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PLURICANONICAL SYSTEMS OF PROJECTIVE VARIETIES OF GENERAL TYPE I

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Abstract

Assuming the minimal model program, we prove that there exists a positive integer ν_n depending only on n such that for every smooth projective n-fold of general type X defined over complex numbers, $|mK_X|$ gives a birational rational map from X into a projective space for every $m \geq \nu_n$.

Contents

1. Introduction

Let X be a smooth projective variety and let K_X be the canonical bundle of X. X is said to be a general type, if there exists a positive integer m such that the pluricanonical system $|mK_X|$ gives a birational (rational) embedding of X. The following problem is fundamental to study projective varieties of general type.

Probrem. Find a positive integer ν_n depending only on n such that for every smooth projective n-fold X of general type, $|mK_X|$ gives a birational rational map from X into a projective space for every $m \geqq \nu_n$.

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If X is a smooth projective curve of genus $\geqq 2$, it is well known that $|3K_X|$ gives a projective embedding. In the case that X is a smooth projective surface of general type, E. Bombieri showed that $|5K_X|$ gives a birational rational map from X into a projective space ([2]). But for the case of $\dim X \geqq 3$, very little is known about the above problem.

The main purpose of this article is to prove the following theorems *assuming MMP* (*minimal model program*). The proof without assuming MMP will be published in the subsequent paper [23] which is the transcription of the latter half of [22].

Theorem 1.1. *There exists a positive integer ν_n which depends only on n such that for every smooth projective n-fold X of general type defined over complex numbers, $|mK_X|$ gives a birational rational map from X into a projective space for every $m \geqq \nu_n$.*

Let us explain MMP. It has been conjectured that for every nonuniruled smooth projective variety X, there exists a projective variety $X_{\min}$ such that

1. $X_{\min}$ is birationally equivalent to X,
2. $X_{\min}$ has only **Q**-factorial terminal singularities,
3. $K_{X_{\min}}$ is a nef **Q**-Cartier divisor.

$X_{\min}$ is called a minimal model of X. To construct a minimal model, the minimal model program (MMP) has been proposed (cf. [11, p.96]). The minimal model program was completed in the case of 3-folds by S. Mori ([12]).

The proof of Theorem 1.1 can be very much simplified, if we assume the existence of minimal models for projective varieties of general type. The proof for the general case is modeled after the proof under the existence of minimal models by using the theory of AZD (cf. [23]). The only essential difference is the use of an extension theorem (the subadjunction theorem) instead of the Serre vanishing theorem here.

We should also note that even if we assume the existence of minimal models for projective varieties of general type, Theorem 1.1 is quite nontrivial because the indices of minimal models of ([11, p.159, Definition 5.19]) can be arbitrarily large. Conversely if we assume MMP and restrict ourselves to the case of smooth projective n-folds which have minimal models with indices less than some positive integer, say r, then for such an X, by the method in [1, 20] it is easy to prove that $|(1 + rn(n + 1))K_X|$ gives a birational embedding of X into a projective space. But since the set of indices of minimal 3-folds of general type is unbounded, Theorem 1.1 is quite nontrivial even in the case of $\dim X = 3$. Hence in this sense the major difficulty of the proof of Theorem 1.1 is to find "*a* (*universal*) lower bound" *of the positivity of* K_X. In fact Theorem 1.1 is equivalent to the following theorem (see the last part of Section 3).

Theorem 1.2. *For a smooth projective n-fold X over complex numbers, we define the volume $\mu(X, K_X)$ of X with respect to K_X by*

$$\mu(X, K_X) := n! \cdot \overline{\lim_{m\to\infty}} m^{-n} \dim H^0(X, \mathcal{O}_X(mK_X)).$$

Then there exists a positive number C_n depending only on n such that for every smooth projective n-fold X of general type, the inequality:

$$\mu(X, K_X) \geqq C_n$$

holds.

We note that $\mu(X, K_X)$ is equal to the intersection number K_X^n for a minimal projective n-fold X of general type (cf. Proposition 4.1 and Remark 4.2 in Appendix). In Theorems 1.1 and 1.2, the numbers ν_n and C_n have not yet been computed effectively.

The relation of Theorems 1.1 and 1.2 is as follows. Theorem 1.2 means that there exists a universal lower bound of the positivity of canonical bundle of smooth projective variety of general type with a fixed dimension. On the other hand, for a smooth projective variety of general type X, the lower bound of m such that $|mK_X|$ gives a birational embedding depends on the positivity of K_X on subvarieties which appear as the strata of the filtrations as in [20, 1] (cf. Section 3.2).

The positivity of K_X on the subvarieties can be related to the positivity of the canonical bundles of the smooth models of the subvarieties via the subadjunction theorem due to Kawamata ([7]). We note that there exists a nonempty Zariski open subset U_0 of X in *countable Zariski topology* such that any subvarieties passing through a point in U_0 should be of general type. Here the countable Zariski topology means that the topology on X whose closed sets are at most countable union of subvarieties of X.

The organization of the paper is as follows.

In Section 2, we review the relation between multiplier ideal sheaves and singularities of divisors. And we review Kawamata's subadjunction theorem which is essential in our proofs.

In Section 3, we prove Theorems 1.1 and 1.2 assuming the existence of minimal models for projective varieties of general type. For the proofs we use the induction on dimension. Section 3.2 is similar to the argument in [20, 1]. The essential part of Section 3 consists of Section 3.4. In Section 3.4, we use the subadjunction theorem of Kawamata to relate the canonical divisor of centers of log canonical singularities and the canonical divisor of the ambient space. And we prove that the minimal projective n-fold X of general type with $K_X^n \leqq 1$ can be embedded birationally into a projective space as a variety with degree $\leqq C^n$, where C is a positive constant depending only on n (defined in Lemma 3.11). Using this fact we finish the proofs of Theorems 1.1 and 1.2 assuming the existence of minimal models.

In this paper all the varieties are defined over $\mathbf{C}$.

2. Preliminaries

2.1. Multiplier ideal sheaves and singularities of divisors. In this subsection we shall review the relation between multiplier ideal sheaves and singularities of divisors. Throughout this subsection L will denote a holomorphic line bundle on a complex manifold M.

DEFINITION 2.1. A singular hermitian metric h on L is given by

$$h = e^{-\varphi} \cdot h_0,$$

where h_0 is a C^∞-hermitian metric on L and $\varphi \in L^1_{\mathrm{loc}}(M)$ is an arbitrary function on M. We call φ the weight function of h with respect to h_0.

The curvature current Θ_h of the singular hermitian line bundle (L, h) is defined by

$$\Theta_h := \Theta_{h_0} + \sqrt{-1}\,\partial\bar{\partial}\varphi,$$

here $\partial\bar{\partial}$ is taken in the sense of a current. The L^2-sheaf $\mathcal{L}^2(L, h)$ of the singular hermitian line bundle (L, h) is defined by

$$\mathcal{L}^2(L, h)(U) := \left\{\sigma \in \Gamma(U, \mathcal{O}_M(L)) \;\middle|\; h(\sigma, \sigma) \in L^1_{\mathrm{loc}}(U)\right\},$$

where U runs over the open subsets of M. In this case there exists an ideal sheaf $\mathcal{I}(h)$ such that

$$\mathcal{L}^2(L, h) = \mathcal{O}_M(L) \otimes \mathcal{I}(h)$$

holds. We call $\mathcal{I}(h)$ the *multiplier ideal sheaf* of (L, h). If we write h as

$$h = e^{-\varphi} \cdot h_0,$$

where h_0 is a C^∞ hermitian metric on L and $\varphi \in L^1_{\mathrm{loc}}(M)$ is the weight function, we see that

$$\mathcal{I}(h) = \mathcal{L}^2(\mathcal{O}_M, e^{-\varphi})$$

holds. For $\varphi \in L^1_{\mathrm{loc}}(M)$ we define the multiplier ideal sheaf of φ by

$$\mathcal{I}(\varphi) := \mathcal{L}^2(\mathcal{O}_M, e^{-\varphi}).$$

EXAMPLE 2.2. Let m be a positive integer. Let $\sigma \in \Gamma(X, \mathcal{O}_X(mL))$ be a global section. Then

$$h := \frac{1}{|\sigma|^2} = \frac{h_0}{\left(h_0^m(\sigma, \sigma)\right)^{1/m}}$$

is a singular hemitian metric on L, where h_0 is an arbitrary C^∞-hermitian metric on L (the righthand side is obviously independent of h_0). The curvature Θ_h is given by

$$\Theta_h = \frac{2\pi\sqrt{-1}}{m}(\sigma)$$

where (σ) denotes the current of integration over the divisor of σ.

DEFINITION 2.3. L is said to be pseudoeffective, if there exists a singular hermitian metric h on L such that the curvature current Θ_h is a closed positive current.

Also a singular hermitian line bundle (L, h) is said to be pseudoeffective, if the curvature current Θ_h is a closed positive current.

Let m be a positive integer and $\{\sigma_i\}$ a finite number of global holomorphic sections of mL. Let ϕ be a C^∞-function on M. Then

$$h := e^{-\phi} \cdot \frac{1}{\left(\sum_i |\sigma_i|^2\right)^{1/m}}$$

defines a singular hermitian metric on L. We call such a metric h a singular hermitian metric on L with *algebraic singularities*. Singular hermitian metrics with algebraic singularities are particulary easy to handle, because its multiplier ideal sheaf of the metric can be controlled by taking a suitable modification $f : N \to M$ of the base scheme $\bigcap_i (\sigma_i)$.

Let $D = \sum a_i D_i$ be an effective **Q**-divisor on X. Let σ_i be a section of $\mathcal{O}_X(D_i)$ with divisor D_i respectively. Then we define

$$\mathcal{I}(D) := \mathcal{I}\left(\sum_i a_i \log h_i(\sigma_i, \sigma_i)\right)$$

and call it the multiplier ideal sheaf of the divisor D, where h_i denotes a C^∞-hermitian metric of $\mathcal{O}_X(D_i)$ respectively. It is clear that $\mathcal{I}(D)$ is independent of the choice of the hermitian metrics $\{h_i\}$.

Let us consider the relation between $\mathcal{I}(D)$ and singularities of D. As is seen below, the multiplier ideal sheaf $\mathcal{I}(D)$ can be computed in terms of log resolution of the pair (X, D).

DEFINITION 2.4. Let X be a normal variety and $D = \sum_i d_i D_i$ an effective **Q**-divisor such that $K_X + D$ is **Q**-Cartier. If $\mu : Y \to X$ is a log resolution of the pair (X, D), i.e., μ is a composition of successive blowing ups with smooth centers such that Y is smooth and $(f^*D)_{\mathrm{red}}$ is a divisor with normal crossings, then we can write

$$K_Y + \mu_*^{-1} D = \mu^*(K_X + D) + F$$

with $F = \sum_j e_j E_j$ for the exceptional divisors $\{E_j\}$, where $\mu_*^{-1}D$ denotes the strict transform of D. We call F the discrepancy and $e_j \in \mathbf{Q}$ the discrepancy coefficient for E_j. We regard $-d_i$ as the discrepancy coefficient of D_i.

The pair (X, D) is said to have only *Kawamata log terminal singularities* (*KLT*) (resp. *log canonical singularities* (*LC*)), if $d_i < 1$ (resp. $\leqq 1$) for all i and $e_j > -1$ (resp. $\geqq -1$) for all j for a log resolution $\mu\colon Y \to X$. One can also say that (X, D) is KLT (resp. LC), or $K_X + D$ is KLT (resp. LC), when (X, D) has only KLT (resp. LC). The pair (X, D) is said to be KLT (resp. LC) at a point $x_0 \in X$, if $(U, D|_U)$ is KLT (resp. LC) for some neighbourhood U of x_0.

The following proposition is a dictionary between algebraic geometry and the L^2-method.

Proposition 2.5. *Let D be an effective $\mathbf{Q}$-divisor normal n-fold X. Then (X, D) is KLT at $x \in X_{\mathrm{reg}}$, if and only if $\mathcal{I}(D)_x$ is trivial* $(= \mathcal{O}_{X,x})$.

In particular, $\mathrm{mult}_x D \geqq n$ *implies $\mathcal{I}(D)$ is nontrivial at $x \in X$. holds.*

The proof is trivial and left to the reader. The last assertion follows from the fact that $\left(\sum_{i=1}^n |z_i|^2\right)^{-n}$ is not locally integrable around $O \in \mathbf{C}^n$.

For a multiplier ideal sheaf $\mathcal{I}(h)$, the support of $\mathcal{O}_X/\mathcal{I}(h)$ is called the co-support of $\mathcal{I}(h)$. To locate the co-support of a multiplier ideal sheaf of effective $\mathbf{Q}$-divisors, the following notion is useful.

DEFINITION 2.6. A subvariety W of X is said to be a *center of log canonical singularities* for the pair (X, D), if there is a log resolution $\mu\colon Y \to X$ and a prime divisor E on Y with the discrepancy coefficient $e \leqq -1$ such that $\mu(E) = W$.

By definition $W \subset \mathrm{Supp}\, D$ holds. The set of all the centers of log canonical singularities is denoted by $CLC(X, D)$. For a point $x_0 \in X$, we define $CLC(X, x_0, D) := \{W \in CLC(X, D) \mid x_0 \in W\}$. We quote the following proposition to introduce the notion of the minimal center of log canoical singularities.

Proposition 2.7 ([8, p.494, Proposition 1.5]). *Let X be a normal variety and D an effective $\mathbf{Q}$-Cartier divisor such that $K_X + D$ is $\mathbf{Q}$-Cartier. Assume that X is KLT and (X, D) is LC. If $W_1, W_2 \in CLC(X, D)$ and W an irreducible component of $W_1 \cap W_2$, then $W \in CLC(X, D)$. This implies that if (X, D) is not KLT, then there exists a unique minimal element of $CLC(X, D)$. Also if (X, D) is LC but not KLT at a point $x_0 \in X$, then there exists the unique minimal element of $CLC(X, x_0, D)$.*

We call these minimal elements the *minimal center of LC singularities* of (X, D) and the *minimal center of LC singularities* of (X, D) at x_0 respectively.

2.2. Kawamata's subadjunction theorem. The following subadjunction theorem is crucial in our proof.

Theorem 2.8 ([7, Theorem 1]). *Let X be a normal projective variety and $x \in X_{\mathrm{reg}}$. Let D° and D be effective $\mathbf{Q}$-divisors on X such that $D^\circ < D$, (X, D) is KLT at x and (X, D) is LC at x. Let W be the minimal center of LC singularities at x for (X, D). Let $\pi : \overline{W} \to W$ be the desingularization of W. Let H be an ample Cartier divisor on X and ϵ a positive rational number.*

Then there exists an effective $\mathbf{Q}$-divisor $D_{\overline{W}}$ on $\overline{W}$ such that

$$\pi^*(K_X + D + \epsilon H) \sim_{\mathbf{Q}} K_{\overline{W}} + D_{\overline{W}}.$$

REMARK 2.9. The above theorem is a little bit different from the original Kawamata's subadjunction theorem [7, Theorem 1]. In fact we only assume that W is a local minimal center at x. But the proof of Theorem 2.8 is contained in Kawamata's by just replacing "minimal center of LC singularities" by "local minimal center" whenever necessary. And the main difference to Kawamata's subadjunction is that local minimal center W is not necessarily normal everywhere, hence it is not clear what K_W should be.

Roughly speaking, Theorem 2.8 implies that $K_X + D|_W$ (almost) dominates K_W.

2.3. Several remarks on singular hermitian line bundles on minimal algebraic varieties. Since minimal algebraic varieties are singular in general, we cannot apply the theory of singular hermitian line bundles directly. Here I would like to explain the modifications we need.

Let X be a minimal projective n-fold of general type, i.e., X has only $\mathbf{Q}$-factorial terminal singularities and the canonical divisor K_X is nef.

For a reduced complex space Y, we define the space of C^∞-functions (resp. plurisubharmonic functions) on Y as a space of continuous functions (resp. plurisubharmonic functions) on the regular part of Y which are locally extendable to C^∞-functions (resp. plurisubharmonic functions) on an ambient space with respect to some local embedding of Y into an open subset of a complex Euclidean space ("some local embedding" is enough for our purposes).

Let r be a positive integer such that rK_X is Cartier. Then rK_X admits a C^∞-hermitian metric h_0, where C^∞-hermitian metric means that it is locally expressed by a C^∞-function with respect to a local holomorphic frame. Then the r-th root $\sqrt[r]{h_0}$ is well defined. We consider $\sqrt[r]{h_0}$ as a C^∞ hermitian metric on K_X.

Let h be a singular hermitian metric on $(m-1)K_X$ such that

1. h has algebraic singularities, i.e.,

$$h = e^{-\phi} \cdot \frac{1}{\left(\sum_{j=1}^{N} |\sigma_j|^2\right)^{1/a}},$$

where ϕ is a C^∞-function on X, a is a positive integer and

$$\sigma_j \in H^0(X, \mathcal{O}_X(a(m-1)K_X)) \quad (1 \leqq j \leqq N)$$

(for the notation $|\sigma_j|^2$, see Example 2.2).

2. The curvature current Θ_h is strictly positive in the sense that it dominates a positive multiple of a Kähler form which is induced by a projective embedding of X, i.e. Θ_h is locally extendable to a closed positive current on the projective embedding which dominates a positive multiple of the Kähler form.

Later we will consider slightly more general situation, i.e., h is a product of singular hermitian metrics with algebraic singularities. But the argument below is identical also in this more general case.

Let

$$\pi : \tilde{X} \to X$$

be a resolution of singularities such that the exceptional set F is a divisor with normal crossings.

h defines a singular hermitian metric π^*h on $(m-1)K_{\tilde{X}}$. Here we should note that we have identified π^*h as a metric on $(m-1)K_{\tilde{X}}$ not of $(m-1)\pi^*K_X$. The reason is that $(m-1)K_{\tilde{X}}$ is a line bundle and is easier to handle. We note that since X has only canonical singularities, $K_{\tilde{X}} - \pi^*K_X$ is effective. Hence π^*h has semipositive curvature current on $\tilde{X}$ and strictly positive on $\pi^{-1}(X_{\mathrm{reg}})$, where X_{reg} denotes the regular locus of X.

Let $F = \sum_k F_k$ be the irreducible decomposition of the exceptional divisor F of π and let σ_{F_k} be a nontrivial global holomorphic section of $\mathcal{O}_{\tilde{X}}(F_k)$ with divisor F_k. Let h_k be a C^∞-hermitian metric on $\mathcal{O}_{\tilde{X}}(F_k)$. Let $\tilde{h}$ be a singular hermitian metric on $(m-1)K_{\tilde{X}}$ defined by

$$\tilde{h} = \frac{\pi^*h}{\prod_k \|\sigma_{F_k}\|^{2c_k}}$$

for some positive rational numbers $\{c_k\}$. Since Θ_h is strictly positive on X, we may and do choose $\{h_k\}$ and $\{c_k\}$ so that the curvature current $\Theta_{\tilde{h}}$ of $\tilde{h}$ is strictly positive on $\tilde{X}$. Then for a sufficiently small positive number $\varepsilon \ll 1$, $(\pi^*h^{1-\varepsilon}) \cdot \tilde{h}^\varepsilon$ has strictly positive curvature on $\tilde{X}$ and

$$\mathcal{I}\left((\pi^*h^{1-\varepsilon}) \cdot \tilde{h}^\varepsilon\right) = \mathcal{I}(\pi^*h)$$

holds. This follows from Proposition 2.5, since h has algebraic singularities. Then by Nadel's vanishing theorem ([13, p.561]), we have that

$$H^q\big(\tilde{X}, \mathcal{O}_{\tilde{X}}(K_{\tilde{X}} + (m-1)K_{\tilde{X}}) \otimes \mathcal{I}(\pi^* h)\big) = 0$$

holds for every $q \geqq 1$. We set $\omega_X := \pi_* \mathcal{O}_X(K_{\tilde{X}})$ and call it the L^2-dualizing sheaf of X. ω_X is nothing but the sheaf of germs of L^2-holomorphic canonical forms on X. Hence it is independent of the choice of the resolution. Since X has only canonical singualities, the L^2-dualizing sheaf ω_X is isomorphic to $\mathcal{O}_X(K_X)$.

Since

$$R^p \pi_*\big(\mathcal{O}_{\tilde{X}}(K_{\tilde{X}} + (m-1)K_{\tilde{X}}) \otimes \mathcal{I}(\pi^* h)\big) = 0$$

holds for every $p \geqq 1$ by the standard L^2-vanishing theorem on holomorphically convex manifolds (cf. [6], this is nothing but the local Nadel's vanishing theorem), we have that

$$H^q\big(X, \mathcal{O}_{\tilde{X}}(K_X + (m-1)K_X) \otimes \mathcal{I}(h)\big) = 0$$

holds for every $q \geqq 1$, where

$$\mathcal{O}_X(K_X + (m-1)K_X) \otimes \mathcal{I}(h) := \pi_*\big(\mathcal{O}_{\tilde{X}}(K_{\tilde{X}} + (m-1)K_{\tilde{X}}) \otimes \mathcal{I}(\pi^* h)\big).$$

It is clear that $\mathcal{O}_{\tilde{X}}(K_X + (m-1)K_X) \otimes \mathcal{I}(h)$ is independent of the choice of the resolution π. Here we note that $\mathcal{I}(h)$ may not be well defined, if mK_X is not Cartier. But $\mathcal{O}_{\tilde{X}}(K_X + (m-1)K_X) \otimes \mathcal{I}(h)$ is well defined.

3. Proofs of Theorems 1.1 and 1.2 assuming MMP

In this section we prove Theorems 1.1 and 1.2 assuming the minimal model program (MMP). Since the minimal model program is established in the case of 3-folds, the proof under this assumption provides the full proofs of Theorems 1.1 and 1.2 for the case of projective varieties of general type of $\dim X \leqq 3$.

3.1. Construction of a filtration. Let X be a minimal projective n-fold of general type, i.e., X has only $\mathbf{Q}$-factorial terminal singularities and the canonical divisor K_X is nef. We set

$$X^\circ = \{x \in X_{\mathrm{reg}} \mid x \notin \mathrm{Bs}|mK_X| \text{ and } \Phi_{|mK_X|} \text{ is a biholomorphism on a neighbourhood of } x \text{ for some } m \geqq 1\}.$$

Then X° is a nonempty Zariski open subset of X.

In this subsection we shall construct a filtration as follows.

Lemma 3.1. *Let x and x' be distinct points on X°. Then there exists a filtration*:

$$X = X_0 \supset X_1 \supset \cdots \supset X_r \supset X_{r+1} = x \quad \textit{or} \quad x'$$

of X by a strictly decreasing sequence of subvarieties $\{X_i\}_{i=0}^{r+1}$ for some r (depending on x and x'), effective **Q**-*divisors*

$$D_0, \ldots, D_r$$

which are **Q**-*lineraly equivalent to K_X and invariants*:

$$\alpha_0, \alpha_1, \ldots, \alpha_r \in \mathbf{Q}^+,$$
$$n =: n_0 > n_1 > \cdots > n_r \quad (n_i = \dim X_i, \ \ i = 0, \ldots, r)$$

and

$$\mu_0, \mu_1, \ldots, \mu_r \quad (\mu_i = K_X^{n_i} \cdot X_i, \ \ i = 0, \ldots, r)$$

with the estimates

$$\alpha_i \leqq \frac{n_i \sqrt[n_i]{2}}{\sqrt[n_i]{\mu_i}} + \delta \quad (0 \leqq i \leqq r),$$

where δ is a fixed positive number less than $1/n$ and α_i is defined inductively by:

$$\alpha_i = \inf \left\{ \alpha > 0 \;\middle|\; \left(X, \sum_{j=0}^{i-1} (\alpha_j - \varepsilon_j) D_j + \alpha D_i \right) \textit{ is KLT at neither } x \textit{ nor } x' \right\},$$

where $\varepsilon_0, \ldots, \varepsilon_{i-1}$ are small positive rational numbers which can be taken arbitrarily small. Here each filter X_i $(1 \leqq i \leqq r)$ is the minimal center of log canonical singularities of $\left(X, \sum_{j=0}^{i-2} (\alpha_j - \varepsilon_j) D_j + \alpha_{i-1} D_{i-1}\right)$ at x or x' (if $i = 1$, we consider $\sum_{j=0}^{i-2} (\alpha_j - \varepsilon_j) D_j = 0$).

$\varepsilon_0, \ldots, \varepsilon_{i-1}$ will be specified during the constrution of the filtration.

Roughly the construction of the filtration is as follows.

First we set $X_0 = X$. Suppose that we have already constructed the filtration up to X_i, i.e., we have constructed the filtration:

$$X = X_0 \supset X_1 \supset \cdots \supset X_i,$$

divisors $D_0, \ldots, D_{i-1}$ and so on. Then one of the following two cases occurs. Here one has to split off the construction of D_i.

CASE 1. For every sufficiently small positive number λ, $\left(X, \sum_{j=0}^{i-2} (\alpha_j - \varepsilon_j) D_j + (\alpha_{i-1} - \lambda) D_{i-1}\right)$ is KLT at both x and x'.

CASE 2. For every sufficiently small positive number λ, $\left(X, \sum_{j=0}^{i-2}(\alpha_j - \varepsilon_j)D_j + (\alpha_{i-1} - \lambda)D_{i-1}\right)$ is KLT at exactly one of x or x' say x.

In Case 1, we construct an effective **Q**-divisor D_i which is **Q**-linearly equivalent to K_X such that

1. Supp D_i does not contain X_i.
2. $D_i|X_i$ has "high multiplicities" both at x and x' (for the precise meaning of "high multiplicities," see the detailed construction below).
3. Around x, Supp D_i is smooth outside X_i and D_i has sufficiently low multiplicities on $X - X_i$.

We choose a sufficiently small positive rational number ε_{i-1} and define

$$\alpha_i = \inf\left\{\alpha > 0 \;\middle|\; \left(X, \sum_{j=0}^{i-1}(\alpha_j - \varepsilon_j)D_j + \alpha D_i\right) \text{ is KLT at neither } x \text{ nor } x'\right\}.$$

Then we define X_{i+1} to be the minimal center of log canonical singularities at x or x'. In general X_{i+1} may not be unique, when $\left(X, \sum_{j=0}^{i-1}(\alpha_j - \varepsilon_j)D_j + \alpha_i D_i\right)$ is log canonical both x and x'. Since Supp D_i is smooth around x and x', the minimal center X_{i+1} is a proper subvariety of X_i.

We set $n_{i+1} = \dim X_{i+1}$ and $\mu_{i+1} = K_X^{n_{i+1}} \cdot X_{i+1}$.

In Case 2, we construct the D_i so that D_i has relatively large multiplicities at x instead of at both x and x'. We note that if we encounter Case 2, in the following steps, we encounter only Case 2, i.e., we may concentrate ourselves around a single point.

We continue the construction until X_{r+1} is a point.

Now we shall describe the construction more closely. The construction of a filtration below is similar to that in [20, 1]. The only difference is the fact that we deal with the **Q**-Cartier divisor K_X which is not Cartier in general. Of course this difference is very minor as long as we work on the regular locus of X. The only essential difference is that the intersection number of a power of K_X and the subvarieties of X is a rational number in general.

We set

$$\mu_0 := K_X^n.$$

Lemma 3.2. *We set*

$$\mathcal{M}_{x,x'} := \mathcal{M}_x \cdot \mathcal{M}_{x'},$$

where $\mathcal{M}_x, \mathcal{M}_{x'}$ denote the maximal ideal sheaves of the points x and x' respectively. Let ε be a positive rational number less than 1. Then

$$H^0(X, \mathcal{O}_X(mK_X) \otimes \mathcal{M}_{x,x'}^{\left\lceil \sqrt[n]{\mu_0}(1-\varepsilon)\frac{m}{\sqrt[n]{2}} \right\rceil}) \neq 0$$

for every sufficiently large m (independent of x, x'), where for a real number a, $\lceil a \rceil$ denotes the smallest integer greater or equal to a.

Proof. Let us consider the exact sequence:

$$\begin{aligned} 0 \to H^0(X, \mathcal{O}_X(mK_X) \otimes \mathcal{M}_{x,x'}^{\left\lceil \sqrt[n]{\mu_0}(1-\varepsilon)\frac{m}{\sqrt[n]{2}} \right\rceil}) &\to H^0(X, \mathcal{O}_X(mK_X)) \\ \to H^0(X, \mathcal{O}_X(mK_X) \otimes \mathcal{O}_X/\mathcal{M}_{x,x'}^{\left\lceil \sqrt[n]{\mu_0}(1-\varepsilon)\frac{m}{\sqrt[n]{2}} \right\rceil}). \end{aligned}$$

We note that

$$n! \cdot \varlimsup_{m\to\infty} m^{-n} \dim H^0(X, \mathcal{O}_X(mK_X)) = \mu_0$$

holds, since K_X is nef and big (cf. Proposition 4.1 and Remark 4.2 in Appendix).

Then since

$$n! \cdot \varlimsup_{m\to\infty} m^{-n} \dim H^0(X, \mathcal{O}_X(mK_X) \otimes \mathcal{O}_X/\mathcal{M}_{x,x'}^{\left\lceil \sqrt[n]{\mu_0}(1-\varepsilon)\frac{m}{\sqrt[n]{2}} \right\rceil}) = \mu_0(1-\varepsilon)^n < \mu_0$$

hold, by the above exact sequence we complete the proof of Lemma 3.2. □

Let $\varepsilon > 0$ be as in Lemma 3.2. Let us take a sufficiently large positive integer m_0 so that

$$H^0(X, \mathcal{O}_X(m_0K_X) \otimes \mathcal{M}_{x,x'}^{\left\lceil \sqrt[n]{\mu_0}(1-\varepsilon)\frac{m_0}{\sqrt[n]{2}} \right\rceil}) \neq 0$$

holds as in Lemma 3.2 and let σ_0 be a general nonzero element of $H^0(X, \mathcal{O}_X(m_0K_X) \otimes \mathcal{M}_{x,x'}^{\lceil \sqrt[n]{\mu_0}(1-\varepsilon)m_0/\sqrt[n]{2} \rceil})$. We define the effective **Q**-divisor D_0 by

$$D_0 = \frac{1}{m_0}(\sigma_0).$$

We define the positive number α_0 by

$$\alpha_0 := \inf\{\alpha > 0 \mid (X, \alpha D_0) \text{ is KLT at neither } x \text{ nor } x'\},$$

where KLT is short for Kawamata log terminal (cf. Definition 2.4). Let $\mu\colon Y \to X$ be a log resolution of (X, D) and for $\alpha > 0$ let

$$K_Y + \mu_*^{-1}(\alpha D) = \mu^*(K_X + \alpha D) + F(\alpha),$$

where $F(\alpha)$ denotes the discrepancy depending on α. Then α_0 is the infimum of α such that the discrepancy $F(\alpha)$ has a component whose coefficient is less than or equal to -1. Hence by the construction α_0 is a rational number.

Considering the multiplicities of D_0 at x and x', by Proposition 2.5, we see that

$$\alpha_0 \leqq \frac{n\sqrt[n]{2}}{\sqrt[n]{\mu_0}(1-\varepsilon)}$$

holds.

Let us fix an arbitrary positive number $\delta \ll 1/n$. Let us take $\varepsilon > 0$ sufficiently small so that

$$\alpha_0 \leqq \frac{n\sqrt[n]{2}}{\sqrt[n]{\mu_0}} + \delta$$

holds. Then one of the following two cases occurs.

CASE 1. For every sufficiently small positive number λ, $(X, (\alpha_0 - \lambda)D_0)$ is KLT at both x and x'.

CASE 2. For every sufficiently small positive number λ, $(X, (\alpha_0 - \lambda)D_0)$ is KLT at exactly one of x or x' say x.

We define the next stratum X_1 as

$$\begin{aligned} X_1 := &\text{ the minimal center of log canonical singularities of } (X, \alpha_0 D_0) \\ &\text{ at } x \text{ (cf. Section 2).} \end{aligned}$$

Let n_1 denote the dimension of X_1. Let us define the volume μ_1 of X_1 with respect to K_X by

$$\mu_1 := K_X^{n_1} \cdot X_1.$$

If X_1 is a point, we stop the construction of the filtration. Suppose that X_1 is not a point.

Case 1 divides into the following two subcases.

CASE 1.1. X_1 passes through both x and x'.

CASE 1.2. X_1 passes through exactly one of x and x' (by the above assumption in Case 2, X_1 passes through x).

First we shall consider Case 1.1. In this case X_1 is not isolated at x. Since $x \in X^\circ$, we see that $\mu_1 > 0$ holds. The proof of the following lemma is identical to that of Lemma 3.2.

Lemma 3.3. *Let ε' be a positive rational number less than* 1 *and let x_1 and x_2 be distinct regular points on X_1. Then for a sufficiently large $m > 1$ (indendent of x_1, x_2),*

$$H^0(X_1, \mathcal{O}_{X_1}(mK_X) \otimes \mathcal{M}_{x_1,x_2}^{\lceil \sqrt[n_1]{\mu_1}(1-\varepsilon')\frac{m}{\sqrt[n_1]{2}} \rceil}) \neq 0$$

holds.

Let x_1 and x_2 be distinct regular points of $X_1 \cap X^\circ$. Let ε' be a positive rational number as in Lemma 3.3. Let m_1 be a sufficiently large positive integer so that

$$H^0(X_1, \mathcal{O}_{X_1}(m_1K_X) \otimes \mathcal{M}_{x_1,x_2}^{\left\lceil \sqrt[n_1]{\mu_1}(1-\varepsilon')\frac{m_1}{\sqrt[n_1]{2}} \right\rceil}) \neq 0$$

as in Lemma 3.3 and let

$$\sigma'_{1,x_1,x_2} \in H^0(X_1, \mathcal{O}_{X_1}(m_1K_X) \otimes \mathcal{M}_{x_1,x_2}^{\left\lceil \sqrt[n_1]{\mu_1}(1-\varepsilon')\frac{m_1}{\sqrt[n_1]{2}} \right\rceil})$$

be a nonzero element.

By Kodaira's lemma [10, Appendix] there is an effective **Q**-divisor E such that $K_X - E$ is ample. By the definition of X°, we may assume that the support of E contains neither x nor x'. In fact this can be verified as follows. Let H be an arbitrary ample divisor on X. Then by the definition of X°, $|aK_X - H|$ is base point free at x and x' for every sufficiently large a. Fix such an a and take a member E' of $|aK_X - H|$ which contains neither x nor x'. Then we may take E to be $a^{-1}E'$.

Let l_1 be a sufficiently large positive integer which will be specified later such that

$$L_1 := l_1(K_X - E)$$

is Cartier.

Lemma 3.4. *If we take l_1 sufficiently large, then*

$$\phi_m \colon H^0(X, \mathcal{O}_X(mK_X + L_1)) \to H^0(X_1, \mathcal{O}_{X_1}(mK_X + L_1))$$

is surjective for every $m \geqq 0$.

Proof. K_X is nef **Q**-Cartier divisor by the assumption. Let r be the index of X, i.e. r is the minimal positive integer such that rK_X is Cartier. Then for every locally free sheaf $\mathcal{E}$, by Lemma 4.3 in Appendix, there exists a positive integer k_0 depending on $\mathcal{E}$ such that if $l_1 \geqq k_0$ holds, then

$$H^q(X, \mathcal{O}_X((1 + mr)K_X + L_1) \otimes \mathcal{E}) = 0$$

holds for every $q \geqq 1$ and $m \geqq 0$. Let us consider the exact sequences

$$0 \to \mathcal{K}_j \to \mathcal{E}_j \to \mathcal{O}_X(jK_X) \otimes \mathcal{I}_{X_1} \to 0$$

for some locally free sheaf $\mathcal{E}_j$ for every $0 \leqq j \leqq r-1$, where $\mathcal{I}_{X_1}$ denotes the ideal sheaf associated with X_1. Then noting the above fact, we can prove that if we take l_1 sufficiently large,

$$H^q(X, \mathcal{O}_X(mK_X + L_1) \otimes \mathcal{I}_{X_1}) = 0$$

holds for every $q \geqq 1$ and $m \geqq 0$ by exactly the same manner as the standard proof of Serre's vanishing theorem (cf. [5, p.228, Theorem 5.2]). This implies the desired surjection. $\square$

Note that for l_1 sufficiently large, the surjectivity is true for every $m \geqq 0$. Let l_1 be as in Lemma 3.4. Let τ be a general section in $H^0(X, \mathcal{O}_X(L_1))$. Then by Lemma 3.4 we see that

$$\sigma'_{1,x_1,x_2} \otimes \tau \in H^0(X_1, \mathcal{O}_{X_1}(m_1 K_X + L_1) \otimes \mathcal{M}_{x_1,x_2}^{\left\lceil \sqrt[n_1]{\mu_1}(1-\varepsilon')\frac{m_1}{\sqrt[n_1]{2}} \right\rceil})$$

extends to a section

$$\sigma_{1,x_1,x_2} \in H^0(X, \mathcal{O}_X((m_1 + l_1)K_X)).$$

We may assume that the divisor $(\sigma_{1,x_1.x_2})$ is smooth on the neighbourhood $X_{\mathrm{reg}} \backslash (X_1 \cup \operatorname{Supp} E)$ of x and x' by Bertini's theorem. This is because if we take l_1 sufficiently large, as in the proof of Lemma 3.4 (see also the proof of Lemma 4.3),

$$(\flat) \qquad H^0(X, \mathcal{O}_X(mK_X + L_1)) \to H^0(X, \mathcal{O}_X(mK_X + L_1) \otimes \mathcal{O}_X/\mathcal{I}_{X_1} \cdot \mathcal{M}_y)$$

is surjective for every $y \in X_{\mathrm{reg}} \backslash X_1$ and $m \geqq 0$ (we may and do assume that l_1 is independent of y and m, since X is projective algebraic). We set

$$D_1(x_1, x_2) = \frac{1}{m_1 + l_1}(\sigma_{1,x_1,x_2}).$$

Let $X_{1,\mathrm{reg}}$ denote the regular locus of X_1. We may construct the divisors $\{D_1(x_1, x_2)\}$ as an algebraic family over $(X_{1,\mathrm{reg}} \times X_{1,\mathrm{reg}}) \backslash \Delta_{X_1}$, where Δ_{X_1} denotes the diagonal of $X_1 \times X_1$. Since in Lemma 3.4 we may take L_1 independent of x_1, x_2, the construction of the algebraic family is possible. Letting x_1 and x_2 tend to x and x' respectively, we obtain a **Q**-divisor D_1 on X which is $(m_1 + l_1)^{-1}$ times a divisor of a global holomorphic section

$$\sigma_1 \in H^0(X, \mathcal{O}_X((m_1 + l_1)K_X)).$$

By the construction, we may and do assume that (σ_1) is smooth on the neighbourhood $X_{\mathrm{reg}} \backslash (X_1 \cup \operatorname{Supp} E)$ of x and x'. In fact this follows from the surjectivity of $(\flat)$ (which is independent of x_1, x_2) and Bertini's theorem.

Let ε_0 be a positive rational number with $\varepsilon_0 < \alpha_0$. And we define the positive numbers $\alpha_1(x_1, x_2)$ and α_1 by

$$\alpha_1(x_1, x_2) := \inf\{\alpha > 0 \mid (\alpha_0 - \varepsilon_0)D_0 + \alpha D_1(x_1, x_2) \text{ is KLT at neither } x_1 \text{ nor } x_2\}$$

and

$$\alpha_1 := \inf\{\alpha > 0 \mid (\alpha_0 - \varepsilon_0)D_0 + \alpha D_1 \text{ is KLT at neither } x \text{ nor } x'\}$$

respectively. We shall estimate α_1. We note that m_1 is independent of l_1 (cf. Lemma 3.4).

Lemma 3.5. *Let δ be the fixed positive number as above. Then we may assume that*

$$\alpha_1 \leqq \frac{n_1 \sqrt[n_1]{2}}{\sqrt[n_1]{\mu_1}} + \delta$$

holds, if we take ε', l_1/m_1 and ε_0 sufficiently small.

Proof. To prove Lemma 3.5, we need the following elementary lemma.

Lemma 3.6 ([20, p.12, Lemma 6]). *Let a, b be positive numbers and n_1 a positive integer. Then*

$$\int_0^1 \frac{r_2^{2n_1-1}}{\left(r_1^2 + r_2^{2a}\right)^b}\, dr_2 = r_1^{2n_1/a-2b} \int_0^{r_1^{-2a}} \frac{r_3^{2n_1-1}}{\left(1 + r_3^{2a}\right)^b}\, dr_3$$

holds, where

$$r_3 = \frac{r_2}{r_1^{1/a}}.$$

First suppose that both x and x' are *nonsingular points* on X_1. Then we may set $x_1 = x$, $x_2 = x'$, i.e., we do not need the limiting process to define the divisor D_1.

Let $(z_1, \ldots, z_n)$ be a local coordinate system on a neighbourhood U of x in X such that

$$U \cap X_1 = \{q \in U \mid z_{n_1+1}(q) = \cdots = z_n(q) = 0\}.$$

We set $r_1 = \left(\sum_{i=n_1+1}^{n} |z_1|^2\right)^{1/2}$ and $r_2 = \left(\sum_{i=1}^{n_1} |z_i|^2\right)^{1/2}$. Fix an arbitrary C^∞-hermitian metric h_X on K_X. Then there exists a positive constant C such that

$$(\star) \qquad \|\sigma_1\|^2 \leqq C(r_1^2 + r_2^{2\left\lceil \sqrt[n_1]{\mu_1}(1-\varepsilon')\frac{m_1}{\sqrt[n_1]{2}} \right\rceil})$$

holds on a neighbourhood of x, where $\|\ \|$ denotes the norm with respect to $h_X^{m_1+l_1}$.

Let us apply Lemma 3.6 by taking

$$a := \left\lceil \sqrt[n_1]{\mu_1}(1-\varepsilon')\frac{m_1}{\sqrt[n_1]{2}} \right\rceil.$$

Then by Lemma 3.6 and the estimate $(\star)$, we see that for every

$$b > \frac{n_1}{\left\lceil \sqrt[n_1]{\mu_1}(1-\varepsilon')m_1 / \sqrt[n_1]{2} \right\rceil}.$$

$\|\sigma_1\|$ produces a singularity greater than or equal to $r_1^{2n_1/a-b}$, if we average the singularity in terms of the volume form in $z_1, \ldots, z_{n_1}$ direction.

On the other hand, there exists a positive integer M such that

$$\|\sigma_0\|^{-2} = O\left(r_1^{-M}\right)$$

holds on a neighbourhood of the generic point of $U \cap X_1$, where $\| \ \|$ denotes the norm with respect to $h_X^{m_0}$ and c is a positive constant.

Hence by the definition of α_0, by Proposition 2.5 we have the inequality:

$$\alpha_1 \leqq \left(\frac{m_1 + l_1}{m_1}\right) \frac{n_1 \sqrt[n_1]{2}}{\sqrt[n_1]{\mu_1}(1 - \varepsilon')} + M \frac{m_1 + l_1}{m_0} \varepsilon_0.$$

We note that since one l_1 works for all $m \geqq 0$, l_1/m_1 can be made arbitrary small. Taking ε', l_1/m_1 and ε_0 sufficiently small, we obtain that

$$\alpha_1 \leqq \frac{n_1 \sqrt[n_1]{2}}{\sqrt[n_1]{\mu_1}} + \delta$$

holds.

Next we consider the case that x or x' is a singular point on X_1. We need the following lemma.

Lemma 3.7. *Let φ be a plurisubharmonic function on $\Delta^n \times \Delta$. Let φ_t $(t \in \Delta)$ be the restriction of φ on $\Delta^n \times \{t\}$. Assume that $e^{-\varphi_t}$ does not belong to $L^1_{\mathrm{loc}}(\Delta^n, O)$ for any $t \in \Delta^*$.*

Then $e^{-\varphi_0}$ is not locally integrable at $O \in \Delta^n$.

Lemma 3.7 is an immediate consequence of the L^2-extension theorem [15, p.20, Theorem].

Using Lemma 3.7 and Lemma 3.6, letting $x_1 \to x$ and $x_2 \to x'$, we see that

$$\alpha_1 \leqq \liminf_{x_1 \to x,\, x_2 \to x'} \alpha_1(x_1, x_2)$$

holds. Hence Lemma 3.5 holds also in this case. □

Let X_2 be the minimal center of LC singularities of $(X, (\alpha_0 - \varepsilon_0)D_0 + \alpha_1 D_1)$ at x. Since $(X, (\alpha_0 - \varepsilon_0)D_0)$ is KLT by the definition of α_0 and D_1 is smooth on $X_{\mathrm{reg}} \backslash (X_1 \cup \operatorname{Supp} E)$, if we take m_1 sufficiently large, we may and do assume that X_2 is a proper subvariety of X_1.

Next we consider Case 2. The remaining case Case 1.2 will be considered later. In Case 2, for every sufficiently small positive number λ, $(X, (\alpha_0 - \lambda)D_0)$ is KLT at x and not KLT at x'. In Case 2, instead of Lemma 3.3, we use the following simpler lemma.

Lemma 3.8. *Let ε' be a positive number less than* 1 *and let x_1 be a regular point on X_1. Then for a sufficiently large $m > 1$,*

$$H^0\left(X_1, \mathcal{O}_{X_1}(mK_X) \otimes \mathcal{M}_{x_1}^{\lceil \sqrt[n_1]{\mu_1}(1-\varepsilon')m\rceil}\right) \neq 0$$

holds.

Let x_1 be a regular point of X_1. Using Lemma 3.8, let us take a nonzero element σ'_{1,x_1} in

$$H^0\left(X_1, \mathcal{O}_{X_1}(m_1K_X) \otimes \mathcal{M}_{x_1}^{\lceil \sqrt[n_1]{\mu_1}(1-\varepsilon')m_1\rceil}\right),$$

for a sufficiently large m_1. Let l_1 be as in Lemma 3.4 and let τ be a general nonzero section in $H^0(X, \mathcal{O}_X(L_1))$ as before, where L_1 is the line bundle as in Lemma 3.4. By Lemma 3.4, we may extend $\sigma_{1,x_1} \otimes \tau$ to a section

$$\sigma_{1,x_1} \in H^0(X, \mathcal{O}_X((m_1 + l_1)K_X)).$$

As in Case 1.1, taking l_1 sufficiently large, we may assume that (σ_{1,x_1}) is smooth on the neighbourhood $X_{\text{reg}}\backslash(X_1 \cup \operatorname{Supp} E)$ of x and x'. We set

$$D_1(x_1) = \frac{1}{m_1 + l_1}(\sigma_{1,x_1}).$$

Let $X_{1,\text{reg}}$ denote the regular locus of X_1. We may construct the divisors $\{D_1(x_1)\}$ as an algebraic family over $X_{1,\text{reg}}$. Letting x_1 tend to x, we obtain a **Q**-divisor D_1 on X which is $(m_1 + l_1)^{-1}$-times a divisor of a global holomorphic section

$$\sigma_1 \in H^0(X, \mathcal{O}_X((m_1 + l_1)K_X)).$$

By the construction, we may and do assume that (σ_1) is smooth on the neighbourhood $X_{\text{reg}}\backslash(X_1 \cup \operatorname{Supp} E)$ of x and x'.

Let ε_0 be a sufficiently small positive rational number with $\varepsilon_0 < \alpha_0$ such that $(\alpha_0 - \varepsilon_0)D_0$ is not KLT at x' (this is possible because we are considering Case 2).

And we define $\alpha_1(x_1)$ and α_1 by

$$\alpha_1(x_1) := \inf\{\alpha > 0 \mid (\alpha_0 - \varepsilon_0)D_0 + \alpha D_1(x_1) \text{ is not KLT at } x_1\}.$$

and

$$\alpha_1 := \inf\{\alpha > 0 \mid (\alpha_0 - \varepsilon_0)D_0 + \alpha D_1 \text{ is KLT at neither } x \text{ nor } x'\}$$

respectively. The definition of α_1 is the same as in Case 1.1. But we note that $(\alpha_0 - \varepsilon_0)D_0$ is already not KLT at x'. We shall estimate α_1. The proof of the following lemma is similar to that of Lemma 3.5.

Lemma 3.9. *Let δ be the fixed positive number as above. Then we may assume that*

$$\alpha_1 \leqq \frac{n_1}{\sqrt[n_1]{\mu_1}} + \delta$$

holds, if we take ε', l_1/m_1 and ε_0 sufficiently small.

This estimate is better than Lemma 3.5. Then we may define the proper subvariety X_2 of X_1 as the minimal center of log canonical singularities of $(X, (\alpha_0 - \varepsilon_0)D_0 + \alpha_1 D_1)$ at x or x' as we have defined X_1.

Lastly in Case 1.2 the construction of the filtration reduces to Case 2 as follows. In Case 1.2, X_1 does not pass through x'. Hence in this case the minimal center of LC singularities X_1' at x' does not pass through x.

Let a_1 be a sufficiently large positive integer such that

$$H^0\big(X, \mathcal{O}_X(a_1 K_X) \otimes \mathcal{I}_{X_1'}\big) \neq 0.$$

Let τ' be a general nonzero section of $H^0(X, \mathcal{O}_X(a_1 K_X) \otimes \mathcal{I}_{X_1'})$.

We note that there exists an effective **Q**-divisor G on X such that

1. $K_X - G$ is ample,
2. x is not contained in $\operatorname{Supp} G$.

In fact this can be verified as follows. Let H be an arbitrary ample divisor on X. Then by the definition of X°, $|bK_X - H|$ is base point free at x for every sufficiently large b. Fix such a b and take a member G' of $|bK_X - H|$ which does not contain x. Then we may take G to be $b^{-1}G'$.

Let a_1 be a sufficiently large positive integer such that $a_1(K_X - G)$ and $a_1 G$ are Cartier. By 1, it follows there exists $\tau'' \in H^0(X, \mathcal{O}_X(a_1(K_X - G))$ such that $\tau''(X_1') = 0$ and $\tau''(x) \neq 0$. By tensoring the global section of $\mathcal{O}_X(a_1 G)$ with divisor $a_1 G$ to τ'', if we take a_1 sufficiently large, we may assume that the divisor (τ') does not contain x.

In this case instead of σ_0, we shall use $\sigma_0^e \otimes \tau'$, where e is a positive integer. Let $D_0' := (m_0 e + a_1)^{-1}(\sigma_0^e \otimes \tau')$. Let us define the positive rational number α_0' for (X, D_0') similar to α_0. Then since $\tau'(X_1') = 0$ and $\tau'(x) \neq 0$, the minimal center of LC singularities of $(X, \alpha_0' D_0')$ at x is X_1 and $(X, \alpha_0' D_0')$ is not LC at x'. Also we can make α_0' arbitrary close to α_0 by taking e sufficiently large. Hence we may assume that α_0' satisfies the same estimate:

$$\alpha_0' \leqq \frac{n\sqrt[n]{2}}{\sqrt[n]{\mu_0}} + \delta$$

as α_0. In this way we can reduce Case 1.2 to Case 2.

In any case, we construct the next stratum X_2 as the minimal center of log canonical singularities of $(X, (\alpha_0 - \varepsilon_0)D_0 + \alpha_1 D_1)$ at x. If X_2 is a point, then we stop the construction of the filtration. If X_2 is not a point, we continue exactly the same procedure replacing X_1 by X_2. And we continue the procedure as long as the new center of log canonical singularities $(X_1, X_2, \dots)$ is not a point. As a result, for any distinct points $x, x' \in X^\circ$, we construct a strictly decreasing sequence of subvarieties:

$$X = X_0 \supset X_1 \supset \cdots \supset X_r \supset X_{r+1} = x \quad \text{or} \quad x',$$

effective **Q**-divisors

$$D_0, \dots, D_r$$

numerically equivalent to K_X and invariants:

$$\alpha_0, \alpha_1, \dots, \alpha_r,$$
$$n =: n_0 > n_1 > \cdots > n_r \quad (n_i = \dim X_i,\ \ i = 0, \dots, r)$$

and

$$\mu_0, \mu_1, \dots, \mu_r \quad (\mu_i = K_X^{n_i} \cdot X_i,\ \ i = 0, \dots, r)$$

depending on small positive rational numbers $\varepsilon_0, \dots, \varepsilon_{r-1}$, large positive integers $m_0, m_1, \dots, m_r$, positive integers $0 =: l_0, l_1, \dots, l_r$,

$$\sigma_i \in H^0(X, \mathcal{O}_X((m_i + l_i)K_X)) \quad (i = 0, \dots, r),$$
$$D_i = \frac{1}{m_i + l_i}(\sigma_i) \quad (i = 0, \dots, r),$$

etc.

Here each X_i $(1 \leqq i \leqq r)$ is the minimal center of log canonical singularities of $\left(X, \sum_{j=0}^{i-1}(\alpha_j - \varepsilon_j)D_j\right)$ at x or x'.

By Nadel's vanishing theorem ([13, p.561]) we have the following lemma.

Lemma 3.10. *For every positive integer $m > 1+\sum_{i=0}^{r}\alpha_i$, $\Phi_{|mK_X|}$ separates x and x'. And we may assume that*

$$\alpha_i \leqq \frac{n_i \sqrt[n_i]{2}}{\sqrt[n_i]{\mu_i}} + \delta$$

holds for every $0 \leqq i \leqq r$.

Proof. For $i = 0, 1, \dots, r$, let h_i be the singular hermitian metric on K_X defined by

$$h_i := \frac{1}{|\sigma_i|^{2/(m_i+l_i)}} := \frac{h_X}{\left(h_X^{m_i+l_i}(\sigma_i, \sigma_i)\right)^{1/(m_i+l_i)}},$$

where we have set $l_0 = 0$ and h_X is a C^∞-hermitian metric on K_X (the righthand side does not depend on the choice of h_X). As before, using Kodaira's lemma ([10, Appendix]), let G be an effective $\mathbf{Q}$-divisor such that $K_X - G$ is ample. As before we may assume that $\operatorname{Supp} G$ contains neither x nor x'. Let m be a positive integer such that $m > 1 + \sum_{i=0}^{r} \alpha_i$ holds. Let h_L is a C^∞-hermitian metric on the ample $\mathbf{Q}$-line bundle

$$L := \left(m - 1 - \left(\sum_{i=0}^{r-1}(\alpha_i - \varepsilon_i)\right) - \alpha_r\right) K_X - \delta_L G$$

with strictly positive curvature, where δ_L be a sufficiently small positive rational number and we shall consider h_L as a singular hermitian metric on $\left(m - 1 - \left(\sum_{i=0}^{r-1}(\alpha_i - \varepsilon_i)\right) - \alpha_r\right)K_X$, i.e., we identify h_L and the singular hermitian metric

$$\frac{h_L}{|\sigma_G|^{2\delta_L}}$$

on $\left(m - 1 - \left(\sum_{i=0}^{r-1}(\alpha_i - \varepsilon_i)\right) - \alpha_r\right)K_X$, where σ_G is a multi-holomorphic section of the $\mathbf{Q}$-line bundle G with divisor G. Let us define the singular hermitian metric $h_{x,x'}$ of $(m-1)K_X$ defined by

$$h_{x,x'} = \left(\prod_{i=0}^{r-1} h_i^{\alpha_i - \varepsilon_i}\right) \cdot h_r^{\alpha_r} \cdot h_L.$$

Then we see that $\mathcal{I}(h_{x,x'})$ defines a subscheme of X with isolated support around x or x' by the definition of the invariants $\{\alpha_i\}$'s. By the construction the curvature current $\Theta_{h_{x,x'}}$ is strictly positive on X. Then by Nadel's vanishing theorem ([13, p.561]) we see that

$$H^1(X, \mathcal{O}_X(mK_X) \otimes \mathcal{I}(h_{x,x'})) = 0$$

holds (see Section 2.3). Hence

$$H^0(X, \mathcal{O}_X(mK_X)) \to H^0(X, \mathcal{O}_X(mK_X) \otimes \mathcal{O}_X/\mathcal{I}(h_{x,x'}))$$

is surjective. Since by the construction of $h_{x,x'}$ (if we take δ_L sufficiently small) $\operatorname{Supp}(\mathcal{O}_X/\mathcal{I}(h_{x,x'}))$ contains both x and x' and is isolated at least at one of x or x'. Hence by the above surjection, there exists a section $\sigma \in H^0(X, \mathcal{O}_X(mK_X))$ such that

$$\sigma(x) \neq 0, \quad \sigma(x') = 0$$

or

$$\sigma(x) = 0, \quad \sigma(x') \neq 0$$

holds. This implies that $\Phi_{|mK_X|}$ separates x and x'. The proof of the last statement is similar to the proof of Lemma 3.5 $\square$

3.2. Estimate of the degree. To relate μ_0 and the degree of the pluricanonical image of X, we need the following lemma.

Lemma 3.11. *If $\Phi_{|mK_X|}$ is a birational rational map onto its image, then*

$$\deg \Phi_{|mK_X|}(X) \leqq \mu_0 \cdot m^n$$

holds.

Proof. Let $p\colon \tilde{X} \to X$ be the resolution of the base locus of $|mK_X|$ and let

$$p^*|mK_X| = |P_m| + F_m$$

be the decomposition into the free part $|P_m|$ and the fixed component F_m. We have

$$\deg \Phi_{|mK_X|}(X) = P_{m,}^n$$

holds.

We note that $\mathcal{O}_{\tilde{X}}(\nu P_m)$ is globally generated on $\tilde{X}$. This implies that for every $\nu \geqq 1$ we have the injection

$$\mathcal{O}_{\tilde{X}}(\nu P_m) \to p^*\mathcal{O}_X(m\nu K_X).$$

Hence there exists a natural morphism

$$H^0\big(\tilde{X}, \mathcal{O}_{\tilde{X}}(\nu P_m)\big) \to H^0(X, \mathcal{O}_X(m\nu K_X))$$

for every $\nu \geqq 1$. This morphism is clearly injective. This implies that

$$\mu_0 \geqq m^{-n}\mu\big(\tilde{X}, P_m\big)$$

holds. Since P_m is nef and big on $\tilde{X}$, we see that

$$\mu\big(\tilde{X}, P_m\big) = P_m^n$$

holds. Hence

$$\mu_0 \geqq m^{-n} P_m^n$$

holds. This implies the desired inequality:

$$\deg \Phi_{|mK_X|}(X) \leqq \mu_0 \cdot m^n$$

holds. $\square$

3.3. Use of Kawamata's subadjunction theorem. Let X be a minimal projective n-fold X of general type and let X° be the Zariski open subset of X defined by

$$X^\circ = \{x \in X_{\mathrm{reg}} \mid x \notin \mathrm{Bs}\,|mK_X| \text{ and } \Phi_{|mK_X|} \text{ is a biholomorphism on a neighbourhood of } x \text{ for some } m \geqq 1\}$$

as in the beginning of Section 3. Let x, x' be distinct points on X°. Let us consider again the sequence of numbers α_j, divisors D_j and the filtration

$$X \supset X_1 \supset \cdots \supset X_r \supset X_{r+1} = \{x\} \quad \text{or} \quad \{x'\}$$

which were defined in Section 3.1. For $1 \leqq j \leqq r$, let $\pi_j \colon W_j \to X_j$ be a desingularization of X_j. Let us fix $1 \leqq j \leqq r$. Applying Theorem 2.8 to (X, D) where

$$D := (\alpha_0 - \varepsilon_0)D_0 + \cdots + (\alpha_{j-2} - \varepsilon_{j-2})D_{j-2} + \alpha_{j-1}D_{j-1},$$

we get

$$(\sharp) \qquad \mu(W_j, K_{W_j}) \leqq \left(1 + \sum_{i=0}^{j-1} \alpha_i\right)^{n_j} \cdot \mu_j$$

holds, where

$$\mu(W_j, K_{W_j}) := n_j! \cdot \overline{\lim_{m\to\infty}}\, m^{-n_j} \dim H^0(W_j, \mathcal{O}_{W_j}(mK_{W_j})).$$

In fact by Theorem 2.8 and Remark 2.9, we see that

$$(K_X + D)|_{X_j} - (\pi_j)_* K_{W_j}$$

is pseudoeffective. Hence

$$\mu(W_j, K_{W_j}) \leqq \mu\big(W_j, \pi_j^*(K_X + D)\big)$$

holds. Here we have defined $\mu\big(W_j, \pi_j^*(K_X + D)\big)$ by

$$\mu\big(W_j, \pi_j^*(K_X + D)\big) := c^{-n_j} \cdot \mu\big(W_j, a \cdot \pi_j^*(K_X + D))\big),$$

where c is a positive integer such that $c(K_X + D)$ is Cartier. It is easy to see that this definition is independent of the choice of c (cf. Remark 4.2). Also we note that since every D_i $(1 \leqq i \leqq j - 1)$ is $\mathbf{Q}$-linearly equivalent to K_X, $K_X + D$ is $\mathbf{Q}$-linearly equivalent to

$$1 + \left(\alpha_{j-1} + \sum_{i=0}^{j-2}(\alpha_i - \varepsilon_i)\right) K_X.$$

Then combining the above facts, by Proposition 4.1 (see also Remark 4.2) and the definition $\mu_j := K_X^{n_j} \cdot X_j$, we have the desired inequality ($\sharp$).

We note that X cannot be dominated by a family of varieties of nongeneral type. In fact if there exists a dominant family of subvarieties of nongeneral type, then this contradicts the assumption that X is of general type. Hence there exists a nonempty open set U_0 of X° in *countable Zariski topology* such that for every $x \in U_0$, any subvariety of X passing through x is of general type.

We shall prove Theorem 1.2 by induction on n. Suppose that Theorem 1.2 holds for projective varieties of general type of dimension less than or equal to $n-1$ (the case of $n = 1$ is trivial), i.e., for every positive integer $k < n$ there exists a positive number $C(k)$ such that for every smooth projective variety W of general type of dimension k,

$$\mu(W, K_W) \geqq C(k)$$

holds. Let us assume that (x, x') belongs to $(U_0 \times U_0) \backslash \Delta_X$. Then X_j is of general type by the definition of U_0 and by the above inequality ($\sharp$) and the definition of $C(n_j)$,

$$C(n_j) \leqq \left(1 + \sum_{i=0}^{j-1} \alpha_i\right)^{n_j} \cdot \mu_j$$

holds. Since

$$\alpha_i \leqq \frac{\sqrt[n_i]{2}\, n_i}{\sqrt[n_i]{\mu_i}} + \delta$$

holds for every $0 \leqq i \leqq r$ by Lemma 3.10, we see that

$$\frac{1}{\sqrt[n_j]{\mu_j}} \leqq \left(2 + \sum_{i=0}^{j-1} \frac{\sqrt[n_i]{2}\, n_i}{\sqrt[n_i]{\mu_i}}\right) \cdot C(n_j)^{-1/n_j}$$

holds for every $j \geqq 1$. We note that the stricly decreasing sequence $\{n, n_1, \ldots, n_r\}$ has finitely many possibilities. Then using the above inequality inductively, we have the following lemma.

Lemma 3.12. *Suppose that $\mu_0 \leqq 1$ holds. Then there exists a positive constant C depending only on n such that for every $(x, x') \in (U_0 \times U_0) \backslash \Delta_X$ the corresponding invariants $\{\mu_0, \ldots, \mu_r\}$ and $\{n_1, \ldots, n_r\}$ depending on (x, x') (r may also depend on (x, x')) satisfies the inequality*:

$$2 + \left\lceil \sum_{i=0}^{r} \frac{\sqrt[n_i]{2}\, n_i}{\sqrt[n_i]{\mu_i}} \right\rceil \leqq \left\lfloor \frac{C}{\sqrt[n]{\mu_0}} \right\rfloor,$$

where for a real number a, $\lfloor a \rfloor$ denotes the largest integer less than or equal to a.

By Lemmas 3.10 and 3.12 we see that if $\mu_0 \leqq 1$ holds, for

$$m := \left\lfloor \frac{C}{\sqrt[n]{\mu_0}} \right\rfloor,$$

$|mK_X|$ gives a birational embedding of X and

$$\deg \Phi_{|mK_X|}(X) \leqq C^n \tag{1}$$

holds by Lemma 3.11, where C is the positive constant in Lemma 3.12. Also

$$\dim H^0(X, \mathcal{O}_X(mK_X)) \leqq n + 1 + \deg \Phi_{|mK_X|}(X)$$

holds by the semipositivity of the Δ-genus ([3]). Hence we have that if $\mu_0 \leqq 1$,

$$\dim H^0(X, \mathcal{O}_X(mK_X)) \leqq n + 1 + C^n \tag{2}$$

holds.

Since C is a positive constant depending only on n, combining the above two inequalities (1) and (2), we have that there exists a positive constant $C(n)$ depending only on n such that

$$\mu_0 = K_X^n \geqq C(n)$$

holds.

More precisely we argue as follows. Let $\mathcal{H}$ be the union of the irreducible components of the Hilbert scheme parametrizing subschemes of degree $\leqq C^n$ in projective spaces of dimension $\leqq n + C^n$.

By the general theory of Hilbert schemes ([4, exposé 221]), $\mathcal{H}$ consists of finitely many irreducible components. Let $\mathcal{H}_0$ be the Zariski open subset of $\mathcal{H}$ which parametrizes irreducible subvarieties. Then there exists a finite stratification of $\mathcal{H}_0$ by Zariski locally closed subsets such that on each stratum, there exists a simultaneous resolution of the universal family on the stratum. We note that the volume of the canonical bundle of the resolution (for the definition of the volume see Theorem 1.2) is constant on each stratum by the invariance of plurigenera ([21, 14]). Hence there exists a positive constant $C(n)$ depending only on n such that

$$\mu(X, K_X) \geqq C(n)$$

holds for every projective n-fold X of general type with $\mu(X, K_X) \leqq 1$. This completes the proof of Theorem 1.2 assuming MMP. □

Now let us prove Theorem 1.1. By Lemmas 3.10 and 3.12, Theorem 1.2 implies that there exists a positive integer ν_n depending only on n such that for every projective n-fold X of general type, $|mK_X|$ gives a birational embedding into a projective space for every $m \geqq \nu_n$. This completes the proof of Theorem 1.1 assuming MMP. $\square$

4. Appendix

4.1. Volume of nef and big line bundles. The following fact seems to be well known. But for the completeness, I would like to include the proof.

Proposition 4.1. *Let M be a smooth projective n-fold and let L be a nef and big line bundle on M. Then*

$$n! \cdot \overline{\lim_{m\to\infty}} m^{-n} \dim H^0(M, \mathcal{O}_M(mL)) = L^n$$

holds.

Proof. Since L is big, there exists an effective $\mathbf{Q}$-divisor F such that $L - F$ is ample. Let a be a positive integer such that $A := a(L - F)$ is a very ample Cartier divisor and $A - K_X$ is ample. Then by the Kodaira vanishing theorem, for every $q \geqq 1$,

$$H^q(M, \mathcal{O}_M(A + mL)) = 0$$

holds for every $m \geqq 0$. By the Riemann-Roch theorem, we have that

$$n! \cdot \overline{\lim_{m\to\infty}} m^{-n} \dim H^0(M, \mathcal{O}_M(A + mL)) = L^n$$

holds. By the definition of A, we see that

$$n! \cdot \overline{\lim_{m\to\infty}} m^{-n} \dim H^0(M, \mathcal{O}_M(mL)) = L^n$$

holds. This completes the proof. $\square$

REMARK 4.2. Let X be a minimal projective n-fold of general type and let r be a positive integer such that rK_X is Cartier. Let Y be a subvariety of X. Let $\varpi : \tilde{Y} \to Y$ be a resolution of singularities. Then $r\varpi^*K_X$ is a nef Cartier divisor on $\tilde{Y}$. $\varpi^*\mathcal{O}_X(mK_X)$ is a sheaf on $\tilde{Y}$ for every $m \geqq 1$. We define

$$\mu(Y, K_X|_Y) = (\dim Y)! \cdot \overline{\lim_{m\to\infty}} m^{-\dim Y} \dim H^0\bigl(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(\varpi^*(mK_X))\bigr)$$

as above. Suppose that $\mu(Y, K_X|_Y) > 0$ holds, i.e., $K_X|_Y$ is big.

We note that by Kodaira's lemma, there exists a positive integer a_0 such that for every positive integer $a \geqq a_0$, $H^0\bigl(\tilde{Y}, \varpi^*\mathcal{O}_Y(aK_X)\bigr) \neq 0$ holds. In particular, there exists a positive integer b_0 such that

$$H^0\bigl(\tilde{Y}, \varpi^*\mathcal{O}_Y((b_0 + j)K_X)\bigr) \neq 0$$

for every $j = 0, 1, \ldots, r-1$. Hence there exists an injection

$$H^0\big(\tilde{Y}, \varpi^*\mathcal{O}_Y(mK_X)\big) \to H^0\big(\tilde{Y}, \varpi^*\mathcal{O}_Y((m+b_0+j)K_X)\big)$$

for every $0 \leqq j \leqq r-1$.

This implies that

$$\mu(Y, K_X|_Y) = r^{-n} \cdot \mu\big(\tilde{Y}, r\varpi^* K_X\big)$$

holds.

Then by Proposition 4.1, we see that

$$\mu(Y, K_X|_Y) = r^{-n} \cdot \mu\big(\tilde{Y}, r\varpi^* K_X\big) = r^{-n} \cdot \big((\varpi^*(rK_X))^{\dim Y} \cdot \tilde{Y}\big) = K_X^{\dim Y} \cdot Y$$

holds.

4.2. A Serre type vanishing theorem.

Lemma 4.3. *Let X be a projective variety with only canonical singularities* (*cf*. [11, *p*.56, Definition 2.34]). *Let E be a vector bundle on X and let L be a nef line bundle on X. Let A be an ample line bundle on X. Then there exsists a positive integer k_0 depending only on E such that for every $k \geqq k_0$*

$$H^q(X, \mathcal{O}_X(K_X + mL + kA) \otimes E) = 0$$

holds for every $m \geqq 0$ and $q \geqq 1$.

Proof. Let ω_X be the L^2-dualizing sheaf of X, i.e., the direct image sheaf of the canonical sheaf of a resolution of X. Since X has only canonical singularities, we see that ω_X is isomorphic to $\mathcal{O}_X(K_X)$. Since L is nef and A is ample, there exists a positive integer k_0 such that for every $k \geqq k_0$, $(mL + kA) \otimes E$ admits a C^∞-hermitian metric with (strictly) Nakano positive curvature.

Then by exactly the same way as in Section 2.3, we see that

$$H^q(X, \omega_X \otimes \mathcal{O}_X(mL + kA) \otimes E) = 0$$

holds for every $m \geqq 0$ and $q \geqq 1$.

Since ω_X is isomorphic to $\mathcal{O}_X(K_X)$, we have that

$$H^q(X, \mathcal{O}_X(K_X + mL + kA) \otimes E) = 0$$

holds for every $m \geqq 0$ and $q \geqq 1$. This completes the proof. $\square$

NOTE ADDED IN PROOFS. Very recently the following two papers appeared and proved the same result in this paper and [23].

[H-M] C. Hacon and J. McKernan: *Boundedness of pluricanonical maps of varieties of general type*, Invent. Math. **166** (2006), 1–25.

[Ta] S. Takayama: *Pluricanonical systems of varieties of general type*, Invent. Math. **165** (2006), 551–587.

Apparently they have followed the strategy and the arguments in this paper and [23] as they mentioned in their papers. Actually as in [23], the crucial tools in their proofs (Section 4 in [H-M], Theorem 4.1 in [Ta]) are also the extension theorems of sections of multi adjoint bundles from the subvariety to the ambient variety which follow the subadjunction theorem, Theorem 2.23 in [23]. Theorem 2.23 in [23] and their corresponding extension theorems follow from entirely the same argument which appeared in the paper: Y.-T. Siu, Invariance of plurigenera, Invent. Math **134** (1998), 661–673. Actually all the proofs of extension theorems are completely parallel to the proof of invariance of plurigenera in Siu's paper.

The only difference between their proofs and the one in [23] is that the extension theorem is from a divisor in their proofs, while in my proof the extension is from a subvariety of arbitrary codimension, because I have used the L^2-extension theorem of Ohsawa ([21]) instead of the Kawamata-Viehweg vanishing theorem. Hence I do not see anything essentially new in their proofs, although their proofs require only algebraic tools.

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