<table>
<thead>
<tr>
<th><strong>Title</strong></th>
<th>On congruent axioms in linearly ordered spaces. II</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Author(s)</strong></td>
<td>Terasaka, Hidetaka; Katayama, Shigeru</td>
</tr>
<tr>
<td><strong>Citation</strong></td>
<td>Osaka Journal of Mathematics. 4(1) P.111-P.132</td>
</tr>
<tr>
<td><strong>Issue Date</strong></td>
<td>1967</td>
</tr>
<tr>
<td><strong>Text Version</strong></td>
<td>publisher</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="https://doi.org/10.18910/5749">https://doi.org/10.18910/5749</a></td>
</tr>
<tr>
<td><strong>DOI</strong></td>
<td>10.18910/5749</td>
</tr>
<tr>
<td><strong>rights</strong></td>
<td></td>
</tr>
</tbody>
</table>
6. Model \( M(R, C, I) \)

\( M(R, C, I) \): A model of a geometry in which Axioms \( R, C \) and \( I \) alone hold besides Axiom \( E \). (Notice that \( I \) follows automatically from \( E, R \) and \( C \).)

The construction of \( M(R, C, I) \) is quite different from those of other models, and its exposition here may be too long, but it seems to the authors appropriate to provide it with a full proof. It depends essentially upon Lemma below, and we will begin by introducing some definitions and auxiliary axioms needed in it.

Let \( A \) be a finite number of linearly ordered points, in which congruence relations are supposed to hold among some of the segments, and let \( P, Q, P' \) etc.

**Definition.** We write

\[ PQ \cong Q'P' \quad \text{or} \quad Q'P' \cong PQ, \]

if and only if

\[ PQ = Q'P' \quad \text{and} \quad Q'P' = PQ, \]

at the same time.

**Axiom \( E_w \):** If \( PQ = Q'P' \) and \( PQ = Q'P'' \), then \( P' = P'' \).

**Axiom \( C^+ (= \text{Axiom } C) \)**

\[
\begin{align*}
P < Q < R, \\
R' < Q' < P', \\
PQ = Q'P', \\
QR = R'Q'
\end{align*}
\] \[ \Rightarrow \quad PR = R'P'. \]

**Axiom \( C^+ \)**

\[
\begin{align*}
P < Q < R, \\
R' < Q' < P', \\
PQ \cong Q'P', \\
QR \cong R'Q'
\end{align*}
\] \[ \Rightarrow \quad \begin{cases} PR \cong R'P' \\ QR \cong R'Q' \end{cases}. \]

---

Axiom $C^-$

\[
P < R < Q , \quad \begin{cases} \ P \neq Q'P' , \\ RQ \equiv Q'R' \end{cases} \Rightarrow PR \equiv R'P' .
\]

Axiom $\bar{C}^-$:

\[
P < R < Q , \quad \begin{cases} \ P \neq Q'P' , \\ RQ \equiv Q'R' \end{cases} \Rightarrow \begin{cases} PR \equiv R'P' , \\ QP \equiv Q'P' , \end{cases}
\]

The following is an important consequence of $\bar{C}^-$, and will sometimes be denoted by $e^-$. 

\[
e^- : \quad PQ \equiv Q'P' , \quad Q' < P \Rightarrow PQ \equiv Q'P' , \quad Q'P \equiv P'Q .
\]

Proof.

\[
Q' < P < Q, \quad Q'Q \equiv Q'Q , \quad PQ \equiv Q'P' \quad \Rightarrow \begin{cases} PQ \equiv Q'P' , \\ Q'P \equiv P'Q . \end{cases}
\]

Definition. A segment $PQ$ will be called elementary, if there is no point $X$ with $P < X < Q$.

Lemma. Let $A_{n-1} = \{A_\lambda | \lambda = 1, 2, \cdots, n-1\}$ be a finite number of points in some linear order such that they satisfy Axioms $E_n$, $R$, $C^+$, $\bar{C}^+$, $C^-$, and $\bar{C}^-$. Then, for a given elementary segment $A_1A_j$ and a given point $A_k$ such that the equality

\[
A_iA_j = A_kA_1
\]

has no solution in $A_i \in A_{n-1}$, a new point $A_n$ can be introduced, so that

\[
A_iA_j = A_kA_n
\]

holds and the linearly ordered points $A_n = \{A_1, A_2, \cdots, A_{n-1}, A_n\}$ satisfy the same Axioms from $E_n$ to $\bar{C}^-$. 

Proof. Points as well as notations such as $A$, $P$, $X$, $P'$ etc. will mean in this proof points of $A_{n-1}$, except for $A'$ which will be introduced below as a new point $A_n$. If two segments are equal it is convenient to write the corresponding end points counterwise with and without dashes such as $PQ = Q'P'$, since several axioms of the type of $C$ are involved.

For the sake of simplicity, set $A_i = A$, $A_j = B$ and $A_k = B'$. 
Thus by assumption there is no point $X$ with $AB = B'X$

**Definition of the new point $A'(=A_n)$ and of ordering.**

Let $A'$ be introduced as a new point such that $\{A_1, \ldots, A_{n-1}, A'\}$ satisfy the following linear ordering:

(i) $B'<A'$.
(ii) If $X < B'$, then $X < A'$ for any point $X \in A_{n-1}$.
(iii) If $B' < X$, then $A' < X$ for any point $X \in A_{n-1}$.

**Definition of the basic equality.** The following is the basic congruence relation:

(i) $AB \approx B'A'$,

i.e. $AB = B'A'$ and $B'A' = AB$ at the same time, if and only if there exist some $X$ and $X'$ such that $XB \approx B'X'$.

In particular, $AB \approx B'A'$ if $B' < A$, since $B'B \approx B'B$.

(ii) Otherwise $AB = B'A'$ but $B'A' \neq AB$, that is, $B'A' = AB$ is not defined.

**Definition of other equalities.** Besides the above basic congruence relation we must define other new congruence relations in order to make the system of points $A_n = \{A_1, \ldots, A_{n-1}, A_n\}$ satisfy all axioms from $E_n$ to $C'$.

To insure Axiom R we only need

**Definition 0.** *For any $X \in A_{n-1}$: $A'X = A'X$ and $XA' =XA'$.*

In the following are defined all the equalities between old segments and new ones with one end point $A'$. They are classified into four types according to the position of $A'$.

Some of them are redundant, such as $AB = B'A'$, $AA' = AA'$ and $A'A = A'A$, but are included for the sake of completeness.

**Definition 1.** $AP = P'A'$, if and only if

(i) $P = B$, $P' = B'$, i.e., $AB = B'A'$,

or (ii) $BP = P'B'$,

or (iii) $P = A'$, $P' = A$, i.e., $AA' = AA'$. 


DEFINITION 2. \( P'A' = AP \), if and only if
\[(i) \ P' = B', \ P = B, \ i.e., \ B'A' = AB, \]
or
\[(ii) \ P' < A \ (or \ B'A' = AB) \ and \ P'B' = BP, \]
or
\[(iii) \ P' = A, \ P = A', \ i.e., AA' = AA'. \]

DEFINITION 3. \( PA = A'P' \), if and only if
\[(i) \ PB = B'P', \]
or
\[(ii) \ P = A, \ P' = A \ (or \ A' = A'). \]

DEFINITION 4. \( A'P' = PA \), if and only if
\[(i) \ B'P' \approx PB, \]
or
\[(ii) \ P' = A, \ P = A', \ i.e., A'A = A'A. \]

Having thus defined all congruence relations between old segments and new ones with one end point \( A' \), we are now going to verify Axioms \( E_u, C^+, C^- \) and \( C_- \) one by one.

The verification will be done after a pattern: each equality under consideration is first classified according to its type, and then dealt with by Definitions 1, 2, 3 and 4 accordingly almost mechanically. Verbal explanations in detail will be omitted.

VERIFICATION OF \( E_u \).

Type 1.
\[ AP = P'A', \ AP = P'X \Rightarrow X = A'. \]

Proof. According to Definition 1, we divide the proof into three cases.

Case (i). \( P = B, \ P' = B' \): \( AB = B'A' \).
Then \( AB = B'X \) is impossible for \( X \in A_{n-1} \).

Case (ii). \( BP = B'P' \).
\[ A < B < P, \ AP = P'X, \ BP = P'B' \quad (C^-) \Rightarrow AB = B'X, \]
which is impossible for any old point \( X \in A_{n-1} \).

Case (iii). \( P = A', \ P' = A \): \( AA' = AA' \).
Then \( AA' = AX \) is impossible for any old point \( X \in A_{n-1} \).

Type 2.
\[ P'A' = AP, \ P'A' = AX \Rightarrow X = P \]

Proof. Divide into three cases by Definition 2.

Case (i). \( P' = B', \ P = B \): \( B'A' = AB \).
Then \( B'A' = AX \) is only possible for \( X = B \) by Definition 2.

Case (ii). \( P' < A \ (or \ B'A' = AB) \) and \( P'B' = BP \) and \( P'B' = BX \).
Then \( X = P \) by Axiom \( E_u \) applied to old congruence relations.
Case (iii).  \( P' = A, P = A' \):  \( AA' = AA' \),  
\( AA' = AX \).  Then \( X = A' \) by Definition 2.

**Type 3.**

\[ PA = A'P', PA = A'X \Rightarrow X = P' \]

Proof. Divide into two cases by Definition 3.

Case (i).  \( PB = B'P' \).  Then

\[ PB = B'P', PB = B'X \overset{(Ea)}{\Longrightarrow} X = P' \]

Case (ii).  \( P = A', P' = A \).  Then

\[ A'A = A'A', A'A = A'X \Rightarrow X = A \quad \text{by Definition 3.} \]

**Type 4.**

\[ A'P' = PA, A'P' = PX \Rightarrow X = A \]

Proof. Divide into two cases by Definition 4.

Case (i).  \( B'P' \approx PB \).  Then

\[ B'P' \approx PB, A'P' = PX \Rightarrow X = A \quad \text{by Definition 4.} \]

Case (ii).  \( P' = A, P = A' \).

\[ A'A = A'A, A'A = A'X \Rightarrow X = A \quad \text{by Definition 4.} \]

**Verification of \( C^+ \).**

To show that Axiom \( C^+ \) is satisfied for \( A_n = \{ A_i, \ldots, A_n = A' \} \) we consider six types of equalities.

**Type 1.**

\[
\begin{align*}
A &< P < Q, \quad Q' < P' < A', \\
AP & = P'A' \quad (1), \\
EQ' & = Q'P' \quad (2)
\end{align*}
\]

Proof. We divide the proof into three cases, according to (1); cf. Definition 1.

Case (i).  \( P = B, P' = B' \).  Then from (2),

\[ BQ = Q'B' \overset{(Def.1(ii))}{\Longrightarrow} AQ = Q'A' \]

Case (ii).  \( BP = B'B' \).

\[
\begin{align*}
B &< P < Q, \quad Q' < P' < B', \\
BP & = P'B', \quad PQ = Q'P' \quad (C^+) \overset{(Def.1(ii))}{\Longrightarrow} BQ = Q'B' \overset{(Def.1(ii))}{\Longrightarrow} AQ = Q'A'.
\end{align*}
\]

Case (iii).  \( P = A', P' = A \).  Then (2) becomes

\[ A'Q = Q'A' \quad (2') \]

Divide into two subcases according to (2'); cf. Definition 4.

Subcase (i).  \( B'Q \approx Q'B' \).
$Q'<B'$, $B'Q\approx Q'B \xrightarrow{(\varepsilon^{-})} BQ = Q'B' \xrightarrow{(\text{Def.}1)} AQ = Q'A'$.

Subcase (ii). $Q=A$, $Q'=A'$. This is impossible, since $A<Q$.

Type 2.

$Q'<P'<A'$, $A<P<Q$,

\[ P'A'=AP \quad (1) \quad \text{and} \quad Q'P'=PQ \quad (2) \]

Divide into three cases by (1); cf. Definition 2.

Case (i). $P'=B'$, $P=B$: $B'A'=AB$.

$b'A'=AB$, $Q'B'=BQ \quad \xrightarrow{\text{Def.2} (ii)} \quad Q'A'=AQ$.

Case (ii). $P'<A$ (or $B'A'=AB$) and $P'B'=BP$.

$P'B'=BP$, $Q'P'=PQ \quad \xrightarrow{\text{(C')}} \quad Q'B'=BQ \quad \xrightarrow{\text{(Def.2} (ii))} \quad Q'A'=AQ$.

Case (iii). $P'=A$, $P=A'$: $A<A'$. Then (2) becomes

$Q'A=A'Q$ \quad (2)'$

We divide into two subcases according to (2)'; cf. Definition 3.

Subcase (i). $Q'B=B'Q$.

Since $A<A'$ and since $AB$ and $B'A'$ are elementary, either $B<B'$ or $B=B'$.

If $B<B'$,

$Q'B=B'Q \quad \xrightarrow{\text{C'}} \quad Q'B'=BQ \quad \xrightarrow{Q'<A} \quad Q'A'=AQ$.

If $B=B'$,

$Q'<A$, $Q'B'=BQ \quad \xrightarrow{\text{Def.2} (ii)} \quad Q'A'=AQ$.

Subcase (ii). $Q'=A'$, $Q=A$.

This case is impossible, since $Q'<A$.

Type 3.

$P<Q<A$, $A'<Q'<P'$,

\[ QA = A'Q' \quad (1) \quad \text{and} \quad PQ = Q'P' \quad (2) \]

\[ \Rightarrow PA = A'P' \quad . \]
We divide into two cases by (1); cf. Definition 3.

Case (i). \( QB = B'Q' \).

\[
QB = B'Q', \ PQ = Q'P' \quad \text{(C)} \quad \Rightarrow \quad PB = B'P' \quad \text{(Def. 3)} \quad PA = A'P'.
\]

Case (ii). \( Q = A', Q' = A : A' < A \). Then (2) becomes

\[
PA' = AP'.
\]

Divide into three subcases by (2)'; cf. Definition 2.

Subcase (i). \( P = B', P' = B \). Since \( B' < A' < A < B \),

\[
B'P = B'B \quad \text{(Def. 3)} \quad \Rightarrow \quad B'A = A'B, \ i.e., \ PA = A'P'.
\]

Subcase (ii). \( P < A \) (or \( B'A' = AB \)) and \( PB' = BP' \).

\[
PB' = BP', \ B'B = B'B \quad \text{(C)} \quad \Rightarrow \quad PB = B'P' \quad \text{(Def. 3)} \quad PA = A'P'.
\]

Subcase (iii). \( P = A, P' = A' \). This case is impossible, since \( A' < P' \).

Type 4.

\[
\begin{align*}
A' < Q' < P', & \quad P < Q < A, \\
A'Q' = QA & \quad \text{(1)}, \\
Q'P' = PQ & \quad \text{(2)}
\end{align*}
\]

Proof. Divide into two cases by (1); cf. Definition 4.

Case (i). \( B'Q' \approx QB \).

\[
B'Q' \approx QB, \ PQ = PQ \quad \text{(C)} \quad \Rightarrow \quad B'P' \approx PB \quad \text{(Def. 4)} \quad AP' = PA' .
\]

Case (ii). \( Q' = A, Q' = A' : A' < A \). Then (2) becomes

\[
AP' = PA' .
\]

Divide into three subcases by (2)'; cf. Definition 1.

Subcase (i). \( P' = B, P = B' \). Then \( B' < B \), since \( B' < A' < A < B \). Then

\[
B'P = B'B \quad \text{(Def. 4)} \quad \Rightarrow \quad A'B = B'A, \ i.e., \ A'P' = PA .
\]

Subcase (ii).

\[
BP' = PB', \ B'B = B'B \quad \text{(C)} \quad \Rightarrow \quad B'P' \approx PB \quad \text{(Def. 4)} \quad AP' = PA .
\]

Subcase (iii). \( P' = A', P = A \). Impossible, since \( A' < P' \).

Type 5.

\[
\begin{align*}
P < A < Q, & \quad Q' < A' < P', \\
PA = A'P' & \quad \text{(1)}, \\
AQ = Q'A' & \quad \text{(2)}
\end{align*}
\]

Proof. Divide into two cases by (1); cf. Definition 3.
Case (i). \( PB = B'P' \).

Divide into three subcases by (2); cf. Definition 1.

Subcase (i). \( Q = B, Q' = B' \). Then

\[ PB = B'P' \]  gives  \[ PQ = Q'P' \].

Subcase (ii). \( BQ = Q'B' \).

\[ PB = B'P', BQ = Q'B' \]  \( \Longleftrightarrow \)  \[ PQ = Q'P' \].

Subcase (iii). \( Q = A', Q' = A \).

This case has been treated in Type 2.

Case (ii). \( P = A', P' = A \).

Proved in Type 4.

Type 6.

\[ \begin{align*}
Q' &< A' < P', P < A < Q, \\
Q'A' & = AQ \quad (1), \\
A'P' & = PA \quad (2)
\end{align*} \]

\[ \Rightarrow Q'P' = PQ. \]

Proof. Divide into two cases by (2); cf. Definition 4.

Case (i). \( B'P' \approx PB \).

Divide into three subcases by (1); cf. Definition 2.

Subcase (i). \( Q' = B', Q = B \). Then

\[ B'P' = PB \]  gives  \[ Q'P' = PQ \]

Subcase (ii). \( Q' < A \) (or \( B'A' = AB \)) and \( Q'B' = BQ \).

\[ Q'B' = BQ, B'P' = PB \]  \( \Longleftrightarrow \)  \[ Q'P' = PQ \].

Subcase (iii). \( Q' = A, Q = A' \).

This case has been proved in Type 1.

Case (ii). \( P = A, P' = A' \).

Has been proved in Type 3.

Verification of \( C^+ \).

Type 1.

\[ \begin{align*}
A &< P < Q, \ Q' < P' < A', \\
AP &\approx P'A' \quad (1), \\
PQ &\approx Q'P' \quad (2)
\end{align*} \]

\[ \Rightarrow \begin{cases} 
PQ \approx Q'P', \\
AQ \approx Q'A'.
\end{cases} \]

Proof. Divide into three cases by (1); cf. Definition 1.

Case (i). \( P = B, P' = B' \): \( AB \approx B'A' \).

\[ AB \approx B'A' \]  \( (\text{Def}) \)  \[ \exists X, X' \]: \( XB \approx B'X' \).

\[ XB \approx B'X', BQ = Q'B' \]  \( \overline{C^+} \)  \[ BQ \approx Q'B' \], i.e., \( PQ \approx Q'P' \).
ON CONGRUENT AXIOMS IN LINEARLY ORDERED SPACES, II

\[ PQ \approx Q'P', \ AB \approx B'A' \quad (\text{Def. 1, 2}) \quad A \approx Q'A' \]

\[
\begin{array}{c|c|c|c}
L & A & B=P & Q \\
\hline
L & Q' & P'=B' & A' & X' \\
\end{array}
\]

Case (ii). \( P' < A \) (or \( B'A' = AB \)) and \( P'B' \approx BP \)

\[
BP \approx P'B', \ PQ \approx Q'P' \quad (\text{Def. 3, 4}) \quad BQ \approx Q'B', \quad \begin{cases} \text{Case (ii).} \\
Q' < A \end{cases} \quad (\text{or } B'A' = AB) \quad (\text{Def. 1, 2}) \quad A \approx Q'A'.
\]

Case (iii). \( P = A', P' = A \).

\[
A'Q = Q' A. \\
\text{(2)'}
\]

Divide into two subcases by (2)'; cf. Definition 4.

Subcase (i). \( B'Q \approx Q'B \).

\[
B'Q \approx Q'B \quad (\text{Def. 3, 4}) \quad Q' < A \Rightarrow B' < B', Q'A = A'Q, \text{ i.e., } Q'P' \approx PQ.
\]

Since \( A < A' \), \( Q' < P' = A < B' \).

\[
Q' < B', B'Q \approx Q'B \quad (\text{Def. 3, 4}) \quad BQ \approx Q'B' \Rightarrow A \approx Q'A'.
\]

Subcase (ii). \( Q = A, Q' = A' \). Impossible, since \( A < Q \).

Type 2.

\[
\begin{cases} \text{Case (i).} \\
Q' < P' < A', A < P < Q, \quad (1) \quad \Rightarrow \quad Q'A' \approx AQ, \\
P'A' = AP \quad (2) \quad \Rightarrow \quad P'A' \approx AP.
\end{cases}
\]

Proof. Divide into three cases by (2); cf. Definition 2.

Case (i). \( P' = B', P = B \); \( B'A' = AB \).

\[
Q'B' \approx BQ, \ AB \approx B'A' \quad (\text{Def. 1, 2}) \quad A \approx Q'A'. \\
B'A' \approx AB \quad \text{gives} \quad P'A' \approx AP.
\]

Case (ii). \( P' < A \) (or \( B'A' = AB \)) and \( P'B' = BP \).

\[
Q'P' \approx PQ, \ P'B' = BP \quad (\text{Def. 1, 2}) \quad Q'B' \approx BQ \quad \Rightarrow \quad P'B' \approx BP \Rightarrow P'A' \approx AP.
\]

Case (iii). \( P' = A, P = A'; A < A' \). Then (1) becomes

\[
Q'A' \approx A'Q. \\
\text{(1)'}
\]
Divide into two subcases by (1)'; cf. Definition 3,4.
Subcase (i). \( Q'B \approx B'Q \).
Since \( A < A' \), \( Q' < P' = A < B \leq B' \).

\[
\begin{align*}
Q' < B', & \quad Q'B \approx B'Q \\
& \quad \xrightarrow{(\xi^-)} Q'B' \approx BQ \\
& \quad \quad \rightarrow Q'A' \approx AQ .
\end{align*}
\]

\( AP \approx P'A' \) is evident.

Subcase (ii). \( Q' = A' \), \( Q = A \). Impossible, since \( Q' < A' \).

**Type 3.**

\[
\begin{align*}
P & < Q < A, \quad A' < Q' < P' , \\
PQ & \approx Q'P' \quad (1) , \\
QA & = A'Q' \quad (2) ,
\end{align*}
\]

**Proof.** Divide into two cases by (2); cf. Definition 3.

Case (i). \( QB = B'Q' \) \xrightarrow{(\xi^+)} \( PB \approx B'P' \; (\text{Def. 3,4}) \; PA \approx A'P' .

\( PQ \approx Q'P' \) \xrightarrow{(\xi^-)} \( QB \approx B'Q' \; (\text{Def. 3,4}) \; QA \approx A'Q' .

Case (ii). \( Q = A' \), \( Q' = A' \). \( A' < A \). (1) becomes

\[ PA' \approx AP' \;
\]

**Proof.** Divide into two subcases by (1)'; cf. Definition 1,2.

Subcase (i). \( P = B' \), \( P' = B' \). \( B' A' \approx AB .

\[ A'A \approx A'A \; \text{gives} \; QA \approx A'Q' .
\]

Since \( B' < A' < A < B \),

\[ B'B \approx B'B \; (\text{Def. 3,4}) \; B' \approx B' , \text{i.e.,} \; PA \approx A'P' .
\]

Subcase (ii). \( BP' \approx PB' \). Since \( P < A < B \),

\[ BP' \approx PB', \; P < B \; (\xi^-) \; PB \approx B'P' \rightarrow PA \approx A'P' .
\]

Subcase (iii). \( P = A \), \( P' = A' \). Impossible, since \( P < A \).

**Type 4.**

\[
\begin{align*}
A' & < Q' < P', \; P < Q < A , \\
A'Q' & \approx QA \quad (1) , \\
Q'P' & = PQ \quad (2) ,
\end{align*}
\]

**Proof.** Divide into two cases by (1); cf. Definition 4.

Case (i). \( B'Q' \approx QB \).

\[
\begin{align*}
B'Q' & \approx QB, \; Q'P' = PQ \; (\xi^-) \; \rightarrow \quad PB \approx B'P' \rightarrow A'P' \approx PA .
\end{align*}
\]
Case (ii). \( Q' = A, \ Q = A' : A' < A \). (2) becomes
\[ AP' = PA' \quad (2)' \]
Divide into three subcases by (2)'; cf. Definition 1.

Subcase (i). \( P' = B, \ P = B' \). Since \( B' < A' < A < B \),
\[ B' B \approx B' B' \xRightarrow{(\text{Def.}, 3, 4)} A' B \approx B' A', \text{ i.e., } A' P' \approx PA \cdot \]
\[ AB = B' A', \ B' B \approx B' B \xRightarrow{(\text{Def.})} AB \approx B' A', \text{ i.e., } Q' P' \approx PQ . \]

Subcase (ii). \( BP' = PB' \). Since \( P < B' < A' < A < B \),
\[ P < B, \ BP' = PB' \xRightarrow{(\text{C} - 1)} \begin{cases} PB \approx B' P' \xRightarrow{(\text{Def.}, 3, 4)} A' P' \approx PA \cdot \\ BP' \approx PB' \end{cases} \xRightarrow{(\text{Def.} 1, 2)} AP' \approx PA' \cdot \text{ i.e., } Q' P' \approx PQ . \]

Subcase (iii). \( P' = A', \ P = A \). Impossible, since \( A' < P' \).

Type 5.
\[ P < A < Q, \ Q' < A' < P' , \]
\[ PA \approx A' P' \quad (1) , \]
\[ A' Q = Q' A' \quad (2) \]
Proof. Divide into two cases by (1); cf. Definition 3.

Case (i). \( B' P' \approx PB' \).
Divide into three subcases by (2); cf. Definition 1.

Subcase (i). \( Q = B, \ Q' = B' \).
\[ B' P' \approx PB \text{ gives } Q' P' \approx PQ . \]
\[ B' P' \approx PB \xRightarrow{(\text{Def.})} AB \approx B' A' , \text{ i.e., } A' Q = Q' A' . \]

Subcase (ii). \( BQ = Q' B' \).
\[ BQ = Q' B' , \ PB \approx B' P' \xRightarrow{(\text{C} - 1)} PQ \approx Q' P' . \]
\[ PB \approx B' P' \xRightarrow{(\text{Def.})} B' A' \approx AB \xRightarrow{(\text{Def.} 1, 2)} A' Q = Q' A' . \]

Subcase (iii). \( Q = A', \ Q' = A \). Proved in Type 2.
Case (ii). \( P = A', \ P' = A \). Proved in Type 4.

Type 6.
\[ Q' < A' < P' , \ P < A < Q , \]
\[ Q' A' \approx A' Q \quad (1) , \]
\[ A' P' = PA \quad (2) \]
Proof. Divide into two cases by (2); cf. Definition 4.
Case (i). \( B'P' \approx PB \).

Divide into three subcases by (1); cf. Definition 2.

Subcase (i). \( Q' = B' \), \( Q = B \):

\[ B'P' \approx PB \] gives \( Q'P' \approx PQ \).

\[ B'P' \approx PB \xrightarrow{(\text{Def.3,4})} PA \approx A'P' \].

Subcase (ii). \( Q' < A \) (or \( B'A' = AB \)) and \( Q'B' \approx BQ \).

\[ B'P' \approx PB, \ Q'B' \approx BQ \xrightarrow{(C')} Q'P' \approx PQ \].

\[ B'P' \approx PB \xrightarrow{(\text{Def.3,4})} PA \approx A'P' \].

Subcase (iii). \( Q = A' \), \( Q' = A \). Proved in Type 1.

Case (ii). \( P' = A \), \( P = A' \). Proved in Type 3.

**Verification of \( C^- \).**

*Type 1.*

\[ A < P < Q, \]
\[ AQ = Q'A' \quad (1), \]
\[ PQ = Q'P' \quad (2) \]

Proof. Divide into three cases by (1); cf. Definition 1.

Case (i). \( Q = B \), \( Q' = B' \). Impossible, since \( AB \) is elementary.

Case (ii). \( BQ = Q'B' \).

\[ B < P < Q, \]
\[ BQ = Q'B', \ PQ = Q'P' \xrightarrow{(C')} BP = P'B' \Rightarrow AP = P'A'. \]

Divide into three subcases by (2'); cf. Definition 2.

Subcase (i). \( P = B' \), \( P' = B \). Since \( AB \) is elementary,

\[ A < B \leq B' < A'. \]

If \( B < B' \),

\[ BB' = BB' \xrightarrow{(\text{Def.1(ii)})} AB' = BA', \ i.e., AP = P'A'. \]

If \( B = B' \), \( AB = B'A' \) gives \( AP = P'A' \).
ON CONGRUENT AXIOMS IN LINEARLY ORDERED SPACES, II 123

Subcase (ii). \( P < A \) (or \( B' A' = AB \)) and \( PB' = BP' \).
Since \( A < P < A' \) and since \( AB \) is elementary,
\[
B \leq P,
\]
If \( B < P \),
\[
PB' = BP' \quad (\varepsilon^{-}) \quad \Rightarrow \quad BP = P'B' \Rightarrow AP = P'A'.
\]
If \( B = P \), then \( B' = P' \), so \( AP = P'A' \).
Subcase (iii). \( P = A, P' = A' \). Impossible, since \( A < P \).

Type 2.
\[
\begin{align*}
Q' &< P' < A' , \\
Q' A' & = AQ \quad (1) , \quad \Rightarrow \quad Q' P' = PQ . \\
P' A' & = AP \quad (2) 
\end{align*}
\]

Proof. Divide into three cases by (2); cf. Definition 2.
Case (i). \( P' = B' , P = B \). \( B' A' = AB \).
Divide into three subcases by (1); cf. Definition 2.
   Subcase (i). \( Q' = B' , Q = B \). \( Q' < P' \). Impossible, since \( Q' < P' \).
   Subcase (ii). \( Q' < A \) (or \( B' A' = AB \)) and \( Q' B' = BQ \).
   \[
   Q' B' = BQ \quad \text{gives} \quad Q' P' = PQ
   \]
   Subcase (iii). \( Q = A , Q' = A \). \( A < A' \).
If \( B < B' \),
\[
BB' = BB' \quad (\text{Def.} 1) \quad \Rightarrow \quad AB' = BA' , \quad \text{i.e.,} \quad Q' P' = PQ.
\]
If \( B = B' , AB = B' A' \) gives \( Q' P' = PQ \).
Case (ii). \( P' < A \) (or \( B' A' = AB \)) and \( P' B' = BP \).
Divide into three subcases by (1); cf. Definition 2.
   Subcase (i). \( Q' = B' , Q = B \). \( Q' < P' < B' \). Impossible, since \( Q' < P' < B' \).
   Subcase (ii). \( Q' < A \) (or \( B' A' = AB \)) and \( Q' B' = BQ \).
   \[
   Q' < P' < B' , \quad Q' B' = BQ , \quad P' B' = BP \quad (\text{C}) \quad \Rightarrow \quad Q' P' = PQ .
   \]
   Subcase (iii). \( Q' = A , Q = A' \). \( A < P' < A' \), since \( Q' < P' < A' \).
If \( B < P' \),
\[
B < P' , \quad P' B' = BP' \quad (\varepsilon^{-}) \quad \Rightarrow \quad BP' = PB' \Rightarrow AP' = P A' , \quad \text{i.e.,} \quad Q' P' = PQ .
\]
If \( B = P' \), then \( B' = P \) and \( AB = B' A' \) gives \( Q' P' = PQ \).
Case (iii). \( P' = A , P = A' \). \( AA' = AA' \).
Divide into three cases by (1); cf. Definition 2.
   Subcase (i). \( Q' = B' , Q = B \). \( Q' < P' < A' \), since \( Q' < P' < A' \).
   Subcase (ii). \( Q' < A \) (or \( B' A' = AB \)) and \( Q' B' = BQ \).
If \( B < B' \),
\[
B < B', \ Q'B' = BQ \xrightarrow{(C^-)} Q'B = B'Q \xrightarrow{(\text{Def. 3})} Q'A = A'Q, \ i.e., \ Q'P' = PQ.
\]
If \( B = B' \),
\[
Q'B' = BQ \Rightarrow Q'A = A'Q, \ i.e., \ Q'P' = PQ.
\]
Subcase (iii). \( Q' = A, \ Q = A' \). Impossible, since \( Q' < P' \).

Type 3.
\[
\begin{align*}
P<Q<A, \\
P'A = A'P' \quad (1), & \quad \Rightarrow PQ = Q'P'. \\
Q'A = A'Q' \quad (2)
\end{align*}
\]

Proof. Divide into two cases by (1); cf. Definition 3.
Case (i). \( PB = B'P' \).
Divide into two subcases by (2); cf. Definition 3.
Subcase (i). \( QB = B'Q' \).
\[
P<Q<B, \ QB = B'Q', \ PB = B'P' \xrightarrow{(C^-)} PQ = Q'P'.
\]
Subcase (ii). \( Q = A', \ Q' = A : A' < A \). Then \( P \leq B' \), since \( P < Q \) and since \( B' A' \) is elementary.
If \( P < B' \),
\[
P < B' < B \xrightarrow{(C^-)} PB' = BP' \xrightarrow{(\text{Def. 2(ii)})} PA' = AP', \ i.e., \ PQ = Q'P'.
\]
If \( P = B' \), then \( B = P' \) and
\[
B'B = B'B' \xrightarrow{(\text{Def.})} B'A' = AB, \ i.e., \ PQ = Q'P'.
\]
Case (ii). \( P = A', \ P' = A : A' < A \).
Divide into two subcases by (2); cf. Definition 3.
Subcase (i). \( QB = B'Q' \).
\[
B' < A' = P < Q, \xrightarrow{\text{(C^-)}} B'Q = Q'B \xrightarrow{(\text{Def. 4})} A'Q = QA, \ i.e., \ PQ = Q'P'.
\]
Subcase (ii). \( Q = A', \ Q' = A \). Impossible, since \( P < Q \).

Type 4.
\[
\begin{align*}
A' < Q' < P', \\
A'P' = PA \quad (1), & \quad \Rightarrow A'Q' = QA. \\
Q'P' = PQ \quad (2)
\end{align*}
\]

Proof. Divide into two cases by (1); cf. Definition 4.
Case (i). \( B'P' \approx PB \).
ON CONGRUENT AXIOMS IN LINEARLY ORDERED SPACES, II

\[
\begin{align*}
B' < A' &< Q' < P' \\
B'P' &\cong PB, \; Q'P' = PQ \\
\left\{ \begin{array}{c}
\text{(C--)} \\
\text{(Def. 4)}
\end{array} \right\} \Rightarrow B'Q' &\cong QB \Rightarrow A'Q' = QA .
\end{align*}
\]

Case (ii). \( P' = A, \; P = A' \); \( A' < A \). Then (2) becomes

\[
Q'A = A'Q . \quad (2)'
\]

Divide into two subcases by \((2)'\); cf. Definition 3.

Subcase (i). \( Q'B = B'Q \).

\[
\begin{align*}
B' < A' &< Q', \; Q'B = B'Q \rightarrow B'Q' &\cong QB \Rightarrow A'Q' = QA .
\end{align*}
\]

Subcase (ii). \( Q' = A', \; Q = A \). Impossible, since \( A' < Q' \).

Type 5.

\[
\begin{align*}
P &< A < Q , \\
PQ &= Q'P' \quad (1) , \\
AQ &= Q'A' \quad (2) \\
\end{align*}
\]

Proof. Divide into three cases by \((2)\); cf. Definition 1.

Case (i). \( BQ = Q'B' \).

\[
\begin{align*}
P &< B < Q , \\
BQ &= Q'B', \; PQ = Q'P' \rightarrow \left\{ \begin{array}{c}
\text{(C--)} \\
\text{(Def. 3)}
\end{array} \right\} PB = B'P' \Rightarrow PA = A'P' .
\end{align*}
\]

Case (ii). \( Q = B, \; Q' = B' \). Then

\[
PB = B'P' \Rightarrow PA = A'P' .
\]

Case (iii). \( Q = A', \; Q' = A \). Proved in Type 2.

Type 6.

\[
\begin{align*}
Q' &< A' < P' , \\
Q'P' &= PQ \quad (1) , \\
A'P' &= PA \quad (2) \\
\end{align*}
\]

Proof. Divide into two cases by \((2)\); cf. Definition 4.

Case (i). \( B'P' \cong PB \).

If \( Q' < B' \)

\[
\begin{align*}
B'P' &\cong PB, \; Q'P' = PQ \rightarrow \left\{ \begin{array}{c}
\text{(C--)} \\
\text{(Def. 2(ii))}
\end{array} \right\} Q'B' = BQ , \\
B'P' &\cong PB \rightarrow B'A' = AB \\
\end{align*}
\]

If \( Q' = B' \), then \( Q = B \) and

\[
B'P' \cong PB \Rightarrow B'A' = AB , \; \text{i.e.,} \; Q' A' = AQ .
\]

Case (ii). \( P' = A, \; P = A' \). Proved in Type 3.
**Verification of \( \tilde{C}^- \).**

**Type 1.**

\[
\begin{align*}
A & < P < Q, \\
A & Q \approx Q'A', \quad (1) \\
P Q & = Q'P', \quad (2)
\end{align*}
\]

\[
\Rightarrow \begin{cases} 
P Q \approx Q'P', \\
A P \approx P'A'.
\end{cases}
\]

**Proof.** Divide into three cases by (1); cf. Definition 1.

Case (i). \( Q = B \), \( Q' = B' \).

Impossible, since \( A < P < Q \) and since \( AB \) is elementary.

Case (ii). \( Q' < A \) (or \( B' A' = AB \)) and \( B Q \approx Q'B' \).

Note that if \( Q' < A \) then \( Q' < B \) and

\[
B Q \approx Q'R' \quad \Rightarrow \quad Q'B \approx B'Q \Rightarrow B'A' = AB.
\]

Now \( B \leq P \), since \( A < P \).

If \( B < P \),

\[
B Q \approx Q'B', \quad P Q = Q'P' \quad \Rightarrow \begin{cases} 
P B \approx P'B', \\
& (B'A' = AB)
\end{cases} \Rightarrow A P \approx P'A'.
\]

Evident, if \( B = P \).

Case (iii). \( Q = A' \), \( Q' = A \): \( A < A' \). Then (2) becomes

\[
PA' = AP' \quad \text{(2)'}
\]

Divide into three subcases by (2)'; cf. Definition 2.

Subcase (i). \( P = B' \), \( P' = B \): \( B'A' = AB \).

If \( B < B' \),

\[
B B' \approx B'B', \quad B'A' = AB \quad \Rightarrow \quad AB' \approx B'A, \quad \text{i.e.,} \quad A P \approx P'A'.
\]

If \( B = B' \), evident.

Subcase (ii). \( B'A' = AB \) and \( P B' = B P' \).

If \( B < P \),

\[
P B' = B P' \quad \Rightarrow \quad \begin{cases} 
P' B' \approx B P', \quad B'A' = AB \Rightarrow A P \approx P'A'. \\
& (P B' \approx B P', \quad B'A' = AB \Rightarrow A P' \approx P A', \quad \text{i.e.,} \quad Q'P' \approx P Q).
\end{cases}
\]

If \( B = P \), evident.

Subcase (iii). \( P = A \), \( P' = A' \). Impossible, since \( A < P \).
Type 2.

\[
\begin{align*}
Q' &< P' < A' , \\
Q' A' &\cong AQ \quad (1), \\
P' A' &\cong AP \quad (2)
\end{align*}
\]

\[\Rightarrow \begin{cases} 
P' A' \cong AP \quad (1), \\
Q' P' \cong PQ \quad (2)
\end{cases}\]

Proof. Divide into three cases by (2); cf. Definition 2.

Case (i). \( P'=B', P=A; B' A'=AB \).

Divide into three subcases by (1); cf. Definition 2.

Subcase (i). \( Q'=B', Q=A \). Impossible, since \( Q'<P' \).

Subcase (ii). \( Q'B' \cong BQ \).

\[Q'B' \cong BQ \quad \text{gives} \quad Q' P' \cong PQ .\]

\[A B \cong B' A' \quad \text{gives} \quad A P \cong P' A' .\]

Subcase (iii). \( Q'=A, Q=A' \). Evident.

Case (ii). \( P'<A \) (or \( B' A'=AB \)) and \( P' B'=BP \).

Divide into three subcases by (1); cf. Definition 2.

Subcase (i). \( Q'=B', Q=A \). Impossible, since \( Q'<P' \).

Subcase (ii). \( Q'B' \cong BQ \).

\[Q'B' \cong BQ \quad \text{gives} \quad Q' P' \cong PQ ,\]

\[P' B' \cong BP \quad \text{gives} \quad P' B' \cong BP \quad \Rightarrow \quad A P \cong P' A' .\]

Subcase (iii). \( Q'=A, Q=A' \): \( A< A' \).

If \( B<P' \).

\[P' B'=BP \quad \Rightarrow \quad P' B' \cong BP , \quad B' A'=AB \Rightarrow A P \cong P' A' .\]

If \( B=P' \), evident.

Case (iii). \( P'=A, P=A' \): \( A<A' \).

Divide into three subcases by (1); cf. Definition 2.

Subcase (i). \( Q'=B', Q=A \). Impossible, since \( B' A' \) is elementary.

Subcase (ii). \( Q'<A \) (or \( B' A'=AB \)) and \( Q'B' \cong BQ \).

Since \( Q'<P'=A< B \),

\[Q' B' \cong BQ \quad \Rightarrow \quad Q' B \cong B' Q \Rightarrow Q' A \cong A' Q , \text{ i.e., } Q' P' \cong PQ .\]

\[AA' \cong AA' \quad \text{gives} \quad A P \cong P' A' .\]

Subcase (iii). \( Q'=A, Q=A' \). Impossible, since \( Q'<P' \).

Type 3.

\[
\begin{align*}
P &< Q < A , \\
PA &\cong A' P' \quad (1), \\
QA &\cong A' Q' \quad (2)
\end{align*}
\]

\[\Rightarrow \begin{cases} 
A' Q' \cong QA , \\
P Q \cong Q' P' .
\end{cases}\]
Proof. Divide into two cases by (1); cf. Definition 3.

Case (i). \( B'P' \approx PB \).

Divide into two subcases by (2); cf. Definition 3.

Subcase (i). \( QB = B'Q' \).

\[
\begin{align*}
P &< Q < B \\
PB &\approx B'P', QB = B'Q' \\
\Rightarrow & \{ PQ \approx Q'P' \}
\end{align*}
\]

\( QB = B'Q' \Rightarrow QA = A'Q' \).

Subcase (ii). \( Q = A', Q' = A \).

Since \( P \leq B' < A' \),

\[
\begin{align*}
B'P' \approx PB & \Rightarrow PB' \approx BP' \\
P &< A \\
\Rightarrow & \{ AP' \approx PA' \}, \ i.e., Q'P' \approx PQ.
\end{align*}
\]

\( QA = A'Q' \) is evident.

Case (ii). \( P = A', P' = A : A'A \approx A'A \).

Divide into two subcases by (1); cf. Definition 3.

Subcase (i). \( QB = B'Q' \).

Since \( B' < Q' \), \( Q' < P' \), \( PQ \approx Q'P' \).

\[
\begin{align*}
QB &\approx B'Q' \\
\Rightarrow & \{ AP' \approx PA' \}, \ i.e., Q'P' \approx PQ.
\end{align*}
\]

Subcase (ii). \( Q = A', Q' = A \). Impossible, since \( P < Q \).

Type 4.

\[
\begin{align*}
A' &< Q' < P' \\
A'P' &\approx PA \quad (1) \\
Q'P' &\approx PQ \quad (2)
\end{align*}
\]

\[
\Rightarrow \begin{cases} 
PQ \approx Q'P' \\
A'Q' \approx QA
\end{cases}
\]

Proof. Divide into two cases by (1); cf. Definition 4.

Case (i). \( B'P' \approx PB \).

\[
\begin{align*}
B' &< Q' < P' \\
B'P' &\approx PB, Q'P' = PQ \\
\Rightarrow & \{ PQ \approx Q'P' \}
\end{align*}
\]

\( B'Q' \approx QB \Rightarrow A'Q' \approx QA \).

Case (ii). \( P' = A, P = A' : A' < A \). \ (2) becomes

\[Q'A = A'Q \]

Divide into two subcases by (2'); cf. Definition 3.

Subcase (i). \( Q'B = B'Q \).

\[
\begin{align*}
B' &< Q' \Rightarrow \{ B'Q' \approx QB \Rightarrow A'Q' \approx QA \} \\
Q'B &\approx B'Q \Rightarrow \{ B'Q \approx Q'B \Rightarrow A'Q \approx Q'A, \ i.e., PQ \approx Q'P' \}
\end{align*}
\]
Subcase (ii). \( Q'=A', Q=A \). Impossible, since \( Q'<P' \) and \( A'<A \).

**Type 5.**

\[
\begin{align*}
P<A<Q, \\
PQ&\approx Q'P' \quad (1) \implies \{ \begin{aligned} &AQ \approx Q'A', \\
&PA \approx A'P'. \end{aligned} \}
\end{align*}
\]

Proof. Divide into three cases by (2); cf. Definition 1.

Case (i). \( Q=B, Q'=B' \). Then

\[
\begin{align*}
(1): PB&\approx B'P' \implies \{ \begin{aligned} &PA \approx A'P', \\
&AB \approx B'A', \text{ i.e., } AQ \approx Q'A'. \end{aligned} \}
\end{align*}
\]

Case (ii). \( BQ=Q'B' \).

\[
\begin{align*}
P&<B, \\
PQ&\approx Q'P', \quad (\varepsilon-), \quad \begin{aligned} &PB \approx B'P' \implies PA \approx A'P', \\
&BQ \approx Q'B', \quad (\xi-) \implies AQ \approx Q'A'. \end{aligned}
\end{align*}
\]

Case (iii). \( Q=A', Q'=A \). Proved in Type 2.

**Type 6.**

\[
\begin{align*}
Q'<A'<P', \\
Q'P'&\approx PQ, \quad (1), \quad \begin{aligned} &A'P' \approx PA, \\
&A'P' \approx PA \quad (2) \implies \{ \begin{aligned} &Q'A' \approx AQ. \end{aligned} \}
\end{aligned}
\end{align*}
\]

Proof. Divide into two cases by (2); cf. Definition 4.

Case (i). \( B'P' \approx PB \). Then

\[
Q' \leq B', \quad \text{since} \quad Q'<A'.
\]

If \( Q'<B' \), then

\[
\begin{align*}
Q'P'&\approx PQ, \quad (\varepsilon-), \quad \begin{aligned} &Q'B' \approx BQ, \quad (\xi-) \implies Q'A' \approx AQ. \\
&B'P' \approx PB \implies \{ \begin{aligned} &AB \approx B'A', \\
&PA \approx A'P'. \end{aligned} \}
\end{aligned}
\end{align*}
\]

If \( Q'=B' \), evident.

Case (ii). \( P'=A, P=A' \). Proved in Type 3.

Thus the proof of Lemma is complete.

We are now in a position to construct a model \( M(R, C, I) \) on the basis of Lemma.

First take all the triples of natural numbers \((i, j, k)\), make a numbering \( N \) on them such that different triples \((i, j, k)\) and \((i', j', k')\) have different numbers \( N(i, j, k) \neq N(i', j', k') \).
Suppose a system $A_{ni}$ of $n_i$ different points $A_1, A_2, \ldots, A_{ni}$ has been already defined such that points are linearly ordered and that it satisfies Axioms $E, R, C^+, C^+, C^-$ and $C^-$. Call a triple of points $(A_i, A_j, A_k)$ $(1 \leq i, j, k \leq n_i)$ with $A_i < A_j$ saturated if the equality

$$A_iA_j = A_kA_l$$

has a solution in $A_i \in A_{ni}$, and insaturated if not, and let $(A_p, A_q, A_r)$ be the insaturated triple with the smallest $N(p, q, r)$.

For the sake of simplicity, set $A_p = A_{m}, A_q = A_{l}, A_r = A_1$ and choose points $P_{m-1}, P_{m-2}, \ldots, P_1$ of $A_{ni}$ such that

$$A_p = P_m < P_{m-1} < \cdots < P_2 < P_1 = A_q$$

and that the consecutive segments

$$P_mP_{m-1}, P_{m-1}P_{m-2}, \ldots, P_2P_1$$

are all elementary.

If there is any saturated triple $(P_m, P_1, P')$, let $(P_{s-1}, P_s, P_s')$ be such a one with the largest $s$. Then there must be a point $P_{s-1} \in A_{ni}$ such that

$$P_{s-1}P_s = P_s'P_{s-1} \tag{1}$$

If there is no saturated triple, set $s=2$. Introduce then $m-s+1$ new points $P'_s, P'_{s+1}, \ldots, P'_m$

and define the linear ordering

$$P'_s < P'_s < \cdots < P'_m < P''$$

where either $P'_s < P'' \ (P'' \in A_{ni})$ is an elementary segment or $P''$ is to be regarded as the point at infinity, if there is no point $X \in A_{ni}$ with $P'_{s-1} < X$.

Repeated applications of Lemma beginning with the successive introduction of basic congruence relations

$$P_{s-1}P_s = P_s'P_{s-1},$$

$$P_{s+1}P_s = P_s'P_{s+1},$$

$$P_mP_{m-1} = P_{m-1}P_m \tag{2}$$

lead us to a system of points $A_1, \ldots, A_{ni}, P'_s, \ldots, P'_m$ in a linear order, satisfying Axioms $E, R, C^+, C^+, C^-$ and $C^-$. Then we have from (1) and (2) on account of $C^+$

$$P_mP_1 = P_1'P_m' \tag{3}$$

If we set
we have by (3)
\[ A_p A_q = A_r A_{n+1}, \]
and \((A_p, A_q, A_r)\) becomes a saturated triple in the system of points
\[ A_{n+1} = \{A_1, A_2, \ldots, A_{n+1}\}. \]

Now let \(n\) be equal to 4 and let \(A_{n_1}\) be defined as a system of four points
\[ A_1, A_2, A_3, A_4 \]
in a linear order
\[ A_1 < A_2 < A_3 < A_4 \]
with the following congruence relations:

i) \(A_i A_j = A_i A_j\) for all \(i, j = 1, \ldots, 4\),
   provided \(A_i < A_j\),

ii) \(A_i A_j = A_2 A_3\) but \(A_2 A_3 = A_1 A_4\),

iii) \(A_2 A_3 = A_1 A_4\),

and

iv) \(A_i A_j = A_2 A_3, A_i A_2 = A_4 A_3, A_4 A_3 = A_1 A_2\).

In \(A_{n_1}\) all Axioms \(E, R, C^+ (= C), C^-\) are seen to be fulfilled. Thus we see by induction that in each \(A_{ni} (i = 1, 2, 3, \ldots)\) all Axioms from \(E_u\) to \(C^-\) are fulfilled, so that in particular Axioms \(E, R, C\) are satisfied in the system of points
\[ A = \bigcup_{i=1}^{\infty} A_{ni}. \]

If \(A_p, A_q, A_r\) is any triple of points with \(A_p < A_q\) in \(A\), then there is by the way of introducing new points of \(A_{n+1}\) into each \(A_{ni} (i = 1, 2, 3, \ldots)\) a natural number \(n_j\) such that the equality
\[ A_p A_q = A_r A_{n_j} \]
is satisfied by an \(A_s \in A_{n_j}\). Thus Axiom E is satisfied in \(A\).

Recalling the fact seen in the proof of Lemma that when the point \(A_n\) is added to the set \(A_{n-1}\) as a new point to obtain \(A_n\), the new congruence relations introduced with it are confined to those between some old segments and new ones having \(A_n\) as an end point, so we see that the relation \(A_2 A_3 = A_1 A_2\) in \(A_{n_1}\) remains true throughout all \(A_{n_i}\). Thus in \(A\):

→ S: Axiom S fails to be satisfied, for \(A_1 A_2 = A_2 A_3\) but
\[ A_1 A_2 = A_1 A_3. \]

→ T: Axiom T fails to be satisfied, for \(A_1 A_2 = A_2 A_3\),
\[ A_2 A_3 = A_1 A_4 \]
but \(A_2 A_3 = A_1 A_4\) by Axiom \(E_u\).

→ A: Axiom A fails to be satisfied, for if A holds, then by Theorem 11 (see Part I) Axiom S would hold good too.
Thus \( A \) is the desired model \( M(R, C, I) \) in which Axioms R, C, and I alone hold besides Axiom E.

NIIHAMA TECHNICAL COLLEGE, NIIHAMA
SOPHIA UNIVERSITY, TOKYO