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DEHN FILLINGS ON A TWO TORUS BOUNDARY COMPONENTS 3-MANIFOLD

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1. Introduction

In this paper we are interested in incompressible, boundary-incompressible planar surfaces, properly embedded in a 3-manifold X with boundary. Much has already been published for the case where X has one torus boundary component T. (see in [1], [5], [6].)

We recall the definition from [6] of *planar boundary-slopes*: Let $(P, \partial P) \subset (X, T)$ be an essential (i.e., properly embedded, incompressible, and boundary-incompressible) planar surface in X. All the components of $\partial P \cap T$ have the same slope r on T. We call this value the *planar boundary-slope*. The *distance* $\Delta(r, s)$ between two slopes r and s is their minimal geometric intersection number.

In [8], Gordon and Luecke have proved that distance between planar boundary-slopes is bounded by 1. Our goal is to obtain similar results when the 3-manifold X has two torus boundary components. In this case, for each planar surface, we have a pair of boundary-slopes. We give a bound for at least one of the two distances between boundary-slopes, depending on the numbers of boundary components of the surfaces. The first approach to this problem was to study the following question: Is it possible to produce S^3 by a non-trivial surgery on a 2-component link in S^3? The case of reducible links in S^3 (a 2-sphere separates the two components) is already treated in [7]: these links never yield S^3 by surgery. But there are many known examples of links for which it is possible (see [1], [3]). Berge has annnounced in a preprint ([2]) that there is an infinity of "non-trivial" (each component is non-trivial and there is no essential annulus joining the two components) 2-component links in S^3 yielding S^3 by surgery with distances between the meridians arbitrarily large. In this paper we give some conditions for the realization of Berge's conjecture, which is the following:
For all integer n, there exists a "non-trivial" link $L = (k_1, k_2)$ in S^3, such that $M_L(\beta_1, \beta_2) \simeq S^3$, and $\Delta(\beta_1, \alpha_1) > n$, $\Delta(\beta_2, \alpha_2) > n$, where α_1 (respectively α_2) is a meridian slope of k_1 (resp. k_2).

We exclude from our study reducible links. We consider only the irreducible "non-trivial" links:

- the components of the link are non-trivial, and
- no annulus cobounds two essential circles on the two boundary components of the

complement of the link, respectively.

The thin position of the link and Cerf theory, as in [7], give us a pair of planar surfaces, P and Q, properly embedded in the link space. By combinatorial analysis of graphs of the intersection $P \cap Q$, we find a bound, depending on the link, for one of the two distances between meridians. The basic idea of this analysis comes from [6].

In Section 2, we introduce definitions and notation necessary for the theorems and we state the results. Section 3 gives elements of intersection graph theory. In Section 4, we give some combinatorial lemmas and the proof of Theorem 1. Section 5 treats the case of S^3.

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2. Preliminaries

Let X be a 3-manifold with two boundary components: $\partial X = T_1 \cup T_2$, where T_i is an incompressible torus in X, $i = 1, 2$. (Throughout, when X is said to be a 3-manifold, it will also be compact, connected and orientable). Let $(P, \partial P) \subset (X, \partial X)$ be a planar surface in X with boundary components on T_1 and T_2. Let a_i be the number of boundary components of P on T_i. We will always assume that $a_1 > 0$ and $a_2 > 0$. All the components of $\partial P \cap T_i$ have the same slope α_i on T_i, so we can assign to the surface P a pair of slopes $\alpha = (\alpha_1, \alpha_2)$, and a pair of positive integers $a = (a_1, a_2)$. Sometimes we shall call α the *slope* of P and a the number of boundary components of P. Finally, the pair (α, a) will denote the *parameters* of P on (T_1, T_2).

Let $X(\alpha)$ be the manifold obtained from X by attaching a solid torus V_1 and a solid torus V_2 along T_1 and T_2 respectively so that α_i bounds a meridian disc in V_i, $i = 1, 2$. Then the manifold $X(\alpha)$ contains a 2-sphere $\widehat{P}$ which intersects V_i in a_i meridian discs, $i = 1, 2$.

In the same way, let W_1 and W_2 denote the two Dehn filling solid tori of the manifold $X(\beta)$, where $\beta = (\beta_1, \beta_2)$ is the slope of a planar surface $(Q, \partial Q) \subset (X, \partial X)$ with number of boundary components $b = (b_1, b_2)$ on (T_1, T_2), $b_i > 0$, $i = 1, 2$. Then the manifold $X(\beta)$ contains a 2-sphere $\widehat{Q}$ intersecting W_i in b_i meridian discs.

The *distance* $\Delta(\alpha, \beta)$ between two slopes $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ on ∂X is the pair $(\Delta(\alpha_1, \beta_1), \Delta(\alpha_2, \beta_2))$. In the following Δ_i will stand for $\Delta(\alpha_i, \beta_i)$.

DEFINITION 1. We shall say that P and Q have *graph properties* if P and Q are in general position, intersect transversely in a finite disjoint union of circles and properly embedded arcs such that no properly embedded arc is boundary parallel in either P or Q, and each component of $\partial P \cap T_i$ intersects transversely each component of $\partial Q \cap T_i$ in Δ_i points, $i = 1, 2$.

REMARK. Let G_P and G_Q be the planar intersection graphs $(P \cap Q \subset \widehat{P})$ and $(P \cap Q \subset \widehat{Q})$ respectively, defined as usual. In the case where P and Q have graph properties, there is no *trivial loop* (disc-face with one single edge in its boundary) in either G_P or G_Q.

Theorem 1. *Let X be a 3-manifold with two incompressible torus boundary components: $\partial X = T_1 \cup T_2$, such that:*
i) *X contains no essential annulus with one boundary component on T_1 and the other on T_2; and*
ii) *None of the manifolds $(X(\alpha_1), T_2)$, $(X(\alpha_2), T_1)$ contains a properly embedded Möbius band.*
Let P be a planar surface properly embedded in X with parameters (α, a), such that $a_1 \geq a_2$. If Q is a planar surface, properly embedded in X, with slope β on ∂X, such that P and Q have graph properties, then either $\Delta(\alpha_1, \beta_1) < 30$, or $\Delta(\alpha_2, \beta_2) < 30a_1/a_2$.

REMARK. It suffices to exchange the roles of T_1 and T_2 to have all the cases.

We shall now examine the analogous situation in S^3. Throughout, we will consider only irreducible links (there is no 2-sphere separating the two components). Let $L = (k, l)$ be a 2-component link in S^3 and M_L denote its complement, $S^3 \setminus (\mathrm{Int}\, N(k) \cup \mathrm{Int}\, N(l))$, where $N(k)$ and $N(l)$ are tubular neighbourhoods of k and l respectively. The manifold M_L has two boundary components $\partial N(k)$ and $\partial N(l)$, each homeomorphic to a torus. Let α_1 and α_2 be the slopes of a meridian of k and l respectively.

We adapt here the definition of *thin position* for a link. (see [4] or [7].)

DEFINITION 2. Note that $S^3 = S^2 \times \mathbf{R} \cup \{+\infty, -\infty\}$. We define $h\colon S^2 \times \mathbf{R} \to \mathbf{R}$ to be the projection onto the second factor. We shall say that L has a *generic presentation* if $L \subset S^2 \times \mathbf{R}$, and $h_{|L}$ is a Morse function. By an isotopy of L, we may always assume that L has a generic presentation. Choose a real number t_i between each pair of adjacent critical values of $h_{|L}$. The *complexity* of this presentation of L is the sum $\sum_i |L \cap h^{-1}(t_i)|$. A *thin presentation* of L is a generic presentation of minimal complexity.

Now choose a thin presentation for L. Let $(x_1, y_1), \ldots, (x_n, y_n)$ be the pairs of adjacent critical values of $h_{|L}$ such that $x_i < y_i$, x_i corresponds to a local minimum, and y_i to a local maximum. To each *level sphere* $\widehat{P}_t = h^{-1}(t)$ we can assign its *ratio* $r_t = M_t/m_t \geq 1$, where $M_t = \max(|\widehat{P}_t \cap k|, |\widehat{P}_t \cap l|)$ and $m_t = \min(|\widehat{P}_t \cap k|, |\widehat{P}_t \cap l|)$. Every level sphere in a *middle slab* $\{\widehat{P}_t, t \in]x_i, y_i[\}$ has the same ratio, because they all intersect k and l the same number of times. Then each middle slab $\{\widehat{P}_t, t \in]x_i, y_i[\}$ has a ratio r_i, defined as $r_i = r_t$ for some $t \in]x_i, y_i[$. The *linking ratio* r of this thin presentation of L is the minimum: $\min_i\ r_i$. We may define the linking ratio $r(L)$ of L

to be the minimum number of the linking ratios r, over all thin presentations for L.

REMARKS. (1) The ratio r_i may be infinite.
(2) Since L is irreducible, in a thin presentation there always exist middle slabs which meet both components of the link. So $r(L) < \infty$.
(3) We may suppose that $r(L)$ corresponds to the ratio $r = r_j = r_t = |\widehat{P}_t \cap k|/|\widehat{P}_t \cap l|$. There is no loss of generality because it suffices to exchange the roles of l and k.

Corollary 1. *Let $L = (k, l)$ be a link in S^3 with linking ratio $r(L)$, α a pair of meridian slopes of L, and β a pair of slopes on ∂M_L, with $\beta_1 \neq \alpha_1$, $\beta_2 \neq \alpha_2$, such that*
(i) $M_L(\beta_1, \beta_2) \simeq S^3$;
(ii) *there is no essential annulus cobording the two boundary components of M_L;*
(iii) *$M_L(\beta_1)$, $M_L(\beta_2)$, $M_L(\alpha_1)$, and $M_L(\alpha_2)$ are boundary irreducible.*
Then either $\Delta(\alpha_1, \beta_1) < 30$, or $\Delta(\alpha_2, \beta_2) < 30r(L)$.

Notice that property (iii) implies no component of the link is trivial. By [7, Theorem 2], property (i) implies L is irreducible.

The link L_β in S^3 shall be the core of the β-surgery on L.

Recall Berge's conjecture. *For any integer n, there exists a link $L = (k, l)$ in S^3, such that M_L has property* (ii) *of* Corollary 1, *l and k are non-trivial, $M_L(\beta_1, \beta_2) \simeq S^3$, and $\Delta(\beta_1, \alpha_1) > n$, $\Delta(\beta_2, \alpha_2) > n$, where α_1 (respectively α_2) is a meridian slope of k_1 (resp. k_2).*

By Corollary 1, if the link verifies Berge's Conjecture for $n \geq 30$, and if the components of the core are non-trivial, then the link must have a "sufficiently large" linking ratio. More precisely, $r(L)$ must be $> n/30$.

Let V_i be a solid torus in S^3 with core knot k_i ($i = 1, 2$). Suppose that there is an annulus A connecting ∂V_1 and ∂V_2; $\partial A = c_1 \cup c_2$ and c_i wraps p_i-times in longitudinal direction of V_i. Without loss of generality, we may assume $0 \leq p_1 \leq p_2$. We divide into several cases depending on the pair p_1, p_2.

If $p_1 = p_2 = 0$, there is a 2-sphere in S^3 intersecting k_i in one point, a contradiction.

If $p_1 = 0$ and $p_2 = 1$, then k_2 is a trivial knot.

Assume that $p_1 = 0$ and $p_2 \geq 2$, then we can find a lens space summand in S^3, a contradiction.

If $p_1 = p_2 = 1$, then k_1 and k_2 are parallel.

If $p_1 = 1$ and $p_2 \geq 2$, then k_1 is a cable of k_2.

Finally suppose that both $p_i \geq 2$. Let us consider the 3-manifold $M = V_1 \cup N(A) \cup V_2$; it is a Seifert fiber space over the disk with two exceptional fibers k_1 and k_2 of indices p_1 and p_2. Note that M is boundary-irreducible, then since ∂M is a torus, $S^3 - \operatorname{Int} M$ is a solid torus whose core is a non-trivial torus knot. Thus the two exceptional fibers k_1 and k_2 form a Hopf link (cf. [10, Theorem 11]).

Hence, if $L = (k_1, k_2)$ is a link in S^3 with k_i non-trivial ($i = 1, 2$) such that M_L contains an annulus A, then either $p_1 = p_2 = 1$ and k_1 and k_2 are parallel, or $p_1 = 1$ and $p_2 \geq 2$ and k_1 is a cable of k_2.

This gives us a new corollary:

Corollary 2. *Let $L = (k, l)$ be a link in S^3 with linking ratio $r(L)$, such that k and l are non-trivial. Let α be a pair of meridian slopes of L, and β a pair of slopes on ∂M_L, with $\beta_1 \neq \alpha_1$, $\beta_2 \neq \alpha_2$.*

Suppose $M_L(\beta_1, \beta_2) \simeq S^3$, and the two components of L_β are non-trivial. If $\Delta(\alpha_1, \beta_1) \geq 30$, and $\Delta(\alpha_2, \beta_2) \geq 30r(L)$, then some component of L is a cable of the other one, or both components represent the same knot.

3. Intersection graphs

Let P and Q be two planar surfaces properly embedded in X, with parameters (α, a) and (β, b) on (T_1, T_2). Let $\widehat{P}$ and $\widehat{Q}$ be the 2-spheres in $X(\alpha)$ and $X(\beta)$ such that $P = X \cap \widehat{P}$, $Q = X \cap \widehat{Q}$. As usual we consider a pair (G_P, G_Q) of graphs in $(\widehat{P}, \widehat{Q})$.

Number, from 1 to a_i, the components of $\partial P \cap T_i$, that we denote by $\partial_1^i P$, $\partial_2^i P, \ldots, \partial_{a_i}^i P$, in the order in which they appear on T_i, $i = 1, 2$. Number from 1 to b_i the components of $\partial Q \cap T_i$, that we denote by $\partial_1^i Q$, $\partial_2^i Q, \ldots, \partial_{b_i}^i Q$, in the order in which they appear on T_i, $i = 1, 2$.

Now label the endpoints of the properly embedded arcs in $P \cap Q$. Let e be an arc in $P \cap Q$, and t be an endpoint of e, say $t \in \partial_n^i P \cap \partial_m^i Q$. Then t is labelled n on the component $\partial_m^i Q$ in the surface Q, and t is labelled m on the component $\partial_n^i P$ in the surface P. Thus around each component of $\partial P \cap T_i$, we see the labels $1, 2, \ldots, b_i$ appearing in cyclic order, and around each component of $\partial Q \cap T_i$ we see the labels $1, 2, \ldots, a_i$, these sequences being repeated Δ_i times, $i = 1, 2$.

Assigning (arbitrarily) orientations to P and Q, we induce an orientation on each component of ∂P and each component of ∂Q. The orientation of X induces an orientation for T_1 and T_2. Here we choose a positive orientation for each unoriented simple closed curve with slope α_i and each one with slope β_i, respecting the orientation of T_i, $i = 1, 2$.

We assign a sign + to a component x of $\partial P \cap T_i$ or $\partial Q \cap T_i$ if its induced orientation is the same as the positive orientation of the closed curves on T_i defined previously, and assign the sign $-$ otherwise. We shall say that two components x and y of $\partial P \cap T_i$ (or $\partial Q \cap T_i$) are *parallel* if they have the same sign and *antiparallel* if they have opposite signs. Notice that signs given to components of $\partial P \cap T_1$ (respectively $\partial Q \cap T_1$) are independant of the signs of the components of $\partial P \cap T_2$ (respectively

$\partial Q \cap T_2$).

Cap off the components of $\partial P \cap T_i$ (respectively $\partial Q \cap T_i$) with discs, we regard these discs as "fat" vertices of type i of the graph G_P (respectively G_Q), $i = 1, 2$. Thus there are two types of vertices in G_P and G_Q, just as there are two types of edges: the *simple edges* of G_P (respectively G_Q) correspond to arcs of $P \cap Q$ in P (respectively Q) whose boundary components are in the same boundary component of ∂X, the *mixed edges* of G_P (respectively G_Q) correspond to arcs of $P \cap Q$ in P (respectively Q) with one boundary component on T_1 and the other on T_2. In the following, we shall consider edges of G_P and G_Q as arcs in $P \cap Q$ and keep the same notation for both, and the labels and signs assigned to boundary components will be kept the same for the vertices.

If P and Q have graph properties, then there is no *trivial loop* (disc-face with one single edge in its boundary) in either G_P or G_Q and the parity rule still works for simple edges:

If a simple edge joins parallel vertices in G_P, it joins antiparallel vertices in G_Q and vice versa.

Two edges e and e' in a graph G are *directly parallel* if they connect the two same vertices, and cobound a disc-face in G. They are *parallel* if there exist a finite set $\{e_1 = e, e_2, \ldots, e_n = e'\}$ of edges of G such that e_i and e_{i+1} are directly parallel, for $i \in \{1, \ldots, n-1\}$.

Recall that the *reduced graph* $\widehat{G}$ of a graph G is obtained from G by replacing each family of parallel edges by a single edge. We shall use $\mathrm{val}(v, G)$ to denote the valency of a vertex v in the graph G.

4. Proof of Theorem 1

Lemma 1 ([6, Lemma 4.1]). *Let Γ be a finite graph in the 2-sphere with no 1-sided faces. Suppose every vertex of Γ has order ≥ 6. Then Γ has two parallel edges.*

Suppose $\mathcal{E}$ is a family of edges of the pair (G_Q, G_P), then $G_P(\mathcal{E})$ is the subgraph of G_P consisting of all edges of $\mathcal{E}$ and their attached vertices.

Lemma 2. *Suppose $a_1 \geq a_2$ and $\Delta_1 \geq 30$, $\Delta_2 \geq 30a_1/a_2$. Then G_Q has a family of parallel edges $\mathcal{E}$, and $G_P(\mathcal{E})$ has two parallel edges.*

Proof. Let $\widehat{G_Q}$ be the reduced graph of G_Q. By Lemma 1, $\widehat{G_Q}$ has a vertex w_0 so that $\mathrm{val}(w_0, \widehat{G_Q}) \leq 5$. But

$$\mathrm{val}(w_0, G_Q) = \begin{cases} \Delta_1 a_1 & \text{if } w_0 \text{ is of type 1} \\ \Delta_2 a_2 & \text{if } \mathrm{w}_0 \text{ is of type 2} \end{cases} .$$

Hence $\mathrm{val}(w_0, G_Q) \geq 30a_1$, and $\widehat{G_Q}$ has an edge $\mathcal{E}$ of order$\geq 6a_1$ which is incident to w_0. The edge $\mathcal{E}$ in $\widehat{G_Q}$ is a family of at least $6a_1$ parallel edges in G_Q.

Suppose the edges of $\mathcal{E}$ are mixed. We rename u and v the vertices of type 1 and 2 respectively, attached to $\mathcal{E}$. Since $a_1 \geq a_2$, for each i in $\{1, 2, \ldots, a_1\}$ there are at least 6 endpoints of $\mathcal{E}$ labelled i on ∂u, and for each j in $\{1, 2, \ldots, a_2\}$ there are at least 6 endpoints of $\mathcal{E}$ labelled j on ∂v. Hence every vertex in $G_P(\mathcal{E})$ has valency≥ 6. By Lemma 1, $G_P(\mathcal{E})$ has two parallel edges. Notice that they are mixed.

Now suppose the edges of $\mathcal{E}$ are simple. They join two vertices of same type i, which we call u and v. Since $a_1 \geq a_2$, $\mathcal{E}$ contains at least $6a_i$ edges. So for each $m_i \in \{1, 2, \ldots, a_i\}$, there are at least 6 endpoints of edges in $\mathcal{E}$ labelled m_i on ∂u, and the same on ∂v. Thus every vertex of $G_P(\mathcal{E})$ is of type i and has valency≥ 12. If there is no trivial loop in $G_P(\mathcal{E})$, we can apply Lemma 1 to show that $G_P(\mathcal{E})$ has two parallel edges.

If $G_P(\mathcal{E})$ contains a trivial loop, then the both endpoints of each edge in $\mathcal{E}$ have the same label on u and v in G_Q. All the edges of $\mathcal{E}$ are loops in $G_P(\mathcal{E})$. By Lemma 3 below, $G_P(\mathcal{E})$ must contain at least two parallel edges (which are loops). In all cases we can choose two edges e_1, e_2 directly parallel in $G_P(\mathcal{E})$. $\square$

Lemma 3. *If every edge of $\mathcal{E}$ is a loop in $G_P(\mathcal{E})$, then $G_P(\mathcal{E})$ contains two parallel edges.*

Proof. For this lemma, we just need $\mathcal{E}$ to be a family of exactly $6a_1$ mutually parallel adjacent edges. The edges of $\mathcal{E}$ are loops in G_P. Thus in G_Q they join two antiparallel vertices. There are two cases, according to whether $\mathcal{E}$ joins in G_Q vertices of type 1, or vertices of type 2.

First assume the vertices of $G_P(\mathcal{E})$ are of type 1. Then $G_P(\mathcal{E})$ (on the sphere $\widehat{P}$) consists of a_1 connected components, each of which is a 6-bouquet. Here, an n-bouquet will be a graph with one vertex and n loops.

Let $\mathcal{F}$ be the set of faces of $G_P(\mathcal{E})$ (as a graph on a sphere), and f_1, f_2 and f_3 be the numbers of disc faces of $G_P(\mathcal{E})$ with one side, two sides and at least three sides, respectively. Then an Euler characteristic calculation gives

$$a_1 - 6a_1 + \sum_{f \in \mathcal{F}} \chi(f) = 2.$$

But

$$\sum_{f \in \mathcal{F}} \chi(f) = f_1 + f_2 + f_3 + \sum_{f \in \mathcal{F}, f \text{ non-disc face}} \chi(f) \leq f_1 + f_2 + f_3.$$

Thus $f_1 + f_2 + f_3 \geq 2 + 5a_1$.

To prove the first case, it is sufficient to show that $f_2 > 0$. So we assume $f_2 = 0$, and reach a contradiction. Then $f_1 + f_3 \geq 2 + 5a_1$. Since G_P has no trivial loop, each 1-sided face of $G_P(\mathcal{E})$ must contain, in G_P, at least one vertex. This vertex is of type 2 because $G_P(\mathcal{E})$ already contains all the vertices of type 1. Thus $f_1 \leq a_2$. Since $a_1 \geq a_2$, we have $f_3 \geq 2 + 4a_1$.

CLAIM 1. A 6-bouquet on a sphere has at most two disc-faces with 3 or more sides.

Proof of Claim 1. Embed a 6-bouquet G on a sphere. Then by Euler's formula, $1 - 6 + \sum \chi(\text{face}) = 2$. Note that all faces are disc.

Let g_1, g_2 and g_3 be the number of disc faces with 1 side, 2 sides, and at least 3 sides, respectively. Then $g_1 + 2g_2 + 3g_3 \leq 2 \times 6 = 12$. Since $g_1 + g_2 + g_3 = \sum \chi(\text{face}) = 7$, we have $2g_3 \leq g_2 + 2g_3 \leq 5$. Thus $g_3 \leq 2$. $\square$

Hence each connected component of $G_P(\mathcal{E})$ has at most two 3-sided disc faces, and so $f_3 \leq 2a_1$. Then we have $2a_1 \geq 2 + 4a_1$, a contradiction.

Next suppose that the vertices of $G_P(\mathcal{E})$ are of type 2. Recall that $\mathcal{E}$ is a family of $6a_1$ parallel edges. Then $G_P(\mathcal{E})$ has a_2 vertices, and a_2 connected subgraphs, each being an n_i-bouquet for some integer $n_i \geq 6$. Note that $\sum_{i=1}^{a_2} n_i = 6a_1$.

A similar proof as the one of Claim 1 gives Claim 2:

CLAIM 2. An n_i-bouquet on a sphere has at most $(n_i - 1)/2$ disc-faces with 3 or more sides.

Keep the same notation as for the above case. In $G_P(\mathcal{E})$ we now have

$$a_2 - 6a_1 + \sum_{f \in \mathcal{F}} \chi(f) = 2,$$

which leads to $f_1 + f_2 + f_3 \geq 2 - a_2 + 6a_1$. Assume $f_2 = 0$ for contradiction. Since $f_1 \leq a_1$ (because $G_P(\mathcal{E})$ contains all the vertices of type 2 and G_P has no trivial loop), then $f_3 \geq 2 - a_2 + 5a_1$. By Claim 2,

$$f_3 \leq \sum_{i=1}^{a_2} \frac{(n_i - 1)}{2} = 3a_1 - \frac{a_2}{2}.$$

Hence

$$2 - a_2 + 5a_1 \leq f_3 \leq 3a_1 - \frac{a_2}{2}.$$

Since $a_2 \leq a_1$, then we have $4 + 3a_1 \leq 0$, a contradiction. $\square$

We shall say that two arcs of $P \cap Q$ are parallel in P if they cut off a disc in P.

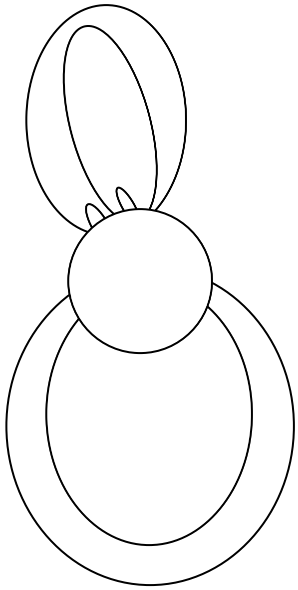

Fig. 1. a 6-bouquet

Lemma 4 ([5, Lemma 2.1]). *Let P and Q be two properly embedded planar surfaces in a* 3*-manifold X such that ∂X contains a torus T, and assume P and Q have boundary components on T. Suppose that P and Q intersect transversely and each component of $\partial P \cap T$ intersects each component of $\partial Q \cap T$ minimaly. Let $\mathcal{A}$, $\mathcal{A}'$ be two arcs of $P \cap Q$, properly embedded in (X, T), and parallel in both P and Q. If $D \cap E = \mathcal{A} \cup \mathcal{A}'$, where D and E are the discs in P and Q respectively, that realize the parallelism of $\mathcal{A}$ and $\mathcal{A}'$, then D and E cannot be identified along $\mathcal{A}$ and $\mathcal{A}'$ as illustrated in* Fig. 2*.*

In the following, $\mathcal{E}$ will always denote this family with at least $6a_1$ parallel edges in G_Q. Under the assumption $\Delta_1 \geq 30$ and $\Delta_2 \geq 30a_1/a_2$, Lemma 2 implies an existence of two parallel edges in $G_P(\mathcal{E})$, which contradicts the assumption (ii) or (i) in Theorem 1 by Lemma 5 or 6 below. This completes the proof of Theorem 1. $\square$

Lemma 5. *Let e_1 and e_2 be parallel edges in $G_P(\mathcal{E})$. Suppose e_1 and e_2 are simple. Then $(X(\alpha_1), T_2)$ or $(X(\alpha_2), T_1)$ contains a properly embedded Möbius band.*

Proof. The edges e_1 and e_2 come from the family $\mathcal{E}$ of edges in G_Q, so they are parallel in both G_Q and $G_P(\mathcal{E})$. First we can assume e_1 and e_2 are directly parallel in $G_P(\mathcal{E})$. Assume now e_1 and e_2 have their boundaries on T_1. Let u, v and x, y denote the vertices attached to e_1 and e_2 in G_P and G_Q respectively. (We can have

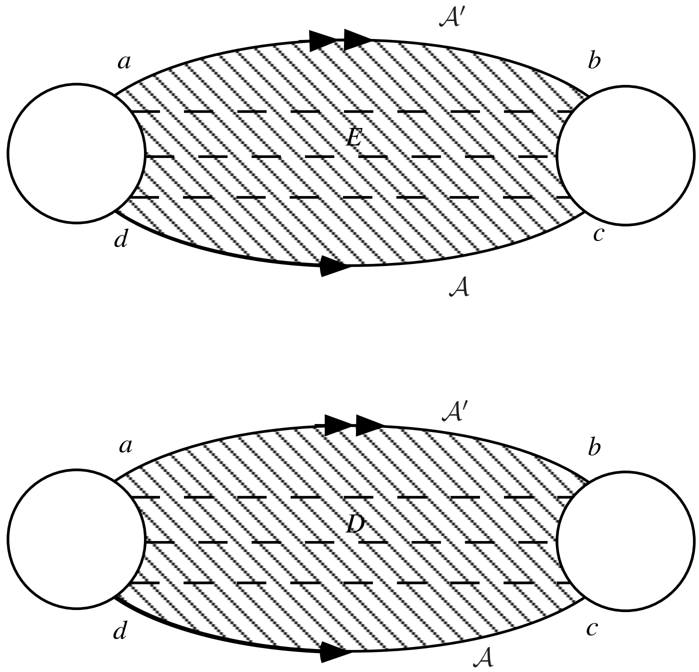

Fig. 2. a, b, c, d are the points of intersection between the arcs and the boundary components of P and Q.

$u = v$ or $x = y$). Since each label i in $\{1, 2, \ldots, a_1\}$ appears as an endpoint of an edge of $\mathcal{E}$ in G_Q, the graph $G_P(\mathcal{E})$ contains all the vertices of type 1 of G_P. Hence the cycle given by the edges e_1 and e_2 bounds a disc D in $\widehat{P}$ such that in the interior of the cycle there are no vertices of type 1. But, there may be vertices of type 2 of G_P in the interior of the cycle. Let E be the disc that realizes the parallelism between the arcs e_1 and e_2 in Q. Each arc of $E \cap P$ corresponds to an edge of $\mathcal{E}$. Then, since e_1 and e_2 are directly parallel in $G_P(\mathcal{E})$, $D \cap E = e_1 \cup e_2 \cup \mathcal{C}$, where $\mathcal{C}$ is a union of circles. By a cut and paste method we can eliminate circles of intersection, and obtain two discs (we shall call them again D and E) such that $D \cap E = e_1 \cup e_2$. There are two possibilities for the way in which E and D are identified along e_1 and e_2, illustrated by Fig. 3 and Fig. 4. Notice that the disc D is in $P(\alpha_2) \subset X(\alpha_2)$. The surfaces $P(\alpha_2)$ and Q are transverse and their boundary components on T_1 intersect minimaly. By Lemma 4, case (i) is impossible. In case (ii), $E \cup D$ is a Möbius band properly embedded in $X(\alpha_2)$. $\square$

REMARK. Suppose $u = v$, $\mathcal{M} = D \cup E$ is a Möbius band properly embedded in $(X(\alpha_2), T_1)$. But $\partial\mathcal{M}$ also bounds a Möbius band $\mathcal{M}'$ in $X(\alpha_1) = X \cup V_1$, where $\mathcal{M}'$ is the union of the meridian disc u of V_1 and a disc Δ on T_1 (see Fig. 5). The

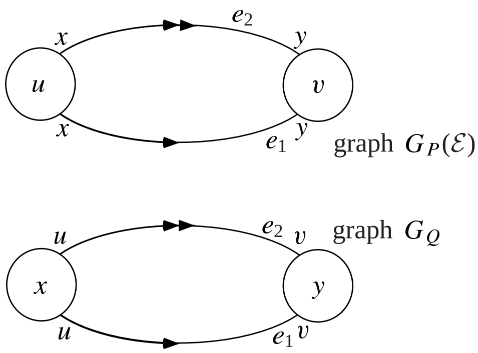

Fig. 3. case (i)

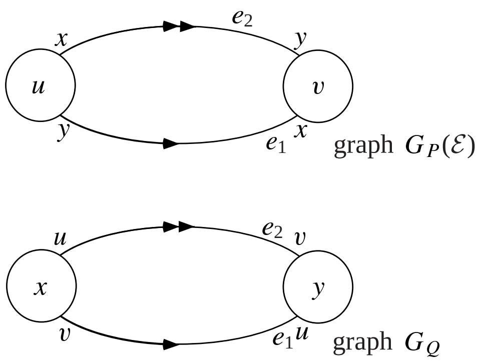

Fig. 4. case (ii)

union of the two Möbius bands is a Klein bottle $K = \mathcal{M} \bigcup_{\partial \mathcal{M}} \mathcal{M}'$ in the manifold $X(\alpha_1, \alpha_2)$. The Dehn filling solid torus V_1 intersects K in a single component.

Lemma 6. *Let e_1 and e_2 be parallel edges in both G_Q and $G_P(\mathcal{E})$. Suppose e_1 and e_2 are mixed. Then X contains an essential annulus, such that one of the boundary components is in T_1 and the other in T_2.*

Proof. Assume e_1, e_2 are directly parallel in $G_P(\mathcal{E})$. Let u, v and x, y be the pairs of vertices in G_P and G_Q respectively, attached to the parallel edges e_1, e_2. Suppose u and x are of type 1, while v and y are of type 2. First notice that $\mathcal{E}$ is a family of at least $6a_1$ parallel edges. Hence each i in $\{1, 2, \dots, a_1\}$ labels an endpoint on ∂x of some edge in $\mathcal{E}$, and every label of type 2 is an endpoint on ∂y of some edge in $\mathcal{E}$. Then $G_P(\mathcal{E})$ contains all the vertices of G_P. If e_1 and e_2 are non parallel in G_P, then e_1 and e_2 cut off in G_P a subgraph which contains at least one vertex of G_P. By the

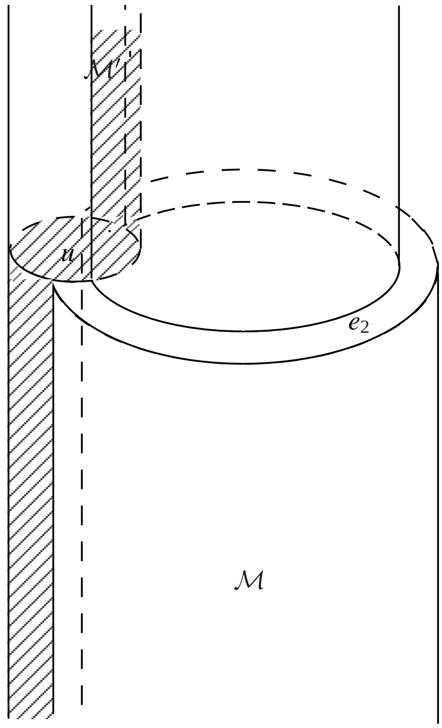

Fig. 5. case $u = v$

previous remark, this vertex is also a vertex of $G_P(\mathcal{E})$, which contradicts the fact that e_1 and e_2 are parallel in $G_P(\mathcal{E})$. Therefore the edges e_1 and e_2 are also parallel in G_P, (but in general they are not directly parallel), so let D (respectively E) be the disc in P (respectively Q) that realizes the parallelism between e_1 and e_2. If the discs E and D contain circles in their intersection, by cut and paste methods we may build two new discs (let's call these discs E and D again), such that they intersect only along e_1 and e_2. Since the edges e_1, e_2 are mixed, the only possibility for the way in which E and D are identified along e_1 and e_2 is illustrated in Fig. 6. The union $E \bigcup_{e_1,e_2} D$ is an annulus A with two boundary components $\partial_+ A \subset T_1$ and $\partial_- A \subset T_2$, where $\partial_+ A = C \cup C'$, and $\partial_- A = \delta \cup \delta'$ (see Fig. 6).

Let $\mid \alpha \cap \beta \mid$ be the geometric intersection number between the two oriented curves α and β, and $\alpha.\beta$ be their algebraic intersection number.

We suppose without loss of generality that $C'.\partial u \geq 0$. We have

$$\partial_+ A.\partial u = C'.\partial u - 1,$$

and $\mid C' \cap \partial u \mid \geq 2$. Then $\partial_+ A$ intersects the meridian ∂u of T_1 at least once and always with the same orientation. The annulus A joins T_1 to T_2 and its boundary components

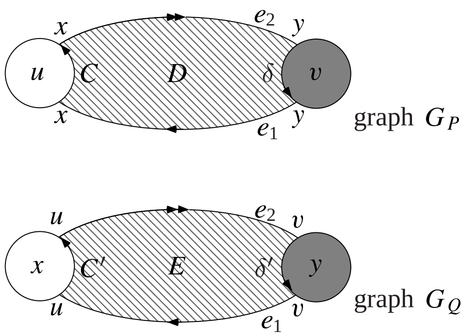

Fig. 6. vertices u and x of type 1, vertices v and y of type 2

are non-trivial on T_1 and T_2 respectively. Therefore A is essential. □

5. Proof of Corollary 1

For the proof of Corollary 1, we'll follow the same argument as for Theorem 1. It suffices to find two planar surfaces P and Q which verify hypothesis of Theorem 1 in the case where $X = M_L$.

For a proof by contradiction, we consider a link $L = (k, l)$ without trivial components which produces S^3 by a non-trivial surgery, such that the cores of the surgery are non-trivial. We may suppose it is an irreducible link because if it were reducible, it wouldn't give S^3 by a non-trivial surgery. (This is an immediate consequence of Theorem 2 in [7]).

Proposition 1. *Let $L = (k, l)$ be a link in S^3 such that k and l are non-trivial, and with a linking ratio $r(L)$ in some thin presentation. Let α be a pair of meridian slopes of L. If $M_L(\beta)$ is homeomorphic to S^3 for a slope $\beta \neq \alpha$ on ∂M_L, and $M_L(\beta_1)$, $M_L(\beta_2)$ are boundary-irreducible, then there exist two properly embedded planar surfaces P and Q in M_L, such that P has parameters (α, a), and Q has parameters (β, b) on ∂M_L, $r(L) = a_1/a_2$, and P and Q have graph properties.*

Proof. We consider a thin presentation for L in S^3 such that the linking ratio of this presentation is $r(L)$. Since $M_L(\beta) \simeq S^3$, the link is irreducible, so there is at least one middle slab $\{\widehat{P}_t, t \in]x_j, y_j[\}$ such that each level sphere $\widehat{P}_t$ in this middle slab intersects both k and l. Its ratio $r_t = r_j$ is finite, and suppose $r_j = r(L)$, that is to say this middle slab realizes $r(L)$.

Now we choose a thin presentation for the core of the surgery $L_\beta = (k_\beta, l_\beta)$ in the copy $M_L(\beta) = {S^3}_\beta$ of S^3. The link L_β is irreducible. As above, we choose a middle slab $\{\widehat{Q}_t, t \in]\overline{x_j}, \overline{y_j}[\}$ which intersects both k_β and l_β.

Applying the method of [7, Proposition 1] to a 2-component link in S^3 with non-trivial components, we find two surfaces P and Q, where $P = \widehat{P}_t \cap M_L$ for some t in $]x_j, y_j[$ and Q is homeomorphic to $\widehat{Q}_t \cap ({S^3}_\beta \setminus \operatorname{Int} N(L_\beta))$, with t in $]\bar{x_j}, \bar{y_j}[$. In order to apply the general definitions of Section 1 and Section 2, we replace X by M_L, T_1 by $\partial N(k)$ and T_2 by $\partial N(l)$. The planar surfaces P and Q are properly embedded in M_L, with parameters (α, a), (β, b) respectively $(a_i > 0, b_i > 0)$, P realizes $r(L)$ and they have graph properties. $\square$

We may assume $r(L) = |\widehat{P}_t \cap k| / |\widehat{P}_t \cap l|$, so $a_1 \geq a_2$. We apply Theorem 1, which leads us to the following conclusion: Either M_L contains an essential annulus that joins its two boundary components or one of the two manifolds $M_L(\alpha_1)$, $M_L(\alpha_2)$ contains a properly embedded Möbius band. The first case contradicts hypothesis i) of Corollary 2. We will see that the second case is impossible too.

The Möbius band we have built in the proof of Theorem 1 leads to a contradiction in each case. We shall describe exactly what happens.

Consider the case (ii) of Fig. 4, and we suppose without loss of generality that u and v are meridian discs for $\partial N(k)$. In fact, we shall divide the case (ii) in the three following subcases:

(ii).1 $u = v$

(ii).2 $u \neq v$, u and v are antiparallel

(ii).3 $u \neq v$, u and v are parallel.

We are going to see that all the subcases are impossible.

Keep the same notation (discs E and D, vertices u, v, x and y) as in the general case, changing the vertices of type 1 (respectively of type 2) into vertices corresponding to boundary components on $\partial N(k)$ (respectively $\partial N(l)$). First, for these three cases, change E and D by cut and paste or isotopy if necessary to eliminate singular surfaces as previously.

(ii).1 The Möbius band $\mathcal{M} = D \bigcup_{e_1, e_2} E$ is properly embedded in $S^3 \setminus \operatorname{Int} N(k)$. But $\partial\mathcal{M}$ also bounds a Möbius band $\mathcal{M}'$ properly embedded in $M_L(\alpha_1) \simeq S^3 \setminus \operatorname{Int} N(l)$: $\mathcal{M}'$ is the union of the meridian disc of u of k and a disc on $\partial N(k)$, (see Fig. 5). The union $K = \mathcal{M} \bigcup_{\partial\mathcal{M}} \mathcal{M}'$ is a Klein bottle embedded in S^3, which is impossible.

(ii).2 There is a Möbius band B in $M_L(\alpha) \simeq S^3$: B is the union of the disc $D \subset S^3 \setminus \operatorname{Int} N(k)$, a meridian disc (in fact u or v) of k and a disc of $\partial N(k)$ (see [6], Fig. 4). The disc E in Q has the same boundary as the Möbius band B, then $B \cup E$ is a projective plane embedded in S^3, which is impossible.

(ii).3 Then, by the parity rule, x and y are antiparallel in G_Q. As in the previous case, we can build a Möbius band B' which is the union of the disc E, a disc on $\partial N(k_\beta)$ and a meridian disc of $N(k_\beta)$. The Möbius band B' is properly embedded in $M_L(\beta_1) \simeq S^3 - \operatorname{Int} N(l_\beta)$, and it has the same boundary as D. The projective plane $D \cup B'$ is embedded in S^3, which is impossible. $\square$

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