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A PROOF OF H. KUMANO-GO–TANIGUCHI THEOREM FOR MULTI-PRODUCTS OF FOURIER INTEGRAL OPERATORS

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0. Introduction

We denote the Fourier integral operators on $\boldsymbol{R}^n$ with phase function ϕ_j and symbol $p_j \in \boldsymbol{S}^{m_j}_\rho$ by $I(\phi_j, p_j)$, $j = 1, 2, \dots, L+1$. If all the canonical maps ω_j associated with phase functions ϕ_j are sufficiently close to the identity, the composite canonical map $\omega_{L+1}\omega_L \cdots \omega_1$ is also near the identity. Moreover, we have

$$I(\phi, q) = I(\phi_{L+1}, p_{L+1}) I(\phi_L, p_L) \cdots I(\phi_1, p_1), \tag{0.1}$$

for some phase function ϕ and some symbol $q \in \boldsymbol{S}^{\sum_{j=1}^{L+1} m_j}_\rho$ (cf. L. Hörmander [6]). Here the correspondence of the symbols $(p_{L+1}, p_L, \dots, p_1) \to q$ is multi-linear. In [9], [10] and [12], H. Kumano-go–Taniguchi theorem gives the following estimate for the symbol q; that is, for any non-negative integers l, l', there exist a positive constant $C_{l,l'}$ and positive integers l_1, l_1' such that

$$|q|^{(\sum_{j=1}^{L+1} m_j)}_{l,l'} \le (C_{l,l'})^L \prod_{j=1}^{L+1} |p_j|^{(m_j)}_{l_1, l_1'}, \tag{0.2}$$

where $|\cdot|^{(m)}_{l,l'}$ denotes the semi-norm of $\boldsymbol{S}^m_\rho$.

This estimate is useful in the calculus of Fourier integral operators. In [9], [10] and [12], this estimate was applied to construct a fundamental solution for hyperbolic systems. Slight modification of this estimate was applied to construct a fundamental solution for Schrödinger equations (cf. D. Fujiwara [1]–[4], H. Kitada and H. Kumano-go [8], N. Kumano-go [11]). However, in their proofs, they used the inverse of the Fourier integral operators whose symbols are equal to 1. Therefore, the canonical maps associated with phase functions ϕ_j must be very close to the identity. Recently, in [5], D. Fujiwara, N. Kumano-go and K. Taniguchi have given a more direct proof and relaxed the condition for the canonical maps associated with phase functions ϕ_j in the case for Schrödinger equations. However they are not successful in the original case for hyperbolic systems. The aim of this paper is to give a proof similar to theirs and to relax the condition for the canonical maps associated with phase functions ϕ_j in the original case for hyperbolic systems.

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1. Statement of results

In order to state our main theorems, we recall some definitions for Fourier integral operator in H. Kumano-go and K. Taniguchi [9], [10] and [12].

DEFINITION 1.1. Let $m \in \boldsymbol{R}$ and $1/2 \leq \rho \leq 1$. We say that a C^∞-function $p(x,\xi)$ on $\boldsymbol{R}^n_x \times \boldsymbol{R}^n_\xi$ belongs to the class of symbols $\boldsymbol{S}^m_\rho$, if, for any α, β, there exists a positive constant $C_{\alpha,\beta}$ such that

$$|\partial_x^\beta \partial_\xi^\alpha p(x,\xi)| \leq C_{\alpha,\beta} \langle\xi\rangle^{m+(1-\rho)|\beta|-\rho|\alpha|}, \tag{1.1}$$

where $\langle\xi\rangle = \sqrt{1+|\xi|^2}$.

REMARK. For $p \in \boldsymbol{S}^m_\rho$, we define semi-norms $|p|^{(m)}_{l,l'}$, $l, l' = 0,1,2,\ldots$ by

$$|p|^{(m)}_{l,l'} = \max_{|\alpha|\leq l, |\beta|\leq l'} \sup_{(x,\xi)} \frac{|\partial_x^\beta \partial_\xi^\alpha p(x,\xi)|}{\langle\xi\rangle^{m+(1-\rho)|\beta|-\rho|\alpha|}}. \tag{1.2}$$

Then $\boldsymbol{S}^m_\rho$ is a Fréchet space with these semi-norms.

DEFINITION 1.2. Let $\{\kappa_l\}_{l=0}^\infty$ be an increasing sequence of positive constants and $t > 0$. We say that a real-valued C^∞-function $\phi(x,\xi)$ on $\boldsymbol{R}^n_x \times \boldsymbol{R}^n_\xi$ belongs to the class of phase functions $\boldsymbol{P}_\rho(t, \{\kappa_l\}_{l=0}^\infty)$, if $\phi(x,\xi)$ satisfies the following:

$$|\partial_x^\beta \partial_\xi^\alpha \phi(x,\xi)| \leq \kappa_{|\alpha+\beta|} t \langle\xi\rangle^{1-|\alpha|} \quad (|\alpha+\beta| \leq 1), \tag{1.3}$$

$$|\partial_x^\beta \partial_\xi^\alpha \phi(x,\xi)| \leq \kappa_{|\alpha+\beta|} t \langle\xi\rangle^{2\rho-1+(1-\rho)|\beta|-\rho|\alpha|} \quad (|\alpha+\beta| \geq 2). \tag{1.4}$$

REMARK. Usually, "phase function" refers to $(x-y)\xi + \phi(x,\xi)$ in (1.5). However, in the present paper, our phase functions will always be of the form $(x-y)\xi + \phi(x,\xi)$. Thus, in the present paper, we call $\phi(x,\xi)$ "phase function".

DEFINITION 1.3. Let $\phi \in \boldsymbol{P}_\rho(t, \{\kappa_l\}_{l=0}^\infty)$ and $p \in \boldsymbol{S}^m_\rho$. We define the Fourier integral operator $I(\phi,p)$ with phase function ϕ and symbol p by

$$I(\phi,p)u(x) = \int_{\boldsymbol{R}^{2n}} e^{i\{(x-y)\xi+\phi(x,\xi)\}} p(x,\xi)u(y)dy\,đ\xi \quad (đ\xi = (2\pi)^{-n}d\xi), \tag{1.5}$$

for $u \in \mathcal{S}$, where $\mathcal{S}$ denotes the Schwartz class of rapidly decreasing C^∞-functions on $\boldsymbol{R}^n$, and i denotes $\sqrt{-1}$.

The integrals of the right hand side do not necessarily converge absolutely. We understand integrals of this type as oscillatory integrals (cf. H. Kumano-go [9]).

Let $I(\phi_j, p_j)$, $j = 1, 2, \dots, L+1$ be Fourier integral operators. Then, the composite of these Fourier integral operators is given by

$$\begin{aligned} I(\phi_{L+1}, p_{L+1}) I(\phi_L, p_L) \cdots I(\phi_1, p_1) u(x_{L+1}) \\ = \int_{\boldsymbol{R}^{2n}} e^{i(x_{L+1}-x_0)\xi_0} K(x_{L+1}, \xi_0) u(x_0) dx_0 đ\xi_0, \end{aligned} \tag{1.6}$$

where

$$K(x_{L+1}, \xi_0) = \int_{\boldsymbol{R}^{2nL}} e^{i\Phi} \prod_{j=1}^{L+1} p_j(x_j, \xi_{j-1}) \prod_{j=1}^{L} dx_j đ\xi_j, \tag{1.7}$$

and

$$\Phi = \sum_{j=1}^{L} (x_{j+1} - x_j)(\xi_j - \xi_0) + \sum_{j=1}^{L+1} \phi_j(x_j, \xi_{j-1}). \tag{1.8}$$

In order to discuss the oscillatory integrals in (1.7) more generally, we will consider oscillatory integrals in the following form:

$$\begin{aligned} \mathbb{I}(\Phi, p)(x_{L+1}, \xi_0) \\ = \int_{\boldsymbol{R}^{2nL}} e^{i\Phi} p(x_{L+1}, \xi_L, x_L, \dots, \xi_1, x_1, \xi_0) \prod_{j=1}^{L} dx_j đ\xi_j, \end{aligned} \tag{1.9}$$

which is defined by the multiple symbol $p = p(x_{L+1}, \xi_L, x_L, \dots, \xi_1, x_1, \xi_0)$ in $\boldsymbol{S}_\rho^{\widetilde{m}_{L+1}}$. Here, $\boldsymbol{S}_\rho^{\widetilde{m}_{L+1}}$ is as follows.

DEFINITION 1.4. Let $\widetilde{m}_{L+1} = (m_{L+1}, m_L, \dots, m_1) \in \boldsymbol{R}^{L+1}$ and $1/2 \le \rho \le 1$. We say that a C^∞-function $p = p(x_{L+1}, \xi_L, x_L, \dots, \xi_1, x_1, \xi_0)$ on $\boldsymbol{R}^{2n(L+1)}$ belongs to the class of multiple symbols $\boldsymbol{S}_\rho^{\widetilde{m}_{L+1}}$, if, for any $\widetilde{\alpha} = (\alpha_L, \alpha_{L-1}, \dots, \alpha_0)$ and $\widetilde{\beta} = (\beta_{L+1}, \beta_L, \dots, \beta_1)$, there exists a positive constant $C_{\widetilde{\alpha}, \widetilde{\beta}}$ such that

$$\begin{aligned} \left| \left(\prod_{j=1}^{L+1} \partial_{x_j}^{\beta_j} \partial_{\xi_{j-1}}^{\alpha_{j-1}} \right) p(x_{L+1}, \xi_L, x_L, \dots, \xi_1, x_1, \xi_0) \right| \\ \le C_{\widetilde{\alpha}, \widetilde{\beta}} \prod_{j=1}^{L+1} \langle \xi_{j-1} \rangle^{m_j + (1-\rho)|\beta_j| - \rho|\alpha_{j-1}|}. \end{aligned} \tag{1.10}$$

REMARK.

(1) For $p \in \boldsymbol{S}_\rho^{\widetilde{m}_{L+1}}$, we define semi-norms $|p|_{l,l'}^{(\widetilde{m}_{L+1})}$, $l, l' = 0, 1, 2, \ldots$ by

$$(1.11)\quad |p|_{l,l'}^{(\widetilde{m}_{L+1})} = \max_{\substack{|\alpha_{j-1}| \le l, \\ |\beta_j| \le l', \\ j = 1, 2, \cdots, L+1}} \sup_{\boldsymbol{R}^{2n(L+1)}} \frac{\left|\left(\prod_{j=1}^{L+1} \partial_{x_j}^{\beta_j} \partial_{\xi_{j-1}}^{\alpha_{j-1}}\right) p\right|}{\prod_{j=1}^{L+1} \langle \xi_{j-1} \rangle^{m_j + (1-\rho)|\beta_j| - \rho|\alpha_{j-1}|}}.$$

Then $\boldsymbol{S}_\rho^{\widetilde{m}_{L+1}}$ is a Fréchet space with these semi-norms.

(2) For $p_j \in \boldsymbol{S}_\rho^{m_j}$, $j = 1, 2, \ldots, L+1$, if we set

$$(1.12)\quad p = \prod_{j=1}^{L+1} p_j(x_j, \xi_{j-1}),$$

then we have $p \in \boldsymbol{S}_\rho^{\widetilde{m}_{L+1}}$. Furthermore we have

$$(1.13)\quad |p|_{l,l'}^{(\widetilde{m}_{L+1})} \le \prod_{j=1}^{L+1} |p_j|_{l,l'}^{(m_j)}.$$

Now, our first main theorem is the following:

Theorem 1.5. *Let $\{\kappa_l\}_{l=0}^\infty$ be an increasing sequence of positive constants and $M \ge 0$. Set $T = \min\{1/(7\sqrt{n}\kappa_1), 1/(4n\kappa_2)\}$. Then there exists a positive constant C such that*

$$(1.14)\quad |\mathbb{I}(\Phi, p)(x_{L+1}, \xi_0)| \le C^L |p|_{l_0, l_0'}^{(\widetilde{m}_{L+1})} \langle \xi_0 \rangle^{\sum_{j=1}^{L+1} m_j},$$

for $\sum_{j=1}^{L+1} t_j \le T$, $\sum_{j=1}^{L+1} |m_j| \le M$, $p \in \boldsymbol{S}_\rho^{\widetilde{m}_{L+1}}$ and $\phi_j \in \boldsymbol{P}_\rho(t_j, \{\kappa_l\}_{l=0}^\infty)$, where $l_0 = n+1$, $l_0' = [2M] + 2n + 1$, and the positive constant C depends only on M, $\{\kappa_l\}_{l=0}^\infty$ and n, not L.

In order to state our second main theorem, we state the following proposition.

Proposition 1.6. *Let $\{\kappa_l\}_{l=0}^\infty$ be an increasing sequence of positive constants.*

(1) *Assume that $\sum_{j=1}^{L+1} t_j \le 1/(4n\kappa_2)$ and $\phi_j \in \boldsymbol{P}_\rho(t_j, \{\kappa_l\}_{l=0}^\infty)$, $j = 1, 2, \ldots, L+1$. Then, for $(x, \xi) \in \boldsymbol{R}^{2n}$, the equations*

$$(1.15)\quad \begin{cases} 0 = -(x_j - x_{j+1}) + \partial_{\xi_j} \phi_{j+1}(x_{j+1}, \xi_j), \\ 0 = -(\xi_j - \xi_{j-1}) + \partial_{x_j} \phi_j(x_j, \xi_{j-1}), \\ j = 1, 2, \ldots, L, \quad x_{L+1} = x, \quad \xi_0 = \xi, \end{cases}$$

have a unique solution $\{x_j, \xi_j\}_{j=1}^{L} = \{x_j^*, \xi_j^*\}_{j=1}^{L}(x, \xi)$.

(2) *Let* Φ^* *be the function defined by*

$$\Phi^*(x, \xi) = \sum_{j=1}^{L}(x_{j+1}^* - x_j^*)(\xi_j^* - \xi_0^*) + \sum_{j=1}^{L+1} \phi_j(x_j^*, \xi_{j-1}^*), \tag{1.16}$$

with $x_{L+1}^* = x$ *and* $\xi_0^* = \xi$.

Then there exists an increasing sequence of positive constants $\{\kappa_l'\}_{l=0}^{\infty}$ *such that*

$$\Phi^* \in \boldsymbol{P}_\rho\left(\sum_{j=1}^{L+1} t_j, \{\kappa_l'\}_{l=0}^{\infty}\right), \tag{1.17}$$

for $\sum_{j=1}^{L+1} t_j \leq 1/(4n\kappa_2)$ *and* $\phi_j \in \boldsymbol{P}_\rho(t_j, \{\kappa_l\}_{l=0}^{\infty})$, *where the increasing sequence of positive constants* $\{\kappa_l'\}_{l=0}^{\infty}$ *depends only on* $\{\kappa_l\}_{l=0}^{\infty}$ *and* n, *not* L.

Our second main theorem is the following:

Theorem 1.7. *Let* $\{\kappa_l\}_{l=0}^{\infty}$ *be an increasing sequence of positive constants and* $M \geq 0$. *Set* $T = \min\{1/(7\sqrt{n}\kappa_1), 1/(4n\kappa_2)\}$.

(1) *For* $\sum_{j=1}^{L+1} t_j \leq T$, $p \in \boldsymbol{S}_\rho^{\widetilde{m}_{L+1}}$ *and* $\phi_j \in \boldsymbol{P}_\rho(t_j, \{\kappa_l\}_{l=0}^{\infty})$, *set*

$$q(x_{L+1}, \xi_0) = e^{-i\Phi^*(x_{L+1}, \xi_0)} \mathbb{I}(\Phi, p)(x_{L+1}, \xi_0). \tag{1.18}$$

Then we have $q \in \boldsymbol{S}_\rho^{\sum_{j=1}^{L+1} m_j}$.

(1) *For any non-negative integers* l, l', *there exists a positive constant* $C_{l,l'}$ *such that*

$$|q|_{l,l'}^{(\sum_{j=1}^{L+1} m_j)} \leq (C_{l,l'})^L |p|_{l_1, l_1'}^{(\widetilde{m}_{L+1})}, \tag{1.19}$$

for $\sum_{j=1}^{L+1} t_j \leq T$, $\sum_{j=1}^{L+1} |m_j| \leq M$, $p \in \boldsymbol{S}_\rho^{\widetilde{m}_{L+1}}$ *and* $\phi_j \in \boldsymbol{P}_\rho(t_j, \{\kappa_l\}_{l=0}^{\infty})$, *where* $l_1 = n + 1 + 2l + 2l'$, $l_1' = [2M] + 2n + 1 + 2l + 3l'$, *and the positive constant* $C_{l,l'}$ *depends only on* M, $\{\kappa_l\}_{l=0}^{\infty}$ *and* n, *not* L.

From the theorem above, we can relax the condition for the canonical maps associated with phase functions ϕ_j of H. Kumano-go–Taniguchi theorem in the following form.

Theorem 1.8. *Let* $\{\kappa_l\}_{l=0}^{\infty}$ *be an increasing sequence of positive constants and* $M \geq 0$. *Set* $T = \min\{1/(7\sqrt{n}\kappa_1), 1/(4n\kappa_2)\}$.

(1) *For* $\sum_{j=1}^{L+1} t_j \leq T$, $p_j \in \boldsymbol{S}_\rho^{m_j}$ *and* $\phi_j \in \boldsymbol{P}_\rho(t_j, \{\kappa_l\}_{l=0}^{\infty})$, *there exists a symbol* $q \in \boldsymbol{S}_\rho^{\sum_{j=1}^{L+1} m_j}$ *such that*

$$I(\Phi^*, q) = I(\phi_{L+1}, p_{L+1}) I(\phi_L, p_L) \ldots I(\phi_1, p_1). \tag{1.20}$$

(1) *For any non-negative integers l, l', there exists a positive constant $C_{l,l'}$ such that*

$$|q|_{l,l'}^{(\sum_{j=1}^{L+1} m_j)} \leq (C_{l,l'})^L \prod_{j=1}^{L+1} |p_j|_{l_1,l_1'}^{(m_j)}, \tag{1.21}$$

for $\sum_{j=1}^{L+1} t_j \leq T$, $\sum_{j=1}^{L+1} |m_j| \leq M$, $p_j \in \boldsymbol{S}_\rho^{m_j}$ and $\phi_j \in \boldsymbol{P}_\rho(t_j, \{\kappa_l\}_{l=0}^\infty)$, where $l_1 = n + 1 + 2l + 2l'$, $l_1' = [2M] + 2n + 1 + 2l + 3l'$, and the positive constant $C_{l,l'}$ depends only on M, $\{\kappa_l\}_{l=0}^\infty$ and n, not L.

REMARK. The condition $\sum_{j=1}^{L+1} t_j \leq T$ implies how close to the identity the canonical maps associated with phase functions ϕ_j need to be. In our proof, the right hand side T of this inequality depends only on κ_1, κ_2 and n. However, in the original proof, T depends on κ_1, κ_2, ..., κ_k and n, with some large integer $k > 2$ depending on n. Moreover, T must be chosen very small. Therefore, the canonical maps with phase functions ϕ_j must be very close to the identity.

2. Some Lemmas

In this section, we state two important lemmas needed later. First lemma is found in H. Kumano-go and K. Taniguchi [9], [10].

Lemma 2.1. *Let $A = (a_{jk})$ be an $L \times L$ real matrix. If there exists a positive constant $0 \leq c < 1$ such that*

$$\sum_{k=1}^{L} |a_{jk}| \leq c, \tag{2.1}$$

for any $j = 1, 2, \ldots, L$, then we have

$$(1 - c)^L \leq \det(I_L - A) \leq (1 + c)^L, \tag{2.2}$$

where I_L denotes the $L \times L$ unit matrix.

Proof. By induction. See Proposition 5.3 in Chapter 10 §5 of H. Kumano-go [9]. □

Second lemma is slight modification of Proposition 3.3 in D. Fujiwara, N. Kumano-go and K. Taniguchi [5].

Let N and L be positive integers and $x \in \boldsymbol{R}^N$. For $j = 1, 2, \ldots, L + 1$, let P_j be the first-order partial differential operator with smooth coefficients given by

$$P_j = \sum_{\beta_j \le \gamma_j, |\beta_j| \le 1} a_{j,\beta_j}(x) \partial_x^{\beta_j}, \tag{2.3}$$

where $\gamma_j \in \{0,1\}^N \subset \boldsymbol{N}_0^N$ and $a_{j,\beta_j}(x) \in C^\infty(\boldsymbol{R}^N)$. Furthermore, we assume the following properties:

1° There exists a positive integer Γ independent of N and of L such that

$$|\gamma_j| \le \Gamma, \tag{2.4}$$

for $j = 1, 2, \dots, L+1$.

2° There exists a positive integer K independent of N and of L such that

$$\sharp \Big\{ j = 1, 2, \dots, k; \quad \partial_x^{\beta_{k+1}} a_{j,\beta_j}(x) \not\equiv 0 \Big\} \le K, \tag{2.5}$$

for $k = 1, 2, \cdots, L$, $\beta_j \le \gamma_j$, $|\beta_j| \le 1$, $j = 1, 2, \dots, k$ and $0 \ne \beta_{k+1} \le \gamma_{k+1}$. Then we get the following lemma:

Lemma 2.2.

(1) *The product of operators* $P_{L+1} P_L \cdots P_1$ *is of the form*

$$\begin{aligned} &P_{L+1} P_L \cdots P_1 \\ &= \sum\nolimits'_{\{\beta_j\}_{j=1}^{L+1}} \sum\nolimits''_{\{\alpha_j\}_{j=0}^{L+1}} C(\{\beta_j\}_{j=1}^{L+1}, \{\alpha_j\}_{j=0}^{L+1}) \Big(\prod_{j=1}^{L+1} \partial_x^{\alpha_j} a_{j,\beta_j}(x) \Big) \partial_x^{\alpha_0}, \end{aligned} \tag{2.6}$$

where $\sum'_{\{\beta_j\}_{j=1}^{L+1}}$ *is the summation with respect to* $\{\beta_j\}_{j=1}^{L+1}$ *such that* $\beta_j \le \gamma_j$ *and* $|\beta_j| \le 1$ *for* $j = 1, 2, \dots, L+1$, $\sum''_{\{\alpha_j\}_{j=0}^{L+1}}$ *is the summation with respect to* $\{\alpha_j\}_{j=0}^{L+1}$ *such that* $\sum_{j=0}^{L+1} \alpha_j = \sum_{j=1}^{L+1} \beta_j$ *and* $\alpha_{L+1} = 0$, *and* $C(\{\beta_j\}_{j=1}^{L+1}, \{\alpha_j\}_{j=0}^{L+1})$ *is a non-negative integer.*

(2) *Furthermore, there exists a positive integer* C *independent of* N *and of* L *such that*

$$\sum\nolimits'_{\{\beta_j\}_{j=1}^{L+1}} \sum\nolimits''_{\{\alpha_j\}_{j=0}^{L+1}} C(\{\beta_j\}_{j=1}^{L+1}, \{\alpha_j\}_{j=0}^{L+1}) \le C^{L+1}. \tag{2.7}$$

We can choose $C \le (1 + \Gamma(K+1))$.

Proof. By induction. Proposition 3.3 in D. Fujiwara, N. Kumano-go and K. Taniguchi [5]. □

3. Proof of Theorem 1.5

In this section, we prove Theorem 1.5.

Proof of Theorem 1.5.

1°. From (1.8), for $j = 1, 2, \ldots, L$, we have

$$\begin{aligned} \partial_{\xi_j}\Phi &= -(x_j - x_{j+1}) + \partial_{\xi_j}\phi_{j+1}(x_{j+1}, \xi_j), \\ \partial_{x_j}\Phi &= -(\xi_j - \xi_{j-1}) + \partial_{x_j}\phi_j(x_j, \xi_{j-1}). \end{aligned} \tag{3.1}$$

Set

$$\begin{aligned} M_j &= \frac{1 - i\langle\xi_j\rangle^{1/2}(\partial_{\xi_j}\Phi)\langle\xi_j\rangle^{1/2}\partial_{\xi_j}}{1 + |\langle\xi_j\rangle^{1/2}(\partial_{\xi_j}\Phi)|^2}, \\ N_j &= \frac{1 - i\langle\xi_{j-1}\rangle^{-1/2}(\partial_{x_j}\Phi)\langle\xi_{j-1}\rangle^{-1/2}\partial_{x_j}}{1 + |\langle\xi_{j-1}\rangle^{-1/2}(\partial_{x_j}\Phi)|^2}. \end{aligned} \tag{3.2}$$

We denote the adjoint operators of M_j and of N_j respectively by M_j^* and by N_j^*. Then we can write

$$\begin{aligned} M_j^* &= a_j^1(x_{j+1}, \xi_j, x_j)\partial_{\xi_j} + a_j^0(x_{j+1}, \xi_j, x_j), \\ N_j^* &= b_j^1(\xi_j, x_j, \xi_{j-1})\partial_{x_j} + b_j^0(\xi_j, x_j, \xi_{j-1}), \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} a_j^1(x_{j+1}, \xi_j, x_j) &= \frac{i\langle\xi_j\rangle^{1/2}(\partial_{\xi_j}\Phi)}{1 + |\langle\xi_j\rangle^{1/2}(\partial_{\xi_j}\Phi)|^2}\langle\xi_j\rangle^{1/2}, \\ a_j^0(x_{j+1}, \xi_j, x_j) &= \frac{1}{1 + |\langle\xi_j\rangle^{1/2}(\partial_{\xi_j}\Phi)|^2} \\ &\quad + \partial_{\xi_j}\left(\frac{i\langle\xi_j\rangle^{1/2}(\partial_{\xi_j}\Phi)}{1 + |\langle\xi_j\rangle^{1/2}(\partial_{\xi_j}\Phi)|^2}\langle\xi_j\rangle^{1/2}\right), \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} b_j^1(\xi_j, x_j, \xi_{j-1}) &= \frac{i\langle\xi_{j-1}\rangle^{-1/2}(\partial_{x_j}\Phi)}{1 + |\langle\xi_{j-1}\rangle^{-1/2}(\partial_{x_j}\Phi)|^2}\langle\xi_{j-1}\rangle^{-1/2}, \\ b_j^0(\xi_j, x_j, \xi_{j-1}) &= \frac{1}{1 + |\langle\xi_{j-1}\rangle^{-1/2}(\partial_{x_j}\Phi)|^2} \\ &\quad + \partial_{x_j}\left(\frac{i\langle\xi_{j-1}\rangle^{-1/2}(\partial_{x_j}\Phi)}{1 + |\langle\xi_{j-1}\rangle^{-1/2}(\partial_{x_j}\Phi)|^2}\langle\xi_{j-1}\rangle^{-1/2}\right). \end{aligned} \tag{3.5}$$

2°. We note the formula $\langle\xi + \eta\rangle \leq |\eta| + \langle\xi\rangle$. Then, when $|\xi_j - \xi_{j-1}| \leq (1/2)\langle\xi_{j-1}\rangle$, we have

$$2^{-1}\langle\xi_{j-1}\rangle \leq \langle\xi_j\rangle \leq 2\langle\xi_{j-1}\rangle. \tag{3.6}$$

And when $|\xi_j - \xi_{j-1}| > (1/2)\langle\xi_{j-1}\rangle$, we have

$$\begin{aligned}(3.7)\qquad |\partial_{x_j}\Phi| &\geq |\xi_j - \xi_{j-1}| - \sqrt{n}\kappa_1 t_j\langle\xi_{j-1}\rangle \\ &\geq (1 - 2\sqrt{n}\kappa_1 t_j)|\xi_j - \xi_{j-1}| \\ &\geq (1 - 2\sqrt{n}\kappa_1 T)3^{-1}\langle\xi_j\rangle.\end{aligned}$$

Using (3.6) and (3.7), we get the following estimates for derivatives of b_j^1 and b_j^0: For any α_j, β_j, α_{j-1}, there exists a positive constant $C_{\alpha_j,\beta_j,\alpha_{j-1}}$ independent of j such that

$$\begin{aligned}(3.8)\quad |\partial_{\xi_j}^{\alpha_j}\partial_{x_j}^{\beta_j}\partial_{\xi_{j-1}}^{\alpha_{j-1}}b_j^1(\xi_j,x_j,\xi_{j-1})| &\leq C_{\alpha_j,\beta_j,\alpha_{j-1}}\frac{1}{(1+\langle\xi_{j-1}\rangle^{-1}|\partial_{x_j}\Phi|^2)^{1/2}} \\ &\quad\times\langle\xi_j\rangle^{-|\alpha_j|/2}\langle\xi_{j-1}\rangle^{-1/2+|\beta_j|/2-|\alpha_{j-1}|/2}, \\ |\partial_{\xi_j}^{\alpha_j}\partial_{x_j}^{\beta_j}\partial_{\xi_{j-1}}^{\alpha_{j-1}}b_j^0(\xi_j,x_j,\xi_{j-1})| &\leq C_{\alpha_j,\beta_j,\alpha_{j-1}}\frac{1}{(1+\langle\xi_{j-1}\rangle^{-1}|\partial_{x_j}\Phi|^2)^{1/2}} \\ &\quad\times\langle\xi_j\rangle^{-|\alpha_j|/2}\langle\xi_{j-1}\rangle^{|\beta_j|/2-|\alpha_{j-1}|/2}.\end{aligned}$$

Furthermore, we get the following estimates for derivatives of a_j^1 and a_j^0: For any α_j, there exists a positive constant C_{α_j} independent of j such that

$$\begin{aligned}(3.9)\qquad |\partial_{\xi_j}^{\alpha_j}a_j^1(x_{j+1},\xi_j,x_j)| &\leq C_{\alpha_j}\frac{1}{(1+\langle\xi_j\rangle|\partial_{\xi_j}\Phi|^2)^{1/2}}\langle\xi_j\rangle^{1/2-|\alpha_j|/2}, \\ |\partial_{\xi_j}^{\alpha_j}a_j^0(x_{j+1},\xi_j,x_j)| &\leq C_{\alpha_j}\frac{1}{(1+\langle\xi_j\rangle|\partial_{\xi_j}\Phi|^2)^{1/2}}\langle\xi_j\rangle^{-|\alpha_j|/2}.\end{aligned}$$

3°. We take $\chi \in C_0^\infty(\boldsymbol{R}^n)$ such that

$$(3.10)\qquad 0 \leq \chi \leq 1 \quad\text{and}\quad \chi(x) = \begin{cases}1 & (|x| \leq 1/3) \\ 0 & (|x| \geq 1/2)\end{cases}.$$

For simplicity, when $k > k'$, we set $\prod_{j=k}^{k'}\cdots = 1$.

For $R = 0, 1, 2, \ldots, L$ and $0 = j_0 < j_1 < \ldots < j_R < j_{R+1} = L+1$, let

$$\begin{aligned}(3.11)\qquad \chi_{j_0,j_1,\ldots,j_R} &= \prod_{r=1}^{R+1}\prod_{j=j_{r-1}+1}^{j_r-1}\chi\Big((\xi_j - \xi_{j_{r-1}})/\langle\xi_{j_{r-1}}\rangle\Big) \\ &\quad\times\prod_{r=1}^{R}\Big(1-\chi\Big((\xi_{j_r} - \xi_{j_{r-1}})/\langle\xi_{j_{r-1}}\rangle\Big)\Big).\end{aligned}$$

We divide $\mathbb{I}(\Phi, p)$ into 2^L terms as follows:

$$(3.12)\qquad \mathbb{I}(\Phi, p) = \sum_{R=0}^{L}\sum_{0=j_0<j_1<\ldots<j_R<j_{R+1}=L+1}\mathbb{I}(\Phi, \chi_{j_0,j_1,\ldots,j_R}p).$$

4°. We consider $\mathbb{I}(\Phi, \chi_{j_0,j_1,\dots,j_R}p)$. Set $J=[2M]+2n+1$. Integrating by parts, we have

$$\mathbb{I}(\Phi, \chi_{j_0,j_1,\dots,j_R}p) = \mathbb{I}(\Phi, p^{\circ}_{j_0,j_1,\dots,j_R}), \tag{3.13}$$

where

$$\begin{aligned} p^{\circ}_{j_0,j_1,\dots,j_R} &= (M_L^*)^{n+1}(M_{L-1}^*)^{n+1}\cdots(M_1^*)^{n+1} \\ &\quad \circ (N_L^*)^J(N_{L-1}^*)^J\cdots(N_1^*)^J\chi_{j_0,j_1,\dots,j_R}p. \end{aligned} \tag{3.14}$$

Therefore, by Lemma 2.2, there exists a positive constant C_1 such that

$$\begin{aligned} |p^{\circ}_{j_0,j_1,\dots,j_R}| &\le (C_1)^L|p|^{(\widetilde{m}_{L+1})}_{n+1,J}\langle\xi_0\rangle^{m_1} \\ &\times \prod_{r=1}^{R+1}\prod_{j=j_{r-1}+1}^{j_r-1}\left\{\frac{1}{(1+\langle\xi_j\rangle|\partial_{\xi_j}\Phi|^2)^{(n+1)/2}}\cdot\frac{1}{(1+\langle\xi_{j-1}\rangle^{-1}|\partial_{x_j}\Phi|^2)^{J/2}}\langle\xi_j\rangle^{m_{j+1}}\right\} \\ &\times \prod_{r=1}^{R}\left\{\frac{1}{(1+\langle\xi_{j_r}\rangle|\partial_{\xi_{j_r}}\Phi|^2)^{(n+1)/2}}\cdot\frac{1}{(1+\langle\xi_{j_r-1}\rangle^{-1}|\partial_{x_{j_r}}\Phi|^2)^{J/2}}\langle\xi_{j_r}\rangle^{m_{j_r+1}}\right\}. \end{aligned} \tag{3.15}$$

5°. For $r=1,2,\dots,R+1$ and $j=j_{r-1}+1, j_{r-1}+2,\dots,j_r-1$, we note that

$$|\xi_j-\xi_{j_{r-1}}| \le \frac{1}{2}\langle\xi_{j_{r-1}}\rangle, \tag{3.16}$$

on the support of $p^{\circ}_{j_0,j_1,\dots,j_R}$. Using the formula $\langle\xi+\eta\rangle \le |\eta|+\langle\xi\rangle$, we have

$$2^{-1}\langle\xi_{j_{r-1}}\rangle \le \langle\xi_j\rangle \le 2\langle\xi_{j_{r-1}}\rangle, \tag{3.17}$$

for $r=1,2,\dots,R+1$ and $j=j_{r-1}+1, j_{r-1}+2,\dots,j_r-1$ on the support of $p^{\circ}_{j_0,j_1,\dots,j_R}$. Therefore, there exists a positive constant C_2 such that

$$\begin{aligned} |p^{\circ}_{j_0,j_1,\dots,j_R}| &\le (C_2)^L|p|^{(\widetilde{m}_{L+1})}_{n+1,J}\langle\xi_0\rangle^{\sum_{j=1}^{L+1}m_j} \\ &\times \prod_{j=j_R+1}^{L}\left\{\frac{1}{(1+\langle\xi_{j_R}\rangle|\partial_{\xi_j}\Phi|^2)^{(n+1)/2}}\cdot\frac{1}{(1+\langle\xi_{j_R}\rangle^{-1}|\partial_{x_j}\Phi|^2)^{J/2}}\right\} \\ &\times \prod_{r=1}^{R}\prod_{j=j_{r-1}+1}^{j_r-1}\left\{\frac{1}{(1+\langle\xi_{j_{r-1}}\rangle|\partial_{\xi_j}\Phi|^2)^{(n+1)/2}}\cdot\frac{1}{(1+\langle\xi_{j_{r-1}}\rangle^{-1}|\partial_{x_j}\Phi|^2)^{(J-2M)/4}}\right\} \\ &\times \prod_{r=1}^{R}\frac{1}{(1+\langle\xi_{j_r}\rangle|\partial_{\xi_{j_r}}\Phi|^2)^{(n+1)/2}}\cdot\prod_{r=1}^{R}\langle\xi_{j_r}\rangle^{-(J-2M)/4} \\ &\times \prod_{r=1}^{R}\frac{\langle\xi_{j_r}\rangle^{\sum_{j=j_r+1}^{j_{r+1}}m_j+(J-2M)/4}\lambda_r(\xi_0)}{\prod_{j=j_{r-1}+1}^{j_r}(1+\langle\xi_{j_{r-1}}\rangle^{-1}|\partial_{x_j}\Phi|^2)^{M+(J-2M)/4}}, \end{aligned} \tag{3.18}$$

where $\lambda_1(\xi_0) = \langle \xi_0 \rangle^{-\sum_{j=j_1+1}^{L+1} m_j}$ and $\lambda_r(\xi_0) = 1$ for $r \neq 1$.
6°. For $r = 1, 2, \dots, R$, we note that

$$|\xi_{j_r} - \xi_{j_{r-1}}| \geq \frac{1}{3} \langle \xi_{j_{r-1}} \rangle, \tag{3.19}$$

on the support of $p^\circ_{j_0,j_1,\dots,j_R}$. Using the formula $\langle \xi + \eta \rangle \leq |\eta| + \langle \xi \rangle$, we have

$$|\xi_{j_r} - \xi_{j_{r-1}}| \geq \frac{1}{4} \langle \xi_{j_r} \rangle, \tag{3.20}$$

for $r = 1, 2, \dots, R$ on the support of $p^\circ_{j_0,j_1,\dots,j_R}$. Furthermore, noting (3.17) and (3.19), we have

$$\begin{aligned}
&\prod_{j=j_{r-1}+1}^{j_r} (1 + \langle \xi_{j_{r-1}} \rangle^{-1} |\partial_{x_j} \Phi|^2)^{1/2} \\
&\geq 2^{-(j_r - j_{r-1})/2} \prod_{j=j_{r-1}+1}^{j_r} (1 + \langle \xi_{j_{r-1}} \rangle^{-1/2} |\partial_{x_j} \Phi|) \\
&\geq 2^{-(j_r - j_{r-1})/2} \langle \xi_{j_{r-1}} \rangle^{-1/2} \sum_{j=j_{r-1}+1}^{j_r} |\partial_{x_j} \Phi| \\
&\geq 2^{-(j_r - j_{r-1})/2} \langle \xi_{j_{r-1}} \rangle^{-1/2} \sum_{j=j_{r-1}+1}^{j_r} \Big(|\xi_j - \xi_{j-1}| - \sqrt{n} \kappa_1 t_j \langle \xi_{j-1} \rangle \Big) \\
&\geq 2^{-(j_r - j_{r-1})/2} \langle \xi_{j_{r-1}} \rangle^{-1/2} \sum_{j=j_{r-1}+1}^{j_r} \Big(|\xi_j - \xi_{j-1}| - 2\sqrt{n} \kappa_1 t_j \langle \xi_{j_{r-1}} \rangle \Big) \\
&\geq 2^{-(j_r - j_{r-1})/2} 3^{-1/2} (1 - 6\sqrt{n} \kappa_1 T) |\xi_{j_r} - \xi_{j_{r-1}}|^{1/2},
\end{aligned} \tag{3.21}$$

for $r = 1, 2, \dots, R$ on the support of $p^\circ_{j_0,j_1,\dots,j_R}$.

Therefore, there exists a positive constant C_3 such that

$$\begin{aligned}
&|p^\circ_{j_0,j_1,\dots,j_R}| \leq (C_3)^L |p|^{(\widetilde{m}_{L+1})}_{n+1,J} \langle \xi_0 \rangle^{\sum_{j=1}^{L+1} m_j} \\
&\times \prod_{j=j_R+1}^{L} \left\{ \frac{1}{(1 + \langle \xi_{j_R} \rangle |\partial_{\xi_j} \Phi|^2)^{(n+1)/2}} \cdot \frac{1}{(1 + \langle \xi_{j_R} \rangle^{-1} |\partial_{x_j} \Phi|^2)^{J/2}} \right\} \\
&\times \prod_{r=1}^{R} \prod_{j=j_{r-1}+1}^{j_r - 1} \left\{ \frac{1}{(1 + \langle \xi_{j_{r-1}} \rangle |\partial_{\xi_j} \Phi|^2)^{(n+1)/2}} \cdot \frac{1}{(1 + \langle \xi_{j_{r-1}} \rangle^{-1} |\partial_{x_j} \Phi|^2)^{(J-2M)/4}} \right\} \\
&\times \prod_{r=1}^{R} \frac{1}{(1 + \langle \xi_{j_r} \rangle |\partial_{\xi_{j_r}} \Phi|^2)^{(n+1)/2}} \cdot \prod_{r=1}^{R} \langle \xi_{j_r} \rangle^{-(J-2M)/4}.
\end{aligned} \tag{3.22}$$

7°. For $r=1,2,\dots,R+1$ and $j=j_{r-1}+1,j_{r-1}+2,\dots,j_r-1$, let

$$\begin{aligned} z_j &= \partial_{\xi_j}\Phi = -(x_j-x_{j+1})+\partial_{\xi_j}\phi_{j+1}(x_{j+1},\xi_j),\\ \zeta_j &= \partial_{x_j}\Phi = -(\xi_j-\xi_{j-1})+\partial_{x_j}\phi_j(x_j,\xi_{j-1}). \end{aligned} \tag{3.23}$$

For simplicity, we set $k=j_{r-1}+1$, $k'=j_r-1$ and

$$\begin{aligned} \widetilde{x}_{k,k'} &= (x_k,x_{k+1},\dots,x_{k'}), & \widetilde{\xi}_{k,k'} &= (\xi_k,\xi_{k+1},\dots,\xi_{k'}),\\ \widetilde{z}_{k,k'} &= (z_k,z_{k+1},\dots,z_{k'}), & \widetilde{\zeta}_{k,k'} &= (\zeta_k,\zeta_{k+1},\dots,\zeta_{k'}). \end{aligned} \tag{3.24}$$

Then we have

$$\frac{\partial(\widetilde{z}_{k,k'},\widetilde{\zeta}_{k,k'})}{\partial(\widetilde{x}_{k,k'},\widetilde{\xi}_{k,k'})} = -\begin{pmatrix} \Delta_{k'-k+1} & 0\\ 0 & {}^t\Delta_{k'-k+1}\end{pmatrix} + \begin{pmatrix} \Lambda^1_{k,k'} & \Lambda^2_{k,k'}\\ \Lambda^3_{k,k'} & \Lambda^4_{k,k'}\end{pmatrix}, \tag{3.25}$$

where $\Delta_{k'-k+1}$, $\Lambda^1_{k,k'}$, $\Lambda^2_{k,k'}$, $\Lambda^3_{k,k'}$ and $\Lambda^4_{k,k'}$ are $(n(k'-k+1))\times(n(k'-k+1))$ matrices defined by

$$\Delta_{k'-k+1} = \begin{pmatrix} I_n & -I_n & 0 & \dots & 0\\ 0 & I_n & -I_n & \ddots & \vdots\\ 0 & 0 & I_n & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & -I_n\\ 0 & \dots & 0 & 0 & I_n \end{pmatrix}, \tag{3.26}$$

$$\Lambda^1_{k,k'} = \begin{pmatrix} 0 & \partial_{x_{k+1}}\partial_{\xi_k}\phi_{k+1} & 0 & \dots & 0\\ 0 & 0 & \partial_{x_{k+2}}\partial_{\xi_{k+1}}\phi_{k+2} & \ddots & \vdots\\ 0 & 0 & 0 & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & \partial_{x_{k'}}\partial_{\xi_{k'-1}}\phi_{k'}\\ 0 & \dots & 0 & 0 & 0 \end{pmatrix}, \tag{3.27}$$

$$\Lambda^2_{k,k'} = \begin{pmatrix} \partial^2_{\xi_k}\phi_{k+1} & 0 & 0 & \dots & 0\\ 0 & \partial^2_{\xi_{k+1}}\phi_{k+2} & 0 & \ddots & \vdots\\ 0 & 0 & \partial^2_{\xi_{k+2}}\phi_{k+3} & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & 0\\ 0 & \dots & 0 & 0 & \partial^2_{\xi_{k'}}\phi_{k'+1} \end{pmatrix}, \tag{3.28}$$

$$\Lambda^3_{k,k'} = \begin{pmatrix} \partial^2_{x_k}\phi_k & 0 & 0 & \dots & 0 \\ 0 & \partial^2_{x_{k+1}}\phi_{k+1} & 0 & \ddots & \vdots \\ 0 & 0 & \partial^2_{x_{k+2}}\phi_{k+2} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 0 & \partial^2_{x_{k'}}\phi_{k'} \end{pmatrix}, \tag{3.29}$$

and

$$\Lambda^4_{k,k'} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ \partial_{\xi_k}\partial_{x_{k+1}}\phi_{k+1} & 0 & 0 & \ddots & \vdots \\ 0 & \partial_{\xi_{k+1}}\partial_{x_{k+2}}\phi_{k+2} & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \partial_{\xi_{k'-1}}\partial_{x_{k'}}\phi_{k'} & 0 \end{pmatrix}. \tag{3.30}$$

Furthermore, we can write

$$\begin{aligned} \det \frac{\partial(\widetilde{z}_{k,k'}, \widetilde{\zeta}_{k,k'})}{\partial(\widetilde{x}_{k,k'}, \widetilde{\xi}_{k,k'})} &= (-1)^{2n(k'-k+1)} \det \begin{pmatrix} \Delta_{k'-k+1} & 0 \\ 0 & {}^t\Delta_{k'-k+1} \end{pmatrix} \\ &\times \det \left\{ I_{2n(k'-k+1)} - \begin{pmatrix} \Lambda^5_{k,k'} & \Lambda^6_{k,k'} \\ \Lambda^7_{k,k'} & \Lambda^8_{k,k'} \end{pmatrix} \right\}, \end{aligned} \tag{3.31}$$

where

$$\begin{aligned} \Lambda^5_{k,k'} &= (\Delta_{k'-k+1})^{-1}\Lambda^1_{k,k'}, \\ \Lambda^6_{k,k'} &= \langle \xi_{j_{r-1}} \rangle \cdot (\Delta_{k'-k+1})^{-1}\Lambda^2_{k,k'}, \\ \Lambda^7_{k,k'} &= \langle \xi_{j_{r-1}} \rangle^{-1} \cdot ({}^t\Delta_{k'-k+1})^{-1}\Lambda^3_{k,k'}, \\ \Lambda^8_{k,k'} &= ({}^t\Delta_{k'-k+1})^{-1}\Lambda^4_{k,k'}. \end{aligned} \tag{3.32}$$

Hence, by Lemma 2.1 and (3.17), we have

$$\begin{aligned} &(1 - 3n\kappa_2 T)^{2n(j_r - j_{r-1} - 1)} \\ &\leq \det \frac{\partial(\widetilde{z}_{j_{r-1}+1, j_r - 1}, \widetilde{\zeta}_{j_{r-1}+1, j_r - 1})}{\partial(\widetilde{x}_{j_{r-1}+1, j_r - 1}, \widetilde{\xi}_{j_{r-1}+1, j_r - 1})} \\ &\leq (1 + 3n\kappa_2 T)^{2n(j_r - j_{r-1} - 1)}, \end{aligned} \tag{3.33}$$

for $r = 1, 2, \dots, R+1$ on the support of $p^\circ_{j_0, j_1, \dots, j_R}$.

Therefore, there exists a positive constant C_4 such that

$$\left|p^{\circ}_{j_0,j_1,\dots,j_R}\prod_{r=1}^{R+1}\det\frac{\partial(\widetilde{x}_{j_{r-1}+1,j_r-1},\widetilde{\xi}_{j_{r-1}+1,j_r-1})}{\partial(\widetilde{z}_{j_{r-1}+1,j_r-1},\widetilde{\zeta}_{j_{r-1}+1,j_r-1})}\right|\le (C_4)^L|p|^{(\widetilde{m}_{L+1})}_{n+1,J}\langle\xi_0\rangle^{\sum_{j=1}^{L+1}m_j}$$

$$\times\prod_{j=j_R+1}^{L}\left\{\frac{\langle\xi_{j_R}\rangle^{n/2}}{(1+\langle\xi_{j_R}\rangle|z_j|^2)^{(n+1)/2}}\cdot\frac{\langle\xi_{j_R}\rangle^{-n/2}}{(1+\langle\xi_{j_R}\rangle^{-1}|\zeta_j|^2)^{J/2}}\right\}$$

$$\times\prod_{r=1}^{R}\prod_{j=j_{r-1}+1}^{j_r-1}\left\{\frac{\langle\xi_{j_{r-1}}\rangle^{n/2}}{(1+\langle\xi_{j_{r-1}}\rangle|z_j|^2)^{(n+1)/2}}\cdot\frac{\langle\xi_{j_{r-1}}\rangle^{-n/2}}{(1+\langle\xi_{j_{r-1}}\rangle^{-1}|\zeta_j|^2)^{(J-2M)/4}}\right\}$$

$$\times\prod_{r=1}^{R}\frac{\langle\xi_{j_r}\rangle^{n/2}}{(1+\langle\xi_{j_r}\rangle|x_{j_r}-x_{j_r+1}-\partial_{\xi_{j_r}}\phi_{j_r+1}(x_{j_r+1},\xi_{j_r})|^2)^{(n+1)/2}}$$

$$\times\prod_{r=1}^{R}\langle\xi_{j_r}\rangle^{-n/2-(J-2M)/4}. \tag{3.34}$$

8°. We change the variables:

$$(\widetilde{x}_{j_{r-1}+1,j_r-1},\widetilde{\xi}_{j_{r-1}+1,j_r-1})\Longrightarrow(\widetilde{z}_{j_{r-1}+1,j_r-1},\widetilde{\zeta}_{j_{r-1}+1,j_r-1}),$$

for $r=1,2,\dots,R+1$.

Now, for $r=1,2,\dots,R$, x_{j_r+1} is a function depending only on $x_{j_{r+1}}$, $\widetilde{z}_{j_r+1,j_{r+1}-1}$, $\widetilde{\zeta}_{j_r+1,j_{r+1}-1}$ and ξ_{j_r}, not x_{j_r}.

Keeping this in mind, we integrate in the following order. First we integrate by x_{j_1}, x_{j_2}, …, x_{j_R}. Secondly we integrate by $\widetilde{z}_{j_{r-1}+1,j_r-1}$, $\widetilde{\zeta}_{j_{r-1}+1,j_r-1}$, $r=1,2,\dots,R+1$. Thirdly we integrate by ξ_{j_R}, $\xi_{j_{R-1}}$,…,ξ_{j_1}. Then there exists a positive constant C_5 such that

$$|\mathbb{I}(\Phi,p^{\circ}_{j_0,j_1,\dots,j_R})|\le(C_5)^L|p|^{(\widetilde{m}_{L+1})}_{n+1,J}\langle\xi_0\rangle^{\sum_{j=1}^{L+1}m_j}. \tag{3.35}$$

Therefore, we have

$$\begin{aligned}|\mathbb{I}(\Phi,p)| &\le\sum_{R=0}^{L}\sum_{0=j_0<j_1<\dots<j_R<j_{R+1}=L+1}|\mathbb{I}(\Phi,p^{\circ}_{j_0,j_1,\dots,j_R})|\\ &\le(2C_5)^L|p|^{(\widetilde{m}_{L+1})}_{n+1,J}\langle\xi_0\rangle^{\sum_{j=1}^{L+1}m_j}.\end{aligned} \tag{3.36}$$

□

Now, for $R=0,1,2,\dots,L$ and $0=j_0<j_1<\dots<j_R<j_{R+1}=L+1$, set

$$(3.37)\qquad E_{j_0,j_1,\dots,j_R}=\left\{\begin{array}{l}(x_{L+1},\xi_L,x_L,\xi_{L-1},\dots,x_1,\xi_0);\\ |\xi_j-\xi_{j_{r-1}}|\le\frac{1}{2}\langle\xi_{j_{r-1}}\rangle\\ (1\le r\le R+1,\quad j_{r-1}+1\le j\le j_r-1)\\ |\xi_{j_r}-\xi_{j_{r-1}}|>\frac{1}{2}\langle\xi_{j_{r-1}}\rangle\\ (1\le r\le R)\end{array}\right\}.$$

Looking over the proof of Theorem 1.5 once again, we can get the following corollary.

Corollary 3.1. *Let $\{\kappa_l\}_{l=0}^{\infty}$ be an increasing sequence of positive constants and $M\ge 0$. Set $T=\min\{1/(7\sqrt{n}\kappa_1),1/(4n\kappa_2)\}$. Then there exist positive constants C' and C'' independent of L satisfying the following:*

(1) *Let $\sum_{j=1}^{L+1}t_j\le T$, $\sum_{j=1}^{L+1}|m_j|\le M$, $p\in S_\rho^{\widetilde{m}_{L+1}}$ and $\phi_j\in P_\rho(t_j,\{\kappa_l\}_{l=0}^{\infty})$. If E_0 contains the support of p, we have*

$$(3.38)\qquad |\mathbb{I}(\Phi,p)(x_{L+1},\xi_0)|\le(C')^L|p|_{n+1,n+1}^{(\widetilde{m}_{L+1})}\langle\xi_0\rangle^{\sum_{j=1}^{L+1}m_j}.$$

(2) *Let $\sum_{j=1}^{L+1}t_j\le T$, $\sum_{j=1}^{L+1}|m_j|\le M$, $p\in S_\rho^{\widetilde{m}_{L+1}}$ and $\phi_j\in P_\rho(t_j,\{\kappa_l\}_{l=0}^{\infty})$. Let $R=1,2,\dots,L$ and $0=j_0<j_1<\dots<j_R<j_{R+1}=L+1$. If $E_{j_0,j_1,\dots,j_R}$ contains the support of p, we have*

$$(3.39)\qquad |\mathbb{I}(\Phi,p)(x_{L+1},\xi_0)|\le(C'')^L|p|_{l_0,l_0'}^{(\widetilde{m}_{L+1})}\langle\xi_0\rangle^{-(M-\sum_{j=1}^{L+1}\max\{0,m_j\})},$$

where $l_0=n+1$ and $l_0'=[2M]+2n+1$.

4. Proof of Proposition 1.6

In this section, we prove Proposition 1.6.

Proof of Proposition 1.6.

1°. First we assume that the solution $\{x_j^*,\xi_j^*\}_{j=1}^{L}$ of (1.15) exists. Then we have

$$(4.1)\qquad \begin{aligned}|\xi_j^*-\xi_{j-1}^*|&\le\sqrt{n}\kappa_1t_j\langle\xi_{j-1}^*\rangle\\ &\le\sqrt{n}\kappa_1t_j\left\{\sum_{k=1}^{L}|\xi_k^*-\xi_{k-1}^*|+\langle\xi_0\rangle\right\},\end{aligned}$$

for $j=1,2,\dots,L$. Hence we get

$$\sum_{j=1}^{L}|\xi_j^*-\xi_{j-1}^*| \le \frac{\sqrt{n}\kappa_1\sum_{j=1}^{L}t_j}{1-\sqrt{n}\kappa_1\sum_{j=1}^{L}t_j}\langle\xi_0\rangle \le \frac{1}{2}\langle\xi_0\rangle. \tag{4.2}$$

Therefore, the solution $\{x_j^*,\xi_j^*\}_{j=1}^{L}$ of (1.15) satisfies

$$|\xi_j^*-\xi_0| \le \frac{1}{2}\langle\xi_0\rangle, \tag{4.3}$$

for $j=1,2,\dots,L$.

2°. For $(\widetilde{x}_{1,L},\widetilde{\xi}_{1,L}) \in \boldsymbol{R}^{2nL}$, we introduce the norms $||^t(\widetilde{x}_{1,L},\widetilde{\xi}_{1,L})||_\infty^{\xi_0}$, $||^t(\widetilde{x}_{1,L},\widetilde{\xi}_{1,L})||_1^{\xi_0}$ given by

$$\begin{aligned} ||^t(\widetilde{x}_{1,L},\widetilde{\xi}_{1,L})||_\infty^{\xi_0} &= \max_{j=1,2,\dots,L}|x_j| + \langle\xi_0\rangle^{-1}\max_{j=1,2,\dots,L}|\xi_j|, \\ ||^t(\widetilde{x}_{1,L},\widetilde{\xi}_{1,L})||_1^{\xi_0} &= \sum_{j=1}^{L}\{|x_j|+\langle\xi_0\rangle^{-1}|\xi_j|\}. \end{aligned} \tag{4.4}$$

Let $\Omega_\infty^{\xi_0}$ be the normed space $(\boldsymbol{R}^{2nL},||\cdot||_\infty^{\xi_0})$ and let $\Omega_1^{\xi_0}$ be the normed space $(\boldsymbol{R}^{2nL},||\cdot||_1^{\xi_0})$. Let $\Theta_\infty^{\xi_0}$ be the closed set of $\Omega_\infty^{\xi_0}$ given by

$$\Theta_\infty^{\xi_0} = \left\{(\widetilde{x}_{1,L},\widetilde{\xi}_{1,L}) \in \Omega_\infty^{\xi_0};\ \ |\xi_j-\xi_0| \le \frac{1}{2}\langle\xi_0\rangle,\ \ j=1,2,\dots,L\right\}. \tag{4.5}$$

Let Δ_L be the matrix obtained by putting $k=1$ and $k'=L$ in (3.26).

For $(\widetilde{x}_{1,L},\widetilde{\xi}_{1,L}) \in \Theta_\infty^{\xi_0}$, we consider the mapping $\mathcal{F}:(\widetilde{x}_{1,L},\widetilde{\xi}_{1,L}) \longmapsto (\widetilde{y}_{1,L},\widetilde{\eta}_{1,L})$ given by

$$^t(\widetilde{y}_{1,L},\widetilde{\eta}_{1,L}) = \Delta^{-1}\Psi(x_{L+1},\widetilde{x}_{1,L},\widetilde{\xi}_{1,L},\xi_0), \tag{4.6}$$

where

$$\Delta = \begin{pmatrix} \Delta_L & 0 \\ 0 & {}^t\Delta_L \end{pmatrix}, \tag{4.7}$$

and

$$\Psi(x_{L+1},\widetilde{x}_{1,L},\widetilde{\xi}_{1,L},\xi_0) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x_{L+1} \\ \xi_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} \partial_{\xi_1}\phi_2(x_2,\xi_1) \\ \partial_{\xi_2}\phi_3(x_3,\xi_2) \\ \vdots \\ \partial_{\xi_L}\phi_{L+1}(x_{L+1},\xi_L) \\ \partial_{x_1}\phi_1(x_1,\xi_0) \\ \partial_{x_2}\phi_2(x_2,\xi_1) \\ \vdots \\ \partial_{x_L}\phi_L(x_L,\xi_{L-1}) \end{pmatrix}. \tag{4.8}$$

From (4.5), we have

$$2^{-1}\langle\xi_0\rangle \le \langle\xi_j\rangle \le 2\langle\xi_0\rangle, \tag{4.9}$$

for $j = 1, 2, \ldots, L$. Furthermore, from (4.6), we have

$$\begin{aligned} |\eta_j - \xi_0| &\le \sum_{k=1}^{j} |\partial_{x_k}\phi_k(x_k, \xi_{k-1})| \le \sum_{j=1}^{L} \sqrt{n}\kappa_1 t_j \langle\xi_{j-1}\rangle \\ &\le 2\sqrt{n}\kappa_1 \sum_{j=1}^{L} t_j\langle\xi_0\rangle \le \frac{1}{2}\langle\xi_0\rangle, \end{aligned} \tag{4.10}$$

for $j = 1, 2, \ldots, L$. Therefore, the map $\mathcal{F} : \Theta_\infty^{\xi_0} \to \Theta_\infty^{\xi_0}$ is well-defined.

3°. Let $\Lambda_{1,L}^1$, $\Lambda_{1,L}^2$, $\Lambda_{1,L}^3$ and $\Lambda_{1,L}^4$ be the matrices obtained by putting $k = 1$ and $k' = L$ in (3.27)–(3.30). Set

$$\Lambda(x_{L+1}, \widetilde{x}_{1,L}, \widetilde{\xi}_{1,L}, \xi_0) = \begin{pmatrix} \Lambda_{1,L}^1 & \Lambda_{1,L}^2 \\ \Lambda_{1,L}^3 & \Lambda_{1,L}^4 \end{pmatrix}. \tag{4.11}$$

For $(\widetilde{x}_{1,L}, \widetilde{\xi}_{1,L}), (\widetilde{x}'_{1,L}, \widetilde{\xi}'_{1,L}) \in \Theta_\infty^{\xi_0}$, let

$$\begin{aligned} {}^t(\widetilde{y}_{1,L}, \widetilde{\eta}_{1,L}) &= \Delta^{-1}\Psi(x_{L+1}, \widetilde{x}_{1,L}, \widetilde{\xi}_{1,L}, \xi_0), \\ {}^t(\widetilde{y}'_{1,L}, \widetilde{\eta}'_{1,L}) &= \Delta^{-1}\Psi(x_{L+1}, \widetilde{x}'_{1,L}, \widetilde{\xi}'_{1,L}, \xi_0). \end{aligned} \tag{4.12}$$

Then we have

$$\begin{aligned} &\|{}^t(\widetilde{y}'_{1,L}, \widetilde{\eta}'_{1,L}) - {}^t(\widetilde{y}_{1,L}, \widetilde{\eta}_{1,L})\|_\infty^{\xi_0} \le \|\Delta^{-1}\|_{\Omega_1^{\xi_0} \to \Omega_\infty^{\xi_0}} \\ &\times \left\| \int_0^1 \Lambda(x_{L+1}, \widetilde{x}_{1,L} + \theta(\widetilde{x}'_{1,L} - \widetilde{x}_{1,L}), \widetilde{\xi}_{1,L} + \theta(\widetilde{\xi}'_{1,L} - \widetilde{\xi}_{1,L}), \xi_0)d\theta \right\|_{\Omega_\infty^{\xi_0} \to \Omega_1^{\xi_0}} \\ &\times \|{}^t(\widetilde{x}'_{1,L}, \widetilde{\xi}'_{1,L}) - {}^t(\widetilde{x}_{1,L}, \widetilde{\xi}_{1,L})\|_\infty^{\xi_0}. \end{aligned} \tag{4.13}$$

Clearly we have

$$\|\Delta^{-1}\|_{\Omega_1^{\xi_0} \to \Omega_\infty^{\xi_0}} \le 1. \tag{4.14}$$

Noting that

$$\begin{aligned} \langle\xi_j + \theta(\xi'_j - \xi_j)\rangle &\le (1-\theta)|\xi_j - \xi_0| + \theta|\xi'_j - \xi_0| + \langle\xi_0\rangle, \\ \langle\xi_0\rangle &\le (1-\theta)|\xi_j - \xi_0| + \theta|\xi'_j - \xi_0| + \langle\xi_j + \theta(\xi'_j - \xi_j)\rangle, \end{aligned} \tag{4.15}$$

we have

$$2^{-1}\langle\xi_0\rangle \leq \langle\xi_j+\theta(\xi'_j-\xi_j)\rangle \leq 2\langle\xi_0\rangle, \tag{4.16}$$

for $j=1,2,\ldots,L$ and $0\leq\theta\leq 1$. Hence we get

$$\begin{aligned}&\left\|\int_0^1 \Lambda(x_{L+1},\widetilde{x}_{1,L}+\theta(\widetilde{x}'_{1,L}-\widetilde{x}_{1,L}),\widetilde{\xi}_{1,L}+\theta(\widetilde{\xi}'_{1,L}-\widetilde{\xi}_{1,L}),\xi_0)d\theta\right\|_{\Omega^{\xi_0}_\infty\to\Omega^{\xi_0}_1}\\&\leq 3n\kappa_2\sum_{j=1}^{L+1}t_j<1.\end{aligned} \tag{4.17}$$

By (4.13), (4.14) and (4.17), $\mathcal{F}$ is a contraction. Hence there exists a unique solution $\{x^*_j,\xi^*_j\}_{j=1}^L\in\Theta^{\xi_0}_\infty$ such that

$${}^t(\widetilde{x}^*_{1,L},\widetilde{\xi}^*_{1,L})=\Delta^{-1}\Psi(x_{L+1},\widetilde{x}^*_{1,L},\widetilde{\xi}^*_{1,L},\xi_0). \tag{4.18}$$

Therefore, there exists a unique solution $\{x^*_j,\xi^*_j\}_{j=1}^L\in\Theta^{\xi_0}_\infty$ such that

$$\begin{cases}0=-(x^*_j-x^*_{j+1})+\partial_{\xi_j}\phi_{j+1}(x^*_{j+1},\xi^*_j),\\0=-(\xi^*_j-\xi^*_{j-1})+\partial_{x_j}\phi_j(x^*_j,\xi^*_{j-1}),\\j=1,2,\ldots,L,\quad x^*_{L+1}=x_{L+1},\quad \xi^*_0=\xi_0.\end{cases} \tag{4.19}$$

4°. Clearly, from (4.19), we have

$$\begin{aligned}&|x^*_j-x^*_{j+1}|\leq\sqrt{n}\kappa_1t_{j+1},\\&|\xi^*_j-\xi^*_{j-1}|\leq\sqrt{n}\kappa_1t_j\langle\xi^*_{j-1}\rangle\leq 2\sqrt{n}\kappa_1t_j\langle\xi_0\rangle,\end{aligned} \tag{4.20}$$

for $j=1,2,\ldots,L$. Furthermore, for any α_0, β_{L+1} with $|\alpha_0+\beta_{L+1}|\geq 1$, there exists a positive constant $C_{\alpha_0,\beta_{L+1}}$ such that

$$\begin{aligned}&|\partial^{\beta_{L+1}}_{x_{L+1}}\partial^{\alpha_0}_{\xi_0}(x^*_j-x^*_{j+1})|\leq C_{\alpha_0,\beta_{L+1}}t_{j+1}\langle\xi_0\rangle^{-(1-\rho)+(1-\rho)|\beta_{L+1}|-\rho|\alpha_0|},\\&|\partial^{\beta_{L+1}}_{x_{L+1}}\partial^{\alpha_0}_{\xi_0}(\xi^*_j-\xi^*_{j-1})|\leq C_{\alpha_0,\beta_{L+1}}t_j\langle\xi_0\rangle^{\rho+(1-\rho)|\beta_{L+1}|-\rho|\alpha_0|},\end{aligned} \tag{4.21}$$

for $j=1,2,\ldots,L$. Therefore we get (1.17). □

5. Proof of Theorem 1.7

In this section, we prove Theorem 1.7.

Proof of Theorem 1.7.

1°. For $R = 0, 1, 2, \ldots, L$ and $0 = j_0 < j_1 < \ldots < j_R < j_{R+1} = L+1$, let

$$q_{j_0,j_1,\ldots,j_R} = e^{-i\Phi^*}\mathbb{I}(\Phi, \chi_{j_0,j_1,\ldots,j_R}p). \tag{5.1}$$

Then we have

$$q = \sum_{R=0}^{L} \sum_{0=j_0<j_1<\ldots<j_R<j_{R+1}=L+1} q_{j_0,j_1,\ldots,j_R}. \tag{5.2}$$

2°. First we consider the case where $R \neq 0$. We can write

$$\begin{aligned} \partial_{\xi_0} q_{j_0,j_1,\ldots,j_R} &= -i(\partial_{\xi_0}\Phi^*)e^{-i\Phi^*}\mathbb{I}(\Phi, \chi_{j_0,j_1,\ldots,j_R}p) \\ &\quad + e^{-i\Phi^*}\mathbb{I}(\Phi, i(\partial_{\xi_0}\Phi)\chi_{j_0,j_1,\ldots,j_R}p) \\ &\quad + e^{-i\Phi^*}\mathbb{I}(\Phi, \partial_{\xi_0}(\chi_{j_0,j_1,\ldots,j_R}p)). \end{aligned} \tag{5.3}$$

Note that

$$\partial_{\xi_0}\Phi = -(x_{L+1} - x_1) + \partial_{\xi_0}\phi_1(x_1, \xi_0), \tag{5.4}$$

and

$$i(x_{L+1} - x_1)e^{i\sum_{j=1}^{L}(x_{j+1}-x_j)(\xi_j-\xi_0)} = \left(\sum_{j=1}^{L}\partial_{\xi_j}\right)e^{i\sum_{j=1}^{L}(x_{j+1}-x_j)(\xi_j-\xi_0)}. \tag{5.5}$$

Integrating by parts, we can write

$$\begin{aligned} \partial_{\xi_0} q_{j_0,j_1,\ldots,j_R} &= -i(\partial_{\xi_0}\Phi^*)e^{-i\Phi^*}\mathbb{I}(\Phi, \chi_{j_0,j_1,\ldots,j_R}p) \\ &\quad + e^{-i\Phi^*}\sum_{j=0}^{L}\mathbb{I}(\Phi, i(\partial_{\xi_j}\phi_{j+1})\chi_{j_0,j_1,\ldots,j_R}p) \\ &\quad + e^{-i\Phi^*}\sum_{j=0}^{L}\mathbb{I}(\Phi, \partial_{\xi_j}(\chi_{j_0,j_1,\ldots,j_R}p)). \end{aligned} \tag{5.6}$$

Here we note that

$$(\partial_{\xi_j}\phi_{j+1})\chi_{j_0,j_1,\ldots,j_R}p, \quad \partial_{\xi_j}(\chi_{j_0,j_1,\ldots,j_R}p) \in \boldsymbol{S}_{\rho}^{\widetilde{m_{L+1}}}. \tag{5.7}$$

If we apply Corollary 3.1 (2) to the right hand side of (5.6) with M in Corollary 3.1 (2) replaced by $M + \rho$, then we have the estimate of $\partial_{\xi_0} q_{j_0,j_1,\ldots,j_R}$. Estimates for higher derivatives will be proved in a similar manner. It is enough to take

$l_1 \geq n+1+l$ and $l_1' \geq \left[2M+2\rho l+2l'\right]+2n+1+l'$.

3°. Next we consider the case where $R=0$. We change the variables:

$$y_j = x_j - x_j^*, \qquad \eta_j = \xi_j - \xi_j^*, \tag{5.8}$$

for $j=1,2,\dots,L$. Then we have

$$q_0 = \int_{\boldsymbol{R}^{2nL}} e^{i\Pi} a(x_{L+1}, \eta_L, y_L, \dots, \eta_1, y_1, \xi_0) \prod_{j=1}^{L} dy_j đ\eta_j, \tag{5.9}$$

where

$$\begin{aligned} &a(x_{L+1}, \eta_L, y_L, \dots, \eta_1, y_1, \xi_0) \\ &= (\chi_0 p)(x_{L+1}, \xi_L^* + \eta_L, x_L^* + y_L, \dots, \xi_1^* + \eta_1, x_1^* + y_1, \xi_0), \end{aligned} \tag{5.10}$$

and

$$\begin{aligned} \Pi &= \Phi - \Phi^* \\ &= -\sum_{j=1}^{L} y_j(\eta_j - \eta_{j-1}) \\ &\quad + \sum_{j=1}^{L} y_j \int_0^1 (1-\theta)(\partial_{x_j}^2 \phi_j)(x_j^* + \theta y_j, \xi_{j-1}^* + \theta\eta_{j-1}) d\theta \cdot y_j \\ &\quad + \sum_{j=1}^{L} \eta_j \int_0^1 (1-\theta)(\partial_{\xi_j}^2 \phi_{j+1})(x_{j+1}^* + \theta y_{j+1}, \xi_j^* + \theta\eta_j) d\theta \cdot \eta_j \\ &\quad + \sum_{j=2}^{L} y_j \int_0^1 (1-\theta)(\partial_{\xi_{j-1}} \partial_{x_j} \phi_j)(x_j^* + \theta y_j, \xi_{j-1}^* + \theta\eta_{j-1}) d\theta \cdot \eta_{j-1} \\ &\quad + \sum_{j=1}^{L-1} \eta_j \int_0^1 (1-\theta)(\partial_{x_{j+1}} \partial_{\xi_j} \phi_{j+1})(x_{j+1}^* + \theta y_{j+1}, \xi_j^* + \theta\eta_j) d\theta \cdot y_{j+1}, \end{aligned} \tag{5.11}$$

with $x_{L+1}^* = x_{L+1}$, $\xi_0^* = \xi_0$, $y_{L+1} = 0$ and $\eta_0 = 0$.

For any α, β, we define $a_{\alpha,\beta}(x_{L+1}, \eta_L, y_L, \dots, \eta_1, y_1, \xi_0)$ such that

$$\partial_{x_{L+1}}^{\beta} \partial_{\xi_0}^{\alpha} q_0 = \int_{\boldsymbol{R}^{2nL}} e^{i\Pi} a_{\alpha,\beta}(x_{L+1}, \eta_L, y_L, \dots, \eta_1, y_1, \xi_0) \prod_{j=1}^{L} dy_j đ\eta_j. \tag{5.12}$$

Note that

$$2^{-1}\langle\xi_0\rangle \le \langle\xi_j^* + \theta\eta_j\rangle \le 2\langle\xi_0\rangle, \tag{5.13}$$

for $j = 1, 2, \dots, L$ and $0 \le \theta \le 1$ on the support of $a_{\alpha,\beta}$.

For any α_0, β_{L+1} and non-negative integers K, K', there exists a positive constant C_1 such that

$$\begin{aligned}&\left|\partial_{x_{L+1}}^{\beta_{L+1}}\partial_{\xi_0}^{\alpha_0}\left(\prod_{j=1}^{L}\partial_{y_j}^{\beta_j}\partial_{\eta_j}^{\alpha_j}\right)a_{\alpha,\beta}(x_{L+1},\eta_L,y_L,\dots,\eta_1,y_1,\xi_0)\right| \\ &\le (C_1)^L|p|^{(\widetilde{m}_{L+1})}_{|\alpha+\beta+\alpha_0+\beta_{L+1}|+K,|\alpha+\beta+\alpha_0+\beta_{L+1}|+K'} \\ &\quad\times\langle\xi_0\rangle^{\sum_{j=1}^{L+1}m_j+(1-\rho)|\beta+\beta_{L+1}|-\rho|\alpha+\alpha_0|+\sum_{j=1}^{L}(|\beta_j|/2-|\alpha_j|/2)} \\ &\quad\times\left(1+\langle\xi_0\rangle^{1/2}\max_{j=1,2,\dots,L}|y_j|+\langle\xi_0\rangle^{-1/2}\max_{j=1,2,\dots,L}|\eta_j|\right)^{2|\alpha+\beta|},\end{aligned}\tag{5.14}$$

for any $|\alpha_j| \le K$ and $|\beta_j| \le K'$, $j = 1, 2, \dots, L$.

4°. We restore the variables:

$$\begin{aligned}x_j &= y_j + x_j^*,\\ \xi_j &= \eta_j + \xi_j^*,\end{aligned}\tag{5.15}$$

for $j = 1, 2, \dots, L$. Then we have

$$\int_{\boldsymbol{R}^{2nL}} e^{i\Pi}a_{\alpha,\beta}(x_{L+1},\eta_L,y_L,\dots,\eta_1,y_1,\xi_0)\prod_{j=1}^{L}dy_j\,đ\eta_j = e^{-i\Phi^*}\mathbb{I}(\Phi,p_{\alpha,\beta}),\tag{5.16}$$

where

$$\begin{aligned}&p_{\alpha,\beta}(x_{L+1},\xi_L,x_L,\dots,\xi_1,x_1,\xi_0)\\ &\quad= a_{\alpha,\beta}(x_{L+1},\xi_L-\xi_L^*,x_L-x_L^*,\dots,\xi_1-\xi_1^*,x_1-x_1^*,\xi_0).\end{aligned}\tag{5.17}$$

For any non-negative integers K, K', there exists a positive constant C_2 such that

$$\begin{aligned}&\left|\left(\prod_{j=1}^{L}\partial_{x_j}^{\beta_j}\partial_{\xi_j}^{\alpha_j}\right)p_{\alpha,\beta}(x_{L+1},\xi_L,x_L,\dots,\xi_1,x_1,\xi_0)\right| \\ &\le (C_2)^L|p|^{(\widetilde{m}_{L+1})}_{|\alpha+\beta|+K,|\alpha+\beta|+K'}\langle\xi_0\rangle^{\sum_{j=1}^{L+1}m_j+(1-\rho)|\beta|-\rho|\alpha|+\sum_{j=1}^{L}(|\beta_j|/2-|\alpha_j|/2)} \\ &\quad\times\left(1+\langle\xi_0\rangle^{1/2}\max_{j=1,2,\dots,L}|x_j-x_j^*|+\langle\xi_0\rangle^{-1/2}\max_{j=1,2,\dots,L}|\xi_j-\xi_j^*|\right)^{2|\alpha+\beta|},\end{aligned}\tag{5.18}$$

for any $|\alpha_j| \le K$ and $|\beta_j| \le K'$, $j = 1, 2, \dots, L$.
5°. For $j = 1, 2, \dots, L$, let

$$z_j = \partial_{\xi_j}\Phi, \qquad \zeta_j = \partial_{x_j}\Phi. \tag{5.19}$$

Let $\Psi(x_{L+1}, \widetilde{x}_{1,L}, \widetilde{\xi}_{1,L}, \xi_0)$ be the vector in (4.8) and $\Lambda(x_{L+1}, \widetilde{x}_{1,L}, \widetilde{\xi}_{1,L}, \xi_0)$ the matrix in (4.11). Since

$$\begin{aligned} {}^t(\widetilde{z}_{1,L}, \widetilde{\zeta}_{1,L}) &= -\Delta^t(\widetilde{x}_{1,L}, \widetilde{\xi}_{1,L}) + \Psi(x_{L+1}, \widetilde{x}_{1,L}, \widetilde{\xi}_{1,L}, \xi_0), \\ {}^t(0,0) &= -\Delta^t(\widetilde{x}^*_{1,L}, \widetilde{\xi}^*_{1,L}) + \Psi(x_{L+1}, \widetilde{x}^*_{1,L}, \widetilde{\xi}^*_{1,L}, \xi_0), \end{aligned} \tag{5.20}$$

we can write

$${}^t(\widetilde{z}_{1,L}, \widetilde{\zeta}_{1,L}) = -\Delta\left(I_{2nL} - \Delta^{-1}\int_0^1 \Lambda_\theta d\theta\right){}^t(\widetilde{x}_{1,L} - \widetilde{x}^*_{1,L}, \widetilde{\xi}_{1,L} - \widetilde{\xi}^*_{1,L}), \tag{5.21}$$

where

$$\Lambda_\theta = \Lambda(x_{L+1}, \widetilde{x}^*_{1,L} + \theta(\widetilde{x}_{1,L} - \widetilde{x}^*_{1,L}), \widetilde{\xi}^*_{1,L} + \theta(\widetilde{\xi}_{1,L} - \widetilde{\xi}^*_{1,L}), \xi_0). \tag{5.22}$$

Furthermore, note that

$$||\Delta^{-1}||_{\Omega_1^{\xi_0} \to \Omega_\infty^{\xi_0}} \le 1, \tag{5.23}$$

and

$$\left\|\int_0^1 \Lambda_\theta d\theta\right\|_{\Omega_\infty^{\xi_0} \to \Omega_1^{\xi_0}} \le 3n\kappa_2 \sum_{j=1}^{L+1} t_j \le \frac{3}{4}. \tag{5.24}$$

Hence, we have

$$||{}^t(\widetilde{x}_{1,L} - \widetilde{x}^*_{1,L}, \widetilde{\xi}_{1,L} - \widetilde{\xi}^*_{1,L})||_\infty^{\xi_0} \le 4||{}^t(\widetilde{z}_{1,L}, \widetilde{\zeta}_{1,L})||_1^{\xi_0}. \tag{5.25}$$

Therefore,

$$\begin{aligned} &\left(1 + \langle\xi_0\rangle^{1/2} \max_{j=1,2,\dots,L} |x_j - x_j^*| + \langle\xi_0\rangle^{-1/2} \max_{j=1,2,\dots,L} |\xi_j - \xi_j^*|\right) \\ &\le 4\left(1 + \langle\xi_0\rangle^{1/2} \sum_{j=1}^L |z_j| + \langle\xi_0\rangle^{-1/2} \sum_{j=1}^L |\zeta_j|\right) \\ &\le 4\prod_{j=1}^L (1 + \langle\xi_0\rangle^{1/2}|z_j|) \cdot \prod_{j=1}^L (1 + \langle\xi_0\rangle^{-1/2}|\zeta_j|) \\ &\le 4 \cdot 2^L \prod_{j=1}^L (1 + \langle\xi_0\rangle|z_j|^2)^{1/2} \cdot \prod_{j=1}^L (1 + \langle\xi_0\rangle^{-1}|\zeta_j|^2)^{1/2}. \end{aligned} \tag{5.26}$$

6°. Integrating by parts, we have

$$\mathbb{I}(\Phi, p_{\alpha,\beta}) = \mathbb{I}(\Phi, p^{\circ}_{\alpha,\beta}), \tag{5.27}$$

where

$$\begin{aligned} p^{\circ}_{\alpha,\beta} &= (M_L^*)^{2|\alpha+\beta|+n+1}(M_{L-1}^*)^{2|\alpha+\beta|+n+1}\cdots(M_1^*)^{|\alpha+\beta|+n+1} \\ &\quad \circ (N_L^*)^{2|\alpha+\beta|+n+1}(N_{L-1}^*)^{2|\alpha+\beta|+n+1}\cdots(N_1^*)^{2|\alpha+\beta|+n+1}p_{\alpha,\beta}. \end{aligned} \tag{5.28}$$

Hence, there exists a positive constant C_3 such that

$$\begin{aligned} |\partial^{\beta}_{x_{L+1}}\partial^{\alpha}_{\xi_0}q_0| &= |\mathbb{I}(\Phi, p^{\circ}_{\alpha,\beta})| \\ &\le (C_3)^L|p|^{(\widetilde{m}_{L+1})}_{3|\alpha+\beta|+n+1,3|\alpha+\beta|+n+1}\langle\xi_0\rangle^{\sum_{j=1}^{L+1} m_j+(1-\rho)|\beta|-\rho|\alpha|}. \end{aligned} \tag{5.29}$$

7°. Now, we separate $a_{\alpha,\beta}$ in (5.11) depending on the degree of the term:

$$\Big(1+\langle\xi_0\rangle^{1/2}\max_{j=1,2,\dots,L}|y_j| + \langle\xi_0\rangle^{-1/2}\max_{j=1,2,\dots,L}|\eta_j|\Big),$$

to get a better estimate. Similarly, we can make better estimates for (5.12)–(5.29).
In particular, the new estimates for (5.29) is the following:

$$|\partial^{\beta}_{x_{L+1}}\partial^{\alpha}_{\xi_0}q_0| \le (C_4)^L|p|^{(\widetilde{m}_{L+1})}_{2|\alpha+\beta|+n+1,2|\alpha+\beta|+n+1}\langle\xi_0\rangle^{\sum_{j=1}^{L+1} m_j+(1-\rho)|\beta|-\rho|\alpha|}. \tag{5.30}$$

Therefore, in the case where $R = 0$, it is enough to take $l_1 \ge n+1+2(l+l')$ and $l'_1 \ge n+1+2(l+l')$. □

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