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CURVES IN PROJECTIVE SPACES AND THEIR INDEX OF REGULARITY

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Abstract

For all integers $n \geq 3$ we show the existence of many triples (d, g, ρ) such that there is a smooth non-degenerate curve $C \subset \mathbf{P}^n$ with degree d, genus g and index of regularity ρ. The curve C lies in a smooth $K3$ surface $S \subset \mathbf{P}^n$.

1. Index of regularity

Let $C \subset \mathbf{P}^n$ be a curve, i.e. a locally Cohen-Macaulay pure one-dimensional closed subscheme. Set $\rho(C) := \min\{t : h^1(\mathbf{P}^n, \mathcal{I}_C(x)) = 0 \text{ for every } x \geq t\}$. We will call $\rho(C)$ the index of regularity of C. Since the old works of Castelnuovo, the integer $\rho(C)$ is considered a fundamental invariant of C ([5], [2]). In all cases we will consider in this paper we will have $h^1(C, \mathcal{O}_C(\rho - 1)) = 0$ and hence by Castelnuovo-Mumford lemma in this case the integer $\rho(C)$ will be also the regularity index of the minimal free resolution of C ([2]): another very good reason to consider it a fundamental invariant of C. Thus for any fixed integer $n \geq 3$ it seems nice to show the existence of many triples (d, g, ρ) such that there is a smooth non-degenerate curve $C \subset \mathbf{P}^n$ with degree d, genus g and index of regularity ρ. A weaker, but very important problem, classical problems is to find at least "almost all" pairs (d, g) that may appear as (degree, genus) of a smooth non-degenerate curve $C \subset \mathbf{P}^n$. For this classical problem (when $n = 3$) S. Mori used a $K3$ surface ([4]). Later, A.L. Knutsen extended Mori's idea to the case $n \geq 4$. Using Knutsen's paper it was possible to construct curves C such that certain cohomology groups $h^1(\mathbf{P}^n, \mathcal{I}_C(x))$ vanish ([1]). Here we adapt the proofs in [1] to get results on the index of regularity.

Theorem 1. *Fix integers d, g, n such that $n \geq 3$ and $0 \leq d - n < g < d^2/(4n - 4) - (n - 1)/4$. Set $r := \lfloor (d - \sqrt{d^2 - (4n - 4)g})/(2n - 2) \rfloor$, $d_0 := d - (2n - 2)r$ and $g_0 := (n-1)r^2 - dr + g$. Then $r \geq 1$ and $0 \leq g_0 \leq d_0 - n$. There is a smooth and arithmetically Cohen-Macaulay degree $2n - 2$ $K3$ surface $S \subset \mathbf{P}^n$ with the following properties. Set $H := \mathcal{O}_S(1)$. There is a smooth and connected curve $C_0 \subset S$ such that* $\deg(C_0) = d_0$, $p_a(C_0) = g_0$, $h^1(C_0, \mathcal{O}_{C_0}(1)) = 0$, $h^0(S, \mathcal{O}_S(H - C_0)) = h^0(S, \mathcal{O}_S(C_0 - H)) = 0$, $\mathrm{Pic}(S)$

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is freely generated by the classes of H and C_0, and the general element of $|C_0 + rH|$ is a smooth and connected non-degenerate curve with degree d and genus g. We have $e(C_0) = 0$ if $g_0 > 0$ and $e(C_0) = -1$ if $g_0 = 0$. Take any $C \in |C_0 + rH|$. Then $\rho(C) = \rho(C_0) + r$ and $e(C) = e(C_0) + r$. C is arithmetically normal if and only if C_0 is projectively normal and this is the case if and only if $d_0 = g_0 + n$ and $h^1(\mathbf{P}^n, \mathcal{I}_{C_0}(2)) = 0$. If $d_0 = g_0 + n$ and $n > g_0$, then C_0 is projectively normal.

REMARK 1. Use the notation of Theorem 1. The existence of S was proved in [1], proof of Th.1.4. By [1], Th.1.4, we have $h^1(\mathbf{P}^n, \mathcal{I}_C(r+1)) = d_0 - g_0 - n$ and $h^1(\mathbf{P}^n, \mathcal{I}_C(t)) = 0$ for every integer t such that $0 \leq t \leq r$.

By Theorem 1 the computation of the index of regularity $\rho(C)$ of C is reduced to the computation of the integer $\rho(C_0)$. Since $d \gg d_0$, the following remark may be useful.

REMARK 2. Let $C \subset \mathbf{P}^m$ be an integral degree d non-degenerate curve. If $m = 2$, then $\rho(C) = 0$. However, if $m = 2$, then $h^1(C, \mathcal{O}_C(t)) = 0$ if and only if $t \geq d - 2$. Now assume $m \geq 3$. By [2] we have $\rho(C) \leq d + 1 - m$ and $\rho(C) = d + 1 - m$ if and only if C is smooth and rational and either $d \leq m + 1$ or $d \geq m + 2$ and C has a $(d + 2 - m)$-secant line. Furthermore, $h^1(C, \mathcal{O}_C(z)) = 0$ for all $z \geq d - m$.

We work over an algebraically closed field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) = 0$.

Proof of Theorem 1. The existence of the pair (S, C_0) was checked in [1], proof of Th.1.4 (see in particular the last two lines of that proof for the critical condition $h^1(C_0, \mathcal{O}_{C_0}(1)) = 0$). Since $h^1(C_0, \mathcal{O}_{C_0}(1)) = 0$, we have $e(C_0) \leq 0$. Hence $e(C_0) = 0$ if $g_0 > 0$ and $e(C_0) \in \{-2, -1\}$ if $g_0 = 0$. Since $h^0(S, \mathcal{O}_S(H - C_0)) = 0$, C_0 is not a line and hence $e(C_0) = -1$ if $g_0 = 0$. The construction of the pair (S, C_0) used in an essential way the construction of many curves in suitable $K3$ surfaces due to S. Mori ([4]) for $n = 3$ and to A.L. Knutsen ([3]) for arbitrary n. Fix an integer $a \geq 0$ and any $T \in |C_0 + aH|$. For all integers t we have the following exact sequence

$$0 \to \mathcal{O}_S((t-a)H - C_0) \to \mathcal{O}_S(tH) \to \mathcal{O}_T(t) \to 0 \tag{1}$$

If $t \geq a$ and $(t, a) \neq (0, 0)$, then $h^1(S, \mathcal{O}_S(tH)) = h^2(S, \mathcal{O}_S(tH)) = 0$ and hence $h^1(T, \mathcal{O}_T(t)) = h^2(S, \mathcal{O}_S((t-a)H - C_0)) = h^0(S, \mathcal{O}_S(C_0 + (a-t)H))$. From this relation for $a = 0$ and $a = r$ we get $e(C) = e(T) = e(C_0) + r$. Since this relation is obvious for $r = 0$, we do not need the case $(t, a) = (0, 0)$. By [1], Th.1.4, (its proof does not require the smoothness of C) we have $h^1(\mathbf{P}^n, \mathcal{I}_T(t)) = 0$ if $0 \leq t \leq r$ and $h^1(\mathbf{P}^n, \mathcal{I}_T(r+1)) = d_0 - g_0 - n$. Now assume $a \in \{0, r\}$ and take an arbitrary integer $t \geq r + 2$. Since S is projectively normal, $h^1(\mathbf{P}^n, \mathcal{I}_T(t)) = h^1(S, \mathcal{O}_S((t-a)H - C_0))$. Hence $\rho(C) = \rho(C_0) + r$. The projective normality of a degree d_0 linearly normal em-

bedding of C_0 if $d_0 \geq 2g_0 + 1$ was proved by D. Mumford ([5], Cor. at p.55). □

REMARK 3. The proof of Theorem 1 shows that $\rho(C)$ is the minimal integer t such that $h^1(S, \mathcal{O}_S((t - r - 1)H - C_0)) \neq 0$.

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