see that the right hand side of the above is $\varphi(\theta)$. If we consider a reversed random walk $\{-S_n\}_{n \geq 0}$, we get the assertion for $\psi$ also. By the extended continuity theorem for Laplace transform, (1) yields (2). In view of (2), (3) can be proved by applying (2) of Lemma 3 to the following identity ([6], p. 25).

\[ P(M_n \leq x, M_n - S_n \leq y) = \sum_{j=0}^{n} u_j(x)v_{n-j}(y) \quad n \geq 0. \]

The proof of Lemma 4 is complete.

We give some definitions. Fix a constant $K > 0$. For a function $f$ on $(-\infty, K]$ and a constant $a \in \mathbb{R}$, set
\[
\|f\|_a = \sup_{x \leq K} e^{ax}|f(x)|,
\]
and define a function space $C(a)$:
\[ C(a) = \{f : f \text{ is continuous on } (-\infty, K] \text{ and } f\|_a < \infty \}. \]
$C(a)$ is a Banach space with respect to the norm $\|f\|_a$. From now on we omit $a$ in $\|f\|_a$. By the condition (c), we can take $\delta > 0$ such that $E(e^{\delta \xi}) < \infty$. Let this $\delta > 0$ and arbitrary $\mu > \delta$ be fixed, and set $\lambda = \mu - \delta > 0$. Let non-negative $W \in C(-\mu)$ be fixed. For any $f \in C(\lambda)$, we define a family of transforms $\{R_t\}_{t \geq 0}$ depending on $W$ by
\[ R_t f(x) = E_x \left[ f(\xi) \left(1 - e^{-tW(\xi)} \right); \xi \leq K \right], \]
where $E_x$ denotes the expectation of the random walk starting at $x$. For any $f \in C(-\delta)$, we define a sequence of transforms $\{T_n\}_{n \geq 0}$ by
\[ T_n f(x) = E_x \left[ f(S_n); M_n \leq K \right], \quad n \geq 0. \]
In the sequel, the conditions (a)-(c) are assumed to be satisfied.

**Lemma 5.** Each $R_t$ is a bounded operator from $C(\lambda)$ to $C(-\delta)$.

Proof. Let $f \in C(\lambda)$. The continuity of $R_t f$ is derived from the condition (b) and the continuity of $f$ and $W$. From the assumptions of $f$ and $W$,
\[
|f(x)\left(1 - e^{-tW(x)} \right)| \leq tW(x)|f(x)| \leq t\|W\||f|e^{(\mu-\lambda)x}.
\]
By the definition of \( \lambda \) and (2.2),
\[
|R_t f(x)| \leq t\|W\|\|f\|E(e^{\delta(x+\xi)}; \xi \leq K - x)
\leq t\|W\|\|f\|e^{\delta x} E(e^{\delta \xi}).
\]
Therefore we obtain
\[
\|R_t\| \leq t\|W\|E(e^{\delta \xi}) < \infty.
\]
The proof of Lemma 5 is complete.

**Lemma 6.** Each \( T_n \) is a bounded operator from \( C(-\delta) \) to \( C(\lambda) \), and there exists a constant \( C \) such that \( \|T_n\| \leq Cn^{-3/2} \) for \( n \geq 1 \). Here \( C \) does not depend on \( n \).

**Proof.** Let \( f \in C(-\delta) \). The continuity of \( T_n f \) is derived from the condition (b) and the continuity of \( f \). Direct calculations show \( \|T_0\| = e^{\mu K} \). We will estimate \( \|T_n\| \) for \( n \geq 1 \). Since \( f \in C(-\delta) \), we have \( |f(x)| \leq \|f\|e^{\delta x} \). Applying this and (2.1) to (2.3), we see
\[
|T_n f(x)| \leq \|f\|e^{\delta x} E(e^{\delta S_n}; M_n \leq K - x)
\leq \|f\|e^{\delta x} \sum_{j=0}^n \psi_{n-j}(\delta) \int_0^{K-x} e^{\delta y} du_j(y).
\]
Using Chebyshev's inequality, we see
\[
\int_0^{K-x} e^{\delta y} du_j(y) \leq e^{\delta(K-x)} u_j(K - x) \leq e^{(\delta+\lambda)(K-x)} \varphi_j(\lambda).
\]
Thus we have
\[
|T_n f(x)| \leq \|f\|e^{\mu K - \lambda x} \sum_{j=0}^n \psi_{n-j}(\delta) \varphi_j(\lambda),
\]
which shows
\[
\|T_n\| \leq e^{\mu K} \sum_{j=0}^n \psi_{n-j}(\delta) \varphi_j(\lambda).
\]
Applying (2) of Lemma 3 and (1) of Lemma 4 to the right hand side of the above, we have
\[
C := e^{\mu K} \sup_{n \geq 1} \left\{ n^{3/2} \sum_{j=0}^n \psi_{n-j}(\delta) \varphi_j(\lambda) \right\} < \infty.
\]
This completes the proof of Lemma 6.

From Lemma 6 we get the following corollary.

**Corollary 7.** Let \( S = \sum_{n=0}^{\infty} T_n \). Then \( S \) is a bounded operator from \( C(-\delta) \) to \( C(\lambda) \).

Now we consider the limit of \( n^{3/2}T_n \). Let \( f \in C(-\delta), x \leq K \) and \( L > 0 \) be fixed. Then

\[
T_n f(x) = E_x [f(S_n); M_n \leq K, M_n - S_n \leq L] + E_x [f(S_n); M_n \leq K, M_n - S_n > L] = a_n + b_n.
\]

Applying (3) of Lemma 4 to \( a_n \), we have

\[
\lim_{n \to \infty} n^{3/2} a_n = \int_0^L \int_{K-x}^0 f(x + a - b) d(U(a)v(b) + u(a)V(b)).
\]

Using the method employed in the proof of Lemma 6, we get

\[
|b_n| \leq \|f\| e^{nK-\lambda x} \sum_{j=0}^{\infty} \varphi_{n-j}(\lambda) \int_L^\infty e^{-\delta b} d\nu_j(b).
\]

In view of (1) and (2) of Lemma 4, we apply (2) of Lemma 3 to the right hand side

\[
\limsup_{n \to \infty} n^{3/2} |b_n| \leq \|f\| e^{nK-\lambda x} \left\{ \sum_{n=0}^{\infty} \varphi_n(\lambda) \int_0^\infty e^{-\delta b} d\nu(b) + \varphi(\lambda) \int_0^\infty e^{-\delta b} d\nu(b) \right\}.
\]

The last term goes to 0 as \( L \to \infty \). Without loss of generality, we assume \( f \geq 0 \). Then we have

\[
n^{3/2} a_n \leq n^{3/2} T_n f(x) \leq n^{3/2} a_n + \sup_{m \geq n} (m^{3/2} b_m).
\]

Going to the limit, first with respect to \( n \), and then with respect to \( L \), we obtain

\[
\lim_{n \to \infty} n^{3/2} T_n f(x) = \int_0^\infty \int_0^{K-x} f(x + a - b) d(U(a)v(b) + u(a)V(b)).
\]

Denote by \( Qf(x) \) the right hand side of (2.4). It is clear that \( Qf \) is continuous by (2) and (3) of Lemma 4. The definition of \( Q \) and Lemma 6 show that

\[
|Qf(x)| \leq \limsup_{n \to \infty} n^{3/2} |T_n f(x)| \leq \left( \limsup_{n \to \infty} n^{3/2} \|T_n\| \right) \|f\| e^{-\lambda x}.
\]
Hence we have
\[ \|Q\| \leq \limsup_{n \to \infty} \left( n^{\frac{3}{2}} \|T_n\| \right) \leq C. \]

Collecting the above results, we get the following lemma.

**Lemma 8.** Let \( f \in C(-\delta) \). Then we can define \( Qf(x) = \lim_{n \to \infty} n^{3/2}T_nf(x) \) and \( Q \) is a bounded operator from \( C(-\delta) \) to \( C(\lambda) \).

The following corollary will be used hereafter.

**Corollary 9.** If \( f \in C(-\delta) \) is positive, then \( Qf(0) > 0 \).

**Proof.** From (2.4) and the assumption of \( f \), we have
\[ Qf(0) \geq \int_{0}^{\infty} \int_{0}^{K} f(a-b) dU(a) dv(b). \]

By the definition of \( U(x) \), we see \( U(K) \geq K/\sqrt{2\pi\sigma} > 0 \). The renewal theory ([3], Chap. XI) applied to \( v(x) \) yields \( \lim_{x \to \infty} v(x) = \infty \), which shows the corollary.

---

### 3. Main Lemma

The lemmas established in the previous section enable us to show Theorem 1 if \( h(x) = e^{-tx} \ (t \geq 0) \) and \( W(x) = \infty \) for \( x > K \). That is, we have the following lemma.

**Main Lemma.** For any \( f \in C(-\delta) \), non-negative \( W \in C(-\mu) \) and \( t \geq 0 \), the finite limit
\[ \lim_{n \to \infty} n^{3/2}E \left[ f(S_n) \exp \left\{ -t \sum_{i=1}^{n} W(S_i) \right\} ; M_n \leq K \right] \]
exists.

To prove Main Lemma we need some preparations. Let non-negative \( W \in C(-\mu) \) be fixed. For each \( f \in C(-\delta) \) and \( t \geq 0 \), we put
\[ A_0^{(t)} f(x) = f(x), \ A_n^{(t)} f(x) = E_x \left[ f(S_n) \exp \left\{ -t \sum_{i=1}^{n} W(S_i) \right\} ; M_n \leq K \right], \ n \geq 1. \]
Then $A_n^{(t)} = \{A_1^{(t)}\}^n$ as a transform, for $\{\xi_i\}_{i \geq 1}$ is i.i.d. (2.2) and (2.3) yield $T_1 - A_1^{(t)} = R_t$. Put $f_n^{(t)} = A_n^{(t)}f$. Then $f_n^{(t)} = A_1^{(t)}f_n^{(t)} = (T_1 - R_t)f_n^{(t)}$. By induction we get the following equations in $C(\lambda)$.

\[(3.1) \quad f_0^{(t)} = f, \quad f_n^{(t)} = T_nf - \sum_{j=1}^n T_{n-j}R_tf_{j-1}^{(t)} \quad n \geq 1.\]

Taking account of the definitions of $A_n^{(t)}$, $T_n$ and Lemma 6, we have

\[(3.2) \quad \|f_n^{(t)}\| \leq \|T_n\|\|f\| \leq C\|f\|n^{-\frac{3}{2}}.\]

The operator $SR_t$ maps $C(\lambda)$ to $C(\lambda)$. Hence we can define

\[c_1 = \sup\{t > 0 : \|SR_t\| < 1\}.\]

For a while we assume $0 < t < c_1$ and omit $t$ in $R_t$ and $f_n^{(t)}$. Then there exists $(1 + SR)^{-1}$. By (3.1) and (3.2), $F = \sum_{n=0}^{\infty} f_n = (1 + SR)^{-1}Sf$. Let $g \in C(\lambda)$ satisfy the equation

\[(3.3) \quad g = Qf - QRF - SRg.\]

After some tedious calculations, we rewrite $g$ as follows.

\[g = (1 + SR)^{-1}Q(1 + RS)^{-1}f.\]

We set

\[\tilde{g}(x) = \limsup_{n \to \infty} |n^{\frac{3}{2}}f_n(x) - g(x)|.\]

By (3.2), $\|\tilde{g}\| \leq C\|f\| + \|g\| < \infty$. Now we can start the proof of Main Lemma.

Proof. For fixed $N \in \mathbb{N}$, we have

\[n^{\frac{3}{2}} \sum_{j=1}^n T_{n-j}Rf_{j-1} = n^{\frac{3}{2}} \left[ \sum_{j=1}^N + \sum_{j=N+1}^{n-N} + \sum_{j=n-N+1}^n \right] T_{n-j}Rf_{j-1} = I_n + J_n + K_n.\]

Combining (3.1) and (3.3) with the above, we see

\[(3.4) \quad |n^{\frac{3}{2}}f_n - g| \leq |n^{\frac{3}{2}}T_nf - Qf| + |I_n - QRF| + |K_n - SRg| + |J_n|.\]

We estimate each term of the right hand side of (3.4). Using Lemmas 5, 6 and 8, we have

\[I_n = \sum_{j=1}^N \left( \frac{n}{n-j} \right)^{\frac{3}{2}} (n-j)^{\frac{3}{2}}T_{n-j}Rf_{j-1} \quad n \to \infty.\]
Therefore we get

\[(3.5) \quad \lim_{n \to \infty} (QRF - I_n) = \sum_{j=N}^{\infty} QRF_j.\]

Rewriting \(K_n\), we have

\[K_n = n^{\frac{3}{2}} \sum_{j=1}^{N} T_{j-1} R f_{n-j} = \sum_{j=1}^{N} \left( \frac{n}{n-j} \right)^{\frac{3}{2}} T_{j-1} R \left\{ (n-j)^{\frac{3}{2}} f_{n-j} \right\}.\]

By Corollary 7,

\[|K_n - SRg| \leq \left| K_n - \sum_{j=1}^{N} T_{j-1} R g \right| + \sum_{j=N+1}^{\infty} T_{j-1} R |g|.\]

The first term of the right hand side is bounded from above by the quantity

\[\sum_{j=1}^{N} \left( \frac{n}{n-j} \right)^{\frac{3}{2}} T_{j-1} R \left| (n-j)^{\frac{3}{2}} f_{n-j} - g \right| + \sum_{j=1}^{N} \left\{ \left( \frac{n}{n-j} \right)^{\frac{3}{2}} - 1 \right\} T_{j-1} R |g|.\]

Using Fatou’s lemma, we see

\[(3.6) \quad \limsup_{n \to \infty} |K_n - SRg| \leq \sum_{j=0}^{N-1} T_j R \bar{g} + \sum_{j=N}^{\infty} T_j R |g|.\]

Here we note that \(T_j R \bar{g}\) is well-defined. Applying Lemmas 5, 6 and (3.2) to \(J_n\),

\[\|J_n\| \leq C^2 \|R\| \|f\| n^{\frac{3}{2}} \sum_{j=N+1}^{n-N} (n-j)^{-\frac{3}{2}} (j-1)^{-\frac{3}{2}}.\]

Using (2) of Lemma 3, we have

\[(3.7) \quad \limsup_{n \to \infty} \|J_n\| \leq 2C^2 \|R\| \|f\| \sum_{j=N}^{\infty} j^{-\frac{3}{2}}.\]

Collecting (3.4)-(3.7) and letting \(N \to \infty\), we have \(0 \leq \bar{g} \leq SR\bar{g}\). Thus \(\|\bar{g}\| \leq \|SR\| \|\bar{g}\|\). Since \(t \in [0, c_1]\), this estimate implies \(\|\bar{g}\| = 0\), i.e., \(\bar{g} \equiv 0\), which shows the following: If \(0 \leq t < c_1\), then for each \(f \in C(-\delta)\) and \(x \leq K\),

\[(3.8) \quad \lim_{n \to \infty} n^{\frac{3}{2}} A_n^{(t)} f(x) = (1 + SR_t)^{-1} Q(1 + R_t S)^{-1} f(x).\]
If $c_1 = \infty$, this completes the proof of Main Lemma. We consider the case of $c_1 < \infty$. Let arbitrary $t_0 \in (0, c_1)$ be fixed. For $f \in C(-\delta)$ and $x \leq K$, set

$$\bar{T}_n f(x) = A_n^{(t_0)} f(x) \quad n \geq 0 \quad \text{and} \quad \bar{S} = \sum_{n=0}^{\infty} \bar{T}_n.$$

Since $0 \leq \bar{T}_n f(x) \leq T_n f(x)$ if $f \geq 0$, everything of the above is well-defined. For $f \in C(\lambda)$ and $t \geq 0$, set

$$\bar{R}_t f(x) = E_x \left[ f(\xi) e^{-t_0 W(\xi)} \{1 - e^{-t W(\xi)}\}; \xi \leq K\right].$$

Let $f \in C(-\delta)$. In view of (3.8), we define $\bar{Q} f(x)$ as follows.

$$\bar{Q} f(x) = \lim_{n \to \infty} n^{3/2} \bar{T}_n f(x) = (1 + S R_{t_0})^{-1} Q(1 + R_{t_0} S)^{-1} f(x).$$

Now exactly the same argument as the above is possible if we replace $T_n$, $R_t$, $Q$ by $\bar{T}_n$, $\bar{R}_t$, $\bar{Q}$ respectively. Therefore setting $c_2 = \sup\{t > 0 : \|S R_t\| < 1\}$ where $S R_t = \bar{S}(\bar{R}_t)$, we get the following: If $0 < t < c_2$, then

$$\lim_{n \to \infty} n^{3/2} A_n^{(t+t_0)} f(x) = (1 + S R_{t_0})^{-1} \bar{Q}(1 + \bar{R}_t S)^{-1} f(x).$$

On the other hand, we have the following relations from the definitions.

$$\bar{R}_t = R_{t+t_0} - R_{t_0}, \quad \bar{S} = S - S R_{t_0} \bar{S}.$$  \hspace{1cm} (cf. (3.1))

Using the above relations, we get

$$(1 + S R_{t_0})(1 + S R_t) = (1 + S R_{t_0+t}),$$

$$(1 + \bar{R}_t \bar{S})(1 + R_{t_0} S) = (1 + R_{t_0+t} S).$$

From these identities and the definition of $\bar{Q}$, we see that the right hand side of (3.9) is equivalent to $(1 + S R_{t_0+t})^{-1} Q(1 + R_{t_0+t} S)^{-1} f(x)$. Hence we can rewrite (3.9) in the following way: If $0 < t < c_1 + c_2$, then

$$\lim_{n \to \infty} n^{3/2} A_n^{(t)} f(x) = (1 + S R_t)^{-1} Q(1 + R_t S)^{-1} f(x).$$

Comparing $\bar{S}$, $\bar{R}_t$ with $S$, $R_t$, we easily see $\|S R_t\| \leq \|S R_t\|$. Thus $0 < c_1 \leq c_2$. From this fact, the method on the above can be iterated infinitely often. Therefore (3.10) holds for all $t \geq 0$. This finishes the proof of Main Lemma. \hspace{1cm} $\Box$
4. Proof of Theorem 1

In this section functions $f$, $h$ and $W$ are assumed to satisfy the conditions (i), (ii) and (iii) in Theorem 1. We devide the proof into three parts. For any $K > 0$, set

$$L_n = E \left[ f(e^{S_n}) h \left( \sum_{i=1}^{n} W(S_i) \right) \right]$$

$$= E \left[ f(e^{S_n}) h \left( \sum_{i=1}^{n} W(S_i) \right); M_n \leq K \right] + E \left[ f(e^{S_n}) h \left( \sum_{i=1}^{n} W(S_i) \right); M_n > K \right]$$

$$= H_n(K) + J_n(K).$$

**STEP 1.** For any $K > 0$, the finite limit $\lim_{n \to \infty} n^{3/2} H_n(K)$ exists.

Proof. Let $K > 0$ be fixed. From the conditions (c) and (iii), we can choose two numbers $\mu$ and $\delta$ such that $0 < \delta < \mu < \min(\alpha, \varepsilon)$ and $E(e^{\delta x}) < \infty$. We easily see $f(x) \in C(-\delta)$ and $W \in C(-\mu)$. From the conditions of $h$, there exists a sequence of functions $\{h_m\}_{m \geq 1}$ satisfying $\sup_{x \geq 0} |h(x) - h_m(x)| \leq m^{-1}$ and $h_m(x)$ is a finite linear combinations of $\{e^{-nx}\}_{n \geq 0}$. Hence by Main Lemma we can define $C_m$ as follows.

$$C_m = \lim_{n \to \infty} n^{3/2} E \left[ f(e^{S_n}) h_m \left( \sum_{i=1}^{n} W(S_i) \right); M_n \leq K \right].$$

It is obvious that $\{C_m\}_{m \geq 1}$ is a Cauchy sequence. Set $C_\infty = \lim_{m \to \infty} C_m$. Then

$$|n^{3/2} H_n(K) - C_\infty| \leq \frac{1}{m} n^{3/2} E \left[ |f(e^{S_n})|; M_n \leq K \right]$$

$$+ n^{3/2} E \left[ f(e^{S_n}) h_m \left( \sum_{i=1}^{n} W(S_i) \right); M_n \leq K \right] - C_\infty.$$

Therefore we have

$$\limsup_{n \to \infty} |n^{3/2} H_n(K) - C_\infty| \leq \frac{a}{m} + |C_m - C_\infty| \to 0 \quad \text{as} \quad m \to \infty,$$

where $a = \lim_{n \to \infty} n^{3/2} E[|f(e^{S_n})|; M_n \leq K]$. Thus we obtain the assertion of Step 1.

**STEP 2.** $\limsup_{K \to \infty} \limsup_{n \to \infty} n^{3/2} |J_n(K)| = 0.$

Proof. From the condition (iii), there exist positive constants $A$, $B$ and $K$ which satisfy the following: $|f(x)| \leq Ax^\alpha$, $|h(x)| \leq Ax^{-\beta}$ for $x \geq 0$, and
\( W(x) \geq Be^{nx} \) for \( x \geq K \). Hence we have \( |f(e^{Sn})| \leq Ae^{\alpha Sn} \) and the inequality, 
\( |h(\sum_{i=1}^{n} W(S_i))| \leq AB^{-\beta}e^{-\beta \eta M_n} \), holds on \( (M_n > K) \). Applying these estimates to \( J_n(K) \) and using (2.1), we have

\[
|J_n(K)| \leq \text{const}E(e^{\alpha Sn - \beta \eta M_n}; M_n > K)
\leq \text{const} \sum_{j=0}^{n} \psi_{n-j}(\alpha) \int_{K}^{\infty} e^{-bx} du_j(x),
\]

where const = \( A^2B^{-\beta} \), \( b = \beta \eta - \alpha > 0 \). In view of (1) (2) of Lemma 4, we apply (2) of Lemma 3 to the last term of the above.

\[
\limsup_{n \to \infty} n^{\frac{3}{2}}|J_n(K)| \leq \text{const} \left\{ \sum_{n=0}^{\infty} \psi_n(\alpha) \int_{K}^{\infty} e^{-bx} dU(x) + \psi(\alpha) \int_{K}^{\infty} e^{-bx} du(x) \right\}.
\]

Since the last term goes to 0 as \( K \to \infty \), Step 2 is proved.

**Step 3.** The finite limit \( \lim_{n \to \infty} n^{3/2}L_n \) exists.

Proof. Without loss of generality, we assume that \( f \) and \( h \) are non-negative. By Step 1 we can define \( H(K) = \lim_{n \to \infty} n^{3/2}H_n(K) \). Then \( H(K) \) is a non-decreasing and bounded. Set \( H = \lim_{K \to \infty} H(K) \). Since

\[
n^{\frac{3}{2}}H_n(K) \leq n^{\frac{3}{2}}L_n \leq n^{\frac{3}{2}}H_n(K) + \sup_{m \geq n} \left( m^{\frac{3}{2}}J_m(K) \right),
\]

taking account of Step 2 we easily see that \( n^{3/2}L_n \) goes to \( H \) as \( n \to \infty \). Finally Theorem 1 is established under the conditions (a), (b) and (c).

**Remark.** To prove Theorem 1 under the conditions (a), (c) and (d), we enumerate the modified points. Under the conditions (a) and (d), the local limit theorem

\[
\lim_{n \to \infty} \left[ \sup_{k \in \mathbb{Z}} \left| \sqrt{n}P(S_n = dk) - \frac{d}{\sqrt{2\pi \sigma}} \exp \left\{ \frac{-(dk)^2}{2n\sigma^2} \right\} \right| \right] = 0
\]

holds (See e.g. [3]), which implies, for \( \theta > 0 \)

\[
E(e^{-\theta S_n}; S_n > 0) \sim \frac{d}{\sqrt{2\pi \sigma}} \frac{e^{-\theta}}{1 - e^{-\theta}n^{-\frac{1}{2}}} \quad \text{as} \quad n \to \infty.
\]

This corresponds to Lemma 2. If \( U \) and \( V \) are defined by

\[
U(x) = \frac{d}{\sqrt{2\pi \sigma}} \sum_{0 \leq j \leq i-1} u(dj) \quad (id \leq x < (i+1)d), \quad = 0 \quad (0 \leq x < d),
\]
\[
V(x) = \frac{d}{\sqrt{2\pi \sigma}} \sum_{0 \leq j \leq i} v(dj) \quad (id \leq x < (i+1)d),
\]
then (1) of Lemma 4 holds, and (2) (3) of Lemma 4 hold at every continuous point. Fix $K \in \mathbb{N}$. For a function $f$ on $D = \{id : i \leq K\}$ and $a \in \mathbb{R}$, set $\|f\|_a = \sup_{x \in D} e^{ax}|f(x)|$ and $C(a) = \{f : \|f\|_a < \infty\}$. Then $x$ in the expectation symbol $E_x$ takes value on $D$. Having these modifications, the proof of Theorem 1 under the conditions (a), (c) and (d) are carried out analogously. We give the following corollary used in the next section.

**Corollary 10.** If $f$ is positive on $(0, \infty)$ and $h$ is positive on $[0, \infty)$, then the finite limit in Theorem 1 is positive.

Proof. Set $\nu_n(x) = E[f(e^{S_n}); \sum_{i=1}^n W(S_i) \leq x]$. Then for any $t > 0$,

$$\int_0^{\infty} e^{-tx} d\nu_n(x) = E\left[f(e^{S_n}) \exp\left\{-t \sum_{i=1}^n W(S_i)\right\}\right].$$

Apply Theorem 1 and the extended continuity theorem for Laplace transform to the above. Then there exists a measure $\nu$ satisfying the following: For each $t > 0$,

$$\lim_{n \to \infty} n^{3/2} \int_0^{\infty} e^{-tx} d\nu_n(x) = \int_0^{\infty} e^{-tx} d\nu(x) < \infty.$$ 

On the other hand, (3.10) implies

$$\int_0^{\infty} e^{-tx} d\nu(x) \geq (1 + SRt)^{-1} Q(1 + RtS)^{-1} \hat{f}(0),$$

where $\hat{f}(x) = f(e^x)$. Letting $t \to 0$, we see $\nu(\infty) \geq Q \hat{f}(0) > 0$. The last inequality follows from Corollary 9. Hence, we can choose a positive constant $x$ such that $0 < \nu(x) < \infty$ and

$$\lim_{n \to \infty} n^{3/2} \nu_n(x) = \nu(x).$$

By the assumptions of $h$, we have

$$E\left[f(e^{S_n})h\left(\sum_{i=1}^n W(S_i)\right)\right] = \int_0^{\infty} h(s) d\nu_n(s) \geq \min_{0 \leq s \leq x} h(s) \nu_n(x),$$

which proves Corollary 10. 

**5. Application to a random walk in a random medium**

Let $\{p_i : i \in \mathbb{Z}\}$ be a sequence of i.i.d. random variables with values in $[0, 1]$, and let $\mathcal{F}$ be the $\sigma$-field generated by the sequence $\{p_i\}$. A random walk in a random medium $\{p_i\}$ is a sequence of random variables $\{X_t : t = 0, 1, 2, \cdots\}$ which satisfy the following:

$$X_0 = 0, \quad P(X_{t+1} = X_t + 1|\mathcal{F}, X_0, X_1, \cdots, X_t = i) = p_i,$$

$$P(X_{t+1} = X_t - 1|\mathcal{F}, X_0, X_1, \cdots, X_t = i) = 1 - p_i.$$
Set \( \sigma_i = p_i/(1-p_i) \). In [7] it was shown that the condition \(-\infty \leq E \log \sigma_0 < 0\) implies \( \lim_{t \to \infty} X_t = -\infty \) a.s. Hence \( \max_{t \geq 0} X_t \) is finite a.s. In [1] Afanas'ev showed the asymptotic behavior of the probability \( P_n = P(\max_{t \geq 0} X_t \geq n) \) under the condition \( E\sigma_0 < 1 \) which implies \(-\infty \leq E \log \sigma_0 < 0\) by Jensen's inequality. We consider the same problem under the conditions

\[
(5.1) \quad -\infty \leq E \log \sigma_0 < 0, \quad E\sigma_0 < \infty.
\]

**Theorem 2.** Let \( \sigma_0 \) satisfy (5.1).

1. If \( E\sigma_0 \log \sigma_0 < 0 \), then
   \[
P_n \sim c_1(E\sigma_0)^n \quad \text{as } n \to \infty.
   \]

2. If \( E\sigma_0 \log \sigma_0 = 0 \) and \( E\sigma_0 \log^2 \sigma_0 < \infty \), then
   \[
P_n \sim c_2 n^{-1/2}(E\sigma_0)^n \quad \text{as } n \to \infty.
   \]

3. If \( E\sigma_0 \log \sigma_0 > 0 \) and the distribution of \( \log \sigma_0 \) is continuous or supported on a lattice, then
   \[
P_n \sim c_3 n^{-3/2} \gamma^n \quad \text{as } n \to \infty,
   \]

where \( \gamma = \min_{0 \leq t \leq 1} E\sigma_0^t \) and \( c_1, c_2, c_3 \) are positive numbers independent of \( n \). In case of (1) and (2), \( E\sigma_0 < 1 \) holds automatically.

Proof. Set \( \zeta_i = \log \sigma_{i-1}, \quad \hat{S}_n = \sum_{i=1}^n \zeta_i, \quad n \geq 1 \). Then we have the following identity (See [1]).

\[
P_n = E\rho \left( \rho + \sum_{i=1}^n e^{-\hat{S}_i} \right)^{-1}
\]

where \( \rho = 1 + \sum_{i=1}^{\infty} \sigma_{-1} \sigma_{-2} \cdots \sigma_{-i} < \infty \) a.s.

Since \( \{p_i\} \) are i.i.d., \( \{\zeta_i\}_{i \geq 1} \) are i.i.d. and independent of \( \rho \). Set \( h(x) = E\rho(\rho + x)^{-1} \). Then we can rewrite \( P_n \) using \( h(x) \).

\[
P_n = Eh \left( \sum_{i=1}^n e^{-\hat{S}_i} \right).
\]

Set \( \phi(t) = E\exp t \zeta_1 \). The condition (5.1) ensures that \( \phi(t) \) is well defined on \([0, 1]\), \( \phi'(t) > 0 \) on \((0, 1)\) and \( \lim_{t \to 0} \phi'(t) = E\log \sigma_0 < 0 \). From the assumptions of (1) and (2), \( \lim_{t \to 1} \phi'(t) = E\sigma_0 \log \sigma_0 \leq 0 \). Since \( \phi'(t) \) is monotone increasing on \((0, 1)\), these estimates imply \( \phi(1) = E\sigma_0 < 1 \). Thus (1) and (2) are reduced to Afanas'ev's theorem in [1].
In case of (3), we have $\lim_{t \to 0} \phi'(t) < 0$ and $\lim_{t \to 1} \phi'(t) > 0$. Therefore there exists a unique point $\tau \in (0, 1)$ where $\phi(t)$ attains its minimum $\gamma < 1$, i.e., $\phi'(\tau) = 0$ and $\phi(\tau) = \gamma < 1$. Set a proper distribution function

$$G(x) = \frac{1}{\gamma} \int_{-\infty}^{x} e^{\gamma y} dP(\zeta_1 \leq y).$$

Let $\{\xi_i\}_{i \geq 1}$ be i.i.d. and $-\xi_1$ has the distribution function $G(x)$. By the assumption of (3) and the definition of $G(x)$, we see that $\xi_1$ satisfies the conditions in Section 1. In fact we obtain

$$E e^{\theta \xi_1} = E e^{(\tau - \theta) \xi_1} = \frac{\phi(\tau - \theta)}{\gamma} \quad \text{and} \quad E \xi_1 = -\frac{\phi'(\tau)}{\gamma} = 0.$$

The former of the above shows that the Laplace transform of $\xi_1$ exists in some neighborhood of the origin and $E|\xi_1|^k < \infty$ for all $k \in \mathbb{N}$. Put $S_0 = 0$ and $S_n = \sum_{i=1}^{n} \xi_i$, $n \geq 1$. We express $P_n$ by $S_i$.

(5.2) $P_n = E h \left( \sum_{i=1}^{n} e^{-S_i} \right) = \gamma^n E \left[ e^{\tau S_n} h \left( \sum_{i=1}^{n} e^{S_i} \right) \right].$

Let a number $\beta$ satisfying $\tau < \beta < 1$ and $\phi(\beta) < 1$ be fixed. By the definitions of $\rho$ and $\phi(t)$, $B = E \rho^\beta \leq \sum_{n=0}^{\infty} \phi(\beta)^n < \infty$. Using Chebyshev's inequality, $P(\rho > y) \leq By^{-\beta}$ for $y > 0$. Applying integration by parts to $h(x)$ and using the above result, we have

$$h(x) = \int_{0}^{x} \frac{y}{(y + x)^2} P(\rho > y)dy \leq \int_{0}^{\infty} \frac{x}{(y + x)^2} By^{-\beta} dy = Bx^{-\beta} \int_{0}^{\infty} \frac{t^{-\beta}}{(1+t)^2} dt.$$

Hence for $x > 0$,

(5.3) $0 < h(x) \leq \frac{\pi \beta}{\sin \pi \beta} Bx^{-\beta}$. 

Taking account of (5.2) and (5.3), we can apply Theorem 1 and Corollary 10 to $P_n \gamma^{-n}$. Consequently we have the following positive finite limit

$$c_3 = \lim_{n \to \infty} n^{3/2} \gamma^{-n} P_n = \lim_{n \to \infty} n^{3/2} E \left[ e^{\tau S_n} h \left( \sum_{i=1}^{n} e^{S_i} \right) \right].$$

This finishes the proof of (3).
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References


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