Matsushima, Y.
Osaka Math. J.
14 (1962), 1–20.

ON BETTI NUMBERS OF COMPACT, LOCALLY SYMMETRIC RIEMANNIAN MANIFOLDS

Dedicated to Professor K. Shoda on his sixtieth birthday

By

YOZÔ MATSUSHIMA

This is a continuation of our paper [9]. In [9] we have studied the vanishing of the first Betti number of compact, locally symmetric Riemannian manifolds. We shall study in this paper the p-th Betti number of these manifolds from a somewhat different point of view.

Let X be a simply connected, symmetric Riemannian manifold, all of whose irreducible components are non-euclidean and non-compact. Let G be the identity component of the group of all isometries of X and let Γ be a discrete subgroup of G with compact quotient space G/Γ and without element of finite order different from the identity. The group Γ acts on X discontinuously and the quotient space $M=X/\Gamma$ is a compact, locally symmetric Riemannian manifold. Let A^p be the vector space of all G-invariant p-forms on X. By a well-known theorem of E. Cartan, the covariant derivatives of each form in A^p vanish (see [10]). Since $\Gamma\subset G$, each $\omega\in A^p$ is Γ-invariant and hence there exists a p-form η on M such that $\omega=\eta\circ\rho$, ρ denoting the projection of X onto M. Since ρ is a locally isometric mapping and the covariant derivatives of ω vanish, the covariant derivatives of η also vanish. In particular η is a harmonic p-form. Hence the mapping $\omega\to\eta$ defines an injection of A^p into the vetcor space $\mathfrak{h}^p$ of all harmonic p-forms on M. The purpose of this paper is to study when A^p can be isomrphic to $\mathfrak{h}^p$.

Let $x_0\in X$ and let K be the subgroup of G of all elements which leave fixed the point x_0. It is well-known that K is a maximal compact subgroup of G and X is identified with the quotient space $K\backslash G$. Let $\mathfrak{g}$ denote the Lie algebra of G and let $\mathfrak{k}$ be the subalgebra of $\mathfrak{g}$ corresponding to K. Denote by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form of $\mathfrak{g}$. We have then

$$\mathfrak{g}=\mathfrak{m}+\mathfrak{k},\ [\mathfrak{m},\mathfrak{m}]\subset\mathfrak{k},\ [\mathfrak{k},\mathfrak{m}]\subset\mathfrak{m}\,.$$

Let $\mathfrak{g}^c$ be the complexification of $\mathfrak{g}$ and let G^c be the complex Lie group, with center reduced to the identity, corresponding to the complex Lie algebra $\mathfrak{g}^c$. We may consider G as a subgroup of G^c. Put $\mathfrak{g}_u=\sqrt{-1}\mathfrak{m}+\mathfrak{k}$. Then $\mathfrak{g}_u$ is a compact real form of $\mathfrak{g}^c$ and let G_u be the subgroup of G^c corresponding to the real subalgebra $\mathfrak{g}_u$ of $\mathfrak{g}^c$. G_u is a maximal compact subgroup of G^c containing K and $X_u=K\backslash G_u$ is a simply connected, compact symmetric Riemannian manifold which we shall call the *compact form* of X.

Now we may identify the vector space A^p of all G-invariant p-forms on X with the vector space of all p-forms on the K-module $\mathfrak{m}$ which are invariant by K. On the other hand, the mapping $X\rightarrow\sqrt{-1}X$ defines a K-module isomorphism of $\mathfrak{m}$ onto $\sqrt{-1}\mathfrak{m}$. It follows that the vector space A_u^p of all G_u-invariant p-forms on X_u is isomorphic to A^p. Since X_u is symmetric and G_u is compact, the dimension of the vector space A_u^p equals the p-th Betti number $b_p(X_u)$ of X_u [4]. Therefore the dimension of the vector space A^p is equal to $b_p(X_u)$. On the other hand we have seen that A^p is identified with a subspace of the vector space $\mathfrak{h}^p$ of all harmonic p-forms of M. Thus we get the inequality $b_p(M)\geqq b_p(X_u)$, $b_p(M)$ denoting the p-th Betti number of M. Therefore our problem is stated as follows: Under what condition does the p-th Betti number of M equal the p-th Betti number of X_u?

A condition for this will be given by Theorem 1 in §8. From Theorem 1 we shall obtain the following result.

Theorem 2. *Let X be an irreducible symmetric bounded domain and let X_u be the compact form of X. Then the p-th Betti number of the compact manifold $M=X/\Gamma$ equals the p-th Betti number of X_u in the following cases:*

Type of X	X_u	value of p
$\mathrm{I}_{m,m'}$ $(m\geqq m')$	$U(m+m')/U(m)\times U(m')$	$\frac{m'}{2}>p$ $(m'\geqq 3)$; $p=1$ $(m'=2)$
II_m $(m\geqq 3)$	$SO(2m)/U(m)$	$\frac{m-2}{2}>p$ $(m>4)$; $p=1$ $(m=4)$
III_m $(m\geqq 2)$	$Sp(m)/U(m)$	$\frac{m+2}{4}>p$ $(m>2)$; $p=1$ $(m=2)$
IV_m $(m\geqq 3)$	$SO(n+2)/SO(2)\times SO(n)$	$p=1$
V	$E_6/Spin\ (10)\times T^1$	$p=1$
VI	$E_7/E_6\times T^1$	$p=1,2$

(The notations here are those of [2]).

In § 10 we shall apply these results to the classification of automorphic factors. The method employed in this paper is a natural extension of that of our paper [9].

I express here my hearty thanks to S. Murakami for his friendly cooperations. Especially the results in §§ 2 and 3 owe to him.

§ **1.** Throughout this paper G will denote a connected semi-simple Lie group with center reduced to the identity, all of whose simple components are non-compact; K will denote a maximal compact subgroup of G. We denote by $\mathfrak{g}$ the Lie algebra of all right invariant vector fields on G and by $\mathfrak{k}$ the subalgebra of $\mathfrak{g}$ corresponding to K. Let φ denote the Killing form of $\mathfrak{g}$ and let

$$\mathfrak{m} = \{X \in \mathfrak{g}\,;\ \varphi(X, Y) = 0 \quad \text{for all} \quad Y \in \mathfrak{k}\}\,.$$

Then we have

$$(1.1) \qquad \begin{array}{l} \mathfrak{g} = \mathfrak{m} + \mathfrak{k}\,, \qquad \mathfrak{m} \cap \mathfrak{k} = (0)\,; \\ [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k},\ [\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m},\ [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}\,. \end{array}$$

We know that the restriction of φ onto $\mathfrak{m}$ (resp. $\mathfrak{k}$) defines a positive (resp. negative) definite bilinear form on $\mathfrak{m}$ (resp. $\mathfrak{k}$). Hence we can choose a basis $X_1, \cdots, X_r$ of $\mathfrak{m}$ and $X_{r+1}, \cdots, X_n$ of $\mathfrak{k}$ such that

$$(1.2) \qquad \begin{array}{ll} \varphi(X_i, X_j) = \delta_{ij}\,, & 1 \leqq i,\ j \leqq r\,; \\ \varphi(X_\alpha, X_\beta) = -\delta_{\alpha\beta} & r+1 \leqq \alpha,\ \beta \leqq n. \end{array}$$

Throughout this paper we make the following convention: Latin indices $i, j, k, \cdots$, will range from 1 to r, while Greek indices $\alpha, \beta, \gamma, \cdots$ will range from $r+1$ to n and the indices $\lambda, \mu, \nu, \cdots$ from 1 to n. Let

$$[X_\lambda, X_\mu] = \sum_{\nu=1}^{n} c^\nu{}_{\lambda\mu} \cdot X_\nu\,.$$

By (1.1) among the structure constants $c^\nu{}_{\lambda\mu}$ only the $c^\gamma{}_{\alpha\beta}$, $c^\alpha{}_{ij}$, $c^i{}_{\alpha j}$, $c^i{}_{j\alpha}$ can be $\neq 0$. We shall write $c_{\alpha ij}$ instead of $c^\alpha{}_{ij}$. From the invariance of the Killing form under the adjoint representation follows that

$$(1.3) \qquad c^j{}_{i\alpha} = -c^i{}_{j\alpha} = c^i{}_{\alpha j} = c_{\alpha ij}\,.$$

We get from (1.2):

$$(1.4) \qquad \sum_{k=1}^{r} \sum_{\alpha=r+1}^{n} c_{\alpha ik} c_{\alpha jk} = \frac{1}{2} \delta_{ij}\,.$$

For any $X, Y \in \mathfrak{m}$, we denote by $R(X, Y)$ the endomorphism of the vector space $\mathfrak{m}$ defined by

$$R(X, Y)\cdot Z = [[Y, X], Z]$$

for all $Z \in \mathfrak{m}$. It can be interpreted as the Riemannian curvature tensor for the Riemannian symmetric manifold $K\backslash G$ (cf. [10]). We have

$$R(X_k, X_h)\cdot X_j = \sum_{i=1}^{r} R^i{}_{jkh}\cdot X_i ,$$

where

$$R^i{}_{jkh} = R_{ijkh} = -\sum_{\alpha=r+1}^{n} c_{\alpha ij} c_{\alpha kh} . \tag{1.5}$$

From (1.4) follows:

$$\sum_{k=1}^{r} R_{ikjk} = -\frac{1}{2}\delta_{ij} . \tag{1.6}$$

Let $\omega^\lambda(\lambda=1, \cdots, n)$ be the right invariant 1-forms on G such that $\omega^\lambda(X_\mu) = \delta^\lambda{}_\mu$. The symmetric tensor

$$g = \sum_{\lambda=1}^{n} (\omega^\lambda)^2$$

defines a right invariant Riemannian metric on G.

Now let $X = K\backslash G$ and let $\tilde{\pi}$ be the projection of G onto X. Put $\tilde{\pi}(e) = x_0$, e denoting the identity of G. We may identity the vector space $\mathfrak{m}$ with the tangent vector space of X at the point x_0. We define a G-invariant symmetric tensor h on X by the condition that the value of h at the point x_0 is equal to the restriction of φ onto $\mathfrak{m}$. The tensor h is everywhere positive definite and it defines a G-invariant Riemannian metric on X. X is a simply connected, symmetric Riemannian manifold.

§ **2.** Let Γ be a discrete subgroup of G with compact quotient space G/Γ. We may regard Γ as a discontinuous group of isometries of the symmetric Riemannian manifold X with compact quotient space X/Γ. Put $M = X/\Gamma$. From now on, we assume that Γ cotains no element of finite order different from the identity. Then M is a compact orientable manifold without singularties and G/Γ is a principal fiber bundle of base M and structure group K. We may consider M as the space of double cosets: $M = K\backslash G/\Gamma$. Let π denote the projection of G/Γ onto M. The symmetric tensor h on X being invariant by the action of Γ, it defines a positive definite symmetric tensor on M which we denote by the same letter h. h defines a Riemannian metric on M. We call M a compact, locally symmetric Riemannian manifold.

The right invariant tensor g on G defines a positive definite symmentric tensor on G/Γ which we shall denote also by g. The right invariant vector fields X_λ (resp. 1-forms ω^λ) on G define also the vector fields (reap. 1-forms) on G/Γ which we shall denote by the same letter X_λ (reap. ω^λ). Under these notations, consider the symmetric tensor $\sum_{i=1}^{r}(\omega^i)^2$ on G/Γ. It is easily verified that this tensor is invariant under the action of K on G/Γ. Moreover, the value of this tensor at each point of G/Γ is a symmetric bilinear form on the tangent space which is positive definite modulo the subspace of vectors tangent to the fiber of the bundle G/Γ over M, this latter subspace being spanned by the values of the vector fields X_α ($\alpha=r+1, \cdots, n$). From these considerations follows that the symmetric tensor h on M is related to the tensor $\sum_i(\omega^i)^2$ on G/Γ by the relation

$$h\circ\pi = \sum_{i=1}^{r}(\omega^i)^2. \tag{2.1}$$

Let dm denote the volme element of M determined by the Riemannian metric defined by h. From (2.1) follows that

$$dm\circ\pi = \omega^1\wedge \cdots \wedge\omega^r.$$

We denote by dv the volume element of G/Γ determined by the Riemannian metric g. Then

$$dv = \omega^1\wedge\cdots\wedge\omega^r\wedge\cdots\wedge\omega^n.$$

Further let dk denote the bi-invariant volume element $\omega^{r+1}\wedge\cdots\wedge\omega^n$ of the compact group K, where $\omega^{r+1}, \cdots, \omega^n$ are considered as right invariant 1-forms on K.

Lemma 1. *Let f be a continuous function on M. Then*

$$c\int_M f\,dm = \int_{G/\Gamma}(f\circ\pi)dv,$$

where c denotes the total volume of K measured by dk.

For the proof of this lemma, see [13].

Now let η, ζ be two p-forms on M. We denote by $\langle\eta, \zeta\rangle$ the usual scalar product defined by the Riemannian metric h. The global scalar product (η, ζ) is defined by

$$(\eta, \zeta) = \frac{1}{p!}\int_M \langle\eta, \zeta\rangle dm.$$

In the same way, for any two p-forms ω, θ on G/Γ, we define $\langle\omega, \theta\rangle$

and (ω, θ) using the Riemannian metric g. Put $g_{\lambda_1,\cdots,\lambda_p}=\omega(X_{\lambda_1}, \cdots, X_{\lambda_p})$ and $h_{\lambda_1,\cdots,\lambda_p}=\theta(X_{\lambda_1}, \cdots, X_{\lambda_p})$. Since $X_1, \cdots, X_n$ form an orthonormal basis of the tangent vector space at each point of G/Γ, we have $\langle\omega, \theta\rangle = \sum_{\lambda_1,\cdots,\lambda_p=1}^{n} g_{\lambda_1,\cdots,\lambda_p} h_{\lambda_1,\cdots,\lambda_p}$. Hence

$$(\omega, \theta) = \frac{1}{p!} \sum_{\lambda_1,\cdots,\lambda_p=1}^{n} \int_{G/M} g_{\lambda_1,\cdots,\lambda_p} \cdot h_{\lambda_1,\cdots,\lambda_p} dv . \tag{2.2}$$

Now let η, ζ be two p-forms on M. We see from (2.1) that

$$\langle\eta, \zeta\rangle\circ\pi = \langle\eta\circ\pi, \zeta\circ\pi\rangle . \tag{2.3}$$

It follows then from Lemma 1 the following

Lemma 2. *Let η, ζ be two p-forms on M. Then*

$$c\cdot(\eta, \zeta) = (\eta\circ\pi, \zeta\circ\pi) .$$

We state here the following lemma which will be used frequently in the following.

Lemma 3. *Let f be a C^∞-function on G/Γ. Then*

$$\int_{G/\Gamma} X_\lambda f \cdot dv = 0 , \qquad 1 \leqq \lambda \leqq n .$$

For the proof, see Weil [12].

§3. The projection π of G/Γ onto M defines an injection of the vector space $D^p(M)$ of p-forms on M into the vector space $D^p(G/\Gamma)$ of p-forms on G/Γ. It is well-known that a p-form $\omega \in D^p(G/\Gamma)$ is in the image of $D^p(M)$ if and only if $\theta(X_\alpha)\omega=0$, $i(X_\alpha)\omega=0$ for $\alpha=r+1, \cdots, n$, where θ (resp. i) denotes the Lie derivation (resp. interior product). Let $\eta \in D^p(M)$ and put $\eta'=\eta\circ\pi$. Then $d\eta'=d\eta\circ\pi$ and since $i(X_\alpha)d\eta'=0$, we have $(d\eta')(X_{\lambda_1}, \cdots, X_{\lambda_{p+1}})=0$, if one of the indices $\lambda_1, \cdots, \lambda_p$ is $\geqq r+1$. On the other hand, $(d\eta')(X_{i_1}, \cdots, X_{i_{p+1}})=\sum_{a=1}^{p+1}(-1)^{a-1}X_{i_a}\cdot\eta'(X_{i_1}, \cdots, \hat{X}_{i_a}, \cdots, X_{i_{p+1}})+\sum_{a<b}(-1)^{a+b}\ \eta'([X_{i_a}, X_{i_b}], X_{i_1}, \cdots, \hat{X}_{i_a}, \cdots, \hat{X}_{i_b}, \cdots, X_{i_{p+1}})$. But since $[X_{i_a}, X_{i_b}]=\sum_\alpha c_{\alpha i_a i_b}X_\alpha$ and $i(X_\alpha)\eta'=0$, we get:

$$(d\eta\circ\pi)(X_{i_1}, \cdots, X_{i_{p+1}}) = \sum_{a=1}^{p+1} (-1)^{a-1}X_{i_a}\{(\eta\circ\pi)(X_{i_1}, \cdots, \hat{X}_{i_a}, \cdots, X_{i_{p+1}})\} . \tag{3.1}$$

Let δ denote the operator of codifferentiation on the differential forms on M. For any $\eta \in D^p(M)$ and $\zeta \in D^{p+1}(M)$, we have $(d\eta, \zeta)=(\eta, \delta\zeta)$. For any $\zeta \in D^{p+1}(M)$, put

$$\delta'(\zeta\circ\pi) = \delta\zeta\circ\pi\,, \tag{3.2}$$

Lemma 4. *Let* $\zeta'=\zeta\circ\pi$. *Then*

$$(\delta'\zeta')(X_{i_1},\cdots,X_{i_p}) = -\sum_{k=1}^{r} X_k\cdot\zeta'(X_k, X_{i_1},\cdots,X_{i_p})\,.$$

Proof. For any $\eta\in D^p(M)$, let $\eta'=\eta\circ\pi$. Put $\eta'(X_{i_1},\cdots,X_{i_p})=f_{i_1,\cdots,i_p}$ and $\zeta'(X_{i_1},\cdots,X_{i_{p+1}})=g_{i_1,\cdots,i_{p+1}}$. From (3.1) and (2.2) we get:

$$(d\eta',\zeta') = \frac{1}{(p+1)!}\sum_{a=1}^{p+1}\sum_{i_1,\cdots,i_{p+1}}(-1)^{a-1}\int_{G/\Gamma}(X_{i_a}f_{i_1,\cdots,\hat{i}_a,\cdots,i_{p+1}})\cdot g_{i_1,\cdots,i_{p+1}}dv\,.$$

By Lemma 3,

$$\int_{G/L}(X_{i_a}f_{i_1\cdots\hat{i}_a\cdots i_{p+1}})\cdot g_{i_1\cdots i_{p+1}}dv = (-1)^a\int_{G/L}f_{i_1\cdots\hat{i}_a\cdots i_{p+1}}(X_{i_a}g_{i_ai_1\cdots\hat{i}_a\cdots i_{p+1}})dv\,.$$

Hence we obtain:

$$(d\eta',\zeta') = -\frac{1}{p!}\sum_{j_1,\cdots,j_p}\int_{G/\Gamma}f_{j_1\cdots j_p}(\sum_k X_k g_{kj_1\cdots j_p})dv\,.$$

We define a p-form μ' on G/Γ by the conditions that

$$\mu'(X_{j_1},\cdots,X_{j_p}) = -\sum_{k=1}^{r}X_k g_{kj_1\cdots j_p},\ i(X_\alpha)\mu' = 0\ \ (r+1\leqq\alpha\leqq n)\,.$$

Then

$$(d\eta',\zeta') = (\eta',\mu')\,.$$

Using $i(X_\alpha)\zeta'=0$, $\theta(X_\alpha)\zeta'=0$, we verify easily that $\theta(X_\alpha)\mu'=0$ for all X_α. Hence there exists a p-form μ on M such that $\mu'=\mu\circ\pi$. Then $(d\eta',\zeta')=(\eta\circ\pi,\mu\circ\pi)$. On the other hand, by Lemma 2 we have $(d\eta',\zeta')=c(d\eta,\zeta)$ and $(\eta\circ\pi,\mu\circ\pi)=c(\eta,\mu)$. Hence we get $(d\eta,\zeta)=(\eta,\mu)$. Since η is arbitrary, this implies $\mu=\delta\zeta$ and hence $\mu'=\delta'\zeta'$. Thus $(\delta'\zeta')(X_{j_1},\cdots, X_{j_p})=-\sum_k X_k\cdot\zeta'(X_k,X_{j_1},\cdots,X_{j_p})$.

From (3.1), (3.2) and Lemma 4 follows

Lemma 5. *A p-form ω on G/Γ is a π-image of a harmonic p-form on M if and only if the following three conditions are satisfied*:

1) $i(X_\alpha)\omega = \theta(X_\alpha)\omega = 0,\ r+1\leqq\alpha\leqq n$;

2) $\sum_{a=1}^{p+1}(-1)^{a-1}X_{i_a}\cdot\omega(X_{i_1},\cdots,\hat{X}_{i_a},\cdots,X_{i_{p+1}}) = 0,\ 1\leqq i_1,\cdots,i_{p+1}\leqq r$;

3) $\sum_{k=1}^{r}X_k\cdot\omega(X_k,X_{i_1},\cdots,X_{i_{p-1}}) = 0,\ 1\leqq i_1,\cdots,i_{p-1}\leqq r$.

§ **4.** We retain the notations introduced in the preceding sections. Let ω be a p-form on G/Γ satisfying the conditions in Lemma 5 and put

$$g_{\lambda_1\cdots\lambda_p} = \omega(X_{\lambda_1}, \cdots, X_{\lambda_p}) .$$

Then we have the following equalities:

(4.1) $$g_{\lambda_1,\cdots,\lambda_p} = 0 \text{ if one of the indices is } \geqq r+1 ;$$

(4.2) $$X_\alpha g_{i_1\cdots i_p} = \sum_{a=1}^{p}\sum_{h=1}^{r}(-1)^{a-1}c_{\alpha h i_a} g_{h i_1\cdots \hat{i}_a \cdots i_p} ,$$
$$r+1 \leqq \alpha \leqq n,\ 1 \leqq i_1, \cdots, i_p \leqq r ;$$

(4.3) $$X_k g_{i_1\cdots i_p} = \sum_{a=1}^{p}(-1)^{a-1}X_{i_a} g_{k i_1\cdots \hat{i}_a \cdots i_p},\ 1 \leqq k,\ i_1, \cdots, i_p \leqq r ;$$

(4.4) $$\sum_{k=1}^{r} X_k g_{k i_1\cdots i_{p-1}} = 0,\ 1 \leqq i_1, \cdots, i_{p-1} \leqq r .$$

Now we have

$$X_j X_k g_{i_1\cdots i_p} - X_k X_j g_{i_1\cdots i_p} = \sum_\alpha c_{\alpha j k} X_\alpha g_{i_1\cdots i_p} .$$

From (4.2) and (1.5) follows

$$\sum_\alpha c_{\alpha j k} X_\alpha g_{i_1\cdots i_p} = \sum_{a=1}^{p}(-1)^{a-1}\sum_h R_{j k i_a h} g_{h i_1\cdots \hat{i}_a \cdots i_p} .$$

Thus we get

(4.5) $$X_j X_k g_{i_1\cdots i_p} - X_k X_j g_{i_1\cdots i_p} = \sum_{a=1}^{p}(-1)^{a-1}\sum_{h=1}^{r} R_{j k i_a h} g_{h i_1\cdots \hat{i}_a \cdots i_p} .$$

Now put

(4.6) $$\Phi = \frac{1}{2p}\sum_{i_1,\cdots,i_p=1}^{r}\sum_{j,k=1}^{r}(X_j X_k g_{i_1\cdots i_p} - X_k X_j g_{i_1\cdots i_p})^2 .$$

Lemma 6.

$$\int_{G/\Gamma}\Phi\, dv = -\sum_{t_1,\cdots,t_{p-1}}\sum_{j,k,h,l}\int_{G/\Gamma} R_{jklh}(X_j g_{h t_1\cdots t_{p-1}})(X_k g_{l t_1\cdots t_{p-1}}) dv .$$

Proof. From (4.5) follows

$$\Phi = \frac{1}{2p}\sum_{i_1,\cdots,i_p}\sum_{h,j,k}\sum_{a=1}^{p}(-1)^{a-1}R_{j k i_a h}(X_j X_k g_{i_1\cdots i_p} - X_k X_j g_{i_1\cdots i_p}) g_{h i_1\cdots \hat{i}_a \cdots i_p}$$
$$= \frac{1}{p}\sum_{a=1}^{p}\sum_{i_1,\cdots,i_p}\sum_{h,j,k}(-1)^{a-1}R_{j k i_a h}(X_j X_k g_{i_1\cdots i_p}) g_{h i_1\cdots \hat{i}_a \cdots i_p} .$$

Put

$$\Phi' = \frac{1}{p}\sum_{a=1}^{p}\sum_{i_1,\cdots,i_p}\sum_{h,j,k}(-1)^{a}R_{j k i_a h}(X_k g_{i_1\cdots i_p})(X_j g_{h i_1\cdots \hat{i}_a \cdots i_p}) ,$$

By Lemma 3 we have

(4.7) $$\int_{G/\Gamma}\Phi\, dv = \int_{G/\Gamma}\Phi'\, dv .$$

Now we have

$$\begin{aligned}&\sum_{i_1,\cdots,i_p}\sum_{h,j,k} R_{jki_ah}(X_k g_{i_1\cdots i_p})(X_j g_{hi_1\cdots \hat{i}_a\cdots i_p})\\ &=\sum_{i_1,\cdots,i_p}\sum_{h,j,k} R_{jki_ah}(-1)^{a-1}(X_k g_{i_ai_1\cdots\hat{i}_a\cdots i_p})(X_j g_{hi_1\cdots \hat{i}_a\cdots i_p})\\ &=(-1)^{a-1}\sum_{t_1,\cdots,t_{p-1}}\sum_{h,j,k,l} R_{jklh}(X_k g_{lt_1\cdots t_{p-1}})(X_j g_{ht_1\cdots t_{p-1}}).\end{aligned}$$

Hence

$$\Phi' = -\sum_{t_1,\cdots,t_{p-1}}\sum_{h,j,k,l} R_{jklh}(X_k g_{lt_1\cdots t_{p-1}})(X_j g_{ht_1\cdots t_{p-1}}). \tag{4.8}$$

From (4.8) and (4.7) follows the lemma.

Lemma 7.

$$(\sum_k X_k^2)g_{i_1\cdots i_p} = -\frac{p}{2}g_{i_1\cdots i_p}+2\sum_{b<a}(-1)^{a+b}\sum_{k,h=1}^{r} R_{i_bhi_ak}g_{hki_1\cdots\hat{i}_b\cdots\hat{i}_a\cdots i_p}.$$

Proof. Put

$$F = (\sum_k X_k^2)g_{i_1\cdots i_p}.$$

We let operate X_k on the both sides of (4.3) and summing up on k we get:

$$F=\sum_{a=1}^{p}(-1)^{a-1}\sum_k X_kX_{i_a}g_{ki_1\cdots\hat{i}_a\cdots i_p}.$$

Since $X_kX_{i_a}=X_{i_a}X_k+\sum_\alpha c_{\alpha ki_a}X_\alpha$ and since (4.4) holds, we get

$$F=\sum_{a=1}^{p}(-1)^{a-1}\sum_k\sum_\alpha c_{\alpha ki_a}(X_\alpha g_{ki_1\cdots\hat{i}_a\cdots i_p}). \tag{4.9}$$

By (4.2), $X_\alpha g_{hi_1\cdots\hat{i}_a\cdots i_p}=\sum_h c_{\alpha hk}g_{hi_1\cdots\hat{i}_a\cdots i_p}$

$$+\sum_h\{\sum_{b<a}(-1)^b c_{\alpha hi_b}g_{hki_1\cdots\hat{i}_b\cdots\hat{i}_a\cdots i_p} +\sum_{a<b}(-1)^{b-1}c_{\alpha hi_b}g_{hki_1\cdots\hat{i}_a\cdots\hat{i}_b\cdots i_p}\}.$$

Replacing this in (4.9) and using (1.4) and (1.5) we obtain the lemma.

Lemma 8.

$$\begin{aligned}\sum_{i_1,\cdots i_p}\int_{G/\Gamma} g^2_{i_1\cdots i_p}dv &= \frac{2}{p}\sum_{k,i_1,\cdots,i_p}\int_{G/\Gamma}(X_kg_{i_1\cdots i_p})^2dv\\ &+\frac{4}{p}\sum_{b<a}\sum_{h,k,i_1,\cdots,i_p}(-1)^{a+b}\int_{G/\Gamma} R_{i_bhi_ak}g_{hi_1\cdots\hat{i}_b\cdots i_p}\cdot g_{ki_1\cdots\hat{i}_a\cdots i_p}dv.\end{aligned}$$

Proof. By Lemma 3 we have

$$\sum_k \int_{G/\Gamma} (X_k g_{i_1\cdots i_p})^2 dv = -\int_{G/\Gamma} (\sum_k X_k^2 g_{i_1\cdots i_p}) g_{i_1\cdots i_p} dv .$$

By Lemma 7, $-\int_{G/\Gamma} (\sum_k X_k^2 g_{i_1\cdots i_p}) g_{i_1\cdots i_p} dv$

$$= \frac{p}{2}\int_{G/\Gamma} g^2_{i_1\cdots i_p} dv - 2\sum_{b<a}\sum_{h,k}(-1)^{a+b}\int_{G/\Gamma} R_{i_b h i_a k} g_{h k i_1\cdots \hat{i}_b\cdots \hat{i}_a\cdots i_p} g_{i_1\cdots i_p} dv .$$

It is easily verified that

$$\sum_{i_1,\cdots,i_p}\sum_{h,k} R_{i_b h i_a k} g_{h k i_1\cdots \hat{i}_b\cdots \hat{i}_a\cdots i_p} g_{i_1\cdots i_p}$$
$$= \sum_{i_1,\cdots,i_p}\sum_{h,k} R_{i_a h i_b k} g_{h i_1\cdots \hat{i}_b\cdots i_p} g_{k i_1\cdots \hat{i}_a\cdots i_p} .$$

From these equalities follows the lemma.

§5. From (4.5) and (1.5) follows:

$$\begin{aligned} &X_j X_k g_{i_1\cdots i_p} - X_k X_j g_{i_1\cdots i_p} \\ &= \sum_{a=1}^{p}\sum_{\alpha=r+1}^{n}\sum_h (-1)^a c_{\alpha jk} c_{\alpha i_a h} g_{h i_1\cdots \hat{i}_a\cdots i_p} \end{aligned} \tag{5.1}$$

Calculating $2p\Phi$ directly from (5.1) as in [9], §4, we find:

$$2p\Phi = -\sum_{a,b=1}^{p}(-1)^{a+b}\sum_{\alpha,\beta}\sum_{i_1,\cdots,i_p}\sum_{h,l}\psi(X_\alpha, X_\beta) c_{\alpha i_a h} c_{\beta i_b l} g_{h i_1\cdots \hat{i}_a\cdots i_p} g_{l i_1\cdots \hat{i}_b\cdots i_p} ,$$

where $\psi(X, Y)(X, Y \in \mathfrak{k})$ denotes the bilinear form on $\mathfrak{k}$ defined by

$$\psi(X, Y) = Tr(ad_{\mathfrak{m}}X \cdot ad_{\mathfrak{m}}Y) ,$$

$ad_{\mathfrak{m}}X$ denoting the representation of $\mathfrak{k}$ on the vector space $\mathfrak{m}$ defined by $ad_{\mathfrak{m}}X \cdot Z = [X, Z]$ for all $Z \in \mathfrak{m}$. In particular we have

$$\psi(X_\alpha, X_\beta) = -\sum_{i,j=1}^{r} c_{\alpha ij} c_{\beta ij} .$$

Now let

$$\mathfrak{k} = \mathfrak{z} + \mathfrak{k}_1 + \cdots + \mathfrak{k}_q ,$$

where $\mathfrak{z}$ denotes the center of $\mathfrak{k}$ and $\mathfrak{k}_1, \cdots, \mathfrak{k}_q$ denote the simple ideals of $\mathfrak{k}$. As we have shown in [9], §4, we can choose a basis $\{X_{r+1}, \cdots, X_n\}$ of $\mathfrak{k}$ satisfying the following conditions: 1) each X_α belongs to $\mathfrak{z}$ or to one of the simple ideals; 2) $\varphi(X_\alpha, X_\beta) = -\delta_{\alpha\beta}$; 3) $\psi(X_\alpha, X_\beta) = 0$ for $\alpha \neq \beta$. Moreover, if $X_\alpha \in \mathfrak{z}$, we have $\psi(X_\alpha, X_\alpha) = -1$ and if $X_\alpha \in \mathfrak{k}_s$, $\psi(X_\alpha, X_\alpha) = -a_s$, where a_s a real number such that $0 < a_s < 1$ depending on $\mathfrak{k}_s$ (see [9], §4).

Choose a basis $\{X_{r+1}, \cdots, X_n\}$ of $\mathfrak{k}$ satisfying these conditions. We get then

$$2p\Phi$$

$$(5.2)\qquad = \sum_{a,b=1}^{p}(-1)^{a+b}\sum_{i_1,\cdots,i_p,h,l}\{\sum_{X_\alpha\in\mathfrak{z}} c_{\alpha i_a h}c_{\alpha i_b l}g_{hi_1\cdots\hat{i}_a\cdots i_p}g_{li_1\cdots\hat{i}_b\cdots i_p}$$
$$+\sum_{s=1}^{q} a_s \sum_{X_\alpha\in\mathfrak{k}_s} c_{\alpha i_a h}c_{\alpha i_b l}g_{hi_1\cdots\hat{i}_a\cdots i_p}g_{li_1\cdots\hat{i}_b\cdots i_p}\}\,.$$

§ **6.** From now on we assume that $\mathfrak{g}$ is simple. Let

$$A = \mathrm{Min}\,(a_1, \cdots, a_s)\,.$$

Then $0<A<1$. We know that the center $\mathfrak{z}$ of $\mathfrak{k}$ is (0) or 1-dimensional. Suppose first that $\dim\mathfrak{z}=1$ and let $\mathfrak{z}=\{X_{r+1}\}$. Then

$$2p\Phi$$
$$\geqq A\sum_{a,b=1}^{p}(-1)^{a+b}\sum_{i_1,\cdots,i_p,h,l}\sum_{\alpha} c_{\alpha i_a h}c_{\alpha i_b l}g_{hi_1\cdots\hat{i}_a\cdots i_p}g_{li_1\cdots\hat{i}_b\cdots i_p}$$
$$+(1-A)\sum_{i_1,\cdots,i_p}(\sum_{a=1}^{p}\sum_{h=1}^{r}(-1)^a c_{r+1 i_a h}g_{hi_1\cdots\hat{i}_a\cdots i_p})^2\,.$$

From (1.5) and (4.2) follows:

$$2p\Phi$$

$$(6.1)\qquad \geqq -A\sum_{a,b=1}^{p}\sum_{i_1,\cdots,i_p,h,l}(-1)^{a+b}R_{i_a h i_b l}g_{hi_1\cdots\hat{i}_a\cdots i_p}g_{li_1\cdots\hat{i}_b\cdots i_p}$$
$$+(1-A)\sum_{i_1,\cdots,i_p}(X_{r+1}g_{i_1\cdots i_p})^2\,.$$

Put

$$\Theta = \sum_{b<a}\sum_{i_1,\cdots,i_p,h,l}(-1)^{a+b}R_{i_a h i_b l}g_{hi_1\cdots\hat{i}_a\cdots i_p}g_{li_1\cdots\hat{i}_b\cdots i_p}\,.$$

By an easy calculation we see that the first term of the right hand side of the inequality (6.1) equals $A\frac{p}{2}\sum_{i_1,\cdots,i_p} g^2_{i_1\cdots i_p}-2A\Theta$. Hence we get finally the following inequality:

$$(6.2)\qquad \Phi\geqq\frac{A}{4}\sum_{i_1,\cdots,i_p} g^2_{i_1\cdots i_p}-\frac{A}{p}\Theta\,.$$

It should be noted that, if $\mathfrak{k}=\mathfrak{z}$ (this is the case if and only if $\dim\mathfrak{g}=3$), we have

$$\Phi = \frac{1}{4}\sum_{i_1,\cdots,i_p} g^2_{i_1\cdots i_p}-\frac{1}{p}\Theta\,.$$

Therefore we put $A=1$ in this case.
We see in a similar way that the inequality (6.2) holds also in the case $\mathfrak{z}=(0)$. Moreover, if $\mathfrak{k}$ is simple, the equality holds in (6.2).

§ **7.** Integrating the both sides of (6.2) we get:

$$\int_{G/\Gamma}\Phi\, dv \geqq \frac{A}{4}\sum_{i_1,\cdots,i_p}\int_{G/\Gamma} g^2_{i_1\cdots i_p}dv - \frac{A}{p}\int_{G/\Gamma}\Theta\, dv .$$

On the other hand, we have by Lemma 8

$$\sum_{i_1,\cdots,i_p}\int_{G/\Gamma} g^2_{i_1\cdots i_p}dv$$

$$= \frac{2}{p}\sum_{k,i_1,\cdots,i_p}\int_{G/\Gamma}(X_k g_{i_1\cdots i_p})^2 dv + \frac{4}{p}\int_{G/\Gamma}\Theta\, dv .$$

Hence we get:

$$\int_{G/\Gamma}\Phi\, dv \geqq \frac{A}{2p}\sum_{k,i,\cdots,i_p}\int_{G/\Gamma}(X_k g_{i_1\cdots i_p})^2 dv . \tag{7.1}$$

From (7.1) and Lemma 6 follows

Lemma 9.

$$0 \geqq \frac{A}{2p}\sum_{k,i_1,\cdots,i_p}\int_{G/\Gamma}(X_k g_{i_1\cdots i_p})^2 dv + \sum_{t_1,\cdots,t_{p-1}}\sum_{i,j,k,l}\int_{G/\Gamma} R_{iklj}(X_i g_{jt_1\cdots t_{p-1}})(X_k g_{lt_1\cdots t_{p-1}})dv .$$

If $\mathfrak{g}$ is simple and $\mathfrak{z}\neq(0)$, the symmetric space of G is an irreducible symmetric bounded domain. From the classification of such domain, it is known, except in the case of classical domain of type $I_{m,m'}$ $(m\geqq m'\geqq 2)$, $\mathfrak{k}$ has only one simple factor. In the case, where $\mathfrak{k}$ has only one simple factor, we have shown in [9], § 6, that

$$A = \frac{1}{\dim\mathfrak{k}-1}\left(\frac{\dim\mathfrak{m}}{2}-1\right), \tag{7.2}$$

while in the case where the corresponding domain is of type $I_{m,m'}$ $(m\geqq m'\geqq 2)$,

$$A = \frac{m'}{m+m'} . \tag{7.3}$$

In the case, where $\mathfrak{z}=(0)$ and $\mathfrak{k}$ is simple, we have

$$A = \frac{\dim\mathfrak{m}}{2\dim\mathfrak{k}} \tag{7.4}$$

(see [9], § 7).

We define for each simple non-compact Lie algebra $\mathfrak{g}$ a quadratic form $H_{\mathfrak{g}}^{p}(\xi)$ on the linear space of tensors $\xi=(\xi_{i_1\cdots i_{p+1}})$ by putting

$$H_{\mathfrak{g}}^{p}(\xi)=\frac{A}{2p}\sum_{i_1,\cdots,i_{p+1}}\xi_{i_1\cdots i_{p+1}}^2+\sum_{t_1,\cdots,t_{p-1}}\sum_{i,j,k,l}R_{iklj}\xi_{ijt_1\cdots t_{p-1}}\xi_{klt_1\cdots t_{p-1}}\,.$$

Then by Lemma 9 we have

$$0\geqq\int_{G/\Gamma}H_{\mathfrak{g}}^{p}((X_k g_{i_1\cdots i_p}))dv\,.$$

Remark. In the case $p=1$ we have already defined in [9] a quadratic form $H_{\mathfrak{g}}(\xi)=b(\mathfrak{g})\sum_{i,j}\xi_{ij}^2+\sum_{i,j,k,l}R_{iklj}\xi_{ij}\xi_{kl}$. The constant $b(\mathfrak{g})$ is strictly greater than $\frac{1}{2}A$. This is because we have omitted the factor $\sum_{i_1,\cdots,i_p}(X_{r+1}g_{i_1\cdots i_p})^2$ to obtain (6. 2), while in the case $p=1$ we have a convenient equality $\sum_i(X_{r+1}g_i)^2=\frac{1}{r}\sum_i g_i^2$ under a suitable normalisation of X_{r+1} (see [9], § 5); hence we obtain

$$\Phi\geqq\frac{1}{2}b(\mathfrak{g})\sum_i g_i^2$$

in place of (6. 2), with $b(\mathfrak{g})=\frac{1}{2}A+\frac{1}{r}(1-A)$. Remark that the term containing Θ is missing in the case $p=1$.

§8. We prove now the following theorem.

Theorem 1. *Let X be a simply connected, irreducible symmetric Riemannian manifold which is non-compact and non-euclidean. Let G be the identity component of the group of all isometries of X. Let Γ be a discrete subgroup of G with compact quotient G/Γ and without element of finite order different from the indentity, so that Γ is a discontinuous group of isometries of X with compact quotient $M=X/\Gamma$. Let X_u be the compact form of X. Suppose that the quadaatic form $H_{\mathfrak{g}}^{p}(\xi)$ is positive definite. Then the p-th Betti number $b_p(M)$ of M equals the p-th Betti number $b_p(X_u)$ of X_u.*

Proof. Let A^p denote the vector space of all G-invariant p-form of X. Since $\Gamma\subset G$, each p-form $\alpha\in A^p$ is Γ-invariant and hence there exists a p-form η of M such that $\alpha=\eta\circ\rho$, ρ denoting the projection of X onto M. As we have stated in the introduction, η is a harmonic p-form on M and the mapping $\alpha\to\eta$ defines an injection of A^p into the vector space $\mathfrak{h}^p$ of all harmonic p-form on M. Moreover, we know that the dimension of A^p equals the p-th Betti number $b^p(X_u)$ of X_u (see Introduction). We have $b_p(X_u)=b_p(M)$ if and only if the mapping $\alpha\to\eta$ is a surjection of A^p onto $\mathfrak{h}^p$. Now let $\eta\in\mathfrak{h}^p$ and let $\omega=\eta\circ\pi$, π denoting the projection of G/Γ onto $X/\Gamma=M$. We retain the notations introduced in the preceding sections and put

$$g_{\lambda_1\cdots\lambda_p}=\omega(X_{\lambda_1},\cdots,X_{\lambda_p})\,.$$

Then these $g_{\lambda_1\cdots\lambda_p}$ satisfy the relations (4. 1)-(4. 4). By Lemma 9 we have

$$0\geqq\int_{G/\Gamma}H_{\mathfrak{g}}^{\eta}((X_k g_{i_1\cdots i_p}))dv\,.$$

Since $H_{\mathfrak{g}}^{\eta}(\xi)$ is positive definite by assumption, this implies:

(8. 1) $$X_k g_{i_1\cdots i_p}=0\,,\quad 1\leqq k\,,i_1,\cdots,i_p\leqq r\,.$$

Then $X_jX_kg_{i\cdots i_p}-X_kX_jg_{i_1\cdots i_p}=0$. On the otoer hand $X_jX_kg_{i_1\cdots i_p}-X_kX_jg_{i_1\cdots i_p}=\sum_\alpha c_{\alpha jk}X_\alpha g_{i_1\cdots i_p}$ and hence

(8. 2) $$\sum_{\alpha=r+1}^{n}c_{\alpha jk}X_\alpha g_{i_1\cdots i_p}=0\,,\quad 1\leqq j,\,k,\,i_1,\cdots,i_p\leqq r\,.$$

Now we know that the representation $X\to ad_{\mathfrak{m}}X$ of the Lie algebra $\mathfrak{k}$ on the vector space $\mathfrak{m}$ defined by putting $ad_{\mathfrak{m}}X\cdot Z=[X,Z]$ for all $Z\in\mathfrak{m}$ is faithful. For any real numbers ξ_α, we have $[\sum_\alpha\xi_\alpha X_\alpha, X_k]=\sum_j(\sum_\alpha c_{\alpha jk}\xi_\alpha)X_j$. Hence, if $\sum_\alpha c_{\alpha jk}\xi_\alpha=0$ for $j,k=1,\cdots,r$, we must have $\xi_\alpha=0$. Thus (8. 2) implies

(8. 3) $$X_\alpha g_{i_1\cdots i_p}=0\,,\quad r+1\leqq\alpha\leqq n\,,\quad 1\leqq i_1,\cdots,i_p\leqq r\,.$$

From (8. 1) and (8. 3) follows that $g_{i_1\cdots i_p}$ are constant. On the other hand, by (4. 1) $g_{\lambda_1\cdots\lambda_p}=0$ if one of the indices is $>r$ and hence $g_{\lambda_1\cdots\lambda_p}$ is constant for any indices $\lambda_1,\cdots,\lambda_p$. Let $\tilde\rho$ denote the projection of G onto G/Γ and put $\tilde\omega=\omega\circ\tilde\rho$.

$$\begin{tikzcd}
 & G \arrow[dl, "\tilde\pi"'] \arrow[dr, "\tilde\rho"] & \\
X=K\ G \arrow[dr, "\rho"'] & & G/\Gamma \arrow[dl, "\pi"] \\
 & M &
\end{tikzcd}$$

As in §1, X_λ denotes here the right invariant vector field on G. Then

$\tilde{\omega}(X_{\lambda_1}, \cdots, X_{\lambda_p}) = g_{\lambda_1 \cdots \lambda_p}$ are constant and hence $\tilde{\omega}$ is a right invariant p-form on G. Moreover, since $i(X_\alpha)\omega = \theta(X_\alpha)\omega = 0$, we have also $i(X_\alpha)\tilde{\omega} = \theta(X_\alpha)\tilde{\omega} = 0$. Therefore there exists a G-invariant p-form α on $X = K\backslash G$ such that $\tilde{\omega} = \alpha \circ \tilde{\pi}$, $\tilde{\pi}$ denoting the projection of G onto $K\backslash G$. Since $\pi \circ \tilde{\rho} = \rho \circ \tilde{\pi}$ and $\tilde{\omega} = (\eta \circ \pi) \circ \tilde{\rho}$ we have $\tilde{\omega} = (\eta \circ \rho) \circ \tilde{\pi}$. Since $\tilde{\omega} = \alpha \circ \tilde{\pi}$, we get $\alpha = \eta \circ \rho$ and this shows that the harmonic p-form η is the image of the G-invariant p-form α on X. Since η is arbitrary, the mapping $A^p \to \mathfrak{h}^p$ defined above is surjective. Then we get $b_p(M) = b_p(X_u)$. Theorem 1 is thus proved.

§9. We discuss here when the quadratic form $H_{\mathfrak{g}}^p(\xi)$ is positive definite. For any tensor $\eta = (\eta_{ij})$ put

$$F_p(\eta) = \frac{A}{2p} \sum_{i,j} \eta_{ij}^2 + \sum_{i,j,k,l} R_{iklj} \eta_{ij} \eta_{kl} .$$

Let $\eta = \eta' + \eta''$, where $\eta' = (\eta'_{ij})$ is symmetric and $\eta'' = (\eta''_{ij})$ is alternating in the indices i, j. Using the property $R_{iklj} = R_{jlki}$, we see easily that

$$F_p(\eta) = F_p(\eta') + F_p(\eta'') .$$

(This and the following arguments are those of Weil [12]). Moreover, since η'' is alternating,

$$\sum_{i,j,k,l} R_{iklj} \eta''_{ij} \eta''_{kl} = - \sum_{i,j,k,l} R_{ilkj} \eta''_{ij} \eta''_{kl} = \sum_{i,j,k,l} R_{iljk} \eta''_{ij} \eta''_{kl} .$$

From this and from Bianchi identity $R_{iklj} + R_{iljk} + R_{ijkl} = 0$ follows:

$$2 \sum_{i,j,k,l} R_{iklj} \eta''_{ij} \eta''_{kl} = - \sum_{i,j,k,l} R_{ijkl} \eta''_{ij} \eta''_{kl} = \sum_{\alpha} \Big(\sum_{i,j} c_{\alpha ij} \eta''_{ij}\Big)^2 .$$

Therefore $F_p(\eta'') \geqq 0$ and $F_p(\eta'') = 0$ implies $\eta'' = 0$. Suppose that $F_p(\eta') > 0$ for any symmetric $\eta' \neq 0$ and we show that $H_{\mathfrak{g}}^p(\xi)$ is then positive definite. In fact, we write $\xi = \xi' + \xi''$, where $\xi' = (\xi'_{ijt_1 \cdots t_{p-1}})$ is symmetric and $\xi'' = (\xi''_{ijt_1 \cdots t_{p-1}})$ is alternating in the indices i, j. Since $H_{\mathfrak{g}}^p(\xi) = \sum_{t_1, \cdots, t_{p-1}} F_p((\xi_{ijt_1 \cdots t_{p-1}}))$, we have $H_{\mathfrak{g}}^p(\xi) = H_{\mathfrak{g}}^p(\xi') + H_{\mathfrak{g}}^p(\xi'')$ and $H_{\mathfrak{g}}^p(\xi'') \geqq 0$. Now since $F_p((\xi'_{ijt_1 \cdots t_{p-1}})) \geqq 0$ by assumption, we have $H_{\mathfrak{g}}^p(\xi') \geqq 0$ and hence $H_{\mathfrak{g}}^p(\xi) \geqq 0$. If $H_{\mathfrak{g}}^p(\xi) = 0$, we have $H_{\mathfrak{g}}^p(\xi') = 0$ and $H_{\mathfrak{g}}^p(\xi'') = 0$ and hence $F_p((\xi'_{ijt_1 \cdots t_{p-1}})) = F_p((\xi''_{ijt_1 \cdots t_{p-1}})) = 0$ for any $t_1, \cdots, t_{p-1}$. This implies $\xi'_{ijt_1 \cdots t_{p-1}} = \xi''_{jit_1 \cdots t_{p-1}} = 0$ for any indices $i, j, t_1, \cdots, t_{p-1}$ and hence $\xi = 0$. Thus $H_{\mathfrak{g}}^p(\xi)$ is positve definite.

Let P denote the linear transformation of the vector space of all symmetric tensor η' defined by putting

$$P(\eta')_{ij} = \sum_{k,l} R_{iklj} \eta'_{kl} .$$

Then $(P(\eta'), \zeta') = (\eta', P(\zeta'))$, (η, ζ') denoting usual inner product: (η', ζ')
$= \sum_{i,j} \eta'_{ij}\zeta'_{ij}$.
The linear transformation P of the vector space of η' is thus symmetric. The quadratis form $F_p(\eta')$ is written

$$F_p(\eta') = \frac{A}{2p}(\eta', \eta') + (\eta', P(\eta')) .$$

$F_p(\eta')$ is positive definite, if (and only if) the absolute value of the minimal eigen-value λ_1 of P is strictly smaller than $\frac{A}{2p}$.

We consider the case where X is an irreducible symmetric bounded domain. In this case, the value of A is calculated by (7.2) and (7.3), while the minimal eigen-value λ_1 of P is already known (see [1] [2] and [9], § 11). We obtain the following table.

Type of X	$\frac{A}{2}$	λ_1	$\frac{A}{2p}+\lambda_1$
$\mathrm{I}_{m,m'}$ $(m \geqq m' \geqq 1)$	$\frac{m'}{2(m+m')}$	$-\frac{1}{m+m'}$	$\frac{m'-2p}{2p(m+m')}$
II_m $(m \geqq 3)$	$\frac{m-2}{4(m-1)}$	$-\frac{1}{2(m-1)}$	$\frac{m-2-2p}{4p(m-1)}$
III_m $(m \geqq 2)$	$\frac{m+2}{4(m+1)}$	$-\frac{1}{m+1}$	$\frac{m+2-4p}{4p(m+1)}$
IV_m $(m \geqq 3)$	$\frac{1}{m}$	$-\frac{1}{m}$	
V	$\frac{1}{6}$	$-\frac{1}{12}$	$\frac{2-p}{12p}$
VI	$\frac{1}{6}$	$-\frac{1}{18}$	$\frac{3-p}{18p}$

Therefore $H^p_{\mathfrak{g}}(\xi)$ is positive difinite in the following cases:

Type $\mathrm{I}_{m,m'}$, $\quad \frac{m'}{2} > p$;

Type II_m, $\quad \frac{m-2}{2} > p$;

Type III_m, $\quad \frac{m+2}{4} > p$;

Type V, $\quad p = 1$;

Type VI, $\quad p = 1, 2$.

But we have proved in [9] that, except in the case of type $\mathrm{I}_{m,1}$, the first Betti number of M vanishes. On the other hand, the first Betti number of a compact, simply connected symmetric space vanishes. From

Theorem 1 and from the above results follows Theorem 2 stated in the introduction.

§ **10.** It is known that, for any simply connected compact Hermitian symmetric space X_u, we have $b_{2q+1}(X_u)=0$ and $b_{2q}(X_u)=h_{q,q}(X_u)$, where $h_{r,s}(X_u)$ denotes the dimension of the complex vector space of all harmonic forms of type (r, s) on X_u [5]. Moreover, if X_u is irreducible, we have $b_2(X_u)=h_{1,1}(X_u)=1$. The argument used in proving the inequality $b_p(M) \geqq b_p(X_u)$ shows that $h_{r,s}(M)\geqq h_{r,s}(X_u)$. Since M is Kählerian, we have $b_p(M)=\sum_{r+s=p} h_{r,s}(M)$. Therefore, if $b_p(M)=b_p(X_u)$, we must have $b_p(M)=0$ for odd p and $b_p(M)=h_{q,q}(M)$ for $p=2q$. In particular, if $b_2(M)=b_2(X_u)$ and if X_u (therefore X) is irreducible, we have $b_2(M)=h_{1,1}(M)=1$. From Theorem 2 we obtain the following theorem.

Theorem 3. *Let X be an irreducible symmetric bounded domain of one of the following types*: $\mathrm{I}_{m,m'}(m\geqq m'\geqq 6)$, $\mathrm{II}_m(m\geqq 7)$, III_m $(m\geqq 7)$, VI. *Then we have*

$$b_1(M) = 0\,, \quad b_2(M) = h_{1,1}(M) = 1\,.$$

We apply this result to the classification of automorphic factors. Let X be a symmetric bounded domain and let Γ be a discontinuous group on X without element of finite order different from the identity and with compact quotient space X/Γ. An automorphic factor k (with respect to Γ) is a mapping of $X\times\Gamma$ into $\boldsymbol{C}^*$ such that

$$k(z, \gamma\delta) = k(z\gamma, \delta)k(z, \gamma)$$

for any $z\in X$, γ, $\delta\in\Gamma$ and that $k(z, \gamma)$ is holomorphic in z. If k and k' are automorphic factors, so are the mappings $(z, \gamma)\rightarrow k(z, \gamma)\cdot k'(z, \gamma)$ and $(z, \gamma)\rightarrow k(z, \gamma)^{-1}$. Two automorphic factors k and k' are equivalent ($k\sim k'$), if there exists a non-vanishing holomorphic function f on X such that

$$k'(z, \gamma) = k(z, \gamma)f(z)f(z\gamma)^{-1}$$

for any $(z, \gamma)\in X\times\Gamma$.

The equivalence classes of automorphic factors form a group $\boldsymbol{F}$ with respect to the multiplication defined above.

Now, given an automorphic factor $k(z, \gamma)$, we can define a complex line bundle E_k over the complex manifolds X/Γ as follows. E_k is the quotient of $X\times\boldsymbol{C}$ by the equivalence relation: $(z, \xi)\sim(z\gamma, k(z, \gamma)\cdot\xi)$. It is known that the two line bundle E_k and $E_{k'}$ over X/Γ are isomorphic if and only if the automorphic factor k and k' are equivalent. A line bundle E over X/Γ is defined by an automorphic factor in the above way if and only if the induced bundle ρ^*E over X is analytically trivial, where ρ denotes the projection of X onto X/Γ. (For these facts on line bundles, cf. [11]). Now, since X is a homogeneous bounded domain in C^n, it is a domain of holomorphy by a theorem of Thullen and hence a Stein manifold. Moreover X is homeomorphic to a euclidean space. It follows then from the fundamental theorem (Theorem B) on Stein manifolds (see [6]) that every complex line bundle over X is analytically trivial (this is also a special case of a more general result of Grauert). Therefore every complex line bundle over X/Γ is defined by an automorphic factor and the group of equivalence classes of complex line bunles over X/Γ is isomorphic to the group $\boldsymbol{F}$.

Consider now the exact sequence of sheaves over X/Γ:

$$0 \to \boldsymbol{Z} \to \mathfrak{O} \to \mathfrak{O}^* \to 0\,,$$

where $\boldsymbol{Z}$, $\mathfrak{O}$, $\mathfrak{O}^*$ denote respectively the constant sheaf isomorphic to the additive group of integers, the sheaf of germs of holomorphic functions and the sheaf of germs of non-vanishing holomorphic functions on X/Γ (cf. [7]). We get the exact sequence of cohomologies:

$$\to H^1(X/\Gamma, \mathfrak{O}) \to H^1(X/\Gamma, \mathfrak{O}^*) \to H^2(X/\Gamma, \boldsymbol{Z}) \to H^2(X/\Gamma, \mathfrak{O}) \to$$

Suppose now that $b_1(X/\Gamma)=0$ and $b_2(X/\Gamma)=h_{1,1}(X/\Gamma)=1$. Since X/Γ is Kählerian, the dimensions of the complex vector spaces $H^1(X/\Gamma, \mathfrak{O})$ and $H^2(X/\Gamma, \mathfrak{O})$ equal $h_{0,1}(X/\Gamma)$ and $h_{0,2}(X/\Gamma)$ respectively by a theorem of Dolbeault (see [7]). It follows then that $H^1(X/\Gamma, \mathfrak{O})=H^2(X/\Gamma, \mathfrak{O})=(0)$ and hence $H^1(X/\Gamma, \mathfrak{O}^*) \simeq H^2(X/\Gamma, \boldsymbol{Z})$. The group $H^1(X/\Gamma, \mathfrak{O}^*)$ is identified with the group of equivalence classes of complex line bundles over X/Γ and hence isomorphic to $\boldsymbol{F}$. Thus $\boldsymbol{F} \simeq H^2(X/\Gamma, \boldsymbol{Z})$. Since $b_2(X/\Gamma)=1$, it follows that the group $\boldsymbol{F}$ is a direct product of an infinite cyclic subgroup $\boldsymbol{F}_1$ and a finite subgroup $\boldsymbol{F}_2$. For an automorphic factor k we denote by $[k]$ the equivalence class containing k. Let $[k] \in \boldsymbol{F}_2$. Then there exists an integer $m \geqq 0$ and a non-vanishing holomorphic function f on X such that $k(z, \gamma)^m = f(z)\cdot f(z\cdot\gamma)^{-1}$ for any $(z, \gamma) \in X \times \Gamma$. Since X is simply connected and $f(z) \neq 0$ for any $z \in X$, we can find a holomorphic function h on X such that $h(z)^m = f(z)$ for any $z \in X$. Let $\chi(z, \gamma) = k(z, \gamma)\cdot h(z)^{-1}\cdot h(z\cdot\gamma)$. Then $k \sim \chi$ and $\chi(z, \gamma)^m = 1$. It follows that $\chi(z, \gamma)$ is independent of z and hence $\chi(z, \gamma) = \chi(\gamma)$ is a 1-dimensional unitary representation on Γ, i.e. a character of the finite abelian group Γ/Γ', Γ' denoting the commutator group of Γ (note that Γ/Γ' is finite, because $b_1(X/\Gamma)=0$). Thus each equivalence class in $\boldsymbol{F}_2$ contains a character of Γ/Γ'. Conversely an equivalence class containing a character of Γ/Γ' is clearly of finite order. Now let χ_1 and χ_2 be two characters contained in one and the same equivalence class and let $\chi=\chi_1\cdot\chi_2^{-1}$. Then there exists a non-vanishing holomorphic function f on X such that $f(z\cdot\gamma)=\chi(\gamma)\cdot f(z)$ for any $(z, \gamma)\in X\times\Gamma$. Then $d\log f$ is a holomorphic 1-form on X invariant by Γ and hence it defines a holomorphic 1-form on X/Γ. Since $b_1(X/\Gamma)=0$, we must have $d\log f=0$. Then f is a constant and hence $\chi=1$ and we get $\chi_1=\chi_2$. Thus each equivalence class of finite order contains one and only one character of Γ/Γ'. Therefore $\boldsymbol{F}_2$ is isomorphic to the character group of the finite abelian group Γ/Γ'.

Let k_0 be an automorphic factor such that $[k_0]$ is a generator of the infinite cyclic group $\boldsymbol{F}_1$. For each automorphic factor k, there exist an integer n and a character χ of Γ/Γ' such that $[k]$ is written uniquely in the form:

$$[k] = [k_0^n]\cdot[\chi].$$

Combined with Theorem 3 we get

Theorem 4. *Let X be an irreducible symmetric bounded domain of one of the following types*: $\mathrm{I}_{m,m'}(m\geqq m'\geqq 6)$, $\mathrm{II}_m(m\geqq 7)$, $\mathrm{III}_m(m\geqq 7)$, VI. *Let Γ be a discrete subgroup of the identity component of the automorphism group of X. Suppose that the quotient space X/Γ is compact and that Γ contains no elements of finite order different from the identity. Then the group $\boldsymbol{F}$ of equivalence classes of automorphic factors is direct product of an infinite cyclic group and a finite abelian group, the latter being isomorphic to the character group of the finite abelian group Γ/Γ', where Γ' denotes the commutator group of Γ. Therefore there exists an automorphic factor k_0 with the following property: For each automorphic factor k, there exist a unique character χ of Γ/Γ' and a unique integer n such that $k\sim k_0^n\cdot\chi$.*

It will be an interesting problem to determine a "standard" k_0 for each type of X.

OSAKA UNIVERSITY

(Received November 27, 1961)

Bibliography

[1] A. Borel: *On the curvature tensor of the hermitian symmetric manifolds*, Ann. of Math. **71** (1960), 508–521.

[2] E. Calabi and E. Vesentini: *On compact, locally symmetric Kähler manifolds*, Ann. of Math. **71**(1960), 472–507.

[3] E. Cartan: La théorie des groupes finis et continus et l'analysis situs, Mém. Sci. Math., Gauthier-Villar, Paris, Nouveau tirage (1952).

[4] E. Cartan: *Sur les invariants intégraux de certains espaces homogènes clos et les propriétés topologiques de ces espaces*, Oeuvres complètes, Partie I, Vol. **2**, 1081–1125.

[5] E. Cartan: *Sur les propriétés topologiques des quadriques complexes*, Oeuvres complètes, Partie I, Vol. **2**, 1227–1246.

[6] H. Cartan: *Variétés analytiques complexes et cohomologie*, Colloque sur les fonctions de plusieurs variables, Centre belge de Recherches mathématiques, (1953) 41–45.

[7] F. Hirzebruch: Neue topologische Methoden in der algebraischen Geometrie, Ergebnisse der Math., Springer (1956).

[8] F. Hirzebruch: *Automorphe Formen und der Satz von Riemann-Roch*, Symposium inter. de topologia., Univ. de Mexico 1958, 129–143.

[9] Y. Matsushima: *On the first Betti number of compact quotient spaces of higher-dimensional symmetric spaces*, Ann. of Math. **75** (1962), 312–330.

[10] K. Nomizu: *Invariant affine connections on homogeneous spaces*, Amer. J. Math. **76** (1954), 33–65.

[11] S. Nakano: *An example of deformations of complex analytic bundles*, Memoirs of the College of Sci., Univ. Kyoto **31** (1958), 181–190.

[12] A. Weil: *On discrete subgroups of Lie groups, II*, Ann. of Math. **75** (1962), 578–602.

Added in Proof.

[13] Y. Matsushima and S. Murakami: *On vector bundle valued harmonic forms and automorphic forms on symmetric riemannian manifolds*, to appear.