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# DEHN FILLINGS ON A TWO TORUS BOUNDARY COMPONENTS 3-MANIFOLD

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## 1. Introduction

In this paper we are interested in incompressible, boundary-incompressible planar surfaces, properly embedded in a 3-manifold $X$ with boundary. Much has already been published for the case where $X$ has one torus boundary component $T$. (see in [1], [5], [6].)

We recall the definition from [6] of *planar boundary-slopes*: Let $(P, \partial P) \subset (X, T)$ be an essential (i.e., properly embedded, incompressible, and boundary-incompressible) planar surface in $X$. All the components of $\partial P \cap T$ have the same slope $r$ on $T$. We call this value the *planar boundary-slope*. The *distance* $\Delta(r, s)$ between two slopes $r$ and $s$ is their minimal geometric intersection number.

In [8], Gordon and Luecke have proved that distance between planar boundary-slopes is bounded by 1. Our goal is to obtain similar results when the 3-manifold $X$ has two torus boundary components. In this case, for each planar surface, we have a pair of boundary-slopes. We give a bound for at least one of the two distances between boundary-slopes, depending on the numbers of boundary components of the surfaces. The first approach to this problem was to study the following question: Is it possible to produce $S^3$ by a non-trivial surgery on a 2-component link in $S^3$? The case of reducible links in $S^3$ (a 2-sphere separates the two components) is already treated in [7]: these links never yield $S^3$ by surgery. But there are many known examples of links for which it is possible (see [1], [3]). Berge has annnounced in a preprint ([2]) that there is an infinity of "non-trivial" (each component is non-trivial and there is no essential annulus joining the two components) 2-component links in $S^3$ yielding $S^3$ by surgery with distances between the meridians arbitrarily large. In this paper we give some conditions for the realization of Berge's conjecture, which is the following:
*For all integer $n$, there exists a "non-trivial" link $L = (k_1, k_2)$ in $S^3$, such that $M_L(\beta_1, \beta_2) \simeq S^3$, and $\Delta(\beta_1, \alpha_1) > n$, $\Delta(\beta_2, \alpha_2) > n$, where $\alpha_1$ (respectively $\alpha_2$) is a meridian slope of $k_1$ (resp. $k_2$).*

We exclude from our study reducible links. We consider only the irreducible "non-trivial" links:

- the components of the link are non-trivial, and
- no annulus cobounds two essential circles on the two boundary components of the

complement of the link, respectively.

The thin position of the link and Cerf theory, as in [7], give us a pair of planar surfaces, $P$ and $Q$, properly embedded in the link space. By combinatorial analysis of graphs of the intersection $P \cap Q$, we find a bound, depending on the link, for one of the two distances between meridians. The basic idea of this analysis comes from [6].

In Section 2, we introduce definitions and notation necessary for the theorems and we state the results. Section 3 gives elements of intersection graph theory. In Section 4, we give some combinatorial lemmas and the proof of Theorem 1. Section 5 treats the case of $S^3$.

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## 2.  Preliminaries

Let $X$ be a 3-manifold with two boundary components: $\partial X = T_1 \cup T_2$, where $T_i$ is an incompressible torus in $X$, $i = 1, 2$. (Throughout, when $X$ is said to be a 3-manifold, it will also be compact, connected and orientable). Let $(P, \partial P) \subset (X, \partial X)$ be a planar surface in $X$ with boundary components on $T_1$ and $T_2$. Let $a_i$ be the number of boundary components of $P$ on $T_i$. We will always assume that $a_1 > 0$ and $a_2 > 0$. All the components of $\partial P \cap T_i$ have the same slope $\alpha_i$ on $T_i$, so we can assign to the surface $P$ a pair of slopes $\alpha = (\alpha_1, \alpha_2)$, and a pair of positive integers $a = (a_1, a_2)$. Sometimes we shall call $\alpha$ the *slope* of $P$ and $a$ the *number of boundary components* of $P$. Finally, the pair $(\alpha, a)$ will denote the *parameters* of $P$ on $(T_1, T_2)$.

Let $X(\alpha)$ be the manifold obtained from $X$ by attaching a solid torus $V_1$ and a solid torus $V_2$ along $T_1$ and $T_2$ respectively so that $\alpha_i$ bounds a meridian disc in $V_i$, $i = 1, 2$. Then the manifold $X(\alpha)$ contains a 2-sphere $\widehat{P}$ which intersects $V_i$ in $a_i$ meridian discs, $i = 1, 2$.

In the same way, let $W_1$ and $W_2$ denote the two Dehn filling solid tori of the manifold $X(\beta)$, where $\beta = (\beta_1, \beta_2)$ is the slope of a planar surface $(Q, \partial Q) \subset (X, \partial X)$ with number of boundary components $b = (b_1, b_2)$ on $(T_1, T_2)$, $b_i > 0$, $i = 1, 2$. Then the manifold $X(\beta)$ contains a 2-sphere $\widehat{Q}$ intersecting $W_i$ in $b_i$ meridian discs.

The *distance* $\Delta(\alpha, \beta)$ between two slopes $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ on $\partial X$ is the pair $(\Delta(\alpha_1, \beta_1), \Delta(\alpha_2, \beta_2))$. In the following $\Delta_i$ will stand for $\Delta(\alpha_i, \beta_i)$.

DEFINITION 1.  We shall say that $P$ and $Q$ have *graph properties* if $P$ and $Q$ are in general position, intersect transversely in a finite disjoint union of circles and properly embedded arcs such that no properly embedded arc is boundary parallel in either $P$ or $Q$, and each component of $\partial P \cap T_i$ intersects transversely each component of $\partial Q \cap T_i$ in $\Delta_i$ points, $i = 1, 2$.

REMARK.  Let $G_P$ and $G_Q$ be the planar intersection graphs $(P \cap Q \subset \widehat{P})$ and $(P \cap Q \subset \widehat{Q})$ respectively, defined as usual. In the case where $P$ and $Q$ have graph

properties, there is no *trivial loop* (disc-face with one single edge in its boundary) in either $G_P$ or $G_Q$.

**Theorem 1.** *Let $X$ be a 3-manifold with two incompressible torus boundary components*: $\partial X = T_1 \cup T_2$, *such that*:
i) $X$ *contains no essential annulus with one boundary component on $T_1$ and the other on $T_2$*; *and*
ii) *None of the manifolds $(X(\alpha_1), T_2)$, $(X(\alpha_2), T_1)$ contains a properly embedded Möbius band.*
*Let $P$ be a planar surface properly embedded in $X$ with parameters $(\alpha, a)$, such that $a_1 \geq a_2$. If $Q$ is a planar surface, properly embedded in $X$, with slope $\beta$ on $\partial X$, such that $P$ and $Q$ have graph properties, then either $\Delta(\alpha_1, \beta_1) < 30$, or $\Delta(\alpha_2, \beta_2) < 30a_1/a_2$.*

REMARK. It suffices to exchange the roles of $T_1$ and $T_2$ to have all the cases.

We shall now examine the analogous situation in $S^3$. Throughout, we will consider only irreducible links (there is no 2-sphere separating the two components). Let $L = (k, l)$ be a 2-component link in $S^3$ and $M_L$ denote its complement, $S^3 \setminus (\text{Int } N(k) \cup \text{Int } N(l))$, where $N(k)$ and $N(l)$ are tubular neighbourhoods of $k$ and $l$ respectively. The manifold $M_L$ has two boundary components $\partial N(k)$ and $\partial N(l)$, each homeomorphic to a torus. Let $\alpha_1$ and $\alpha_2$ be the slopes of a meridian of $k$ and $l$ respectively.

We adapt here the definition of *thin position* for a link. (see [4] or [7].)

DEFINITION 2. Note that $S^3 = S^2 \times \mathbf{R} \cup \{+\infty, -\infty\}$. We define $h: S^2 \times \mathbf{R} \to \mathbf{R}$ to be the projection onto the second factor. We shall say that $L$ has a *generic presentation* if $L \subset S^2 \times \mathbf{R}$, and $h_{|L}$ is a Morse function. By an isotopy of $L$, we may always assume that $L$ has a generic presentation. Choose a real number $t_i$ between each pair of adjacent critical values of $h_{|L}$. The *complexity* of this presentation of $L$ is the sum $\sum_i |L \cap h^{-1}(t_i)|$. A *thin presentation* of $L$ is a generic presentation of minimal complexity.

Now choose a thin presentation for $L$. Let $(x_1, y_1), \ldots, (x_n, y_n)$ be the pairs of adjacent critical values of $h_{|L}$ such that $x_i < y_i$, $x_i$ corresponds to a local minimum, and $y_i$ to a local maximum. To each *level sphere* $\widehat{P}_t = h^{-1}(t)$ we can assign its *ratio* $r_t = M_t/m_t \geq 1$, where $M_t = \max(|\widehat{P}_t \cap k|, |\widehat{P}_t \cap l|)$ and $m_t = \min(|\widehat{P}_t \cap k|, |\widehat{P}_t \cap l|)$. Every level sphere in a *middle slab* $\{\widehat{P}_t, t \in ]x_i, y_i[\}$ has the same ratio, because they all intersect $k$ and $l$ the same number of times. Then each middle slab $\{\widehat{P}_t, t \in ]x_i, y_i[\}$ has a ratio $r_i$, defined as $r_i = r_t$ for some $t \in ]x_i, y_i[$. The *linking ratio* $r$ of this thin presentation of $L$ is the minimum: $\min_i r_i$. We may define the linking ratio $r(L)$ of $L$

to be the minimum number of the linking ratios $r$, over all thin presentations for $L$.

REMARKS.   (1)  The ratio $r_i$ may be infinite.
(2)  Since $L$ is irreducible, in a thin presentation there always exist middle slabs which meet both components of the link. So $r(L) < \infty$.
(3)  We may suppose that $r(L)$ corresponds to the ratio $r = r_j = r_t = |\widehat{P}_t \cap k| / |\widehat{P}_t \cap l|$. There is no loss of generality because it suffices to exchange the roles of $l$ and $k$.

**Corollary 1.**   *Let $L = (k, l)$ be a link in $S^3$ with linking ratio $r(L)$, $\alpha$ a pair of meridian slopes of $L$, and $\beta$ a pair of slopes on $\partial M_L$, with $\beta_1 \neq \alpha_1$, $\beta_2 \neq \alpha_2$, such that*
(i)   $M_L(\beta_1, \beta_2) \simeq S^3$;
(ii)  *there is no essential annulus coborded the two boundary components of $M_L$;*
(iii) $M_L(\beta_1)$, $M_L(\beta_2)$, $M_L(\alpha_1)$, *and* $M_L(\alpha_2)$ *are boundary irreducible.*
*Then either $\Delta(\alpha_1, \beta_1) < 30$, or $\Delta(\alpha_2, \beta_2) < 30r(L)$.*

Notice that property (iii) implies no component of the link is trivial. By [7, Theorem 2], property (i) implies $L$ is irreducible.
    The link $L_\beta$ in $S^3$ shall be the core of the $\beta$-surgery on $L$.

Recall Berge's conjecture.   *For any integer $n$, there exists a link $L = (k, l)$ in $S^3$, such that $M_L$ has property* (ii) *of Corollary 1, $l$ and $k$ are non-trivial, $M_L(\beta_1, \beta_2) \simeq S^3$, and $\Delta(\beta_1, \alpha_1) > n$, $\Delta(\beta_2, \alpha_2) > n$, where $\alpha_1$ (respectively $\alpha_2$) is a meridian slope of $k_1$ (resp. $k_2$).*

By Corollary 1, if the link verifies Berge's Conjecture for $n \geq 30$, and if the components of the core are non-trivial, then the link must have a "sufficiently large" linking ratio. More precisely, $r(L)$ must be $> n/30$.
    Let $V_i$ be a solid torus in $S^3$ with core knot $k_i$ ($i = 1, 2$). Suppose that there is an annulus $A$ connecting $\partial V_1$ and $\partial V_2$; $\partial A = c_1 \cup c_2$ and $c_i$ wraps $p_i$-times in longitudinal direction of $V_i$. Without loss of generality, we may assume $0 \leq p_1 \leq p_2$. We divide into several cases depending on the pair $p_1$, $p_2$.
    If $p_1 = p_2 = 0$, there is a 2-sphere in $S^3$ intersecting $k_i$ in one point, a contradiction.
    If $p_1 = 0$ and $p_2 = 1$, then $k_2$ is a trivial knot.
    Assume that $p_1 = 0$ and $p_2 \geq 2$, then we can find a lens space summand in $S^3$, a contradiction.
    If $p_1 = p_2 = 1$, then $k_1$ and $k_2$ are parallel.
    If $p_1 = 1$ and $p_2 \geq 2$, then $k_1$ is a cable of $k_2$.
    Finally suppose that both $p_i \geq 2$. Let us consider the 3-manifold $M = V_1 \cup N(A) \cup V_2$; it is a Seifert fiber space over the disk with two exceptional fibers $k_1$ and $k_2$ of

indices $p_1$ and $p_2$. Note that $M$ is boundary-irreducible, then since $\partial M$ is a torus, $S^3 -$ Int $M$ is a solid torus whose core is a non-trivial torus knot. Thus the two exceptional fibers $k_1$ and $k_2$ form a Hopf link (cf. [10, Theorem 11]).

Hence, if $L = (k_1, k_2)$ is a link in $S^3$ with $k_i$ non-trivial ($i = 1, 2$) such that $M_L$ contains an annulus $A$, then either $p_1 = p_2 = 1$ and $k_1$ and $k_2$ are parallel, or $p_1 = 1$ and $p_2 \geq 2$ and $k_1$ is a cable of $k_2$.

This gives us a new corollary:

**Corollary 2.** *Let $L = (k, l)$ be a link in $S^3$ with linking ratio $r(L)$, such that $k$ and $l$ are non-trivial. Let $\alpha$ be a pair of meridian slopes of $L$, and $\beta$ a pair of slopes on $\partial M_L$, with $\beta_1 \neq \alpha_1$, $\beta_2 \neq \alpha_2$.*

*Suppose $M_L(\beta_1, \beta_2) \simeq S^3$, and the two components of $L_\beta$ are non-trivial. If $\Delta(\alpha_1, \beta_1) \geq 30$, and $\Delta(\alpha_2, \beta_2) \geq 30r(L)$, then some component of $L$ is a cable of the other one, or both components represent the same knot.*

## 3. Intersection graphs

Let $P$ and $Q$ be two planar surfaces properly embedded in $X$, with parameters $(\alpha, a)$ and $(\beta, b)$ on $(T_1, T_2)$. Let $\widehat{P}$ and $\widehat{Q}$ be the 2-spheres in $X(\alpha)$ and $X(\beta)$ such that $P = X \cap \widehat{P}$, $Q = X \cap \widehat{Q}$. As usual we consider a pair $(G_P, G_Q)$ of graphs in $(\widehat{P}, \widehat{Q})$.

Number, from 1 to $a_i$, the components of $\partial P \cap T_i$, that we denote by $\partial_1^i P$, $\partial_2^i P, \ldots, \partial_{a_i}^i P$, in the order in which they appear on $T_i$, $i = 1, 2$. Number from 1 to $b_i$ the components of $\partial Q \cap T_i$, that we denote by $\partial_1^i Q$, $\partial_2^i Q, \ldots, \partial_{b_i}^i Q$, in the order in which they appear on $T_i$, $i = 1, 2$.

Now label the endpoints of the properly embedded arcs in $P \cap Q$. Let $e$ be an arc in $P \cap Q$, and $t$ be an endpoint of $e$, say $t \in \partial_n^i P \cap \partial_m^i Q$. Then $t$ is labelled $n$ on the component $\partial_m^i Q$ in the surface $Q$, and $t$ is labelled $m$ on the component $\partial_n^i P$ in the surface $P$. Thus around each component of $\partial P \cap T_i$, we see the labels $1, 2, \ldots, b_i$ appearing in cyclic order, and around each component of $\partial Q \cap T_i$ we see the labels $1, 2, \ldots, a_i$, these sequences being repeated $\Delta_i$ times, $i = 1, 2$.

Assigning (arbitrarily) orientations to $P$ and $Q$, we induce an orientation on each component of $\partial P$ and each component of $\partial Q$. The orientation of $X$ induces an orientation for $T_1$ and $T_2$. Here we choose a positive orientation for each unoriented simple closed curve with slope $\alpha_i$ and each one with slope $\beta_i$, respecting the orientation of $T_i$, $i = 1, 2$.

We assign a sign $+$ to a component $x$ of $\partial P \cap T_i$ or $\partial Q \cap T_i$ if its induced orientation is the same as the positive orientation of the closed curves on $T_i$ defined previously, and assign the sign $-$ otherwise. We shall say that two components $x$ and $y$ of $\partial P \cap T_i$ (or $\partial Q \cap T_i$) are *parallel* if they have the same sign and *antiparallel* if they have opposite signs. Notice that signs given to components of $\partial P \cap T_1$ (respectively $\partial Q \cap T_1$) are independant of the signs of the components of $\partial P \cap T_2$ (respectively

$\partial Q \cap T_2$).

Cap off the components of $\partial P \cap T_i$ (respectively $\partial Q \cap T_i$) with discs, we regard these discs as "fat" vertices of type $i$ of the graph $G_P$ (respectively $G_Q$), $i = 1, 2$. Thus there are two types of vertices in $G_P$ and $G_Q$, just as there are two types of edges: the *simple edges* of $G_P$ (respectively $G_Q$) correspond to arcs of $P \cap Q$ in $P$ (respectively $Q$) whose boundary components are in the same boundary component of $\partial X$, the *mixed edges* of $G_P$ (respectively $G_Q$) correspond to arcs of $P \cap Q$ in $P$ (respectively $Q$) with one boundary component on $T_1$ and the other on $T_2$. In the following, we shall consider edges of $G_P$ and $G_Q$ as arcs in $P \cap Q$ and keep the same notation for both, and the labels and signs assigned to boundary components will be kept the same for the vertices.

If $P$ and $Q$ have graph properties, then there is no *trivial loop* (disc-face with one single edge in its boundary) in either $G_P$ or $G_Q$ and the parity rule still works for simple edges:

*If a simple edge joins parallel vertices in $G_P$, it joins antiparallel vertices in $G_Q$ and vice versa.*

Two edges $e$ and $e'$ in a graph $G$ are *directly parallel* if they connect the two same vertices, and cobound a disc-face in $G$. They are *parallel* if there exist a finite set $\{e_1 = e, e_2, \ldots, e_n = e'\}$ of edges of $G$ such that $e_i$ and $e_{i+1}$ are directly parallel, for $i \in \{1, \ldots, n-1\}$.

Recall that the *reduced graph* $\widehat{G}$ of a graph $G$ is obtained from $G$ by replacing each family of parallel edges by a single edge. We shall use $\mathrm{val}(v, G)$ to denote the valency of a vertex $v$ in the graph $G$.

## 4.   Proof of Theorem 1

**Lemma 1** ([6, Lemma 4.1]).   *Let $\Gamma$ be a finite graph in the 2-sphere with no 1-sided faces. Suppose every vertex of $\Gamma$ has order $\geq 6$. Then $\Gamma$ has two parallel edges.*

Suppose $\mathcal{E}$ is a family of edges of the pair $(G_Q, G_P)$, then $G_P(\mathcal{E})$ is the subgraph of $G_P$ consisting of all edges of $\mathcal{E}$ and their attached vertices.

**Lemma 2.**   *Suppose $a_1 \geq a_2$ and $\Delta_1 \geq 30$, $\Delta_2 \geq 30a_1/a_2$. Then $G_Q$ has a family of parallel edges $\mathcal{E}$, and $G_P(\mathcal{E})$ has two parallel edges.*

Proof.   Let $\widehat{G_Q}$ be the reduced graph of $G_Q$. By Lemma 1, $\widehat{G_Q}$ has a vertex $w_0$ so that $\mathrm{val}(w_0, \widehat{G_Q}) \leq 5$. But

$$\mathrm{val}(w_0, G_Q) = \begin{cases} \Delta_1 a_1 & \text{if } w_0 \text{ is of type 1} \\ \Delta_2 a_2 & \text{if } w_0 \text{ is of type 2} \end{cases}.$$

Hence $\mathrm{val}(w_0, G_Q) \geq 30a_1$, and $\widehat{G_Q}$ has an edge $\mathcal{E}$ of order$\geq 6a_1$ which is incident to $w_0$. The edge $\mathcal{E}$ in $\widehat{G_Q}$ is a family of at least $6a_1$ parallel edges in $G_Q$.

Suppose the edges of $\mathcal{E}$ are mixed. We rename $u$ and $v$ the vertices of type 1 and 2 respectively, attached to $\mathcal{E}$. Since $a_1 \geq a_2$, for each $i$ in $\{1, 2, \ldots, a_1\}$ there are at least 6 endpoints of $\mathcal{E}$ labelled $i$ on $\partial u$, and for each $j$ in $\{1, 2, \ldots, a_2\}$ there are at least 6 endpoints of $\mathcal{E}$ labelled $j$ on $\partial v$. Hence every vertex in $G_P(\mathcal{E})$ has valency$\geq 6$. By Lemma 1, $G_P(\mathcal{E})$ has two parallel edges. Notice that they are mixed.

Now suppose the edges of $\mathcal{E}$ are simple. They join two vertices of same type $i$, which we call $u$ and $v$. Since $a_1 \geq a_2$, $\mathcal{E}$ contains at least $6a_i$ edges. So for each $m_i \in \{1, 2, \ldots, a_i\}$, there are at least 6 endpoints of edges in $\mathcal{E}$ labelled $m_i$ on $\partial u$, and the same on $\partial v$. Thus every vertex of $G_P(\mathcal{E})$ is of type $i$ and has valency$\geq 12$. If there is no trivial loop in $G_P(\mathcal{E})$, we can apply Lemma 1 to show that $G_P(\mathcal{E})$ has two parallel edges.

If $G_P(\mathcal{E})$ contains a trivial loop, then the both endpoints of each edge in $\mathcal{E}$ have the same label on $u$ and $v$ in $G_Q$. All the edges of $\mathcal{E}$ are loops in $G_P(\mathcal{E})$. By Lemma 3 below, $G_P(\mathcal{E})$ must contain at least two parallel edges (which are loops). In all cases we can choose two edges $e_1$, $e_2$ directly parallel in $G_P(\mathcal{E})$. $\qquad\square$

**Lemma 3.** *If every edge of $\mathcal{E}$ is a loop in $G_P(\mathcal{E})$, then $G_P(\mathcal{E})$ contains two parallel edges.*

Proof. For this lemma, we just need $\mathcal{E}$ to be a family of exactly $6a_1$ mutually parallel adjacent edges. The edges of $\mathcal{E}$ are loops in $G_P$. Thus in $G_Q$ they join two antiparallel vertices. There are two cases, according to whether $\mathcal{E}$ joins in $G_Q$ vertices of type 1, or vertices of type 2.

First assume the vertices of $G_P(\mathcal{E})$ are of type 1. Then $G_P(\mathcal{E})$ (on the sphere $\widehat{P}$) consists of $a_1$ connected components, each of which is a 6-bouquet. Here, an $n$-bouquet will be a graph with one vertex and $n$ loops.

Let $\mathcal{F}$ be the set of faces of $G_P(\mathcal{E})$ (as a graph on a sphere), and $f_1$, $f_2$ and $f_3$ be the numbers of disc faces of $G_P(\mathcal{E})$ with one side, two sides and at least three sides, respectively. Then an Euler characteristic calculation gives

$$a_1 - 6a_1 + \sum_{f \in \mathcal{F}} \chi(f) = 2.$$

But

$$\sum_{f \in \mathcal{F}} \chi(f) = f_1 + f_2 + f_3 + \sum_{f \in \mathcal{F}, f \text{ non-disc face}} \chi(f) \leq f_1 + f_2 + f_3.$$

Thus $f_1 + f_2 + f_3 \geq 2 + 5a_1$.

To prove the first case, it is sufficient to show that $f_2 > 0$. So we assume $f_2 = 0$, and reach a contradiction. Then $f_1 + f_3 \geq 2 + 5a_1$. Since $G_P$ has no trivial loop,

each 1-sided face of $G_P(\mathcal{E})$ must contain, in $G_P$, at least one vertex. This vertex is of type 2 because $G_P(\mathcal{E})$ already contains all the vertices of type 1. Thus $f_1 \leq a_2$. Since $a_1 \geq a_2$, we have $f_3 \geq 2 + 4a_1$.

CLAIM 1.   A 6-bouquet on a sphere has at most two disc-faces with 3 or more sides.

Proof of Claim 1.   Embed a 6-bouquet $G$ on a sphere. Then by Euler's formula, $1 - 6 + \sum \chi(\text{face}) = 2$. Note that all faces are disc.

Let $g_1$, $g_2$ and $g_3$ be the number of disc faces with 1 side, 2 sides, and at least 3 sides, respectively. Then $g_1 + 2g_2 + 3g_3 \leq 2 \times 6 = 12$. Since $g_1 + g_2 + g_3 = \sum \chi(\text{face}) = 7$, we have $2g_3 \leq g_2 + 2g_3 \leq 5$. Thus $g_3 \leq 2$.  □

Hence each connected component of $G_P(\mathcal{E})$ has at most two 3-sided disc faces, and so $f_3 \leq 2a_1$. Then we have $2a_1 \geq 2 + 4a_1$, a contradiction.

Next suppose that the vertices of $G_P(\mathcal{E})$ are of type 2. Recall that $\mathcal{E}$ is a family of $6a_1$ parallel edges. Then $G_P(\mathcal{E})$ has $a_2$ vertices, and $a_2$ connected subgraphs, each being an $n_i$-bouquet for some integer $n_i \geq 6$. Note that $\sum_{i=1}^{a_2} n_i = 6a_1$.

A similar proof as the one of Claim 1 gives Claim 2:

CLAIM 2.   An $n_i$-bouquet on a sphere has at most $(n_i - 1)/2$ disc-faces with 3 or more sides.

Keep the same notation as for the above case. In $G_P(\mathcal{E})$ we now have

$$a_2 - 6a_1 + \sum_{f \in \mathcal{F}} \chi(f) = 2,$$

which leads to $f_1 + f_2 + f_3 \geq 2 - a_2 + 6a_1$. Assume $f_2 = 0$ for contradiction. Since $f_1 \leq a_1$ (because $G_P(\mathcal{E})$ contains all the vertices of type 2 and $G_P$ has no trivial loop), then $f_3 \geq 2 - a_2 + 5a_1$. By Claim 2,

$$f_3 \leq \sum_{i=1}^{a_2} \frac{(n_i - 1)}{2} = 3a_1 - \frac{a_2}{2}.$$

Hence

$$2 - a_2 + 5a_1 \leq f_3 \leq 3a_1 - \frac{a_2}{2}.$$

Since $a_2 \leq a_1$, then we have $4 + 3a_1 \leq 0$, a contradiction.  □

We shall say that two arcs of $P \cap Q$ are parallel in $P$ if they cut off a disc in $P$.
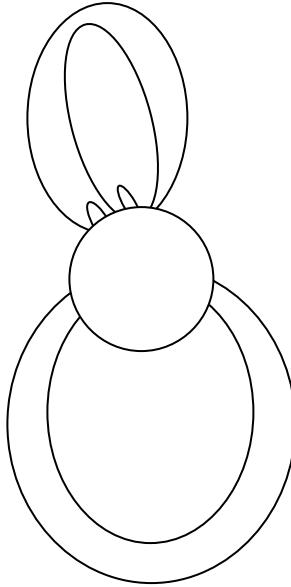
Fig. 1. a 6-bouquet

**Lemma 4** ([5, Lemma 2.1]). *Let $P$ and $Q$ be two properly embedded planar surfaces in a 3-manifold $X$ such that $\partial X$ contains a torus $T$, and assume $P$ and $Q$ have boundary components on $T$. Suppose that $P$ and $Q$ intersect transversely and each component of $\partial P \cap T$ intersects each component of $\partial Q \cap T$ minimaly. Let $\mathcal{A}$, $\mathcal{A}'$ be two arcs of $P \cap Q$, properly embedded in $(X, T)$, and parallel in both $P$ and $Q$. If $D \cap E = \mathcal{A} \cup \mathcal{A}'$, where $D$ and $E$ are the discs in $P$ and $Q$ respectively, that realize the parallelism of $\mathcal{A}$ and $\mathcal{A}'$, then $D$ and $E$ cannot be identified along $\mathcal{A}$ and $\mathcal{A}'$ as illustrated in* Fig. 2.

In the following, $\mathcal{E}$ will always denote this family with at least $6a_1$ parallel edges in $G_Q$. Under the assumption $\Delta_1 \geq 30$ and $\Delta_2 \geq 30a_1/a_2$, Lemma 2 implies an existence of two parallel edges in $G_P(\mathcal{E})$, which contradicts the assumption (ii) or (i) in Theorem 1 by Lemma 5 or 6 below. This completes the proof of Theorem 1. $\qquad\square$

**Lemma 5.** *Let $e_1$ and $e_2$ be parallel edges in $G_P(\mathcal{E})$. Suppose $e_1$ and $e_2$ are simple. Then $(X(\alpha_1), T_2)$ or $(X(\alpha_2), T_1)$ contains a properly embedded Möbius band.*

Proof. The edges $e_1$ and $e_2$ come from the family $\mathcal{E}$ of edges in $G_Q$, so they are parallel in both $G_Q$ and $G_P(\mathcal{E})$. First we can assume $e_1$ and $e_2$ are directly parallel in $G_P(\mathcal{E})$. Assume now $e_1$ and $e_2$ have their boundaries on $T_1$. Let $u$, $v$ and $x$, $y$ denote the vertices attached to $e_1$ and $e_2$ in $G_P$ and $G_Q$ respectively. (We can have
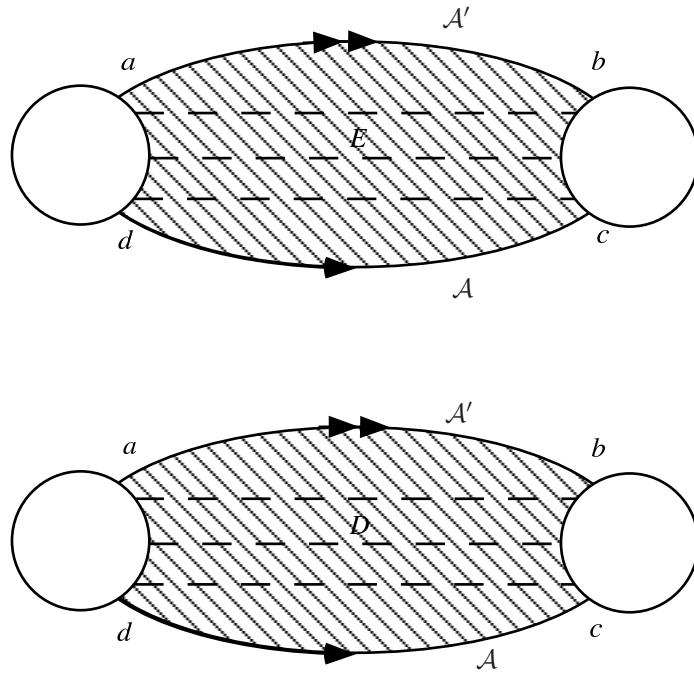
Fig. 2.  $a$, $b$, $c$, $d$ are the points of intersection between the arcs and the boundary components of $P$ and $Q$.

$u = v$ or $x = y$). Since each label $i$ in $\{1, 2, \ldots, a_1\}$ appears as an endpoint of an edge of $\mathcal{E}$ in $G_Q$, the graph $G_P(\mathcal{E})$ contains all the vertices of type 1 of $G_P$. Hence the cycle given by the edges $e_1$ and $e_2$ bounds a disc $D$ in $\widehat{P}$ such that in the interior of the cycle there are no vertices of type 1. But, there may be vertices of type 2 of $G_P$ in the interior of the cycle. Let $E$ be the disc that realizes the parallelism between the arcs $e_1$ and $e_2$ in $Q$. Each arc of $E \cap P$ corresponds to an edge of $\mathcal{E}$. Then, since $e_1$ and $e_2$ are directly parallel in $G_P(\mathcal{E})$, $D \cap E = e_1 \cup e_2 \cup \mathcal{C}$, where $\mathcal{C}$ is a union of circles. By a cut and paste method we can eliminate circles of intersection, and obtain two discs (we shall call them again $D$ and $E$) such that $D \cap E = e_1 \cup e_2$. There are two possibilities for the way in which $E$ and $D$ are identified along $e_1$ and $e_2$, illustrated by Fig. 3 and Fig. 4. Notice that the disc $D$ is in $P(\alpha_2) \subset X(\alpha_2)$. The surfaces $P(\alpha_2)$ and $Q$ are transverse and their boundary components on $T_1$ intersect minimaly. By Lemma 4, case (i) is impossible. In case (ii), $E \cup D$ is a Möbius band properly embedded in $X(\alpha_2)$.                                                                          □

REMARK.  Suppose $u = v$, $\mathcal{M} = D \cup E$ is a Möbius band properly embedded in $(X(\alpha_2), T_1)$. But $\partial \mathcal{M}$ also bounds a Möbius band $\mathcal{M}'$ in $X(\alpha_1) = X \cup V_1$, where $\mathcal{M}'$ is the union of the meridian disc $u$ of $V_1$ and a disc $\Delta$ on $T_1$ (see Fig. 5). The
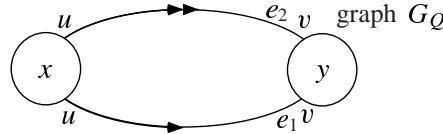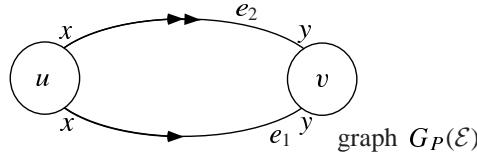
Fig. 3.   case (i)
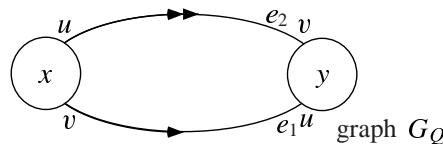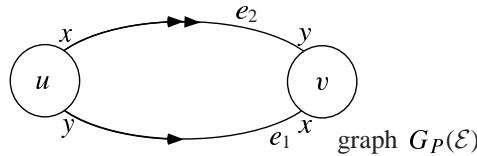


Fig. 4.   case (ii)

union of the two Möbius bands is a Klein bottle $K = \mathcal{M} \bigcup_{\partial \mathcal{M}} \mathcal{M}'$ in the manifold $X(\alpha_1, \alpha_2)$. The Dehn filling solid torus $V_1$ intersects $K$ in a single component.

**Lemma 6.** *Let $e_1$ and $e_2$ be parallel edges in both $G_Q$ and $G_P(\mathcal{E})$. Suppose $e_1$ and $e_2$ are mixed. Then $X$ contains an essential annulus, such that one of the boundary components is in $T_1$ and the other in $T_2$.*

Proof.   Assume $e_1$, $e_2$ are directly parallel in $G_P(\mathcal{E})$. Let $u$, $v$ and $x$, $y$ be the pairs of vertices in $G_P$ and $G_Q$ respectively, attached to the parallel edges $e_1$, $e_2$. Suppose $u$ and $x$ are of type 1, while $v$ and $y$ are of type 2. First notice that $\mathcal{E}$ is a family of at least $6a_1$ parallel edges. Hence each $i$ in $\{1, 2, \ldots, a_1\}$ labels an endpoint on $\partial x$ of some edge in $\mathcal{E}$, and every label of type 2 is an endpoint on $\partial y$ of some edge in $\mathcal{E}$. Then $G_P(\mathcal{E})$ contains all the vertices of $G_P$. If $e_1$ and $e_2$ are non parallel in $G_P$, then $e_1$ and $e_2$ cut off in $G_P$ a subgraph which contains at least one vertex of $G_P$. By the
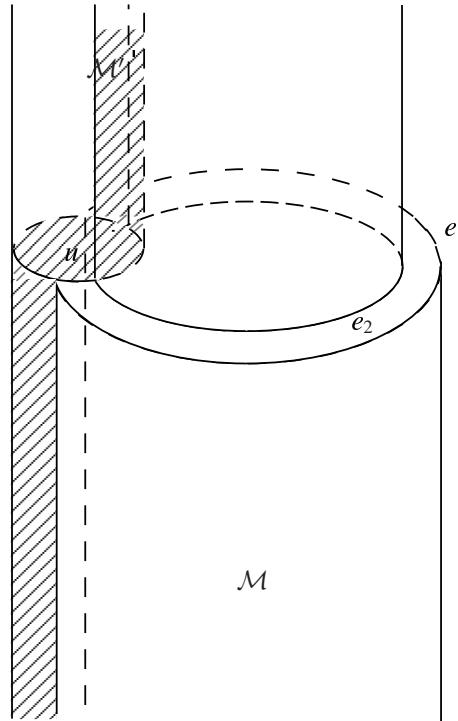
Fig. 5.   case $u = v$

previous remark, this vertex is also a vertex of $G_P(\mathcal{E})$, which contradicts the fact that $e_1$ and $e_2$ are parallel in $G_P(\mathcal{E})$. Therefore the edges $e_1$ and $e_2$ are also parallel in $G_P$, (but in general they are not directly parallel), so let $D$ (respectively $E$) be the disc in $P$ (respectively $Q$) that realizes the parallelism between $e_1$ and $e_2$. If the discs $E$ and $D$ contain circles in their intersection, by cut and paste methods we may build two new discs (let's call these discs $E$ and $D$ again), such that they intersect only along $e_1$ and $e_2$. Since the edges $e_1$, $e_2$ are mixed, the only possibility for the way in which $E$ and $D$ are identified along $e_1$ and $e_2$ is illustrated in Fig. 6. The union $E \bigcup_{e_1, e_2} D$ is an annulus $A$ with two boundary components $\partial_+ A \subset T_1$ and $\partial_- A \subset T_2$, where $\partial_+ A = C \cup C'$, and $\partial_- A = \delta \cup \delta'$ (see Fig. 6).

Let $\mid \alpha \cap \beta \mid$ be the geometric intersection number between the two oriented curves $\alpha$ and $\beta$, and $\alpha.\beta$ be their algebraic intersection number.

We suppose without loss of generality that $C'.\partial u \geq 0$. We have

$$\partial_+ A.\partial u = C'.\partial u - 1,$$

and $\mid C' \cap \partial u \mid \geq 2$. Then $\partial_+ A$ intersects the meridian $\partial u$ of $T_1$ at least once and always with the same orientation. The annulus $A$ joins $T_1$ to $T_2$ and its boundary components
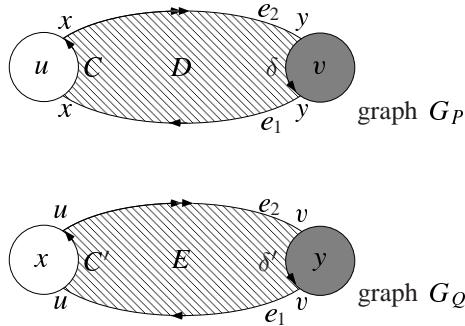
Fig. 6.   vertices $u$ and $x$ of type 1, vertices $v$ and $y$ of type 2

are non-trivial on $T_1$ and $T_2$ respectively. Therefore $A$ is essential.   □

## 5.   Proof of Corollary 1

For the proof of Corollary 1, we'll follow the same argument as for Theorem 1. It suffices to find two planar surfaces $P$ and $Q$ which verify hypothesis of Theorem 1 in the case where $X = M_L$.

For a proof by contradiction, we consider a link $L = (k, l)$ without trivial components which produces $S^3$ by a non-trivial surgery, such that the cores of the surgery are non-trivial. We may suppose it is an irreducible link because if it were reducible, it wouldn't give $S^3$ by a non-trivial surgery. (This is an immediate consequence of Theorem 2 in [7]).

**Proposition 1.**   *Let $L = (k, l)$ be a link in $S^3$ such that $k$ and $l$ are non-trivial, and with a linking ratio $r(L)$ in some thin presentation. Let $\alpha$ be a pair of meridian slopes of $L$. If $M_L(\beta)$ is homeomorphic to $S^3$ for a slope $\beta \neq \alpha$ on $\partial M_L$, and $M_L(\beta_1)$, $M_L(\beta_2)$ are boundary-irreducible, then there exist two properly embedded planar surfaces $P$ and $Q$ in $M_L$, such that $P$ has parameters $(\alpha, a)$, and $Q$ has parameters $(\beta, b)$ on $\partial M_L$, $r(L) = a_1/a_2$, and $P$ and $Q$ have graph properties.*

Proof.   We consider a thin presentation for $L$ in $S^3$ such that the linking ratio of this presentation is $r(L)$. Since $M_L(\beta) \simeq S^3$, the link is irreducible, so there is at least one middle slab $\{\widehat{P}_t, t \in ]x_j, y_j[\}$ such that each level sphere $\widehat{P}_t$ in this middle slab intersects both $k$ and $l$. Its ratio $r_t = r_j$ is finite, and suppose $r_j = r(L)$, that is to say this middle slab realizes $r(L)$.

Now we choose a thin presentation for the core of the surgery $L_\beta = (k_\beta, l_\beta)$ in the copy $M_L(\beta) = S^3{}_\beta$ of $S^3$. The link $L_\beta$ is irreducible. As above, we choose a middle slab $\{\widehat{Q}_t, t \in ]\bar{x}_j, \bar{y}_j[\}$ which intersects both $k_\beta$ and $l_\beta$.

Applying the method of [7, Proposition 1] to a 2-component link in $S^3$ with non-trivial components, we find two surfaces $P$ and $Q$, where $P = \widehat{P}_t \cap M_L$ for some $t$ in $]x_j, y_j[$ and $Q$ is homeomorphic to $\widehat{Q}_t \cap (S^3_\beta \setminus \operatorname{Int} N(L_\beta))$, with $t$ in $]\bar{x}_j, \bar{y}_j[$. In order to apply the general definitions of Section 1 and Section 2, we replace $X$ by $M_L$, $T_1$ by $\partial N(k)$ and $T_2$ by $\partial N(l)$. The planar surfaces $P$ and $Q$ are properly embedded in $M_L$, with parameters $(\alpha, a)$, $(\beta, b)$ respectively $(a_i > 0, b_i > 0)$, $P$ realizes $r(L)$ and they have graph properties.                                                                                    □

We may assume $r(L) = |\widehat{P}_t \cap k| / |\widehat{P}_t \cap l|$, so $a_1 \geq a_2$. We apply Theorem 1, which leads us to the following conclusion: Either $M_L$ contains an essential annulus that joins its two boundary components or one of the two manifolds $M_L(\alpha_1)$, $M_L(\alpha_2)$ contains a properly embedded Möbius band. The first case contradicts hypothesis $i)$ of Corollary 2. We will see that the second case is impossible too.

The Möbius band we have built in the proof of Theorem 1 leads to a contradiction in each case. We shall describe exactly what happens.

Consider the case (ii) of Fig. 4, and we suppose without loss of generality that $u$ and $v$ are meridian discs for $\partial N(k)$. In fact, we shall divide the case (ii) in the three following subcases:

(ii).1 $u = v$

(ii).2 $u \neq v$, $u$ and $v$ are antiparallel

(ii).3 $u \neq v$, $u$ and $v$ are parallel.

We are going to see that all the subcases are impossible.

Keep the same notation (discs $E$ and $D$, vertices $u$, $v$, $x$ and $y$) as in the general case, changing the vertices of type 1 (respectively of type 2) into vertices corresponding to boundary components on $\partial N(k)$ (respectively $\partial N(l)$). First, for these three cases, change $E$ and $D$ by cut and paste or isotopy if necessary to eliminate singular surfaces as previously.

(ii).1 The Möbius band $\mathcal{M} = D \bigcup_{e_1, e_2} E$ is properly embedded in $S^3 \setminus \operatorname{Int} N(k)$. But $\partial \mathcal{M}$ also bounds a Möbius band $\mathcal{M}'$ properly embedded in $M_L(\alpha_1) \simeq S^3 \setminus \operatorname{Int} N(l)$: $\mathcal{M}'$ is the union of the meridian disc of $u$ of $k$ and a disc on $\partial N(k)$, (see Fig. 5). The union $K = \mathcal{M} \bigcup_{\partial \mathcal{M}} \mathcal{M}'$ is a Klein bottle embedded in $S^3$, which is impossible.

(ii).2 There is a Möbius band $B$ in $M_L(\alpha) \simeq S^3$: $B$ is the union of the disc $D \subset S^3 \setminus \operatorname{Int} N(k)$, a meridian disc (in fact $u$ or $v$) of $k$ and a disc of $\partial N(k)$ (see [6], Fig. 4). The disc $E$ in $Q$ has the same boundary as the Möbius band $B$, then $B \cup E$ is a projective plane embedded in $S^3$, which is impossible.

(ii).3 Then, by the parity rule, $x$ and $y$ are antiparallel in $G_Q$. As in the previous case, we can build a Möbius band $B'$ which is the union of the disc $E$, a disc on $\partial N(k_\beta)$ and a meridian disc of $N(k_\beta)$. The Möbius band $B'$ is properly embedded in $M_L(\beta_1) \simeq S^3 - \operatorname{Int} N(l_\beta)$, and it has the same boundary as $D$. The projective plane $D \cup B'$ is embedded in $S^3$, which is impossible.                    □

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