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ON BETTI NUMBERS OF COMPACT, LOCALLY SYMMETRIC RIEMANNIAN MANIFOLDS

Dedicated to Professor K. Shoda on his sixtieth birthday

By

Yozô MATSUSHIMA

This is a continuation of our paper [9]. In [9] we have studied the vanishing of the first Betti number of compact, locally symmetric Riemannian manifolds. We shall study in this paper the \( p \)-th Betti number of these manifolds from a somewhat different point of view.

Let \( X \) be a simply connected, symmetric Riemannian manifold, all of whose irreducible components are non-euclidean and non-compact. Let \( G \) be the identity component of the group of all isometries of \( X \) and let \( \Gamma \) be a discrete subgroup of \( G \) with compact quotient space \( G/\Gamma \) and without element of finite order different from the identity. The group \( \Gamma \) acts on \( X \) discontinuously and the quotient space \( M = X/\Gamma \) is a compact, locally symmetric Riemannian manifold. Let \( A_p \) be the vector space of all \( G \)-invariant \( p \)-forms on \( X \). By a well-known theorem of E. Cartan, the covariant derivatives of each form in \( A_p \) vanish (see [10]). Since \( \Gamma \subset G \), each \( \omega \in A_p \) is \( \Gamma \)-invariant and hence there exists a \( p \)-form \( \eta \) on \( M \) such that \( \omega = \eta \circ \rho \), \( \rho \) denoting the projection of \( X \) onto \( M \). Since \( \rho \) is a locally isometric mapping and the covariant derivatives of \( \omega \) vanish, the covariant derivatives of \( \eta \) also vanish. In particular \( \eta \) is a harmonic \( p \)-form. Hence the mapping \( \omega \to \eta \) defines an injection of \( A_p \) into the vector space \( \mathfrak{h}^p \) of all harmonic \( p \)-forms on \( M \). The purpose of this paper is to study when \( A_p \) can be isomorphic to \( \mathfrak{h}^p \).

Let \( x_0 \in X \) and let \( K \) be the subgroup of \( G \) of all elements which leave fixed the point \( x_0 \). It is well-known that \( K \) is a maximal compact subgroup of \( G \) and \( X \) is identified with the quotient space \( K \backslash G \). Let \( \mathfrak{g} \) denote the Lie algebra of \( G \) and let \( \mathfrak{k} \) be the subalgebra of \( \mathfrak{g} \) corresponding to \( K \). Denote by \( \mathfrak{m} \) the orthogonal complement of \( \mathfrak{k} \) in \( \mathfrak{g} \) with respect to the Killing form of \( \mathfrak{g} \). We have then

\[
\mathfrak{g} = \mathfrak{m} + \mathfrak{k}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}.
\]

Let \( \mathfrak{g}^c \) be the complexification of \( \mathfrak{g} \) and let \( G^c \) be the complex Lie group,
with center reduced to the identity, corresponding to the complex Lie algebra \( g^c \). We may consider \( G \) as a subgroup of \( G^c \). Put \( g_u = \sqrt{-1} m + \mathfrak{t} \). Then \( g_u \) is a compact real form of \( g^c \) and let \( G_u \) be the subgroup of \( G^c \) corresponding to the real subalgebra \( g_u \) of \( g^c \). \( G_u \) is a maximal compact subgroup of \( G^c \) containing \( K \) and \( X_u = K \backslash G_u \) is a simply connected, compact symmetric Riemannian manifold which we shall call the compact form of \( X \).

Now we may identify the vector space \( A^p \) of all \( G \)-invariant \( p \)-forms on \( X \) with the vector space of all \( \mathfrak{g} \)-forms on the \( K \)-module \( m \) which are invariant by \( K \). On the other hand, the mapping \( X \to \sqrt{-1} X \) defines a \( K \)-module isomorphism of \( m \) onto \( \sqrt{-1} m \). It follows that the vector space \( A_u^p \) of all \( G_u \)-invariant \( p \)-forms on \( X_u \) is isomorphic to \( A^p \). Since \( X_u \) is symmetric and \( G_u \) is compact, the dimension of the vector space \( A_u^p \) equals the \( p \)-th Betti number \( b_p(X_u) \) of \( X_u \) [4]. Therefore the dimension of the vector space \( A^p \) is equal to \( b_p(X_u) \). On the other hand we have seen that \( A^p \) is identified with a subspace of the vector space \( \mathfrak{h}^p \) of all harmonic \( p \)-forms of \( M \). Thus we get the inequality \( b_p(M) \geq b_p(X_u) \), \( b_p(M) \) denoting the \( p \)-th Betti number of \( M \). Therefore our problem is stated as follows: Under what condition does the \( p \)-th Betti number of \( M \) equal the \( p \)-th Betti number of \( X_u \)?

A condition for this will be given by Theorem 1 in §8. From Theorem 1 we shall obtain the following result.

**Theorem 2.** Let \( X \) be an irreducible symmetric bounded domain and let \( X_u \) be the compact form of \( X \). Then the \( p \)-th Betti number of the compact manifold \( M = X/\Gamma \) equals the \( p \)-th Betti number of \( X_u \) in the following cases:

<table>
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<tr>
<th>Type of ( X )</th>
<th>( X_u )</th>
<th>value of ( p )</th>
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<tr>
<td>( I_{m, n'} ) ((m \geq m'))</td>
<td>( U(m+m')/U(m) \times U(m') )</td>
<td>( m' &gt; \frac{p}{2} ) ((m' \geq 3); p = 1 ((m' = 2))</td>
</tr>
<tr>
<td>( II_m ) ((m \geq 3))</td>
<td>( SO(2m)/U(m) )</td>
<td>( \frac{m-2}{2} &gt; p ) ((m &gt; 4); p = 1 ((m = 4))</td>
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<tr>
<td>( III_m ) ((m \geq 2))</td>
<td>( Sp(m)/U(m) )</td>
<td>( \frac{m+2}{4} &gt; p ) ((m &gt; 2); p = 1 ((m = 2))</td>
</tr>
<tr>
<td>( IV_m ) ((m \geq 3))</td>
<td>( SO(n+2)/SO(2) \times SO(n) )</td>
<td>( p = 1 )</td>
</tr>
<tr>
<td>( V )</td>
<td>( E_8/Spin(10) \times T^1 )</td>
<td>( p = 1 )</td>
</tr>
<tr>
<td>( VI )</td>
<td>( E_7/E_6 \times T^1 )</td>
<td>( p = 1, 2 )</td>
</tr>
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(The notations here are those of [2]).
In §10 we shall apply these results to the classification of automorphic factors. The method employed in this paper is a natural extension of that of our paper [9].

I express here my hearty thanks to S. Murakami for his friendly cooperations. Especially the results in §§2 and 3 owe to him.

§1. Throughout this paper $G$ will denote a connected semi-simple Lie group with center reduced to the identity, all of whose simple components are non-compact; $K$ will denote a maximal compact subgroup of $G$. We denote by $\mathfrak{g}$ the Lie algebra of all right invariant vector fields on $G$ and by $\mathfrak{k}$ the subalgebra of $\mathfrak{g}$ corresponding to $K$. Let $\varphi$ denote the Killing form of $\mathfrak{g}$ and let

$$m = \{X \in \mathfrak{g}; \varphi(X, Y) = 0 \text{ for all } Y \in \mathfrak{k}\}.$$ 

Then we have

$$\mathfrak{g} = m + \mathfrak{k}, \quad m \cap \mathfrak{k} = (0);$$

$$[m, m] \subset \mathfrak{k}, [\mathfrak{k}, \mathfrak{k}] \subset m, [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}. \tag{1.1}$$

We know that the restriction of $\varphi$ onto $m$(resp. $\mathfrak{k}$) defines a positive (resp. negative) definite bilinear form on $m$ (resp. $\mathfrak{k}$). Hence we can choose a basis $X_1, \ldots, X_r$ of $m$ and $X_{r+1}, \ldots, X_n$ of $\mathfrak{k}$ such that

$$\varphi(X_i, X_j) = \delta_{ij}, \quad 1 \leq i, j \leq r;$$
$$\varphi(X_\alpha, X_\beta) = -\delta_{\alpha\beta}, \quad r + 1 \leq \alpha, \beta \leq n. \tag{1.2}$$

Throughout this paper we make the following convention: Latin indices $i, j, k, \ldots$, will range from 1 to $r$, while Greek indices $\alpha, \beta, \gamma, \ldots$ will range from $r+1$ to $n$ and the indices $\lambda, \mu, \nu, \ldots$ from 1 to $n$. Let

$$[X_\lambda, X_\mu] = \sum_{\nu=1}^r c^\nu_{\lambda\mu} X_\nu. \tag{1.3}$$

By (1.1) among the structure constants $c^\nu_{\lambda\mu}$ only the $c^\nu_{\alpha\beta}, c^\alpha_{ij}, c^i_{a\beta}, c^i_{ja}$ can be $\neq 0$. We shall write $c_{aij}$ instead of $c^a_{ij}$. From the invariance of the Killing form under the adjoint representation follows that

$$c^j_{ia} = -c^j_{ji} = c^i_{aj} = c_{aij}. \tag{1.4}$$

We get from (1.2):

$$\sum_{k=1}^r \sum_{a=r+1}^n c_{ak} c_{ajk} = \frac{1}{2} \delta_{ij}. \tag{1.4}$$

For any $X, Y \in m$, we denote by $R(X, Y)$ the endomorphism of the vector space $m$ defined by
\[ R(X, Y) \cdot Z = [[Y, X], Z] \]

for all \( Z \in \mathfrak{m} \). It can be interpreted as the Riemannian curvature tensor for the Riemannian symmetric manifold \( K \backslash G \) (cf. \([10]\)). We have

\[ R(X_h, X_h) \cdot X_i = \sum_{i=1}^{r} R_{ijkh} \cdot X_i, \]

where

\begin{equation}
R_{ijkh} = R_{ijhk} = -\sum_{a=1}^{n} C_{aij} C_{akh}. \tag{1.5}
\end{equation}

From (1.4) follows:

\begin{equation}
\sum_{i=1}^{r} R_{ikj} = -\frac{1}{2} \delta_{ij}. \tag{1.6}
\end{equation}

Let \( \omega^\lambda(\lambda = 1, \ldots, n) \) be the right invariant 1–forms on \( G \) such that \( \omega^\lambda(X_\mu) = \delta^\lambda_\mu \). The symmetric tensor \( g = \sum_{\lambda=1}^{r} (\omega^\lambda)^2 \) defines a right invariant Riemannian metric on \( G \).

Now let \( X = K \backslash G \) and let \( \overline{\pi} \) be the projection of \( G \) onto \( X \). Put \( \overline{\pi}(e) = x_0 \), \( e \) denoting the identity of \( G \). We may identify the vector space \( \mathfrak{m} \) with the tangent vector space of \( X \) at the point \( x_0 \). We define a \( G \)-invariant symmetric tensor \( h \) on \( X \) by the condition that the value of \( h \) at the point \( x_0 \) is equal to the restriction of \( \phi \) onto \( \mathfrak{m} \). The tensor \( h \) is everywhere positive definite and it defines a \( G \)-invariant Riemannian metric on \( X \). \( X \) is a simply connected, symmetric Riemannian manifold.

§ 2. Let \( \Gamma \) be a discrete subgroup of \( G \) with compact quotient space \( G/\Gamma \). We may regard \( \Gamma \) as a discontinuous group of isometries of the symmetric Riemannian manifold \( X \) with compact quotient space \( X/\Gamma \). Put \( M = X/\Gamma \). From now on, we assume that \( \Gamma \) contains no element of finite order different from the identity. Then \( M \) is a compact orientable manifold without singularities and \( G/\Gamma \) is a principal fiber bundle of base \( M \) and structure group \( K \). We may consider \( M \) as the space of double cosets: \( M = K \backslash G/\Gamma \). Let \( \pi \) denote the projection of \( G/\Gamma \) onto \( M \). The symmetric tensor \( h \) on \( X \) being invariant by the action of \( \Gamma \), it defines a positive definite symmetric tensor on \( M \) which we denote by the same letter \( h \). \( h \) defines a Riemannian metric on \( M \). We call \( M \) a compact, locally symmetric Riemannian manifold.

The right invariant tensor \( g \) on \( G \) defines a positive definite symmetric tensor on \( G/\Gamma \) which we shall denote also by \( g \). The right
invariant vector fields $X_\lambda$ (resp. 1-forms $\omega^\lambda$) on $G$ define also the vector fields (resp. 1-forms) on $G/\Gamma$ which we shall denote by the same letter $X_\lambda$ (resp. $\omega^\lambda$). Under these notations, consider the symmetric tensor $\sum_i (\omega_i)^2$ on $G/\Gamma$. It is easily verified that this tensor is invariant under the action of $K$ on $G/\Gamma$. Moreover, the value of this tensor at each point of $G/\Gamma$ is a symmetric bilinear form on the tangent space which is positive definite modulo the subspace of vectors tangent to the fiber of the bundle $G/\Gamma$ over $M$, this latter subspace being spanned by the values of the vector fields $X_\alpha$ ($\alpha = r + 1, \cdots, n$). From these considerations follows that the symmetric tensor $h$ on $M$ is related to the tensor $\sum_i (\omega_i)^2$ on $G/\Gamma$ by the relation

$$h \circ \pi = \sum_i (\omega_i)^2.$$  

Let $dm$ denote the volume element of $M$ determined by the Riemannian metric defined by $h$. From (2.1) follows that

$$dm \circ \pi = \omega^1 \wedge \cdots \wedge \omega^r.$$  

We denote by $dv$ the volume element of $G/\Gamma$ determined by the Riemannian metric $g$. Then

$$dv = \omega^1 \wedge \cdots \wedge \omega^r \wedge \cdots \wedge \omega^n.$$  

Further let $dk$ denote the bi-invariant volume element $\omega^{r+1} \wedge \cdots \wedge \omega^n$ of the compact group $K$, where $\omega^{r+1}, \cdots, \omega^n$ are considered as right invariant 1-forms on $K$.

**Lemma 1.** Let $f$ be a continuous function on $M$. Then

$$c \int_M f dm = \int_{\omega/\Gamma} (f \circ \pi) dv,$$

where $c$ denotes the total volume of $K$ measured by $dk$.

For the proof of this lemma, see [13].

Now let $\eta, \xi$ be two $p$-forms on $M$. We denote by $\langle \eta, \xi \rangle$ the usual scalar product defined by the Riemannian metric $h$. The global scalar product $(\eta, \xi)$ is defined by

$$(\eta, \xi) = \frac{1}{|K|} \int_M \langle \eta, \xi \rangle dm.$$  

In the same way, for any two $p$-forms $\omega, \theta$ on $G/\Gamma$, we define $\langle \omega, \theta \rangle$
and \((\omega, \theta)\) using the Riemannian metric \(g\). Put
\[ g_{\lambda_1, \cdots, \lambda_p} = \omega(X_{\lambda_1}, \cdots, X_{\lambda_p}) \]
and
\[ h_{\lambda_1, \cdots, \lambda_p} = \theta(X_{\lambda_1}, \cdots, X_{\lambda_p}). \]
Since \(X_1, \cdots, X_n\) form an orthonormal basis of the tangent vector space at each point of \(G/\Gamma\), we have
\[ \langle \omega, \theta \rangle = \sum_{\lambda_1, \cdots, \lambda_p} g_{\lambda_1, \cdots, \lambda_p} h_{\lambda_1, \cdots, \lambda_p}. \]
Hence
\[ \langle \omega, \theta \rangle = \frac{1}{p!} \sum_{\lambda_1, \cdots, \lambda_p} \int_{G/\Gamma} g_{\lambda_1, \cdots, \lambda_p} h_{\lambda_1, \cdots, \lambda_p} dv. \]

Now let \(\eta, \zeta\) be two \(p\)-forms on \(M\). We see from (2.1) that
\[ \langle \eta, \zeta \rangle = \langle \eta \circ \pi, \zeta \circ \pi \rangle. \]

It follows then from Lemma 1 the following

**Lemma 2.** Let \(\eta, \zeta\) be two \(p\)-forms on \(M\). Then
\[ c \cdot (\eta, \zeta) = (\eta \circ \pi, \zeta \circ \pi). \]

We state here the following lemma which will be used frequently in the following.

**Lemma 3.** Let \(f\) be a \(C^\infty\)-function on \(G/\Gamma\). Then
\[ \int_{G/\Gamma} X_\lambda f \cdot dv = 0, \quad 1 \leq \lambda \leq n. \]

For the proof, see Weil [12].

\[ \sum_{\lambda_1, \cdots, \lambda_p} \sum_{a=1}^{n} (-1)^{a-1} X_a \cdot \eta'(X_{\lambda_1}, \cdots, X_{\lambda_p}) = 0. \]

But since
\[ [X_{ia}, X_{ib}] = \sum_{a} c_{ia} a_{ip} X_a \text{ and } i(X_a) \eta' = 0, \]
we get:
\[ (d \eta \cdot \pi)(X_{i_1}, \cdots, X_{i_{p+1}}) = \sum_{a=1}^{n} (-1)^{a-1} X_a \{ (\eta \circ \pi)(X_{i_1}, \cdots, X_{i_p}) \}. \]

Let \(\delta\) denote the operator of codifferentiation on the differential forms on \(M\). For any \(\eta \in D^p(M)\) and \(\zeta \in D^{p+1}(M)\), we have
\[ (d \eta, \zeta) = (\eta, \delta \zeta). \]
For any \(\zeta \in D^{p+1}(M)\), put
Let \( \zeta' = \zeta \circ \pi \). Then
\[
(\delta' \zeta')(X_{i_1}, \ldots, X_{i_p}) = -\sum_{k=1}^{p} X_k \cdot \zeta'(X_k, X_{i_1}, \ldots, X_{i_p}).
\]

**Proof.** For any \( \eta \in D^p(M) \), let \( \eta' = \eta \circ \pi \). Put \( \eta'(X_{i_1}, \ldots, X_{i_p}) = f_{i_1} \cdots i_p \) and \( \zeta'(X_{i_1}, \ldots, X_{i_p}) = g_{i_1} \cdots i_p \). From (3.1) and (2.2) we get:
\[
(d\eta', \zeta') = \frac{1}{(p+1)!} \sum_{\alpha=1}^{p+1} (-1)^{\alpha-1} \int_{\partial \Omega} (X_{i_1} f_{i_1} \cdots i_p \cdots i_{p+1}) \cdot g_{i_1} \cdots i_{p+1} dv.
\]

By Lemma 3,
\[
\int_{\partial \Omega} (X_{i_1} f_{i_1} \cdots i_p \cdots i_{p+1}) \cdot g_{i_1} \cdots i_{p+1} dv = (-1)^{\alpha} \int_{\partial \Omega} f_{i_1} \cdots i_p \cdots i_{p+1} (X_{i_1} g_{i_1} \cdots i_p \cdots i_{p+1}) dv.
\]

Hence we obtain:
\[
(d\eta', \zeta') = -\frac{1}{p!} \sum_{\alpha=1}^{p+1} \int_{\partial \Omega} (\sum_k X_k g_{k i_1} \cdots i_p) dv.
\]

We define a \( p \)-form \( \mu' \) on \( G/\Gamma \) by the conditions that
\[
\mu'(X_{i_1}, \ldots, X_{i_p}) = -\sum_{k=1}^{p} X_k g_{k i_1} \cdots i_p, \quad i(X_a) \mu' = 0 \quad (r+1 \leq \alpha \leq n).
\]

Then
\[
(d\eta', \zeta') = (\eta', \mu').
\]

Using \( i(X_a) \zeta' = 0, \theta(X_a) \zeta' = 0 \), we verify easily that \( \theta(X_a) \mu' = 0 \) for all \( X_a \). Hence there exists a \( p \)-form \( \mu \) on \( M \) such that \( \mu' = \mu \circ \pi \). Then
\[
(d\eta', \zeta') = (\eta \circ \pi, \mu \circ \pi).
\]

On the other hand, by Lemma 2 we have \( (d\eta', \zeta') = c(d\eta, \xi) \) and \( (\eta \circ \pi, \mu \circ \pi) = c(\eta, \mu) \). Hence we get \( (d\eta, \zeta) = (\eta, \mu) \). Since \( \eta \) is arbitrary, this implies \( \mu = \delta \zeta \) and hence \( \mu' = \delta' \zeta' \). Thus \( (\delta' \zeta')(X_{i_1}, \ldots, X_{i_p}) = -\sum_k X_k \cdot \zeta'(X_k, X_{i_1}, \ldots, X_{i_p}). \)

From (3.1), (3.2) and Lemma 4 follows

**Lemma 5.** A \( p \)-form \( \omega \) on \( G/\Gamma \) is a \( \pi \)-image of a harmonic \( p \)-form on \( M \) if and only if the following three conditions are satisfied:
1) \( i(X_a) \omega = \theta(X_a) \omega = 0, \quad r+1 \leq \alpha \leq n; \)
2) \( \sum_{i=1}^{r+1} (-1)^{\alpha-1} X_{i_a} \cdot \omega(X_{i_1}, \ldots, X_{i_a}, \ldots, X_{i_{p+1}}) = 0, \quad 1 \leq i, \ldots, i_{p+1} \leq r; \)
3) \( \sum_{k=1}^{r} X_k \cdot \omega(X_k, X_{i_1}, \ldots, X_{i_{p-1}}) = 0, \quad 1 \leq i, \ldots, i_{p-1} \leq r. \)

§ 4. We retain the notations introduced in the preceding sections. Let \( \omega \) be a \( p \)-form on \( G/\Gamma \) satisfying the conditions in Lemma 5 and put
Then we have the following equalities:

(4.1) \[ g_{\lambda_1 \cdots \lambda_p} = 0 \text{ if one of the indices is } \geq r+1; \]

(4.2) \[ X_{a}^{i_1 \cdots i_p} = \sum_{\alpha=1}^{\rho} \sum_{\alpha=1}^{r} (-1)^{\alpha-1} c_{a \alpha \lambda} g^{i_1 \cdots i_{\alpha-1} \cdots i_{\alpha} \cdots i_p}, \]

\[ r+1 \leq \alpha \leq r, \quad 1 \leq i_1, \cdots, i_p \leq r; \]

(4.3) \[ X_{b}^{i_1 \cdots i_p} = \sum_{\alpha=1}^{r} (-1)^{\alpha-1} X_{a}^{i_1 \cdots i_{\alpha-1} \cdots i_{\alpha} \cdots i_p}, \quad 1 \leq k, i_1, \cdots, i_p \leq r; \]

(4.4) \[ \sum_{\alpha=1}^{\rho} X_{b}^{i_1 \cdots i_{\alpha-1}} = 0, \quad 1 \leq i_1, \cdots, i_{\alpha-1} \leq r. \]

Now we have

\[ X_{j}^{i_1 \cdots i_p} - X_{k}^{i_1 \cdots i_p} = \sum_{a} c_{ajk} X_{a}^{i_1 \cdots i_p}. \]

From (4.2) and (1.5) follows

\[ \sum_{a} c_{ajk} X_{a}^{i_1 \cdots i_p} = \sum_{\alpha=1}^{\rho} (-1)^{\alpha-1} \sum_{\alpha=1}^{r} R_{j \alpha k}^{i_1 \cdots i_{\alpha-1} \cdots i_{\alpha} \cdots i_p}. \]

Thus we get

(4.5) \[ X_{j}^{i_1 \cdots i_p} - X_{k}^{i_1 \cdots i_p} = \sum_{\alpha=1}^{r} (-1)^{\alpha-1} \sum_{\alpha=1}^{r} R_{j \alpha k}^{i_1 \cdots i_{\alpha-1} \cdots i_{\alpha} \cdots i_p}. \]

Now put

(4.6) \[ \Phi = \frac{1}{2p} \sum_{i_1, \cdots, i_p} (X_{j}^{i_1 \cdots i_p} - X_{k}^{i_1 \cdots i_p})^{2}. \]

**Lemma 6.**

\[ \int_{\partial / \Gamma} \Phi \, dv = - \sum_{i_1, \cdots, i_p} \sum_{j, a, k} \int_{\partial / \Gamma} R_{j \alpha k}^{i_1 \cdots i_{\alpha-1} \cdots i_{\alpha} \cdots i_p} (X_{a}^{i_1 \cdots i_{\alpha-1}}) dv. \]

**Proof.** From (4.5) follows

\[ \Phi = \frac{1}{2p} \sum_{i_1, \cdots, i_p} \sum_{j, a, k} (-1)^{\alpha-1} R_{j \alpha k}^{i_1 \cdots i_{\alpha-1} \cdots i_{\alpha} \cdots i_p} g^{i_1 \cdots i_{\alpha-1} \cdots i_{\alpha} \cdots i_p} \]

\[ = \frac{1}{p} \sum_{a=1}^{\rho} \sum_{i_1, \cdots, i_p} \sum_{j, a, k} (-1)^{\alpha-1} R_{j \alpha k}^{i_1 \cdots i_{\alpha-1} \cdots i_{\alpha} \cdots i_p} g^{i_1 \cdots i_{\alpha-1} \cdots i_{\alpha} \cdots i_p}. \]

Put

\[ \Phi' = \frac{1}{p} \sum_{a=1}^{\rho} \sum_{i_1, \cdots, i_p} \sum_{j, a, k} (-1)^{\alpha} R_{j \alpha k}^{i_1 \cdots i_{\alpha-1} \cdots i_{\alpha} \cdots i_p} g^{i_1 \cdots i_{\alpha-1} \cdots i_{\alpha} \cdots i_p}. \]

By Lemma 3 we have

(4.7) \[ \int_{\partial / \Gamma} \Phi \, dv = \int_{\partial / \Gamma} \Phi' \, dv. \]
Now we have

$$\sum_{i_1,\ldots,i_p} \sum_{h,k} R_{jk,ih}(X_k g_{i_1\ldots i_p})(X_j g_{hi_1\ldots i_p}) = \sum_{i_1,\ldots,i_p} \sum_{h,k} R_{jk,ih}(-1)^{a-b}(X_k g_{i_1\ldots i_p})(X_j g_{hi_1\ldots i_p}) = (-1)^{a-b} \sum_{i_1,\ldots,i_p} \sum_{h,k} R_{jk,ih}(X_k g_{i_1\ldots i_p})(X_j g_{hi_1\ldots i_p}).$$

Hence

$$(4.8) \quad \Phi' = - \sum_{i_1,\ldots,i_p} \sum_{h,k} R_{jk,ih}(X_k g_{i_1\ldots i_p})(X_j g_{hi_1\ldots i_p}).$$

From (4.8) and (4.7) follows the lemma.

**Lemma 7.**

$$(\sum X_a^2) g_{i_1\ldots i_p} = -\frac{p}{2} g_{i_1\ldots i_p} + 2 \sum_{a=1}^p (-1)^{a+b} \sum_{h,k} R_{jk,ih}(X_k g_{i_1\ldots i_p}).$$

**Proof.** Put

$$F = (\sum X_a^2) g_{i_1\ldots i_p}.$$

We let operate \(X_k\) on the both sides of (4.3) and summing up on \(k\) we get:

$$F = \sum_{a=1}^p (-1)^{a-1} \sum_k X_k X_a g_{hi_1\ldots i_p}.$$

Since \(X_k X_a = X_a X_k + \sum c_{ah} X_a\) and since (4.4) holds, we get

$$(4.9) \quad F = \sum_{a=1}^p (-1)^{a-1} \sum_k X_k g_{hi_1\ldots i_p}.$$

By (4.2), \(X_a g_{hi_1\ldots i_p} = \sum c_{ab} X_b g_{hi_1\ldots i_p}\)

$$+ \sum_k \left( \sum_{b=1}^p (-1)^b c_{ab} X_b g_{hi_1\ldots i_p} \right) + \sum_{d=1}^p (-1)^{b-1} c_{ab} X_b g_{hi_1\ldots i_p}. $$

Replacing this in (4.9) and using (1.4) and (1.5) we obtain the lemma.

**Lemma 8.**

$$\sum_{i_1,\ldots,i_p} \left( g_{i_1\ldots i_p}^2 \right) dv = \frac{2}{p} \sum_{i_1,\ldots,i_p} \left( X_k g_{i_1\ldots i_p} \right)^2 dv + \frac{4}{p} \sum_{b=1}^p \sum_{h,i_1,\ldots,i_p} (-1)^{a+b} R_{jk,ib}(X_k g_{i_1\ldots i_p}, g_{hi_1\ldots i_p}) dv.$$

**Proof.** By Lemma 3 we have
\[
\sum \int_{\sigma/T} (X_h g_{i_1 \cdots i_p})^2 dv = - \int_{\sigma/T} \left( \sum X_i^2 g_{i_1 \cdots i_p} \right) g_{i_1 \cdots i_p} dv.
\]

By Lemma 7, 
\[
- \int_{\sigma/T} \left( \sum X_i^2 g_{i_1 \cdots i_p} \right) g_{i_1 \cdots i_p} dv = \frac{\rho}{2} \int g_{i_1 \cdots i_p}^2 dv - 2 \sum_{b} \sum_{a, \beta} (-1)^{a+b} \int R_{i_1 i_2 \cdots i_p} g_{i_1 \cdots i_p} g_{i_1 \cdots i_p} dv.
\]

It is easily verified that
\[
\sum \sum R_{i_1 i_2 \cdots i_p} g_{i_1 \cdots i_p} = \sum \sum R_{i_1 i_2 \cdots i_p} g_{i_1 \cdots i_p} g_{i_1 \cdots i_p}.
\]

From these equalities follows the lemma.

§5. From (4.5) and (1.5) follows:

\[
X_j X_h g_{i_1 \cdots i_p} - X_h X_j g_{i_1 \cdots i_p} = \sum_{a=1}^{m} \sum_{a=1}^{n} \sum_{i=1}^{n} (-1)^{a+b} C_{a,b} C_{a,b} g_{i_1 \cdots i_p} g_{i_1 \cdots i_p}.
\]

Calculating \(2\rho \Phi\) directly from (5.1) as in [9], §4, we find:

\[
2\rho \Phi = - \sum_{a=1}^{m} \sum_{a=1}^{n} \sum_{i=1}^{n} (-1)^{a+b} \psi(X_a, X_{\beta}) C_{a_i \beta} C_{a_i \beta} g_{i_1 \cdots i_p} g_{i_1 \cdots i_p},
\]

where \(\psi(X, Y)(X, Y \in \mathfrak{k})\) denotes the bilinear form on \(\mathfrak{k}\) defined by

\[
\psi(X, Y) = Tr(ad_{m} X \cdot ad_{m} Y),
\]

\(ad_{m} X\) denoting the representation of \(\mathfrak{k}\) on the vector space \(m\) defined by \(ad_{m} X \cdot Z = [X, Z]\) for all \(Z \in m\). In particular we have

\[
\psi(X_a, X_{\beta}) = - \sum_{i,j=1}^{n} C_{a_i \beta} C_{b_j}.
\]

Now let

\[
\mathfrak{f} = \mathfrak{g} + \mathfrak{k}_1 + \cdots + \mathfrak{k}_q,
\]

where \(\mathfrak{g}\) denotes the center of \(\mathfrak{f}\) and \(\mathfrak{k}_1, \cdots, \mathfrak{k}_q\) denote the simple ideals of \(\mathfrak{f}\). As we have shown in [9], §4, we can choose a basis \(\{X_{r+1}, \cdots, X_n\}\) of \(\mathfrak{k}\) satisfying the following conditions: 1) each \(X_a\) belongs to \(\mathfrak{g}\) or to one of the simple ideals; 2) \(\psi(X_a, X_{\beta}) = - \delta_{a, \beta}\); 3) \(\psi(X_a, X_{\beta}) = 0\) for \(a \neq \beta\). Moreover, if \(X_a \in \mathfrak{k}\), we have \(\psi(X_a, X_a) = -1\) and if \(X_a \in \mathfrak{k}_r\), \(\psi(X_a, X_a) = - \alpha_s\), where \(\alpha_s\) a real number such that \(0 < \alpha_s < 1\) depending on \(\mathfrak{k}_s\) (see [9], §4).
Choose a basis \( \{X_{r+1}, \ldots, X_n\} \) of \( \mathfrak{f} \) satisfying these conditions. We get then

\[
2p\Phi
\]

\[
= \sum_{a,b=1}^p (-1)^{a+b} \sum_{i_1, \ldots, i_p, h, l} \left\{ \sum_{X_a \in \mathfrak{g}} c_{ai_a h l} c_{ai_b l} g_{i_1} \cdots i_p g_{i_l} \cdots i_p \right\} 
+ \sum_{i_1, \ldots, i_p} a_s \sum_{X_a \in \mathfrak{g}} c_{ai_a h l} c_{ai_b l} g_{i_1} \cdots i_p g_{i_l} \cdots i_p .
\]

§ 6. From now on we assume that \( g \) is simple. Let

\[
A = \text{Min} (a_1, \ldots, a_s).
\]

Then \( 0 < A < 1 \). We know that the center \( \mathfrak{z} \) of \( \mathfrak{f} \) is (0) or 1-dimensional. Suppose first that \( \dim \mathfrak{z} = 1 \) and let \( \mathfrak{z} = \{X_{r+1}\} \). Then

\[
2p\Phi
\]

\[
\geq A \sum_{a,b=1}^p (-1)^{a+b} \sum_{i_1, \ldots, i_p, h, l} c_{ai_a h l} c_{ai_b l} g_{i_1} \cdots i_p g_{i_l} \cdots i_p 
+ (1-A) \sum_{i_1, \ldots, i_p} (\sum_{h,l} (-1)^{a+b} c_{r+i_a h l} g_{i_1} \cdots i_p)^2 .
\]

From (1.5) and (4.2) follows:

\[
2p\Phi
\]

\[
\geq -A \sum_{a,b=1}^p (-1)^{a+b} R_{i_1, \ldots, i_p, h, l} g_{i_1} \cdots i_p g_{i_l} \cdots i_p 
+ (1-A) \sum_{i_1, \ldots, i_p} (X_{r+1} g_{i_1} \cdots i_p)^2 .
\]

Put

\[
\Theta = \sum_{b=1}^p \sum_{i_1, \ldots, i_p, h, l} (-1)^{a+b} R_{i_1, \ldots, i_p, h, l} g_{i_1} \cdots i_p g_{i_l} \cdots i_p .
\]

By an easy calculation we see that the first term of the right hand side of the inequality (6.1) equals \( A \sum_{i_1, \ldots, i_p} g_{i_1} \cdots i_p - 2A\Theta \). Hence we get finally the following inequality:

\[
\Phi \geq \frac{A}{4} \sum_{i_1, \ldots, i_p} g_{i_1} \cdots i_p - \frac{A}{p} \Theta .
\]

It should be noted that, if \( \mathfrak{f} = \mathfrak{z} \) (this is the case if and only if \( \dim g = 3 \)), we have

\[
\Phi = \frac{1}{4} \sum_{i_1, \ldots, i_p} g_{i_1} \cdots i_p - \frac{1}{p} \Theta .
\]
Therefore we put $A=1$ in this case. We see in a similar way that the inequality (6.2) holds also in the case $\mathfrak{z}=(0)$. Moreover, if $\mathfrak{z}$ is simple, the equality holds in (6.2).

§ 7. Integrating the both sides of (6.2) we get:

$$
\int_{\sigma/\Gamma} \Phi dv \geq \frac{A}{4} \sum_{i_1,\ldots,i_p} \int g_{i_1,\ldots,i_p}^2 dv - \frac{A}{p} \int \Theta dv.
$$

On the other hand, we have by Lemma 8

$$
\sum_{i_1,\ldots,i_p} \int g_{i_1,\ldots,i_p}^2 dv = \frac{2}{p} \sum_{k,i_1,\ldots,i_p} \int (X_k g_{i_1,\ldots,t_p})^2 dv + \frac{4}{p} \int \Theta dv.
$$

Hence we get:

$$(7.1) \quad \int_{\sigma/\Gamma} \Phi dv \geq \frac{A}{2p} \sum_{k,i_1,\ldots,i_p} \int (X_k g_{i_1,\ldots,t_p})^2 dv.
$$

From (7.1) and Lemma 6 follows

**Lemma 9.**

$$
0 \geq \frac{A}{2p} \sum_{k,i_1,\ldots,i_p} \int (X_k g_{i_1,\ldots,t_p})^2 dv + \sum_{k,i_1,\ldots,t_p-1} \sum_{l,i_1,\ldots,t_p} \int R_{ktl}(X_k g_{j_1,\ldots,t_p-1}) (X_l g_{i_1,\ldots,t_p}) dv.
$$

If $\mathfrak{g}$ is simple and $\mathfrak{z}=(0)$, the symmetric space of $G$ is an irreducible symmetric bounded domain. From the classification of such domain, it is known, except in the case of classical domain of type $I_{m,m'} (m \geq m' \geq 2)$, $\mathfrak{g}$ has only one simple factor. In the case, where $\mathfrak{g}$ has only one simple factor, we have shown in [9], § 6, that

$$(7.2) \quad A = \frac{1}{\dim \mathfrak{f} - 1} \left( \frac{\dim m}{2} - 1 \right),$$

while in the case where the corresponding domain is of type $I_{m,m'} (m \geq m' \geq 2)$,

$$(7.3) \quad A = \frac{m'}{m + m'}.$$

In the case, where $\mathfrak{z}=(0)$ and $\mathfrak{f}$ is simple, we have

$$(7.4) \quad A = \frac{\dim m}{2 \dim \mathfrak{f}}$$
We define for each simple non-compact Lie algebra \( \mathfrak{g} \) a quadratic form \( H^p(\xi) \) on the linear space of tensors \( \xi = (\xi_{i_1 \cdots i_p}) \) by putting

\[
H^p(\xi) = \frac{A}{2p} \sum_{i_1, \ldots, i_p} \xi_{i_1 \cdots i_p}^2 + \sum_{i_1, \ldots, i_p, j} \sum_{k, l, t} R_{ijkl} \xi_{ij} \xi_{kl} + \xi_{i_{p+1}} \xi_{i_{p+1}} \xi_{k}\ell + \xi_{i_{p+1}} \xi_{i_{p+1}} \xi_{k}\ell.
\]

Then by Lemma 9 we have

\[
0 \geq \int_{\mathbb{G}/\Gamma} H^p((X_k g_{i_1 \cdots i_p}) dv.
\]

Remark. In the case \( p = 1 \) we have already defined in [9] a quadratic form \( H(\xi) = b(\mu) \sum_{i, j} \xi_{ij}^2 + \sum R_{ijkl} \xi_{ij} \xi_{kl} \). The constant \( b(\mu) \) is strictly greater than \( \frac{1}{2} A \). This is because we have omitted the factor \( \sum (X_r + ig_{i_1 \cdots i_p})^2 \) to obtain (6.2), while in the case \( p = 1 \) we have a convenient equality \( \sum (X_r + g_{i_1})^2 = \frac{1}{r} \sum g_i^2 \) under a suitable normalisation of \( X_{r+1} \) (see [9], § 5); hence we obtain

\[
\phi \geq \frac{1}{2} b(\mu) \sum g_i^2
\]

in place of (6.2), with \( b(\mu) = \frac{1}{2} A + \frac{1}{r} (1 - A) \). Remark that the term containing \( \Theta \) is missing in the case \( p = 1 \).

§8. We prove now the following theorem.

**Theorem 1.** Let \( X \) be a simply connected, irreducible symmetric Riemannian manifold which is non-compact and non-euclidean. Let \( G \) be the identity component of the group of all isometries of \( X \). Let \( \Gamma \) be a discrete subgroup of \( G \) with compact quotient \( G/\Gamma \) and without element of finite order different from the identity, so that \( \Gamma \) is a discontinuous group of isometries of \( X \) with compact quotient \( M = X/\Gamma \). Let \( X_u \) be the compact form of \( X \). Suppose that the quadratic form \( H^p(\xi) \) is positive definite. Then the \( p \)-th Betti number \( b_p(M) \) of \( M \) equals the \( p \)-th Betti number \( b_p(X_u) \) of \( X_u \).

**Proof.** Let \( A^p \) denote the vector space of all \( G \)-invariant \( p \)-form of \( X \). Since \( \Gamma \subset G \), each \( p \)-form \( \alpha \in A^p \) is \( \Gamma \)-invariant and hence there exists a \( p \)-form \( \eta \) of \( M \) such that \( \alpha = \eta \circ \rho \), \( \rho \) denoting the projection of \( X \) onto \( M \). As we have stated in the introduction, \( \eta \) is a harmonic \( p \)-form on \( M \) and the mapping \( \alpha \rightarrow \eta \) defines an injection of \( A^p \) into the vector space \( A^p \) of all harmonic \( p \)-form on \( M \). Moreover, we know that the
dimension of \( A^p \) equals the \( p \)-th Betti number \( b_p(X_u) \) of \( X_u \) (see Introduction). We have \( b_p(X_u) = b_p(M) \) if and only if the mapping \( \alpha \rightarrow \eta \) is a surjection of \( A^p \) onto \( \mathfrak{h}^p \). Now let \( \eta \in \mathfrak{h}^p \) and let \( \omega = \eta \circ \pi \), \( \pi \) denoting the projection of \( G/\Gamma \) onto \( X/\Gamma = M \). We retain the notations introduced in the preceding sections and put

\[ g_{\lambda_1 \cdots \lambda_p} = \omega(X_{\lambda_1}, \ldots, X_{\lambda_p}). \]

Then these \( g_{\lambda_1 \cdots \lambda_p} \) satisfy the relations (4.1)-(4.4). By Lemma 9 we have

\[ 0 \geq \int_{\partial T} H^n_0((X_k g_{i_1 \cdots i_p})) dv. \]

Since \( H_0^q(\xi) \) is positive definite by assumption, this implies:

(8.1) \[ X_k g_{i_1 \cdots i_p} = 0, \quad 1 \leq k, i_1, \ldots, i_p \leq r. \]

Then \( X_j X_k g_{i_1 \cdots i_p} - X_k X_j g_{i_1 \cdots i_p} = 0 \). On the other hand \( X_j X_k g_{i_1 \cdots i_p} - X_k X_j g_{i_1 \cdots i_p} = \sum c_{ijk} X_{a} g_{i_1 \cdots i_p} \) and hence

(8.2) \[ \sum_{a=1}^n c_{ijk} X_{a} g_{i_1 \cdots i_p} = 0, \quad 1 \leq j, k, i_1, \ldots, i_p \leq r. \]

Now we know that the representation \( X \rightarrow ad_m X \) of the Lie algebra \( \mathfrak{t} \) on the vector space \( m \) defined by putting \( ad_m X \cdot Z = [X, Z] \) for all \( Z \in m \) is faithful. For any real numbers \( \xi_a \), we have \( \sum_a \xi_a X_a = \sum_j (\sum_a c_{ajk} \xi_a) X_j \).

Hence, if \( \sum_a c_{ajk} \xi_a = 0 \) for \( j, k = 1, \ldots, r \), we must have \( \xi_a = 0 \). Thus (8.2) implies

(8.3) \[ X_{a} g_{i_1 \cdots i_p} = 0, \quad r+1 \leq \alpha \leq n, \quad 1 \leq i_1, \ldots, i_p \leq r. \]

From (8.1) and (8.3) follows that \( g_{i_1 \cdots i_p} \) are constant. On the other hand, by (4.1) \( g_{\lambda_1 \cdots \lambda_p} = 0 \) if one of the indices is \( \geq r \) and hence \( g_{\lambda_1 \cdots \lambda_p} \) is constant for any indices \( \lambda_1, \ldots, \lambda_p \). Let \( \bar{\rho} \) denote the projection of \( G \) onto \( G/\Gamma \) and put \( \bar{\omega} = \omega \circ \bar{\rho} \).

\[ X=K \quad \begin{array}{c} \bar{\pi} \\ \bar{\rho} \end{array} \]

\[ \begin{array}{c} G \quad \begin{array}{c} \rho \end{array} \quad \begin{array}{c} \pi \end{array} \end{array} \quad G/\Gamma \]

As in §1, \( X_\lambda \) denotes here the right invariant vector field on \( G \). Then
\[ \tilde{\omega}(X_\lambda, \cdots, X_\mu) = g_\lambda \cdots g_\mu \] are constant and hence \( \tilde{\omega} \) is a right invariant 3-form on \( G \). Moreover, since \( i(X_\mu)\tilde{\omega} = \theta(X_\mu)\tilde{\omega} = 0 \), we have also \( i(X_\mu)\tilde{\omega} = \theta(X_\mu)\tilde{\omega} = 0 \). Therefore there exists a \( G \)-invariant 3-form \( \alpha \) on \( X = K \setminus G \) such that \( \tilde{\omega} = \alpha \circ \pi \), \( \pi \) denoting the projection of \( G \) onto \( K \setminus G \). Since \( \pi \circ \rho = \rho \circ \pi \) and \( \tilde{\omega} = (\eta \circ \rho) \circ \tilde{\rho} \) we have \( \tilde{\omega} = (\eta \circ \rho) \circ \tilde{\pi} \). Since \( \tilde{\omega} = \alpha \circ \pi \) and this shows that the harmonic 3-form \( \eta \) is the image of the \( G \)-invariant 3-form \( \tilde{\omega} \) defined above is surjective. Then we get \( b_3(M) = b_3(X) \). Theorem 1 is thus proved.

\section{§9.} We discuss here when the quadratic form \( H_\rho(\xi) \) is positive definite. For any tensor \( \eta = (\eta_{ij}) \) put
\[
F_\rho(\eta) = \frac{A}{2 \rho} \sum_{i,j} \eta_{ij}^2 + \sum_{i,j,k,l} R_{ijkl} \eta_{ij} \eta_{kl}.
\]
Let \( \eta = \eta' + \eta'' \), where \( \eta' = (\eta'_{ij}) \) is symmetric and \( \eta'' = (\eta''_{ij}) \) is alternating in the indices \( i, j \). Using the property \( R_{ijkl} = R_{jikl} \), we see easily that
\[
F_\rho(\eta) = F_\rho(\eta') + F_\rho(\eta'').
\]
(This and the following arguments are those of Weil \cite{12}). Moreover, since \( \eta'' \) is alternating,
\[
\sum_{i,j,k,l} R_{ijkl} \eta_{ij}' \eta_{kl}' = - \sum_{i,j,k,l} R_{ijkl} \eta_{ij}'' \eta_{kl}'' = \sum_{i,j,k,l} R_{ijkl} \eta_{ij}'' \eta_{kl}''.
\]
From this and from Bianchi identity \( R_{iklj} + R_{jikl} + R_{ijkl} = 0 \) follows:
\[
2 \sum_{i,j,k,l} R_{ijkl} \eta_{ij}' \eta_{kl}' = - \sum_{i,j,k,l} R_{ijkl} \eta_{ij}'' \eta_{kl}'' = \sum_{i,j} c_{ij} \eta_{ij}'^2.
\]
Therefore \( F_\rho(\eta'') \geq 0 \) and \( F_\rho(\eta'') = 0 \) implies \( \eta'' = 0 \). Suppose that \( F_\rho(\eta') > 0 \) for any symmetric \( \eta' \neq 0 \) and we show that \( H_\rho(\xi) \) is then positive definite. In fact, we write \( \xi = \xi' + \xi'' \), where \( \xi' = (\xi'_{ij}) \) is symmetric and \( \xi'' = (\xi''_{ij}) \) is alternating in the indices \( i, j \). Since \( H_\rho(\xi') = \sum_{t_1, \cdots, t_{p-1}} F_\rho((\xi'_{ij_1 \cdots t_{p-1}})) \), we have \( H_\rho(\xi') = H_\rho(\xi'') + H_\rho(\xi'') \) and \( H_\rho(\xi'') \geq 0 \). Now since \( F_\rho((\xi'_{ij_1 \cdots t_{p-1}})) \geq 0 \) by assumption, we have \( H_\rho(\xi') \geq 0 \) and hence \( H_\rho(\xi) \geq 0 \). If \( H_\rho(\xi') = 0 \), we have \( H_\rho(\xi'') = 0 \) and \( H_\rho(\xi'') = 0 \) and hence \( F_\rho((\xi'_{ij_1 \cdots t_{p-1}})) = F_\rho((\xi''_{ij_1 \cdots t_{p-1}})) = 0 \) for any \( t_1, \cdots, t_{p-1} \). This implies \( \xi'_{ij_1 \cdots t_{p-1}} = 0 \) for any indices \( i, j, t_1, \cdots, t_{p-1} \) and hence \( \xi = 0 \). Thus \( H_\rho(\xi) \) is positive definite.

Let \( P \) denote the linear transformation of the vector space of all symmetric tensor \( \eta' \) defined by putting
\[
P(\eta')_{ij} = \sum_{ij} R_{ijkl} \eta_{ij}' \eta_{kl}'.
\]
Then \((P(\eta'), \zeta') = (\eta', P(\zeta')\), \((\eta, \zeta') = \sum_{i,j} \eta_i \zeta_j\).

The linear transformation \(P\) of the vector space of \(\eta'\) is thus symmetric. The quadratic form \(F_p(\eta')\) is written

\[
F_p(\eta') = \frac{A}{2p} (\eta', \eta') + (\eta', P(\eta')).
\]

\(F_p(\eta')\) is positive definite, if (and only if) the absolute value of the minimal eigen-value \(\lambda_i\) of \(P\) is strictly smaller than \(\frac{A}{2p}\).

We consider the case where \(X\) is an irreducible symmetric bounded domain. In this case, the value of \(A\) is calculated by (7.2) and (7.3), while the minimal eigen-value \(\lambda_i\) of \(P\) is already known (see [1] [2] and [9], § 11). We obtain the following table.

<table>
<thead>
<tr>
<th>Type of (X)</th>
<th>(\frac{A}{2})</th>
<th>(\lambda_1)</th>
<th>(\frac{A}{2p} + \lambda_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(I_{m,m'}) ((m \geq m' \geq 1))</td>
<td>(\frac{m'}{2(m + m')})</td>
<td>(-\frac{1}{m + m'})</td>
<td>(\frac{m' - 2p}{2p(m + m')})</td>
</tr>
<tr>
<td>(II_m) ((m \geq 3))</td>
<td>(-\frac{m - 2}{4(m - 1)})</td>
<td>(-\frac{1}{2(m - 1)})</td>
<td>(\frac{m - 2 - 2p}{4p(m - 1)})</td>
</tr>
<tr>
<td>(III_m) ((m \geq 2))</td>
<td>(\frac{m + 2}{4(m + 1)})</td>
<td>(-\frac{1}{m + 1})</td>
<td>(\frac{m + 2 - 4p}{4p(m + 1)})</td>
</tr>
<tr>
<td>(IV_m) ((m \geq 3))</td>
<td>(\frac{1}{m})</td>
<td>(-\frac{1}{m})</td>
<td>(\frac{2 - p}{12p})</td>
</tr>
<tr>
<td>(V)</td>
<td>(\frac{1}{6})</td>
<td>(-\frac{1}{12})</td>
<td>(\frac{3 - p}{18p})</td>
</tr>
<tr>
<td>(VI)</td>
<td>(\frac{1}{6})</td>
<td>(-\frac{1}{18})</td>
<td>(\frac{3 - p}{18p})</td>
</tr>
</tbody>
</table>

Therefore \(H_{\beta}^*(\xi)\) is positive definite in the following cases:

- Type \(I_{m,m'}\), \(\frac{m'}{2} > p\);
- Type \(II_m\), \(\frac{m - 2}{2} > p\);
- Type \(III_m\), \(\frac{m + 2}{4} > p\);
- Type \(V\), \(p = 1\);
- Type \(VI\), \(p = 1, 2\).

But we have proved in [9] that, except in the case of type \(I_{m,1}\), the first Betti number of \(M\) vanishes. On the other hand, the first Betti number of a compact, simply connected symmetric space vanishes. From
Theorem 1 and from the above results follows Theorem 2 stated in the introduction.

§ 10. It is known that, for any simply connected compact Hermitian symmetric space $X_u$, we have $b_{2q+1}(X_u) = 0$ and $b_{2q}(X_u) = h_{q,q}(X_u)$, where $h_{r,s}(X_u)$ denotes the dimension of the complex vector space of all harmonic forms of type $(r, s)$ on $X_u$ [5]. Moreover, if $X_u$ is irreducible, we have $b_2(X_u) = h_{1,1}(X_u) = 1$. The argument used in proving the inequality $b_p(M) \ge b_p(X_u)$ shows that $h_{r,s}(M) \ge h_{r,s}(X_u)$. Since $M$ is Kählerian, we have $b_p(M) = \sum h_{r,s}(M)$. Therefore, if $b_p(M) = b_p(X_u)$ for odd $p$ and $b_p(M) = h_{2q}(M)$ for $p = 2q$. In particular, if $b_2(M) = b_2(X_u)$ and if $X_u$ (therefore $X$) is irreducible, we have $b_2(M) = h_{1,1}(M) = 1$. From Theorem 2 we obtain the following theorem.

**Theorem 3.** Let $X$ be an irreducible symmetric bounded domain of one of the following types: I$_m$,$n$($m \ge m' \ge 6$), II$_m$($m \ge 7$), III$_m$ ($m \ge 7$), VI. Then we have

$$b_1(M) = 0, \quad b_2(M) = h_{1,1}(M) = 1.$$ 

We apply this result to the classification of automorphic factors. Let $X$ be a symmetric bounded domain and let $\Gamma$ be a discontinuous group on $X$ without element of finite order different from the identity and with compact quotient space $X/\Gamma$. An automorphic factor $k$ (with respect to $\Gamma$) is a mapping of $X \times \Gamma$ into $\mathbb{C}^*$ such that

$$k(z, \gamma \delta) = k(z \gamma, \delta)k(z, \gamma)$$

for any $z \in X$, $\gamma, \delta \in \Gamma$ and that $k(z, \gamma)$ is holomorphic in $z$. If $k$ and $k'$ are automorphic factors, so are the mappings $(z, \gamma) \rightarrow k(z, \gamma) \cdot k'(z, \gamma)$ and $(z, \gamma) \rightarrow k(z, \gamma)^{-1}$. Two automorphic factors $k$ and $k'$ are equivalent ($k \sim k'$), if there exists a non-vanishing holomorphic function $f$ on $X$ such that

$$f'(z, \gamma) = k(z, \gamma)f(z)f(\gamma)^{-1}$$

for any $(z, \gamma) \in X \times \Gamma$.

The equivalence classes of automorphic factors form a group $F$ with respect to the multiplication defined above.

Now, given an automorphic factor $k(z, \gamma)$, we can define a complex line bundle $E_k$ over the complex manifolds $X/\Gamma$ as follows. $E_k$ is the quotient of $X \times \mathbb{C}$ by the equivalence relation: $(z, \xi) \sim (z \gamma, k(z, \gamma) \cdot \xi)$. It is known that the two line bundle $E_k$ and $E_{k'}$ over $X/\Gamma$ are isomorphic if and only if the automorphic factor $k$ and $k'$ are equivalent. A line bundle $E$ over $X/\Gamma$ is defined by an automorphic factor in the above
way if and only if the induced bundle $\rho^*E$ over $X$ is analytically trivial, where $\rho$ denotes the projection of $X$ onto $X/\Gamma$. (For these facts on line bundles, cf. [11]). Now, since $X$ is a homogeneous bounded domain in $C^n$, it is a domain of holomorphy by a theorem of Thullen and hence a Stein manifold. Moreover $X$ is homeomorphic to a euclidean space. It follows then from the fundamental theorem (Theorem B) on Stein manifolds (see [6]) that every complex line bundle over $X$ is analytically trivial (this is also a special case of a more general result of Grauert). Therefore every complex line bundle over $X/\Gamma$ is defined by an automorphic factor and the group of equivalence classes of complex line bundles over $X/\Gamma$ is isomorphic to the group $F$.

Consider now the exact sequence of sheaves over $X/\Gamma$:

$$0 \to \mathcal{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0,$$

where $\mathcal{Z}$, $\mathcal{O}$, $\mathcal{O}^*$ denote respectively the constant sheaf isomorphic to the additive group of integers, the sheaf of germs of holomorphic functions and the sheaf of germs of non-vanishing holomorphic functions on $X/\Gamma$ (cf. [7]). We get the exact sequence of cohomologies:

$$\to H^1(X/\Gamma, \mathcal{Z}) \to H^1(X/\Gamma, \mathcal{O}^*) \to H^2(X/\Gamma, \mathcal{Z}) \to H^2(X/\Gamma, \mathcal{O}) \to \cdots$$

Suppose now that $b_1(X/\Gamma)=0$ and $b_2(X/\Gamma)=h_{1,1}(X/\Gamma)=1$. Since $X/\Gamma$ is Kählerian, the dimensions of the complex vector spaces $H^1(X/\Gamma, \mathcal{O})$ and $H^2(X/\Gamma, \mathcal{O})$ equal $h_{0,1}(X/\Gamma)$ and $h_{0,2}(X/\Gamma)$ respectively by a theorem of Dolbeault (see [7]). It follows then that $H^1(X/\Gamma, \mathcal{O})=H^2(X/\Gamma, \mathcal{O})=(0)$ and hence $H^2(X/\Gamma, \mathcal{O}^*)=H^2(X/\Gamma, \mathcal{Z})$. The group $H^1(X/\Gamma, \mathcal{O}^*)$ is identified with the group of equivalence classes of complex line bundles over $X/\Gamma$ and hence isomorphic to $F$. Thus $F=H^1(X/\Gamma, \mathcal{O}^*)$. Since $b_2(X/\Gamma)=1$, it follows that the group $F$ is a direct product of an infinite cyclic subgroup $F_1$ and a finite subgroup $F_2$. For an automorphic factor $k$ we denote by $[k]$ the equivalence class containing $k$. Let $[k] \in F_2$. Then there exists an integer $m \geq 0$ and a non-vanishing holomorphic function $f$ on $X$ such that $k(z, \gamma)^m = f(z) \cdot f(z \cdot \gamma)^{-1}$ for any $(z, \gamma) \in X \times \Gamma$. Since $X$ is simply connected and $f(z) \neq 0$ for any $z \in X$, we can find a holomorphic function $h$ on $X$ such that $h(z)^m = f(z)$ for any $z \in X$. Let $\chi(z, \gamma) = k(z, \gamma) \cdot h(z)^{-1} \cdot h(z \cdot \gamma)$. Then $k \sim \chi$ and $\chi(z, \gamma)^m = 1$. It follows that $\chi(z, \gamma)$ is independent of $z$ and hence $\chi(z, \gamma) = \chi(\gamma)$ is a 1-dimensional unitary representation on $\Gamma$, i.e. a character of the finite abelian group $\Gamma'/\Gamma''$, $\Gamma''$ denoting the commutator group of $\Gamma$ (note that $\Gamma'/\Gamma''$ is finite, because $b_2(X/\Gamma')=0$). Thus each equivalence class in $F_2$ contains a character of $\Gamma'/\Gamma''$. Conversely an equivalence class containing a character of $\Gamma'/\Gamma''$ is clearly of finite order. Now let $\chi_1$ and $\chi_2$ be two characters contained in one and the
same equivalence class and let $\chi = \chi_1 \cdot \chi_2^{-1}$. Then there exists a non-vanishing holomorphic function $f$ on $X$ such that $f(z \cdot \gamma) = \chi(\gamma) \cdot f(z)$ for any $(z, \gamma) \in X \times \Gamma$. Then $d \log f$ is a holomorphic 1-form on $X$ invariant by $\Gamma$ and hence it defines a holomorphic 1-form on $X/\Gamma$. Since $b_1(X/\Gamma) = 0$, we must have $d \log f = 0$. Then $f$ is a constant and hence $\chi = 1$ and we get $\chi_1 = \chi_2$. Thus each equivalence class of finite order contains one and only one character of $\Gamma/\Gamma'$. Therefore $F_2$ is isomorphic to the character group of the finite abelian group $\Gamma/\Gamma'$.

Let $k_0$ be an automorphic factor such that $[k_0]$ is a generator of the infinite cyclic group $F_l$. For each automorphic factor $k$, there exist an integer $n$ and a character $\chi$ of $\Gamma/\Gamma'$ such that $[k] = [k_0^n] \cdot [\chi]$. Combined with Theorem 3 we get

**Theorem 4.** Let $X$ be an irreducible symmetric bounded domain of one of the following types: $I_{m,m'} (m \geq m' \geq 6)$, $II_m (m \geq 7)$, $III_m (m \geq 7)$, VI. Let $\Gamma$ be a discrete subgroup of the identity component of the automorphism group of $X$. Suppose that the quotient space $X/\Gamma$ is compact and that $\Gamma$ contains no elements of finite order different from the identity. Then the group $F$ of equivalence classes of automorphic factors is direct product of an infinite cyclic group and a finite abelian group, the latter being isomorphic to the character group of the finite abelian group $\Gamma/\Gamma'$, where $\Gamma'$ denotes the commutator group of $\Gamma$. Therefore there exists an automorphic factor $k_0$ with the following property: For each automorphic factor $k$, there exist a unique character $\chi$ of $\Gamma/\Gamma'$ and a unique integer $n$ such that $k \sim k_0^n \cdot \chi$.

It will be an interesting problem to determine a "standard" $k_0$ for each type of $X$.

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**Bibliography**


_A added in proof_.