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## A REMARK ON THE MINLOS-POVZNER TAUBERIAN THEOREM

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In a study of the spectral theory of a random difference operator we utilized without proof a Tauberian theorem on the Laplace transform of a Stieltjes measure supported by  $(-\infty, \infty)$  ([1]; Lemma 2). This is also used in [2] and [3]. In the present note we first prove a nontrivial modification of the Minlos-Povzner Tauberian theorem ([4]; Appendix) and then, as its consequence (Corollary 2), derive the above-stated Tauberian theorem on the bilateral Laplace transform.

1. Let  $\Phi_t(\beta)$  and  $\phi_t(\xi)$  be functions on  $(0, \infty)$  related by, for each  $t$ ,

$$(1.1) \quad \exp \{p(t)\Phi_t(\beta)\} = \int_0^\infty \exp \{p(t)(\beta\xi - \phi_t(\xi))\} d\xi$$

where  $p(t)$  is a non-decreasing function tending to infinity as  $t \rightarrow \infty$ .

**Theorem.** (i) *If  $\Phi_t(\beta)$  converges to a function  $\Phi(\beta)$  as  $t \rightarrow \infty$ ,  $\phi_t(\xi)$  is a non-decreasing function for each  $t$  such that there exists  $\varepsilon(t)$  satisfying  $\lim_{t \uparrow \infty} \varepsilon(t) = 0$ ,  $\lim_{t \uparrow \infty} \frac{\log \varepsilon(t)}{p(t)} = 0$ , and  $\lim_{t \uparrow \infty} |\phi_t(0+) - \phi_t(\varepsilon(t))| = 0$ , then  $\phi_t(\xi)$  has a limit at every regular point  $\xi$  ([4]) of  $\Phi(\beta)$  and*

$$\lim_{t \uparrow \infty} \phi_t(\xi) = \sup_{\beta > 0} \{\beta\xi - \Phi(\beta)\}.$$

(ii) *If a non-decreasing function  $\phi_t(\xi)$  converges to  $\phi(\xi)$  uniformly in any finite interval and there exists a function  $c(\beta)$  such that  $\Phi_t(\beta) < c(\beta)$  for any  $\beta$  and  $t$ , then  $\Phi_t(\beta)$  has a limit and*

$$\lim_{t \uparrow \infty} \Phi_t(\beta) = \sup_{\xi > 0} \{\beta\xi - \phi(\xi)\}.$$

For the proof of the first assertion we prepare four Lemmas.

**Lemma 1.** *For any  $\varepsilon > 0$*

$$(1.2) \quad \beta\xi - \phi_t(\xi) \leq (\beta - \gamma)\xi + K(\gamma; t, \varepsilon), \quad \xi > \varepsilon, \beta > 0, \gamma > 0$$

where  $K(\gamma; t, \varepsilon)$  is such that  $\lim_{\varepsilon \downarrow 0} \lim_{t \uparrow \infty} K(\gamma; t, \varepsilon) = \Phi(\gamma)$

Proof. For  $\xi \geq \varepsilon$

$$\begin{aligned} \exp \{p(t)\Phi_i(\gamma)\} &\geq \int_{\xi-\varepsilon}^{\xi} \exp \{p(t)(\gamma\xi - \phi_i(\xi))\} d\xi \\ &\geq \varepsilon \exp \{p(t)(\gamma(\xi-\varepsilon) - \phi_i(\varepsilon))\}. \end{aligned}$$

Therefore

$$(1.3) \quad -\phi_i(\xi) \leq -\gamma\xi + \Phi_i(\gamma) - \frac{\log \varepsilon}{p(t)} + \gamma\varepsilon.$$

Adding  $\beta\xi$  to each hand side and putting  $K(\gamma; t, \beta) = \Phi_i(\gamma) - \frac{\log \varepsilon}{p(t)} + \varepsilon\gamma$ , we have the Lemma because of the assumption that  $\lim_{t \uparrow \infty} \Phi_i(\gamma) = \Phi(\gamma)$  and  $\lim_{t \uparrow \infty} p(t) = \infty$ .

**Lemma 2.** *There exists a constant  $c$  such that  $-\phi_i(0+) > c$  for all sufficiently large  $t$ .*

$$\begin{aligned} \text{Proof. } \exp \{p(t)\Phi_i(\beta)\} &= \int_0^b \exp \{p(t)(\beta\xi - \phi_i(\xi))\} d\xi \\ &\quad + \int_b^\infty \exp \{p(t)(\beta\xi - \phi_i(\xi))\} d\xi \\ &\equiv J_1(b, \beta, t) + J_2(b, \beta, t). \end{aligned}$$

By making use of (1.2), for  $\beta < \gamma$

$$\begin{aligned} J_2(b, \beta, t) &\leq \int_b^\infty \exp \{p(t)((\beta-\gamma)\xi + K(\gamma; t, \varepsilon))\} d\xi \\ &= \frac{\exp \{p(t)((\beta-\gamma)b + K(\gamma; t, \varepsilon))\}}{(\gamma-\beta)p(t)}, \end{aligned}$$

then

$$J_1(b, \beta, t) \geq \exp \{p(t)\Phi_i(\beta)\} \left[ 1 - \frac{\exp \{p(t)((\beta-\gamma)b + K(\gamma; t, \varepsilon) - \Phi_i(\beta))\}}{(\gamma-\beta)p(t)} \right]$$

By taking sufficiently large  $t$ , it holds that

$$(\beta-\gamma)b + K(\gamma; t, \varepsilon) - \Phi_i(\beta) < 0$$

for all sufficiently large  $t$ . So we have

$$(1.4) \quad J_1(b, \beta, t) \geq \frac{1}{2} \exp \{p(t)\Phi_i(\beta)\}$$

for all sufficiently large  $t$ . On the other hand

$$\begin{aligned} J_1(b, \beta, t) &\leq \int_0^b \exp \{p(t)(\beta\xi - \phi_i(0+))\} d\xi \\ &= \frac{\exp \{\beta b - \phi_i(0+)\}}{\beta p(t)} [1 - \exp \{-p(t)\beta b\}]. \end{aligned}$$

Combining this with (1.4), we have

$$p(t)\Phi_i(\beta) \leq \log \frac{2(1 - \exp\{-p(t)\beta b\})}{p(t)} + p(t)\{\beta b - \phi_i(0+)\}.$$

Then

$$-\phi_i(0+) \geq \Phi_i(\beta) - \beta b + o(1) \quad \text{as } t \rightarrow \infty.$$

Therefore we get our Lemma.

**Lemma 3.**  $\sup_{\xi > 0} \{\beta\xi - \phi_i(\xi)\}$  has a limit as  $t \rightarrow \infty$  and  $\limsup_{t \uparrow \infty} \sup_{\xi > 0} \{\beta\xi - \phi_i(\xi)\} = \Phi(\beta)$ .

Proof. We have

$$\begin{aligned} J_1(a, \beta, t) &\leq a \exp\{P(t) \sup_{\xi > 0} (\beta\xi - \phi_i(\xi))\} \\ &\exp\{-p(t) \sup_{\xi > 0} (\beta\xi - \phi_i(\xi))\} J_2(a, \beta, t) \\ &\leq \exp\{-cp(t)\} \frac{\exp\{p(t)((\beta - \gamma)a + K(\gamma; t, \beta))\}}{(\gamma - \beta)p(t)} \end{aligned}$$

for  $\beta < \gamma$ .

Taking sufficiently large  $a$ , it holds that

$$-c + (\beta - \gamma)a + K(\gamma; t, \varepsilon) < 0$$

for all sufficiently large  $t$ . Therefore

$$\exp\{p(t)\Phi_i(\beta)\} \leq \exp\{p(t) \sup_{\xi > 0} (\beta\xi - \phi_i(\xi))\} \left[ a + \frac{1}{(\gamma - \beta)p(t)} \right]$$

for all sufficiently large  $t$ , that is,

$$\Phi_i(\beta) + o(1) \leq \sup_{\xi > 0} \{\beta\xi - \phi_i(\xi)\} \quad \text{as } t \rightarrow \infty.$$

On the other hand we have, by (1.3),

$$\Phi_i(\beta) + \beta\varepsilon(t) - \frac{\log \varepsilon(t)}{p(t)} \geq \sup_{\varepsilon(t) \leq \xi} \{\beta\xi - \phi_i(\xi)\}.$$

From the assumption with respect to  $\varepsilon(t)$ , it follows that

$$\left| \sup_{\xi > 0} \{\beta\xi - \phi_i(\xi)\} - \sup_{\varepsilon(t) \leq \xi} \{\beta\xi - \phi_i(\xi)\} \right| = o(1) \quad \text{as } t \rightarrow \infty.$$

Hence

$$\Phi_i(\beta) + o(1) \geq \sup_{\xi > 0} \{\beta\xi - \phi_i(\xi)\} \quad \text{as } t \rightarrow \infty.$$

**Lemma 4.** Let  $\xi$  be an  $R$ -point ([4]) and  $\beta_\xi$  be a subordinate to  $\xi$  ([4]), then there exists  $S_i(\beta_\xi)$  such that  $\lim_{t \uparrow \infty} \phi_i(S_i(\beta_\xi)) = \beta_\xi \xi - \Phi(\beta_\xi)$  and  $\lim_{t \uparrow \infty} S_i(\beta_\xi) = \xi$ .

Proof. At first we note that there exists  $A(\beta) < \infty$  such that

$$\sup_{0 < \xi} \{\beta\xi - \phi_i(\xi)\} = \sup_{0 < \xi < A(\beta)} \{\beta\xi - \phi_i(\xi)\}$$

for all sufficiently large  $t$ . We can verify this by making use of (1.2) and Lemma 2 in a similar way as the proof of Lemma 5 in [4]. Therefore we can define  $\bar{S}_i(\beta_\xi)$  such that

$$|\sup_{0 < \zeta} \{\beta_\xi \zeta - \phi_i(\zeta)\} - \{\beta_\xi \bar{S}_i(\beta_\xi) - \phi_i(\bar{S}_i(\beta_\xi))\}| < \frac{1}{t}$$

and

$$\bar{S}_i(\beta_\xi) < A(\beta_\xi)$$

for all sufficiently large  $t$ . Since  $\{\bar{S}_n(\beta_\xi)\}_{n:\text{integer}}$  is bounded, it has a subsequence which converges. We write it  $S_n(\beta_\xi)$  and define  $S_i(\beta_\xi) \equiv S_{[t]}(\beta_\xi)$ , then from Lemma 3 it follows that

$$\lim_{t \uparrow \infty} \{\beta_\xi S_i(\beta_\xi) - \phi_i(S_i(\beta_\xi))\} = \Phi(\beta_\xi).$$

Putting  $\bar{\xi} = \lim_{t \uparrow \infty} S_i(\beta_\xi)$ , we have

$$\lim_{t \uparrow \infty} \phi_i(S_i(\beta_\xi)) = \beta_\xi \bar{\xi} - \Phi(\beta_\xi).$$

In the case that  $\bar{\xi} \neq 0$ , by (1.3) for any  $\gamma$

$$\phi_i(S_i(\beta_\xi)) \geq \gamma S_i(\beta_\xi) - K(\gamma; t, \varepsilon)$$

for all sufficiently large  $t$ . Therefore for any  $\gamma$   $\beta_\xi \bar{\xi} - \Phi(\beta_\xi) \geq \gamma \bar{\xi} - \Phi(\gamma)$ , which means

$$\beta_\xi \bar{\xi} - \Phi(\beta_\xi) = \sup_{0 < \gamma} \{\gamma \bar{\xi} - \Phi(\gamma)\}.$$

Since  $\beta_\xi$  is subordinate to  $\xi$  we get  $\bar{\xi} = \xi$ . In the case that  $\bar{\xi} = 0$  we can define  $S_i(\beta_\xi)$  with additional condition  $S_i(\beta_\xi) \geq \varepsilon(t)$  because  $|\sup_{0 < \zeta} \{\beta_\xi \zeta - \phi_i(\zeta)\} - \sup_{\varepsilon(t) \leq \zeta} \{\beta_\xi \zeta - \phi_i(\zeta)\}| = o(1)$ . Then we get  $\bar{\xi} = \xi$  in the same way as the case  $\bar{\xi} \neq 0$ .

Now we can get the first assertion of our theorem by making use of our lemmas in the same way as in [4].

We turn to the proof of second assertion of our theorem.

Proof of (ii). From (1.3) and the assumptions it follows that

$$\beta\xi - \phi(\xi) \leq (\beta - \gamma)\xi + c(\gamma) + \gamma\varepsilon.$$

Since the right hand side tends to  $-\infty$  as  $\xi \rightarrow \infty$  for fixed  $\beta < \gamma$ , there exists  $A'(\beta)$  such that

$$\sup_{0 < \xi} \{\beta\xi - \phi(\xi)\} = \sup_{0 < \xi \leq A'(\beta)} \{\beta\xi - \phi(\xi)\}.$$

On the other hand,

$$\begin{aligned} \exp \{p(t)\Phi_l(\beta)\} &= J_1(l, \beta, t) + J_2(l, \beta, t) \\ &< l \exp \left\{ p(t) \sup_{0 < \xi < l} (\beta\xi - \phi_l(\xi)) \right\} + \frac{\exp \left\{ p(t) \left( (\beta - \gamma)l + c(\gamma) + \gamma\varepsilon + \frac{\log \varepsilon}{p(t)} \right) \right\}}{p(t)(\gamma - \beta)} \end{aligned}$$

for  $l > \varepsilon$ .

Taking sufficiently large  $l$

$$(\beta - \gamma) + c(\gamma) + \gamma\varepsilon + \frac{\log \varepsilon}{p(t)} < 0$$

for all sufficiently large  $t$ . Put  $\delta(t, l) = \sup_{0 < \xi \leq l} |\phi_l(\xi) - \phi(\xi)|$ , then we have

$$\Phi_l(\beta) \leq \sup_{0 < \xi < l} \{\beta\xi - \phi(\xi)\} + \delta(t, l) + o(1), \text{ as } t \rightarrow \infty.$$

Taking  $l > A'(\beta)$  in advance

$$\Phi_l(\beta) \leq \sup_{0 < \xi} \{\beta\xi - \phi(\xi)\} + o(1), \text{ as } t \rightarrow \infty.$$

The converse inequality also follows from (1.3):

$$\begin{aligned} \Phi_l(\beta) &\geq \sup_{\varepsilon < \xi < A'(\beta)} \{\beta\xi - \phi_l(\xi)\} - \beta\varepsilon + \frac{\log \varepsilon}{p(t)} \\ &> \sup_{\varepsilon < \xi} \{\beta\xi - \phi(\xi)\} - \beta\varepsilon + \frac{\log \varepsilon}{p(t)} - \delta(t, A'(\beta)). \end{aligned}$$

Now take  $\frac{1}{p(t)}$  in the place of  $\varepsilon$ , then

$$\Phi_l(\beta) \geq \sup_{\frac{1}{p(t)} < \xi} \{\beta\xi - \phi(\xi)\} + o(1), \text{ as } t \rightarrow \infty.$$

We arrive at the second statement of our theorem.

2. We now come to the proof of those results on the Laplace transform.

**Corollary 1.** Let  $\rho(\lambda)$  be a non-decreasing function on  $(-\infty, 0]$  with  $\rho(-\infty) = 0$  and

$$(2.1) \quad \kappa(t) \equiv \int_{-\infty}^0 e^{-t\lambda} \rho(\lambda) d\lambda,$$

then following two conditions (2.2) and (2.3) are equivalent:

$$(2.2) \quad \lim_{\lambda \rightarrow -\infty} \frac{\log \rho(\lambda)}{|\lambda|^\alpha} = -A, \alpha > 1, A > 0,$$

$$(2.3) \quad \lim_{t \rightarrow \infty} \frac{\log \kappa(t)}{t^\gamma} = B, \gamma > 1, B > 0,$$

where  $\alpha, \gamma, A, B$  are related by

$$\gamma = \frac{\alpha}{\alpha-1} \left( \alpha = \frac{\gamma}{\gamma-1} \right), \quad B = (\alpha-1) \alpha^{\alpha/1-\alpha} A^{1/\alpha}$$

$$(A = (\gamma-1) \gamma^{\gamma/1-\gamma} B^{1/1-\gamma}).$$

**Corollary 2.** Let  $\rho(\lambda)$  be a non-decreasing function on  $(-\infty, \infty)$  with  $\rho(-\infty)=0$  and  $k(t)$  be its Laplace transform:

$$(2.4) \quad k(t) = \int_{-\infty}^{\infty} e^{-t\lambda} d\rho(\lambda) < \infty,$$

then (2.2) and (2.3) in which we take  $k(t)$  in the place of  $\kappa(t)$ , are equivalent.

Corollary 2 is immediate from Corollary 1. If  $k(t)$  is finite, then  $\lim_{c \uparrow \infty} e^{ct} \rho(-c) = 0$  and so

$$\int_{-\infty}^{\infty} e^{-t\lambda} d\rho(\lambda) = \rho(0) + t \int_{-\infty}^0 e^{-t\lambda} \rho(\lambda) d\lambda + \int_0^{\infty} e^{-t\lambda} d\rho(\lambda).$$

Since the last term decreases as  $t \rightarrow \infty$  we have

$$\frac{\log k(t)}{t^\gamma} \sim \frac{\log \left[ t \int_{-\infty}^0 e^{-t\lambda} \rho(\lambda) d\lambda \right]}{t^\gamma} \sim \frac{\log \kappa(t)}{t^\gamma}$$

$t \rightarrow \infty, \gamma > 0$ . Hence we conclude Corollary 2 from Corollary 1.

Now we will give the proof of Corollary 1.

Proof. (2.2)  $\Rightarrow$  (2.3) Put  $\Phi_t(\beta) = \frac{1}{t^\alpha} \log \kappa(\beta t^{\alpha-1})$  and  $-\phi_t(\xi) = \frac{1}{t^\alpha} \{ \log \rho(-\xi t) + \log t \}$ , then  $\Phi_t(\beta)$  and  $\phi_t(\xi)$  are related by

$$\exp \{ t^\alpha \Phi_t(\beta) \} = \int_0^\infty \exp \{ t^\alpha (\beta \xi - \phi_t(\xi)) \} d\xi.$$

It follows from (2.3) that  $\phi_t(\xi)$  converges to  $\xi^\alpha A$  uniformly on each finite interval. Since  $\rho(\lambda)$  is non-decreasing,  $\phi_t(\xi)$  is non-decreasing in  $\xi$  for each  $t$ . Therefore to apply the second part of our theorem, we have only to verify the existence of  $c(\beta)$  such that  $\Phi_t(\beta) < c(\beta)$  for each  $\beta$  and  $t$ . For any  $\varepsilon > 0$  there exists  $c > 0$  such that

$$\rho(-\lambda) < c \{ \exp \{ -(A-\varepsilon)\lambda^\alpha \} \}, \quad \lambda > 0.$$

Therefore

$$\exp \{ t^\alpha \Phi_t(\beta) \} = \int_0^\infty \exp \{ \beta t^{\alpha-1} \lambda \} \rho(-\lambda) d\lambda$$

$$< \int_0^\infty c \exp \{ \beta t^{\alpha-1} \lambda - (A-\varepsilon)\lambda^\alpha \} d\lambda$$

$$< c \int_0^{\eta t} \exp \{ \beta t^{\alpha-1} \lambda \} d\lambda + c \int_{\eta t}^{\infty} \exp \{ -[(A-\varepsilon)\lambda^{\alpha-1} - \beta t^{\alpha-1}] \lambda \} d\lambda .$$

Taking  $\eta^{\alpha-1} = \frac{\beta+1}{A-\varepsilon}$

$$\begin{aligned} \exp \{ t^\alpha \Phi_t(\beta) \} &\leq \frac{c \exp \{ \eta \beta t^\alpha \}}{\beta t^{\alpha-1}} + c \int_0^{\infty} \exp \{ -[(A-\varepsilon)\eta^{\alpha-1} - \beta] t^{\alpha-1} d\lambda \} d\lambda \\ &\leq \frac{c \exp \{ \eta \beta t^\alpha \}}{t^{\alpha-1}} + \frac{c}{t^{\alpha-1}} , \end{aligned}$$

which means

$$\Phi_t(b) < \eta(\beta) + o(1) \text{ as } t \rightarrow \infty .$$

Since  $\eta$  depends only on  $\beta$  (not on  $t$ ) the existence of  $c(\beta)$  desired is assured. Now it follows from our theorem that

$$\lim_{t \uparrow \infty} \Phi_t(\beta) = \sup_{\xi > 0} \{ \beta \xi - \xi^\alpha A \} = (\alpha-1) \alpha^{\alpha/1-\alpha} A^{1/1-\alpha} B^{\alpha/\alpha-1} .$$

Putting  $\beta=1$  and  $t^{\alpha-1}=x$ , we arrive at (2.3).

(2.3)  $\Rightarrow$  (2.2) Put  $\Phi_t'(\beta) = \frac{1}{t^\gamma} \log \kappa(\beta t)$  and  $-\phi_t'(\xi) = \frac{1}{t^\gamma} \{ \log(-\xi t^{\gamma-1}) + \log t^{\gamma-1} \}$ , then we can see in a similar way as above that they have the relation (1.1) with  $p(t)=t^\gamma$ :

$$\exp \{ t^\gamma \Phi_t'(\beta) \} = \int_0^{\infty} \exp \{ t^\gamma (\beta \xi - \phi_t'(\xi)) \} d\xi .$$

It follows from (2.2) that  $\lim_{t \uparrow \infty} \Phi_t'(\beta) = \beta^\gamma B$ .  $\phi_t'(\xi)$  is clearly non-decreasing in  $\xi$  for each  $t$ . We can easily see that  $\varepsilon(t) = \frac{1}{t^{\gamma-1}}$  satisfies all conditions of the first part of our theorem because  $\gamma > 1$ . Therefore the theorem applies and

$$\lim_{t \uparrow \infty} \phi_t'(\xi) = \sup_{\beta > 0} \{ \beta \xi - \beta^\gamma B \} = (\gamma-1) \gamma^{\gamma/1-\gamma} B^{1/1-\gamma} \xi^{\gamma/\gamma-1}$$

Putting  $\xi=1$  and  $-t^{\gamma-1}=\lambda$ , we get (2.2).

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