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ON CYCLIC SPLITTING COMMUTATIVE RINGS

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Throughout this paper all rings considered are commutative rings with identity and all modules are unital.

A ring R is called cyclic splitting (finitely generated splitting; splitting) if the singular submodule of A is a direct summand of A for all cyclic R-modules (finitely generated R-modules; R-modules) A.

A ring R is splitting if and only if it is a (von Neumann) regular ring such that, for each essential ideal I of R, R/I is a direct sum of fields. This result was shown by Catefories and Sandomierski [3;4]. The study of finitely generated splitting rings was reduced to that of rings with essential socle by Goodearl [7] (cf. [18]). Cyclic splitting rings were studied in Teply [16;17].

This paper is concerned with the study of cyclic splitting rings. By making use of Teply [16, Theorem 3.3], in Theorem 2.2, we give several characterizations of cyclic splitting rings R. One of these is that R is a PP-ring and the Boolean ring of all idempotents in R is cyclic splitting. Our main theorem is Theorem 3.1 in which it is shown that, for certain regular rings R including Boolean rings, the following conditions are equivalent:

(a) R is cyclic splitting.

(b) R is finitely generated splitting.

(c) The singular submodule of every finitely generated R-module is also finitely generated.

Let R be a ring. For a given R-module A, we use $Z(A)$ to denote the set of those elements of A whose annihilator ideals of R are essential ideals. An R-module A is said to be singular (non-singular) provided $Z(A)=A$ ($Z(A)=0$). As is well known, R is non-singular as an R-module if and only if it is a semiprime ring. For the basic properties of singular modules and nonsingular modules, the reader is refered to [8].

For a given ring R, we denote its maximal ring of quotients, its classical ring of quotients and its socle by $Q(R)$, $T(R)$ and $soc(R)$, respectively.

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1. Preliminaries

Let R be a ring. We denote the Boolean ring consisting of all idempotents in R by $B(R)$ and the set of all prime (=maximal) ideals of $B(R)$ by $X(R)$. As is well known, $X(R)$ forms a Boolean space, that is, a totally disconnected compact Hausdorff space with the family $\{U(e)\mid e\in B(R)\}$ as an open-closed basis, where $U(e)=\{x\in X(R)\mid e\in x\}$ (see [15]).

Lemma 1.1. *Let X be a non-empty closed subset of $X(R)$.*

(a) $\bigcap_{x\in X} Ax=A(\bigcap_{x\in X} x)$ *for any R-module A.*

(b) *If Y is a subset of $X(R)$ such that $X\subseteq Y$ and $\bigcap_{x\in X} Rx=\bigcap_{y\in Y} Ry$, then $X=Y$.*

Proof. (a) Let A be an R-module. Clearly $A(\bigcap_{x\in X} x)\subseteq\bigcap_{x\in X} Ax$. Let $a\in\bigcap_{x\in X} Ax$. Then, for each x in X, there exists an element $e(x)$ in x such that $a=ae(x)$. Since X is closed and $X=\bigcup_{x\in X}\{U(e(x))\cap X\}$, we see that $X=\bigcup_{i=1}^{n}\{U(e(x_i))\cap X\}$ for some finite subset $\{x_1, \cdots, x_n\}$ of X. Setting $e=e(x_1)\cdots e(x_n)$, clearly, $e\in\bigcap_{e\in X} x$ and $a=ar$. Thus $a\in A(\bigcap_{x\in X} x)$ as desired.

(b) Suppose that $X\neq Y$ and take y' in $Y-X$. Since X is closed, we can choose e in $B(R)$ such that $X\cap U(e)=\emptyset$ and $y'\in U(e)$. Then $1-e\in\bigcap_{x\in X} Rx=\bigcap_{y\in Y} Ry$ and so $1\in y'$, a contradiction. Thus we must have $X=Y$.

Lemma 1.2. *For a non-empty subset X of $X(R)$,*

(a) $\bigcap_{x\in X} x=\bigcap_{x\in X^-} x$, *where X^- denotes the closure of X,*

(b) $Re+\bigcap_{x\in X} Rx=\bigcap_{x\in X\cap U(e)} Rx$ *for any e in $B(R)$.*

Proof. (a) This follows from the examination of $e\in\bigcap_{x\in X} x\Leftrightarrow U(1-e)\cap X=\emptyset\Leftrightarrow U(1-e)\cap X^-=\emptyset\Leftrightarrow e\in\bigcap_{x\in X^-} x$.

(b) Clearly $Re+\bigcap_{x\in X} Rx\subseteq\bigcap_{x\in X\cap U(e)} Rx$. If $g\in\bigcap_{x\in X\cap U(e)} Rx$, then it is easy to see $g(1-e)\in\bigcap_{x\in X} Rx$. Hence $g\in Re+\bigcap_{x\in X} Rx$ and we obtain the reverse inclusion.

A ring R is called a *PP*-ring if every principal ideal of R is projective or, equivalently, the annihilator ideal of any element of R is generated by an idempotent.

Lemma 1.3. *The following conditions are characterizations for a given ring R to be a PP-ring.*

(a) *$T(R)$ is regular and $B(T(R))=B(R)$.*

(b) *R/Rx is an integral domain for any x in $X(R)$ and supp(r) is an open set in $X(R)$ for any r in R, where supp$(r)=\{x\in X(R)\mid r\notin Rx\}$.*

(c) *The space of minimal prime ideals of R is compact and coincides with* $\{Rx \mid x \in X(R)\}$.

(d) R/Rx *is an integral domain for any x in $X(R)$ and $Q(R)$ is flat as an R-module.*

Proof. The characterization (a) was obtained by Endo [5], (b) appeared in Bergman [2] and (c) was obtained by Kist [9].

(a), (b)$\Rightarrow$(d). Since $T(R)$ is regular, $Q(R)$ is flat as a $T(R)$-module. On the other hand $T(R)$ is always flat as an R-module and therefore $Q(R)$ is flat as an R-module.

(d)$\Rightarrow$(c). Since $Q(R)$ is flat, the first half follows from Mewborn [10, Theorem 3.1] and the latter half follows from the fact that, for any x in $X(R)$, Rx is a prime ideal.

From Lemma 1.3(b) and [15, p.45] we have the following.

Corollary 1.4. *For a Boolean space and an integral domain D, the ring of all continuous functions from X to D is a PP-ring, where D has the discrete topology.*

A ring R is called a Baer ring if every annihilator ideal of R is generated by an idempotent. Clearly Baer rings are semiprime rings. Conversely if R is a semiprime ring, then it has the unique minimal Baer ring of quotients, denoted by $C(R)$. Note that $C(R)$ coincides with the ring generated by $B(Q(R))$ over R in $Q(R)$ ([11, Proposition 2.5]).

Lemma 1.5. *If R is a Baer ring, then every cyclic nonsingular R-module is projective. So it is trivially cyclic splitting.*

Proof. Since R is semiprime, as is well known (e.g. [8]), every cyclic nonsingular R-module can be embedded into $Q(R)$. Hence the lemma follows from the fact that $Q(R)$ is regular and $B(Q(R))=B(R)$.

Lemma 1.6 ([15, Theorem 24.5]). *A regular ring R is self-injective if and only if every finitely generated nonsingular R-module is projective. Hence in this case R is trivially finitely generated splitting.*

2. Cyclic splitting rings

Note that cyclic splitting rings are semiprime rings.

We shall quote Teply's result [16, Theorem 3.3] as follows:

For a given ring R, the following conditions are equivalent:

(a) R is cyclic splitting.

(b) R is a semiprime ring with the property that if I is a closed ideal, then $I=Re\oplus S$ for some e in $B(R)$ and an ideal S contained in $soc(R)$.

However, as is well known ([6, p. 112]), an ideal I of a semiprime ring R is a closed ideal if and only if $I=Q(R)f\cap R$ $(=Rf\cap R)$ for some f in $B(Q(R))$.

Therefore the condition (b) above can be replaced by the condition below:

(Teply's criterion) For any f in $B(Q(R))$, there exists e in $B(R)$ and an ideal S contained in $soc(R)$ for which $Rf \cap R = Re \oplus S$.

Lemma 2.1. *If R is a PP-ring, then the following statements hold;*

(a) $B(Q(R)) = Q(B(R))$.

(b) $Rf \cap R = R(B(R)f \cap B(R))$ *for any* f *in* $B(Q(R))$.

(c) *R is cyclic splitting if and only if $B(R)$ is cyclic splitting.*

Proof. (a) Since $T(R)$ is regular, it is easy to see that $B(Q(R)) = Q(B(T(R)))$. Hence $B(Q(R)) = Q(B(R))$ by Lemma 1.3 (a).

(b) Let $f \in B(Q(R))$. Since $Q(R)$ is regular, it follows that $Q(R)f = \cap \{Q(R)y \mid y \in X(Q(R)), f \in y\}$. Hence

$$\begin{aligned} & Rf \cap R \\ &= \cap \{Q(R)y \cap R \mid y \in X(Q(R)), f \in y\} \\ &= \cap \{Rx \mid x \in \lambda(Y)\} \end{aligned}$$

where $Y = \{y \in X(Q(R)) \mid f \in y\}$ and λ is the continuous (closed) mapping from $X(Q(R))$ to $X(R)$ given by $y \rightarrow y \cap R$. Since $\lambda(Y)$ is closed, by Lemma 1.1 (a), we have further

$$\begin{aligned} & \cap \{Rx \mid x \in \lambda(Y)\} \\ &= R(\cap \{x \mid x \in \lambda(Y)\}) \\ &= R(\cap \{y \cap B(R) \mid f \in y\}) \\ &= R(B(R)f \cap B(R)) \end{aligned}$$

and hence $Rf \cap R = R(B(R)f \cap B(R))$.

(c) This follows from (a), (b) and Teply's criterion.

Now we shall give several characterizations of cyclic splitting rings.

Theorem 2.2. *The following conditions are equivalent for a given ring R:*

(a) *R is cyclic splitting.*

(b) *R is a ring direct sum of two rings H and K such that*

(1) *H is a Baer ring,*

(2) *K is a cyclic splitting ring with essential socle.*

(c) *R is a PP-ring and $B(R)$ is cyclic splitting.*

(d) *R is a PP-ring and $T(R)$ is cyclic splitting.*

(e) *Every R/Rx for x in $X(R)$ is an integral domain and $T(R)$ is cyclic splitting.*

Proof. Clearly all conditions above yield that R is semiprime.

(a)$\Rightarrow$(b). We can take f in $B(Q(R))$ such that $Q(R)(1-f)$ is the injective hull of $soc(R)$ as an R-module. We claim that $B(Q(R)f) \subseteq R$. Let $g \in B(Q(R)f)$. Since $gR \cap soc(R) = 0$, $gR \cap R = Re$ for some e in $B(R)$ by Teply's criterion. If

$g(1-e)\neq 0$, there exists r in R such that $0\neq rg(1-e)$ in R. But $rg(1-e)$ is in Re, a contradiction. Thus $g(1-e)=0$ and $g=e\in R$ as desired. In particular $f\in R$ and $R=Rf\oplus R(1-f)$. Since $Q(Rf)=Q(R)f$ and $B(Q(R)f)=B(Rf)$, we see that Rf is a Baer ring. On the other hand, $R(1-f)$ has essential socle.

(b)$\Rightarrow$(a) follows from Lemma 1.5.

(a)$\Rightarrow$(c). Let $r\in R$. For p in $Q(R)$ such that $rpr=r$, Teply's criterion says that $Rep\cap R=Re\oplus S$ for some e in $B(R)$ and an ideal $S\subseteq soc(R)$. Noting that $Rep\cap R$ is an essential extension of Rr, we have $S=Re_1+\cdots+Re_n$ for some orthogonal idempotents $e_1,\cdots,e_n$ in R. Put $f=e+e_1+\cdots+e_n$. Then $Q(R)r=Q(R)f$ since $Q(R)r=Q(R)rp=E(Rrp\cap R)=E(Re)\oplus E(S)=Q(R)e\oplus Q(R)e_1\oplus\cdots\oplus Q(R)e_n=Q(R)f$, where $E(\#)$ denotes the injective hull of $\#$. Hence it follows that $R(1-f)$ is the annihilator ideal of r in R, and thus R is a PP-ring. The remainder follows from Lemma 2.1.

(c)$\Rightarrow$(a) and (c)$\Leftrightarrow$(d) follow from Lemma 2.1.

(d)$\Rightarrow$(e) follows from Lemma 1.3(b).

(e)$\Rightarrow$(d). Since $T(R)$ is cyclic splitting, by the implication (a)$\Rightarrow$(c), $T(R)$ is a PP-ring. Using Lemma 1.3(d), we can see that R is a PP-ring. This completes the proof.

Combining (c) of Theorem 2.2 with [1, Corollary 1, Theorem A], we have

Corollary 2.3. *A ring R is cyclic splitting if and only if the polynomial ring $R[x]$ is cyclic splitting.*

From (c) of Theorem 2.2 and Corollary 1.4, we have

Corollary 2.4. *Let X be a Boolean space such that its corresponding Boolean ring is cyclic splitting and D an integral domain. Then the ring of all continuous functions from X to D is cyclic splitting.*

From (b) and (c) of Theorem 2.2, the study of cyclic splitting rings can be reduced to that of atomic Boolean rings.

Theorem 2.5. *If R is an atomic Boolean ring, then the following conditions are equivalent:*

(a) *R is cyclic splitting.*

(b) *$X(R)$ satisfies the condition that, for any closed subset X of $X(R)$, there exists an open-closed subset $U\subseteq X(R)$ for which $(X'\cap X)^-=U\cap X$, where X' denotes the set of all isolated points in $X(R)$.*

Proof. (a)$\Rightarrow$(b). Let X be a closed subset of $X(R)$. By [14, Corollary 2.2], it holds that $Z(R/\bigcap_{x\in X} x)=(\bigcap_{x\in X'\cap X} x)/\bigcap_{x\in X} x$. Hence $\bigcap_{x\in X'\cap X} x=Re+\bigcap_{x\in X} x$ for some e in R, and thus we have $(X'\cap X)^-=U(e)\cap X$ by Lemma 1.1(b) and Lemma 1.2.

(b)$\Rightarrow$(a). Let I be an ideal of R. Putting $X=\{x\in X(R)\mid I\subseteq x\}$, we have $I=\bigcap_{x\in X} x$ because R is regular. Let U be an open-closed subset of $X(R)$ such that $(X'\cap X)^{-}=U\cap X$. Since U is open-closed, as is well known, $U=U(e)$ for some e in R. Now Lemma 1.2(b) says that $\bigcap_{x\in(X'\cap X)^{-}} x=\bigcap_{x\in U\cap X} x=Re+\bigcap_{x\in X} x$. Hence, again using [14, Corollary 2.2], we conclude that $Z(R/I)$ is a direct summand of R/I.

The topological space obtained by one point compactification of an infinite discrete space is an example of such a space $X(R)$ which satisfies the condition in (b) of Theorem 2.5.

3. Cyclic splitting regular rings

The purpose of this section is to show the following.

Theorem 3.1. *Let R be a regular ring such that $C(R)=Q(R)$. Then the following conditions are equivalent:*

(a) *R is cyclic splitting.*

(b) *R is finitely generated splitting.*

(c) *The singular submodule of every finitely generated R-module is also finitely generated.*

In order to prove this we need several lemmas.

Lemma 3.2. *Let R be a ring and $e_1, \cdots, e_n$ in $B(R)$. Then there are orthogonal elements $f_1, \cdots, f_m$ in $B(R)$ such that $\sum_{i=1}^{n} Re_i=\sum_{i=1}^{m} Rf_i$ and e_if_j is either 0 or f_j for any i, j.*

This can be easily shown by induction on n. We omit the proof.

Lemma 3.3. *Let R be a regular ring, $p_1, \cdots, p_n$ in $Q(R)$ and T an R-module. If A is a finitely generated R-submodule of the direct product $T\times Rp_1\times\cdots\times Rp_n$, then A can be decomposed into the direct sum of R-modules $A_1, \cdots, A_u$; $B_1, \cdots, B_v$ such that*

(1) *each A_i has a generating set $\{u_1, \cdots, u_m\}$ of the following form in $T\times Rp_1\times\cdots\times Rp_n$*

$$\begin{aligned}
u_1 &= (*, *, 0, 0, 0, \cdots\cdots, 0, *, \cdots, *)\\
u_2 &= (*, 0, *, 0, 0, \cdots\cdots, 0, *, \cdots, *)\\
u_3 &= (*, 0, 0, *, 0, \cdots\cdots, 0, *, \cdots, *)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
u_l &= (*, 0, 0, 0, 0, \cdots, 0, *, *, \cdots, *)\\
u_{l+1} &= (*, 0, \cdots\cdots\cdots\cdots\cdots\cdots, 0)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
u_m &= (*, 0, \cdots\cdots\cdots\cdots\cdots\cdots, 0)
\end{aligned}$$

(2) $\forall i \exists j$: $B_i \subseteq T \times Rp_1 \times \cdots \times \overset{j}{\check{0}} \times \cdots \times Rp_n$.

Proof. Let $A = Ra_1 + \cdots + Ra_m$ and express each a_i in $T \times Rp_1 \cdots \times Rp_n$ as $a_i = (x_i, r_{i1}p_1, \cdots, r_{in}p_n)$ with r_{ij} in R. Let $r'_{i1} \in R$ such that $r_{i1}r'_{i1}r_{i1} = r_{i1}$ and put $e_i = r_{i1}r'_{ij}$, $i = 1, \cdots, m$. By Lemma 3.2, there are orthogonal idempotents $f_1, \cdots, f_s$ in R for which $\sum_{i=1}^{m} Re_i = \sum_{i=1}^{s} Rf_i$ and $e_i f_j = \begin{cases} 0 \\ \text{or} \\ f_j \end{cases}$ for any i, j. Setting $f = f_1 + \cdots + f_s$, we get $A = Af_1 \oplus \cdots \oplus Af_s \oplus A(1-f)$ and $A(1-f) \subseteq T \times 0 \times Rp_2 \times \cdots \times Rp_n$. Thus it sufficies to show the assertion for each Af_k instead of A.

Let $k \in \{1, \cdots, s\}$ and put $B = Af_k$. If $f_k r_{i1} p_1 = 0$ for $i = 1, \cdots, m$, then there is nothing to show since $B \subseteq T \times 0 \times Rp_2 \times \cdots \times Rp_n$. Thus assume that at least one of these is not zero. Without loss of generality we may assume $f_k r_{11} p_1 \neq 0$. Put $b_1 = a_1$, $b_2 = f_k a_2 - r_{21} f_k r'_{11} a_1, \cdots, b_m = f_k a_m - r_{m1} f_k r'_{11} a_1$. Then $\{b_1, \cdots, b_m\}$ is a generating set of B. Express each b_i as $(y_i, s_{i1}p_1, \cdots, s_{in}p_n)$ in $T \times Rp_1 \times \cdots \times Rp_n$ with s_{ij} in R. Then $s_{i1}p_1 = 0$ for all $i \geq 2$, since $f_k e_1 = f_k$. Therefore $\{b_1, \cdots, b_m\}$ is of the following form:

$$\begin{aligned} b_1 &= (y_1, s_{11}p_1, s_{12}p_2, \cdots, s_{1n}p_n) \\ b_2 &= (y_2, \ 0 \ , s_{22}p_2, \cdots, s_{2n}p_n) \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ b_m &= (y_m, \ 0 \ , s_{m2}p_2, \cdots, s_{mn}p_n)\,. \end{aligned}$$

Now we shall discuss the same argument as above for $\{s_{12}p_2, \cdots, s_{m2}p_2\}$. Let $s'_{i2} \in R$ such that $s_{i2}s'_{i2}s_{i2} = s_{i2}$ and put $g_i = s_{i2}s'_{i2}$, $i = 1, \cdots, m$. Again by Lemma 3.2, there are orthogonal idempotents $h_1, \cdots, h_t$ in R such that $\sum_{i=1}^{m} Rg_i = \sum_{i=1}^{t} Rh_i$ and $g_i h_j = \begin{cases} 0 \\ \text{or} \\ h_j \end{cases}$ for any i, j. Setting $h = h_1 + \cdots + h_t$, we have $B = Bh_1 \oplus \cdots \oplus Bh_t \oplus B(1-h)$ and $B(1-h) \subseteq T \times Rp_1 \times 0 \times Rp_3 \times \cdots \times Rp_n$. Thus we may show the assertion for each Bh_l instead of B. If $h_l s_{ij} p_j = 0$ for all $i \geq 2$ and $j \geq 2$, then the proof is completed. If there exist $i \geq 2$ and $j \geq 2$ such that $h_l s_{ij} p_j \neq 0$, we may assume $h_l s_{22} p_2 \neq 0$. Here put $c_1 = h_l b_1 - s_{12} h_l s'_{22} b_2$, $c_2 = b_2$, $c_3 = h_l b_3 - s_{32} h_l s'_{22} b_2, \cdots, c_m = h_l b_m - s_{m2} h_1 s'_{22} b_2$ and express each c_i as $c_i = (z_i, t_{i1}p_1, \cdots, t_{in}p_n)$. $\{c_1, \cdots, c_m\}$ is then a generating set of Bh_l such that

$$\begin{aligned} c_1 &= (z_1, t_{11}p_1, 0, \quad t_{13}p_3, \cdots, t_{1n}p_n) \\ c_2 &= (z_2, \quad 0, \quad t_{22}p_2, t_{23}p_3, \cdots, t_{2n}p_n) \\ c_3 &= (z_3, \quad 0, \quad 0, \quad t_{33}p_3, \cdots, t_{3n}p_n) \\ c_m &= (z_m, \quad 0, \quad 0, \quad t_{m3}p_3, \cdots, t_{mn}p_n)\,. \end{aligned}$$

Repeating this argument, the proof can be established.

Lemma 3.4. *Let R be a cyclic splitting regular ring and T a singular R-module. If A is a finitely generated R-submodule of $T\times Rp_1\times\cdots\times Rp_n$, where $p_1,\cdots,p_n$ in $Q(R)$, with a generating set $\{a_1,\cdots,a_m\}$ of the following form*

$$\begin{aligned}
a_1 &= (*,\ r_1p_1,\ 0,\ \cdots\cdots,\ 0,\ *,\ \cdots,\ *)\\
a_2 &= (*,\ 0,\ r_2p_2,\ 0,\ \cdots,\ 0,\ *,\ \cdots,\ *)\\
&\cdots\cdots\cdots\cdots\cdots\\
a_k &= (*,\ 0,\ \cdots\cdots,\ 0,\ r_kp_k,\ *,\ \ldots,\ *)\\
a_{k+1} &= (*,\ 0,\ \cdots\cdots\cdots\cdots\cdots\cdots,\ 0)\\
&\cdots\cdots\cdots\cdots\cdots\\
a_m &= (*,\ 0,\ \cdots\cdots\cdots\cdots\cdots\cdots,\ 0)\,,
\end{aligned}$$

then $Z(A)$ is a direct summand of A.

Proof. We shall show the lemma by induction on the number n of $\{p_1,\cdots,p_n\}$. Assume that the assertion is true for every finitely generated R-submodule of the type $T=Rq_1\times\cdots\times Rq_{n-1}$ with q_i in $Q(R)$.

Now let $f_i\in B(Q(R))$ such that $Q(R)f_i=Q(R)r_ip_i$, $i=1,\cdots,k$. Since R is cyclic splitting, by Teply's criterion, there are $e_1,\cdots,e_k$ in $B(R)$ and ideals $S_1,\cdots,S_k$ contained in $soc(R)$ such that $R(1-f_i)\cap R=Re_i\oplus S_i$, $i=1,\cdots,k$. Using Lemma 3.2 there are non-zero orthogonal elements $g_1,\cdots,g_l$ in $B(R)$ such that $\sum_{i=1}^{k}Re_i=\sum_{i=1}^{l}Rg_i$ and $e_ig_j=\begin{cases}0\\ \text{or}\\ g_j\end{cases}$ for any i,j. Putting $g=g_1+\cdots+g_l$,

$$\begin{aligned}
A &= Ag\oplus A(1-g)\\
&= Ag_1\oplus\cdots\oplus Ag_l\oplus(1-g)\,.
\end{aligned}$$

For each g_i, there is clearly e_{i_s} in $\{e_1,\cdots,e_k\}$ such that $g_ie_{i_s}=g_i$. Since each $g_ir_{i_s}p_{i_s}=0$, we conclude that $Z(Ag_i)$ is a direct summand of Ag_i by the induction hypothesis. Next we shall show that $Z(A(1-g))$ is a direct summand of $A(1-g)$. Cleary $R(1-g)a_{k+1}+\cdots+R(1-g)a_m$ is a singular R-module. We claim that $\sum_{i=1}^{k}R(1-g)a_i$ is non-singular. To see this let $a=\sum_{i=1}^{k}x_i(1-g)a_i\in Z(\sum_{i=1}^{k}R(1-g)a_i)$, where $x_i\in R$, $i=1,\cdots,k$. Then each $x_i(1-g)r_ip_i$ must be zero because of $Z(\sum_{i=1}^{k}R(1-g)a_i)\subseteq Z(T\times Rp_1\times\cdots\times Rp_n)\subseteq T\times 0\times\cdots\times 0$. This implies that $x_i(1-g)\in Re_i\oplus S_i$, whence $x_i(1-g)\in S_i$, $i=1,\cdots,k$. Noting $soc(R)T=0$, it follows that $a\in 0\times Rp_1\times\cdots\times Rp_n$ and hence $a=0$ as claimed. Thus we get $A(1-g)=\sum_{i=1}^{k}R(1-g)a_i\oplus(R(1-g)a_{k+1}+\cdots+R(1-g)a_m)$ and $Z(A(1-g))=R(1-g)a_{k+1}+\cdots+R(1-g)a_m$. This completes the proof.

Now we shall return to the proof of Theorem 3.1:

(b)$\Rightarrow$(c) is trivial and (c)$\Rightarrow$(a) holds for any regular rings ([15, Proposition 22.5]).

(a)$\Rightarrow$(b). Since $C(R)=Q(R)$, by [12, Lemma 3.6], every finitely generated R-submodule of $Q(R)$ is contained in a submodule of $Q(R)$ which is a direct sum of cyclic modules. On the other hand, every finitely generated R-module A is clearly embedded in an R-module of the type $Z(Q(R)\otimes_R A)\times Q(R)\times\cdots\times Q(R)$ (Lemma 1.6). Hence every finitely generated R-module is embedded into an R-module of the type $T\times Rp_1\times\cdots\times Rp_n$, where T is a singulr R-module and each p_i is in $Q(R)$. Now our assertion is easily verfied from Lemma 3.3 and 3.4 by making use of induction on n.

REMARKS. (1) The condition $C(R)=Q(R)$ holds for Boolean rings R and more generally for the rings R of all continuous functions from Boolean spaces to finite fields ([13]).

(2) The condition $C(R)=Q(R)$ can not be dropped from (a)$\Leftrightarrow$(b) in Theorem 3.1. For example, if R is a regular Baer ring with zero socle but not self-injective, then R is cyclic splitting, but by [7, Proposition 4.11] we can see that it is not finitely generated splitting. The regular ring given in [7, Example 4.5] is cyclic splitting with essential socle but not finitely generated splitting.

(3) It would be of interest to learn whether (c) is equivalent to (a) or (b) in Theorem 3.1 without assuming the condition $C(R)=Q(R)$ (cf. [15, Problem 10]). The answer is not known to the author.

From Corollary 2.4, Theorem 3.1 and Remark (1) we have

Corollary 3.5. *Let X be a Boolean space whose corresponding Boolean ring is cyclic splitting. Then, for any finite field F, the ring of all continuous functions from X to F is finitely generated splitting.*

Finally we shall give an example of a cyclic splitting atomic Boolean ring which is not decomposed into a direct sum of a splitting ring and a self-injective ring.

EXAMPLE. Let $S_1, \cdots, S_n$ be countably infinite sets, say $S_i=\{a_{i1}, a_{i2}, \cdots\}$, and let S be the disjoint union of $S_1, \cdots, S_n$. We denote by Q the atomic Boolean ring consisting of all subsets of S. Let ψ_{ij}: $S_i\rightarrow S_j$ be the mapping given by $a_{is}\rightarrow a_{js}$ $(s=1, 2, \cdots)$, and consider the set R of all x in Q such that $\psi_{ij}((S_i\cap x)\cup F_{i,x}))=(S_j\cap x)\cup F_{j,x}$ with some finite subset $F_{i,x}$ of S_i $(i, j=1, 2, \cdots, n)$. R is then a subring of Q and moreover $Q=Q(R)=RS_1\oplus\cdots\oplus RS_n$ (see [13, Example]). Now, as is easily checked, for any q in Q, $Rq\cap R=Re\oplus T$ for some e in R and some ideal T contained in $soc(R)$. So, Teply's criterion says that R is cyclic splitting. But inasmuch as R has infinitely many essential ideals, we see from [3, Theorem 5] that R is not splitting. Furthermore, when $n \geq 2$, R is not decomposed into a direct sum of a self-injective ring and a splitting ring.

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