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ON CONGRUENT AXIOMS IN LINEARLY ORDERED SPACES, II[1]

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6. Model M(R, C, I)

M(R, C, I): *A model of a geometry in which Axioms* R, C *and* I *alone hold besides Axiom* E. (Notice that I follows automatically from E, R and C.)

The construction of M (R, C, I) is quite different from those of other models, and its exposition here may be too long, but it seems to the authors appropriate to provide it with a full proof. It depends essentially upon Lemma below, and we will begin by introducing some definitions and auxiliarly axioms needed in it.

Let $\boldsymbol{A}$ be a finite number of linearly ordered points, in which congruence relations are supposed to hold among some of the segments, and let P, Q, P' etc. denote points of $\boldsymbol{A}$.

DEFINITION. We write

$$PQ \approx Q'P' \quad \text{or} \quad Q'P' \approx PQ\,,$$

if and only if

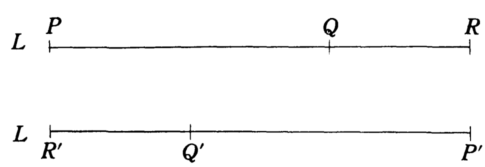
$$PQ = Q'P' \quad \text{and} \quad Q'P' = PQ$$

at the same time.

Axiom E_u: *If* $PQ = Q'P'$ *and* $PQ = Q'P''$, *then* $P' = P''$.

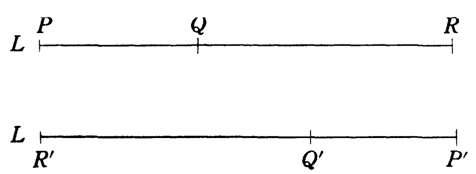
Axiom C^+ (=**Axiom** C)

$$\left.\begin{array}{l} P<Q<R\ , \\ R'<Q'<P'\,, \\ PQ=Q'P'\ , \\ QR=R'Q' \end{array}\right\} \Rightarrow PR=R'P'.$$



Axiom $\tilde{\mathrm{C}}^+$

$$\left.\begin{array}{l} P<Q<R\ , \\ R'<Q'<P'\,, \\ PQ\approx Q'P'\ , \\ QR=R'Q' \end{array}\right\} \Rightarrow \left\{\begin{array}{l} PR\approx R'P' \\ QR\approx R'Q'. \end{array}\right.$$



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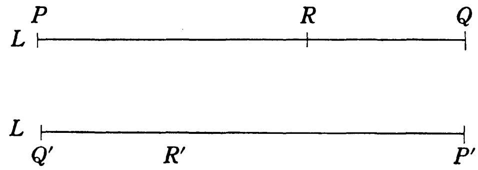
1) Continuation of Part I, this Journal, vol. 3 (1966), 269–292. Referred to as Part I.

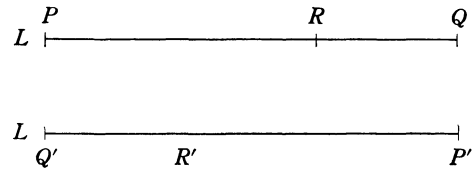
Axiom C^-

$$\left.\begin{array}{l} P<R<Q\ , \\ PQ=Q'P'\ , \\ RQ=Q'R' \end{array}\right\} \Rightarrow\ PR=R'P'\ .$$



Axiom $\tilde{\mathrm{C}}^-$:

$$\left.\begin{array}{l} P<R<Q\ , \\ PQ\approx Q'P'\ , \\ RQ=Q'R' \end{array}\right\} \Rightarrow \left\{\begin{array}{l} PR\approx R'P'\ , \\ RQ\approx Q'R'\ . \end{array}\right.$$

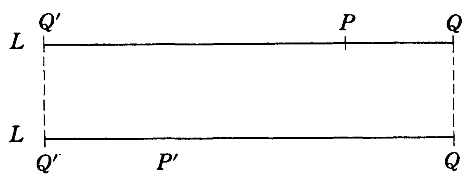


The following is an important consequence of $\tilde{\mathrm{C}}^-$, and will sometimes be denoted by $\tilde{\mathfrak{c}}^-$.

$$\tilde{\mathfrak{c}}^-:\quad PQ=Q'P',\quad Q'<P\ \Rightarrow\ PQ\approx Q'P',\quad Q'P\approx P'Q\,.$$

Proof.

$$\left.\begin{array}{l} Q'<P<Q \\ Q'Q\approx Q'Q \\ PQ\doteq Q'P' \end{array}\right\} \xrightarrow{(\tilde{\mathrm{C}}^-)} \left\{\begin{array}{l} PQ\approx Q'P'\ , \\ Q'P\approx P'Q\ . \end{array}\right.$$



DEFINITION. A segment PQ will be called *elementary*, if there is no point X with $P<X<Q$.

Lemma. *Let* $\boldsymbol{A}_{n-1}=\{A_\lambda\,|\,\lambda=1,2,\cdots,n-1\}$ *be a finite number of points in some linear order such that they satisfy Axioms* E_u, R, C^+, $\tilde{\mathrm{C}}^+$, C^- and $\tilde{\mathrm{C}}^-$. *Then, for a given elementary segment* A_iA_j *and a given point* A_k *such that the equality*

$$A_iA_j=A_kA_l$$

has no solution in $A_l\in\boldsymbol{A}_{n-1}$, *a new point* A_n *can be introduced, so that*

$$A_iA_j=A_kA_n$$

holds and the linearly ordered points $\boldsymbol{A}_n=\{A_1, A_2, \cdots, A_{n-1}, A_n\}$ *satisfy the same Axioms from* E_u *to* $\tilde{\mathrm{C}}^-$.

Proof. Points as well as notations such as A, P, X, P' etc. will mean in this proof points of $\boldsymbol{A}_{n-1}$ except for A' which will be introduced below as a new point A_n. If two segments are equal it is convenient to write the corresponding end points counterwise with and without dashes such as $PQ=Q'P'$, since several axioms of the type of C are involved.

For the sake of simplicity, set $A_i=A$, $A_j=B$ and $A_k=B'$.

Thus by assumption there is no point X with

$$AB=B'X$$

DEFINITION OF THE NEW POINT $A'(=A_n)$ AND OF ORDERING.

Let A' be introduced as a new point such that $\{A_1, \cdots, A_{n-1}, A'\}$ satisfy the following linear ordering:

(i) $B'<A'$.

(ii) If $X<B'$, then $X<A'$ for any point $X\in \boldsymbol{A}_{n-1}$.

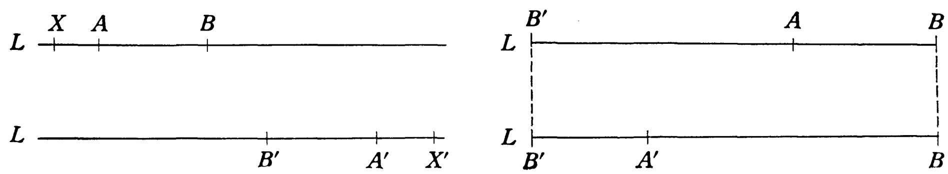
(iii) If $B'<X$, then $A'<X$ for any point $X\in \boldsymbol{A}_{n-1}$.

DEFINITION OF THE BASIC EQUALITY. The following is the basic congruence relation:

(i) $AB\approx B'A'$,

i.e. $AB=B'A'$ and $B'A'=AB$ at the same time, *if and only if there exist some X and X' such that $XB\approx B'X'$.*

In particular, $AB\approx B'A'$ *if* $B'<A$, since $B'B\approx B'B$.



(ii) Otherwise

$$AB=B'A' \quad \text{but} \quad B'A'\neq AB\,,$$

that is, $B'A'=AB$ is not defined.

DEFINITION OF OTHER EQUALITIES. Besides the above basic congruence relation we must define other new congruence relations in order to make the system of points $\boldsymbol{A}_n=\{A_1, \cdots, A_{n-1}, A_n\}$ satisfy all axioms from E_u to $\tilde{\mathrm{C}}^-$.

To insure Axiom R we only need

DEFINITION 0. *For any $X\in \boldsymbol{A}_{n-1}$: $A'X=A'X$ and $XA'=XA'$.*

In the following are defined all the equalities between old segments and new ones with one end point A'. They are classified into four types according to the position of A'.

Some of them are redundant, such as $AB=B'A'$, $AA'=AA'$ and $A'A=A'A$, but are included for the sake of completeness.

DEFINITION 1. $AP=P'A'$, *if and only if*

(i) $P=B$, $P'=B'$, i.e., $AB=B'A'$,

or (ii) $BP=P'B'$,

or (iii) $P=A'$, $P'=A$, i.e., $AA'=AA'$.

DEFINITION 2. *$P'A'=AP$, if and only if*

(i) $P'=B'$, $P=B$, i.e., $B'A'=AB$,

or (ii) $P'<A$ (*or* $B'A'=AB$) *and* $P'B'=BP$,

or (iii) $P'=A$, $P=A'$, i.e., $AA'=AA'$.

DEFINITION 3. *$PA=A'P'$, if and only if*

(i) $PB=B'P'$,

or (ii) $P=A'$, $P'=A$, i.e., $A'A=A'A$.

DEFINITION 4. *$A'P'=PA$, if and only if*

(i) $B'P'\approx PB$,

or (ii) $P'=A$, $P=A'$, i.e., $A'A=A'A$.

Having thus defined all congruence relations between old segments and new ones with one end point A', we are now going to verify Axioms E_u, C^+, $\tilde{C}^+$, C^- and $\tilde{C}^-$ one by one.

The verification will be done after a pattern: each equality under consideration is first classified according to its type, and then dealt with by Definitions 1, 2, 3 and 4 accordingly almost mechanically. Verbal explanations in detail will be omitted.

VERIFICATION OF E_u.

Type 1.

$$AP=P'A',\ AP=P'X \Rightarrow X=A'.$$

Proof. According to Definition 1, we divide the proof into three cases.

Case (i). $P=B$, $P'=B'$: $AB=B'A'$.

Then $AB=B'X$ is impossible for $X\in \boldsymbol{A}_{n-1}$.

Case (ii). $BP=P'B'$.

$$A<B<P,\ AP=P'X,\ BP=P'B' \overset{(C^-)}{\Longrightarrow} AB=B'X\,,$$

which is impossible for any old point $X\in \boldsymbol{A}_{n-1}$.

Case (iii). $P=A'$, $P'=A$: $AA'=AA'$.

Then $AA'=AX$ is impossible for any old point $X\in \boldsymbol{A}_{n-1}$.

Type 2.

$$P'A'=AP,\ P'A'=AX \Rightarrow X=P$$

Proof. Divide into three cases by Definition 2.

Case (i). $P'=B'$, $P=B$: $B'A'=AB$.

Then $B'A'=AX$ is only possible for $X=B$ by Definition 2.

Case (ii). $P'<A$ (or $B'A'=AB$) and $P'B'=BP$ and $P'B'=BX$.

Then $X=P$ by Axiom E_u applied to old congruence relations.

Case (iii). $P'=A$, $P=A'$: $AA'=AA'$,
$AA'=AX$. Then $X=A'$ by Definition 2.

Type 3.

$$PA=A'P',\ PA=A'X \Rightarrow X=P'.$$

Proof. Divide into two cases by Definition 3.
Case (i). $PB=B'P'$. Then

$$PB=B'P',\ PB=B'X \xRightarrow{(\mathrm{E}_u)} X=P'.$$

Case (ii). $P=A'$, $P'=A$. Then
$A'A=A'A$, $A'A=A'X \Rightarrow X=A$ by Definition 3.

Type 4.

$$A'P'=PA,\ A'P'=PX \Rightarrow X=A.$$

Proof. Divide into two cases by Definition 4.
Case (i). $B'P'\approx PB$. Then
$B'P'\approx PB$, $A'P'=PX \Rightarrow X=A$ by Definition 4.
Case (ii). $P'=A$, $P=A'$.
$A'A=A'A$, $A'A=A'X \Rightarrow X=A$ by Definition 4.

VERIFICATION OF C^+.

To show that Axiom C^+ is satisfied for $\boldsymbol{A}_n=\{A_1, \cdots, A_{n-1}, A'\}$ we consider six types of equalities.

Type 1.

$$\left.\begin{array}{ll} A<P<Q, & Q'<P'<A', \\ AP=P'A' & (1)\quad , \\ PQ=Q'P' & (2) \end{array}\right\} \Rightarrow AQ=Q'A'.$$

Proof. We divide the proof into three cases, according to (1); cf. Definition 1.

Case (i). $P=B$, $P'=B'$. Then from (2),

$$BQ=Q'B' \xRightarrow{(\text{Def. }1\text{(ii)})} AQ=Q'A'.$$

Case (ii). $BP=P'B'$.

$$\left.\begin{array}{l} B<P<Q,\ \ Q'<P'<B', \\ BP=P'B',\ PQ=Q'P' \end{array}\right\} \xRightarrow{(\mathrm{C}^+)} BQ=Q'B' \xRightarrow{(\text{Def. }1\text{(ii)})} AQ=Q'A'.$$

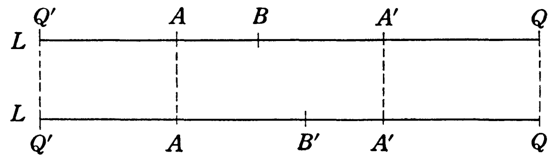
Case (iii). $P=A'$, $P'=A$. Then (2) becomes

$$A'Q=Q'A'. \tag{2$'$}$$

Divide into two subcases according to (2)$'$; cf. Definition 4.

Subcase (i). $B'Q\approx Q'B$.

$$Q'<B',\ B'Q\approx Q'B \overset{(\tilde{\mathrm{c}}^-)}{\Longleftrightarrow} BQ=Q'B' \overset{(\mathrm{Def.\,1})}{\Longrightarrow} AQ=Q'A'.$$



Subcase (ii). $Q=A$, $Q'=A'$. This is impossible, since $A<Q$.

Type 2.

$$\left.\begin{array}{ll} Q'<P'<A',\ A<P<Q\,, & \\ P'A'=AP & (1)\quad , \\ Q'P'=PQ & (2) \end{array}\right\} \Rightarrow Q'A'=AQ\,.$$

Divide into three cases by (1); cf. Definition 2.

Case (i). $P'=B'$, $P=B$: $B'A'=AB$.

$$B'A'=AB,\ Q'B'=BQ \overset{(\mathrm{Def.\,2(ii)})}{\Longrightarrow} Q'A'=AQ\,.$$

Case (ii). $P'<A$ (or $B'A'=AB$) and $P'B'=BP$.

$$\left.\begin{array}{r} P'B'=BP,\ Q'P'=PQ \overset{(\mathrm{C}^+)}{\Longrightarrow} Q'B'=BQ \\ P'<A\ \text{(or } B'A'=AB) \end{array}\right\} \overset{(\mathrm{Def.\,2(ii)})}{\Longrightarrow} Q'A'=AQ\,.$$

Case (iii). $P'=A$, $P=A'$: $A<A'$. Then (2) becomes

$$Q'A=A'Q \tag{2$'$}$$

We divide into two subcases according to (2)$'$; cf. Definition 3.

Subcase (i). $Q'B=B'Q$.

Since $A<A'$ and since AB and $B'A'$ are elementary, either $B<B'$ or $B=B'$.

If $B<B'$,

$$\left.\begin{array}{r} \left.\begin{array}{l} Q'B=B'Q \\ BB'=BB' \end{array}\right\} \overset{(\mathrm{C}^+)}{\Longrightarrow} Q'B'=BQ \\ Q'<A \end{array}\right\} \Rightarrow Q'A'=AQ\,.$$

If $B=B'$,

$$Q'<A,\ Q'B'=BQ \overset{(\mathrm{Def.\,2(ii)})}{\Longrightarrow} Q'A'=AQ\,.$$

Subcase (ii). $Q'=A'$, $Q=A$.

This case is impossible, since $Q'<A'$.

Type 3.

$$\left.\begin{array}{ll} P<Q<A,\ A'<Q'<P', & \\ QA=A'Q' & (1) \\ PQ=Q'P' & (2) \end{array}\right\} \Rightarrow PA=A'P'.$$

We divide into two cases by (1); cf. Definition 3.
Case (i). $QB=B'Q'$.

$$QB=B'Q',\ PQ=Q'P' \overset{(C^+)}{\Longrightarrow} PB=B'P' \overset{(\text{Def. }3)}{\Longrightarrow} PA=A'P'.$$

Case (ii). $Q=A'$, $Q'=A$: $A'<A$. Then (2) becomes

$$PA'=AP' \tag{2}'$$

Divide into three subcases by (2)'; cf. Definition 2.

Subcase (i). $P=B'$, $P'=B$. Since $B'<A'<A<B$,

$$B'B=B'B \overset{(\text{Def. }3)}{\Longrightarrow} B'A=A'B,\ \text{i.e., } PA=A'P'.$$

Subcase (ii). $P<A$ (or $B'A'=AB$) and $PB'=BP'$.

$$PB'=BP',\ B'B=B'B \overset{(C^+)}{\Longrightarrow} PB=B'P' \overset{(\text{Def. }3)}{\Longrightarrow} PA=A'P'.$$

Subcase (iii). $P=A$, $P'=A'$. This case is impossible, since $A'<P'$.

Type 4.

$$\left.\begin{array}{ll} A'<Q'<P',\ P<Q<A\,, & \\ A'Q'=QA & (1)\,, \\ Q'P'=PQ & (2) \end{array}\right\} A'P'=PA\,.$$

Proof. Divide into two cases by (1); cf. Definition 4.
Case (i). $B'Q'\approx QB$.

$$B'Q'\approx QB,\ Q'P'=PQ \overset{(\tilde{C}^+)}{\Longrightarrow} B'P'\approx PB \overset{(\text{Def. }4)}{\Longrightarrow} A'P'=PA\,.$$

Case (ii). $Q'=A$, $Q=A'$: $A'<A$. Then (2) becomes

$$AP'=PA' \tag{2}'$$

Divide into three subcases by (2)'; cf. Definition 1.

Subcase (i). $P'=B$, $P=B'$: $AB=B'A'$. Then
$B'<B$, since $B'<A'<A<B$. Then

$$B'B\approx B'B \overset{(\text{Def. }4)}{\Longrightarrow} A'B=B'A,\ \text{i.e., } A'P'=PA\,.$$

Subcase (ii).

$$BP'=PB',\ B'B\approx B'B \overset{(\tilde{C}^+)}{\Longrightarrow} B'P'\approx PB \overset{(\text{Def. }4)}{\Longrightarrow} A'P'=PA\,.$$

Subcase (iii). $P'=A'$, $P=A$. Impossible, since $A'<P'$.

Type 5.

$$\left.\begin{array}{ll} P<A<Q,\ Q'<A'<P', & \\ PA=A'P' & (1)\,, \\ AQ=Q'A' & (2) \end{array}\right\} \Rightarrow PQ=Q'P'.$$

Proof. Divide into two cases by (1); cf. Definition 3.

Case (i). $PB=B'P'$.
Divide into three subcases by (2); cf. Definition 1.
Subcase (i). $Q=B$, $Q'=B'$. Then

$$PB=B'P' \quad \text{gives} \quad PQ=Q'P'.$$

Subcase (ii). $BQ=Q'B'$.

$$PB=B'P',\ BQ=Q'B' \xrightarrow{(\mathrm{C}^+)} PQ=Q'P'.$$

Subcase (iii). $Q=A'$, $Q'=A$.
This case has been treated in Type 2.
Case (ii). $P=A'$, $P'=A$.
Proved in Type 4.

Type 6.

$$\left.\begin{array}{ll} Q'<A'<P',\ P<A<Q\,, & \\ Q'A'=AQ & (1)\,, \\ A'P'=PA & (2) \end{array}\right\} \Rightarrow Q'P'=PQ\,.$$

Proof. Divide into two cases by (2); cf. Definition 4.
Case (i). $B'P'\approx PB$.
Divide into three subcases by (1); cf. Definition 2.
Subcase (i). $Q'=B'$, $Q=B$. Then

$$B'P'=PB \quad \text{gives} \quad Q'P'=PQ$$

Subcase (ii). $Q'<A$ (or $B'A'=AB$) and $Q'B'=BQ$.

$$Q'B'=BQ,\ B'P'=PB \xrightarrow{(\mathrm{C}^+)} Q'P'=PQ\,.$$

Subcase (iii). $Q'=A$, $Q=A'$.
This case has been proved in Type 1.
Case (ii). $P'=A$, $P=A'$.
Has been proved in Type 3.

VERIFICATION OF $\tilde{\mathrm{C}}^+$.

Type 1.

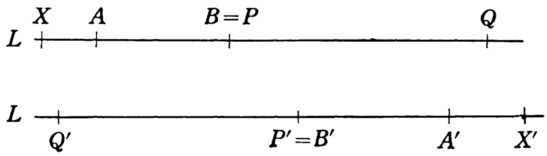
$$\left.\begin{array}{ll} A<P<Q,\ Q'<P'<A'\,, & \\ AP\approx P'A' & (1)\quad, \\ PQ=Q'P' & (2) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} PQ\approx Q'P'\,, \\ AQ\approx Q'A'\,. \end{array}\right.$$

Proof. Divide into three cases by (1); cf. Definition 1.
Case (i). $P=B$, $P'=B'$: $AB\approx B'A'$.

$$AB\approx B'A' \xrightarrow{(\mathrm{Def.})} \exists X, X' \colon XB\approx B'X'\,.$$

$$XB\approx B'X',\ BQ=Q'B' \xrightarrow{(\tilde{\mathrm{C}}^+)} BQ\approx Q'B', \ \text{i.e.,}\ PQ\approx Q'P'\,.$$

$$PQ\approx Q'P',\ AB\approx B'A' \overset{(\text{Def. }1, 2)}{\Longrightarrow} AQ\approx Q'A'$$



Case (ii). $P'<A$ (or $B'A'=AB$) and $P'B'\approx BP$

$$\left.\begin{array}{r} BP\approx P'B',\ PQ=Q'P' \overset{(\tilde{C}^+)}{\Longrightarrow} BQ\approx Q'B', \\ Q'<A \ \text{(or } B'A'=AB) \end{array}\right\} \overset{(\text{Def. }1, 2)}{\Longrightarrow} AQ\approx Q'A'.$$

Case (iii). $P=A'$, $P'=A$.

$$A'Q=Q'A. \tag{2$'$}$$

Divide into two subcases by (2)′; cf. Definition 4.

Subcase (i). $B'Q\approx Q'B$.

$$B'Q\approx Q'B \overset{(\text{Def. }3, 4)}{\Longrightarrow} Q'A\approx A'Q, \text{ i.e., } Q'P'\approx PQ.$$

Since $A<A'$, $Q'<P'=A<B'$.

$$\left.\begin{array}{r} Q'<B',\ B'Q\approx Q'B \overset{(\tilde{c}^-)}{\Longrightarrow} BQ\approx Q'B' \\ Q'<A \end{array}\right\} \overset{(\text{Def. }1, 2)}{\Longrightarrow} AQ\approx Q'A'.$$

Subcase (ii). $Q=A$, $Q'=A'$. Impossible, since $A<Q$.

Type 2.

$$\left.\begin{array}{ll} Q'<P'<A',\ A<P<Q, & \\ Q'P'\approx PQ & (1), \\ P'A'=AP & (2) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} Q'A'\approx AQ, \\ P'A'\approx AP. \end{array}\right.$$

Proof. Divide into three cases by (2); cf. Definition 2.

Case (i). $P'=B'$, $P=B$: $B'A'=AB$.

$$Q'B'\approx BQ,\ AB\approx B'A' \overset{(\text{Def. }1, 2)}{\Longrightarrow} AQ\approx Q'A'.$$
$$B'A'\approx AB \quad \text{gives} \quad P'A'\approx AP.$$

Case (ii). $P'<A$ (or $B'A'=AB$) and $P'B'=BP$.

$$Q'P'\approx PQ,\ P'B'=BP \overset{(\tilde{C}^+)}{\Longrightarrow} \left\{\begin{array}{l} Q'B'\approx BQ \\ P'B'\approx BP \Rightarrow P'A'\approx AP. \end{array}\right.$$

$$Q'<A \text{ (or } B'A'=AB),\ Q'B'\approx BQ \overset{(\text{Def. }1, 2)}{\Longrightarrow} AQ\approx Q'A'.$$

Case (iii). $P'=A$, $P=A'$: $A<A'$. Then (1) becomes

$$Q'A\approx A'Q \tag{1$'$}$$

Divide into two subcases by (1)′; cf. Definition 3,4.

Subcase (i). $Q'B \approx B'Q$.

Since $A < A'$, $Q' < P' = A < B \leqq B'$.

$$\left.\begin{array}{r} Q' < B',\ Q'B \approx B'Q \xrightarrow{(\tilde{c}^-)} Q'B' \approx BQ \\ Q' < A \end{array}\right\} \xrightarrow{(\text{Def. }1,2)} Q'A' \approx AQ\,.$$

$AP \approx P'A'$ is evident.

Subcase (ii). $Q' = A'$, $Q = A$. Impossible, since $Q' < A'$.

Type 3.

$$\left.\begin{array}{ll} P < Q < A,\ A' < Q' < P', & \\ PQ \approx Q'P' & (1), \\ QA = A'Q' & (2) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} PA \approx A'P', \\ QA \approx A'Q'. \end{array}\right.$$

Proof. Divide into two cases by (2); cf. Definition 3.

Case (i). $$\left.\begin{array}{r} QB = B'Q' \\ PQ \approx Q'P' \end{array}\right\} \xrightarrow{(\tilde{C}^+)} \left\{\begin{array}{l} PB \approx B'P' \xrightarrow{(\text{Def. }3,4)} PA \approx A'P'\,. \\ QB \approx B'Q' \xrightarrow{(\text{Def. }3,4)} QA \approx A'Q'\,. \end{array}\right.$$

Case (ii). $Q = A'$, $Q' = A$: $A' < A$. (1) becomes

$$PA' \approx AP' \tag{1$'$}$$

Divide into two subcases by (1)′; cf. Definition 1,2.

Subcase (i). $P = B'$, $P' = B$: $B'A' \approx AB$.

$$A'A \approx A'A \quad \text{gives} \quad QA \approx A'Q'$$

Since $B' < A' < A < B$,

$$B'B \approx B'B \xrightarrow{(\text{Def. }3,4)} B'A \approx A'B, \text{ i.e., } PA \approx A'P'\,.$$

Subcase (ii). $BP' \approx PB'$. Since $P < A < B$,

$$BP' \approx PB',\ P < B \xrightarrow{(\tilde{c}^-)} PB \approx B'P' \Rightarrow PA \approx A'P'\,.$$

Subcase (iii). $P = A$, $P' = A'$. Impossible, since $P < A$.

Type 4.

$$\left.\begin{array}{ll} A' < Q' < P',\ P < Q < A, & \\ A'Q' \approx QA & (1), \\ Q'P' = PQ & (2) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} A'P' \approx PA, \\ Q'P' \approx PQ. \end{array}\right.$$

Proof. Divide into two cases by (1); cf. Definition 4.

Case (i). $B'Q' \approx QB$.

$$B'Q' \approx QB,\ Q'P' = PQ \xrightarrow{(\tilde{C}^+)} \left\{\begin{array}{l} PB \approx B'P' \Rightarrow A'P' \approx PA\,. \\ Q'P' \approx PQ\,. \end{array}\right.$$

Case (ii). $Q'=A$, $Q=A'$: $A'<A$. (2) becomes

$$AP'=PA' \tag{2$'$}$$

Divide into three subcases by (2)$'$; cf. Definition 1.

Subcase (i). $P'=B$, $P=B'$. Since $B'<A'<A<B$,

$$B'B\approx B'B \overset{\text{(Def. 3, 4)}}{\Longrightarrow} A'B\approx B'A, \text{ i.e., } A'P'\approx PA\,.$$

$$AB=B'A',\ B'B\approx B'B \overset{\text{(Def.)}}{\Longrightarrow} AB\approx B'A', \text{ i.e., } Q'P'\approx PQ\,.$$

Subcase (ii). $BP'=PB'$. Since $P<B'<A'<A<B$,

$$P<B,\ BP'=PB' \overset{(\tilde{c}^-)}{\Longrightarrow} \begin{cases} PB\approx B'P' \overset{\text{(Def. 3, 4)}}{\Longrightarrow} A'P'\approx PA\,. \\ \left.\begin{matrix} BP'\approx PB' \\ P<A \end{matrix}\right\} \overset{\text{(Def. 1, 2)}}{\Longrightarrow} AP'\approx PA', \text{ i.e., } Q'P'\approx PQ\,. \end{cases}$$

Subcase (iii). $P'=A'$, $P=A$. Impossible, since $A'<P'$.

Type 5.

$$\left.\begin{matrix} P<A<Q,\ Q'<A'<P', & \\ PA\approx A'P' & (1), \\ AQ=Q'A' & (2) \end{matrix}\right\} \Rightarrow \begin{cases} PQ\approx Q'P', \\ AQ\approx Q'A'. \end{cases}$$

Proof. Divide into two cases by (1); cf. Definition 3.

Case (i). $B'P'\approx PB$.

Divide into three subcases by (2); cf. Definition 1.

Subcase (i). $Q=B$, $Q'=B'$.

$$B'P'\approx PB \quad \text{gives} \quad Q'P'\approx PQ\,.$$

$$B'P'\approx PB \overset{\text{(Def.)}}{\Longrightarrow} AB\approx B'A', \text{ i.e., } AQ=Q'A'\,.$$

Subcase (ii). $BQ=Q'B'$.

$$BQ=Q'B',\ PB\approx B'P' \overset{(\tilde{C}^+)}{\Longrightarrow} PQ\approx Q'P'\,.$$

$$\left.\begin{matrix} PB\approx B'P' \overset{\text{(Def.)}}{\Longrightarrow} B'A'\approx AB \\ BQ\approx Q'B' \end{matrix}\right\} \overset{\text{(Def. 1, 2)}}{\Longrightarrow} AQ\approx Q'A'\,.$$

Subcase (iii). $Q=A'$, $Q'=A$. Proved in Type 2.

Case (ii). $P=A'$, $P'=A$. Proved in Type 4.

Type 6.

$$\left.\begin{matrix} Q'<A'<P',\ P<A<Q, & \\ Q'A'\approx AQ & (1), \\ A'P'=PA & (2) \end{matrix}\right\} \Rightarrow \begin{cases} Q'P'\approx PQ, \\ A'P'\approx PA. \end{cases}$$

Proof. Divide into two cases by (2); cf. Definition 4.

Case (i). $B'P' \approx PB$.
Divide into three subcases by (1); cf. Definition 2.

Subcase (i). $Q'=B'$, $Q=B$:

$$B'P' \approx PB \quad \text{gives} \quad Q'P' \approx PQ\,.$$

$$B'P' \approx PB \overset{\text{(Def. 3, 4)}}{\Longrightarrow} PA \approx A'P'\,.$$

Subcase (ii). $Q'<A$ (or $B'A'=AB$) and $Q'B' \approx BQ$.

$$B'P' \approx PB,\ Q'B' \approx BQ \overset{(\mathrm{C}^+)}{\Longrightarrow} Q'P' \approx PQ\,.$$

$$B'P' \approx PB \overset{\text{(Def. 3, 4)}}{\Longrightarrow} PA \approx A'P'\,.$$

Subcase (iii). $Q=A'$, $Q'=A$. Proved in Type 1.

Case (ii). $P'=A$, $P=A'$. Proved in Type 3.

VERIFICATION OF $\mathbf{C}^-$.

Type 1.

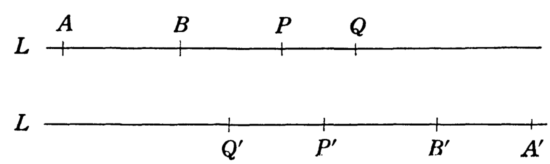
$$\left.\begin{array}{ll} A<P<Q\,, & \\ AQ=Q'A' & (1)\,, \\ PQ=Q'P' & (2) \end{array}\right\} \Rightarrow AP=P'A'\,.$$

Proof. Divide into three cases by (1); cf. Definition 1.

Case (i). $Q=B$, $Q'=B'$. Impossible, since AB is elementary.

Case (ii). $BQ=Q'B'$.

$$\left.\begin{array}{l} B<P<Q\,, \\ BQ=Q'B',\ PQ=Q'P' \end{array}\right\} \overset{(\mathrm{C}^-)}{\Longrightarrow} BP=P'B' \Rightarrow AP=P'A'\,.$$



Case (iii). $Q=A'$, $Q'=A$: $A<A'$. (2) becomes

$$PA'=AP' \tag{2$'$}$$

Divide into three subcases by (2)$'$; cf. Definition 2.

Subcase (i). $P=B'$, $P'=B$. Since AB is elementary,

$$A<B \leqq B'<A'\,.$$

If $B<B'$,

$$BB'=BB' \overset{\text{(Def. 1(ii))}}{\Longrightarrow} AB'=BA',\ \text{ i.e., } AP=P'A'\,.$$

If $B=B'$, $AB=B'A'$ gives $AP=P'A'$.

Subcase (ii). $P<A$ (or $B'A'=AB$) and $PB'=BP'$.
Since $A<P<A'$ and since AB is elementary,

$$B\leqq P,$$

If $B<P$,

$$PB'=BP' \xRightarrow{(\tilde{c}^-)} BP=P'B' \Rightarrow AP=P'A'.$$

If $B=P$, then $B'=P'$, so $AP=P'A'$.

Subcase (iii). $P=A$, $P'=A'$. Impossible, since $A<P$.

Type 2.

$$\left.\begin{array}{ll} Q'<P'<A', & \\ Q'A'=AQ & (1), \\ P'A'=AP & (2) \end{array}\right\} \Rightarrow Q'P'=PQ.$$

Proof. Divide into three cases by (2); cf. Definition 2.
Case (i). $P'=B'$, $P=B$: $B'A'=AB$.
Divide into three subcases by (1); cf. Definition 2.

Subcase (i). $Q'=B'$, $Q=B$. Impossible, since $Q'<P'$.

Subcase (ii). $Q'<A$ (or $B'A'=AB$) and $Q'B'=BQ$.

$$Q'B'=BQ \quad \text{gives} \quad Q'P'=PQ$$

Subcase (iii). $Q=A'$, $Q'=A$; $A<A'$.
If $B<B'$,

$$BB'=BB' \xRightarrow{(\text{Def. }1)} AB'=BA', \text{ i.e., } Q'P'=PQ.$$

If $B=B'$, $AB=B'A'$ gives $Q'P'=PQ$.
Case (ii). $P'<A$ (or $B'A'=AB$) and $P'B'=BP$.
Divide into three subcases by (1); cf. Definition 2.

Subcase (i). $Q'=B'$, $Q=B$. Impossible, since $Q'<P'<B'$.

Subcase (ii). $Q'<A$ (or $B'A'=AB$) and $Q'B'=BQ$.

$$Q'<P'<B',\ Q'B'=BQ,\ P'B'=BP \xRightarrow{(C^-)} Q'P'=PQ.$$

Subcase (iii). $Q'=A$, $Q=A'$. Then $A<P'<A'$, since $Q'<P'<A'$.
If $B<P'$,

$$B<P',\ P'B'=BP \xRightarrow{(\tilde{c}^-)} BP'=PB' \Rightarrow AP'=PA', \text{ i.e., } Q'P'=PQ.$$

If $B=P'$, then $B'=P$ and $AB=B'A'$ gives $Q'P'=PQ$.
Case (iii). $P'=A$, $P=A'$: $AA'=AA'$.
Divide into three cases by (1); cf. Definition 2.

Subcase (i). $Q'=B'$, $Q=B$.

Impossible, since $Q'<P'<A'$ and since $B'A'$ is elementary.

Subcase (ii). $Q'<A$ (or $B'A'=AB$) and $Q'B'=BQ$.

If $B<B'$,

$B<B',\ Q'B'=BQ \overset{(C^-)}{\Longrightarrow} Q'B=B'Q \overset{(\text{Def. }3)}{\Longrightarrow} Q'A=A'Q$, i.e., $Q'P'=PQ$.

If $B=B'$,

$$Q'B'=BQ \Rightarrow Q'A=A'Q, \text{ i.e., } Q'P'=PQ\,.$$

Subcase (iii). $Q'=A,\ Q=A'$. Impossible, since $Q'<P'$.

Type 3.

$$\left.\begin{array}{ll} P<Q<A\,, & \\ PA=A'P' & (1)\,, \\ QA=A'Q' & (2) \end{array}\right\} \Rightarrow PQ=Q'P'\,.$$

Proof. Divide into two cases by (1); cf. Definition 3.
Case (i). $PB=B'P'$.
Divide into two subcases by (2); cf. Definition 3.

Subcase (i). $QB=B'Q'$.

$$P<Q<B,\ QB=B'Q',\ PB=B'P' \overset{(C^-)}{\Longrightarrow} PQ=Q'P'\,.$$

Subcase (ii). $Q=A',\ Q'=A$: $A'<A$. Then
$P\leqq B'$, since $P<Q$ and since $B'A'$ is elementary.
If $P<B'$,

$$\left.\begin{array}{l} P<B'<B \\ PB=B'P',\ B'B=B'B \end{array}\right\} \overset{(C^-)}{\Longrightarrow} PB'=BP' \overset{(\text{Def. }2(\text{ii}))}{\Longrightarrow} PA'=AP', \text{ i.e., } PQ=Q'P'\,.$$

If $P=B'$, then $B=P'$ and

$$B'B\approx B'B \overset{(\text{Def.})}{\Longrightarrow} B'A'=AB, \text{ i.e., } PQ=Q'P'\,.$$

Case (ii). $P=A',\ P'=A$: $A'<A$.
Divide into two subcases by (2); cf. Definition 3.

Subcase (i). $QB=B'Q'$.

$$\left.\begin{array}{l} B'<A'=P<Q\,, \\ QB=B'Q' \end{array}\right\} \overset{(\tilde{c}^-)}{\Longrightarrow} B'Q\approx Q'B \overset{(\text{Def. }4)}{\Longrightarrow} A'Q=Q'A, \text{ i.e., } PQ=Q'P'\,.$$

Subcase (ii). $Q=A',\ Q'=A$. Impossible, since $P<Q$.

Type 4.

$$\left.\begin{array}{ll} A'<Q'<P'\,, & \\ A'P'=PA & (1)\,, \\ Q'P'=PQ & (2) \end{array}\right\} \Rightarrow A'Q'=QA\,.$$

Proof. Divide into two cases by (1); cf. Definition 4.
Case (i). $B'P'\approx PB$.

$$\left.\begin{array}{l} B'<A'<Q'<P' \\ B'P'\approx PB,\ Q'P'=PQ \end{array}\right\} \overset{(\tilde{c}^-)}{\Longrightarrow} B'Q'\approx QB \overset{(\text{Def. }4)}{\Longrightarrow} A'Q'=QA\,.$$

Case (ii). $P'=A$, $P=A'$: $A'<A$. Then (2) becomes

$$Q'A=A'Q\,. \tag{2)'$$

Divide into two subcases by (2)′; cf. Definition 3.

Subcase (i). $Q'B=B'Q$.

$$B'<A'<Q',\ Q'B=B'Q \overset{(\tilde{c}^-)}{\Longrightarrow} B'Q'\approx QB \overset{(\text{Def. }4)}{\Longrightarrow} A'Q'=QA\,.$$

Subcase (ii). $Q'=A'$, $Q=A$. Impossible, since $A'<Q'$.

Type 5.

$$\left.\begin{array}{ll} P<A<Q\,, & \\ PQ=Q'P' & (1)\,, \\ AQ=Q'A' & (2) \end{array}\right\} \Rightarrow PA=A'P'\,.$$

Proof. Divide into three cases by (2); cf. Definition 1.

Case (i). $BQ=Q'B'$.

$$\left.\begin{array}{l} P<B<Q \\ BQ=Q'B',\ PQ=Q'P' \end{array}\right\} \overset{(C^-)}{\Longrightarrow} PB=B'P' \overset{(\text{Def. }3)}{\Longrightarrow} PA=A'P'\,.$$

Case (ii). $Q=B$, $Q'=B'$. Then

$$PB=B'P' \Rightarrow PA=A'P'\,.$$

Case (iii). $Q=A'$, $Q'=A$. Proved in Type 2.

Type 6.

$$\left.\begin{array}{ll} Q'<A'<P'\,, & \\ Q'P'=PQ & (1)\,, \\ A'P'=PA & (2) \end{array}\right\} \Rightarrow Q'A'=AQ\,.$$

Proof. Divide into two cases by (2); cf. Definition 4.

Case (i). $B'P'\approx PB$.

If $Q'<B'$

$$\left.\begin{array}{l} B'P'\approx PB,\ Q'P'=PQ \overset{(C^-)}{\Longrightarrow} Q'B'=BQ\,, \\ B'P'\approx PB \overset{(\text{Def.})}{\Longrightarrow} B'A'=AB \end{array}\right\} \overset{(\text{Def. }2(\text{ii}))}{\Longrightarrow} Q'A'=AQ\,.$$

If $Q'=B'$, then $Q=B$ and

$$B'P'\approx PB \Rightarrow B'A'=AB,\ \text{i.e., } Q'A'=AQ\,.$$

Case (ii). $P'=A$, $P=A'$. Proved in Type 3.

VERIFICATION OF $\tilde{\mathrm{C}}^-$.

Type 1.

$$\left.\begin{array}{ll} A<P<Q\,, & \\ AQ\approx Q'A' & (1)\,, \\ PQ=Q'P' & (2) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} PQ\approx Q'P'\,, \\ AP\approx P'A'\,. \end{array}\right.$$

Proof. Divide into three cases by (1); cf. Definition 1.

Case (i). $Q=B$, $Q'=B'$.

Impossible, since $A<P<Q$ and since AB is elementary.

Case (ii). $Q'<A$ (or $B'A'=AB$) and $BQ\approx Q'B'$.

Note that if $Q'<A$ then $Q'<B$ and

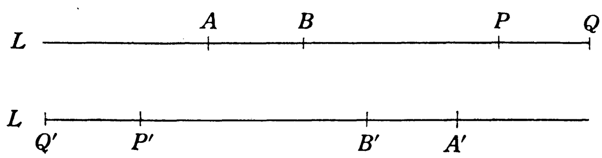
$$BQ\approx Q'B' \xrightarrow{(\tilde{c}^-)} Q'B\approx B'Q \Rightarrow B'A'=AB\,.$$

Now $B\leqq P$, since $A<P$.

If $B<P$,

$$BQ\approx Q'B',\ PQ=Q'P' \xrightarrow{(\tilde{c}^-)} \left\{\begin{array}{l} \left.\begin{array}{l} BP\approx P'B'\,, \\ (B'A'=AB) \end{array}\right\} \Rightarrow AP\approx P'A' \\ Q'P'\approx PQ\,. \end{array}\right.$$

Evident, if $B=P$.



Case (iii). $Q=A'$, $Q'=A$: $A<A'$. Then (2) becomes

$$PA'=AP' \tag{2$'$}$$

Divide into three subcases by (2)$'$; cf. Definition 2.

Subcase (i). $P=B'$, $P'=B$: $B'A'=AB$.

If $B<B'$,

$$BB'\approx BB',\ B'A'=AB \xrightarrow{(\mathrm{Def.}\,1,2)} AB'\approx B'A,\ \text{i.e., } AP\approx P'A'\,.$$

$$B'A'\approx AB \quad \text{gives} \quad PQ\approx Q'P'\,.$$

If $B=B'$, evident.

Subcase (ii). $B'A'=AB$ and $PB'=BP'$.

If $B<P$,

$$PB'=BP' \xrightarrow{(\tilde{c}^-)} \left\{\begin{array}{l} P'B'\approx BP,\ B'A'=AB \Rightarrow AP\approx P'A'\,. \\ PB'\approx BP',\ B'A'=AB \Rightarrow AP'\approx PA',\ \text{i.e. } Q'P'\approx PQ\,. \end{array}\right.$$

If $B=P$, evident.

Subcase (iii). $P=A$, $P'=A'$. Impossible, since $A<P$.

Type 2.

$$\left.\begin{array}{ll} Q'<P'<A'\,, & \\ Q'A'\approx AQ & (1)\,, \\ P'A'=AP & (2) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} P'A'\approx AP\,. \\ Q'P'\approx PQ\,. \end{array}\right.$$

Proof. Divide into three cases by (2); cf. Definition 2.

Case (i). $P'=B'$, $P=B$: $B'A'=AB$.

Divide into three subcases by (1); cf. Definition 2.

Subcase (i). $Q'=B'$, $Q=B$. Impossible, since $Q'<P'$.

Subcase (ii). $Q'B'\approx BQ$.

$$\begin{array}{lll} Q'B'\approx BQ & \text{gives} & Q'P'\approx PQ\,. \\ AB\approx B'A' & \text{gives} & AP\approx P'A'\,. \end{array}$$

Subcase (iii). $Q'=A$, $Q=A'$. Evident.

Case (iii). $P'<A$ (or $B'A'=AB$) and $P'B'=BP$.

Divide into three subcases by (1); cf. Definition 2.

Subcase (i). $Q'=B'$, $Q=B$. Impossible, since $Q'<P'$.

Subcase (ii). $Q'B'\approx BQ$.

$$\left.\begin{array}{l} Q'<P' \\ Q'B'\approx BQ,\ P'B'=BP \end{array}\right\} \xrightarrow{(\tilde{c}^-)} \left\{\begin{array}{l} Q'P'\approx PQ \\ P'B'\approx BP \end{array}\right.$$

$$\left.\begin{array}{r} P'B'\approx BP \\ P'<A \text{ (or } B'A'=AB) \end{array}\right\} \Rightarrow AP\approx P'A'\,.$$

Subcase (iii). $Q'=A$, $Q=A'$: $A<A'$.

If $B<P'$.

$$P'B'=BP \xrightarrow{(\tilde{c}^-)} \left\{\begin{array}{l} BP\approx P'B',\ B'A'=AB \Rightarrow AP\approx P'A'\,. \\ PB'\approx BP',\ B'A'=AB \Rightarrow PA'\approx AP', \text{ i.e., } PQ\approx Q'P'\,. \end{array}\right.$$

If $B=P'$, evident.

Case (iii). $P'=A$, $P=A'$: $A<A'$.

Divide into three subcases by (1); cf. Definition 2.

Subcase (i). $Q'=B'$, $Q=B$. Impossible, since $B'A'$ is elementary.

Subcase (ii). $Q'<A$ (or $B'A'=AB$) and $Q'B'\approx BQ$.

Since $Q'<P'=A<B$,

$$\begin{array}{l} Q'B'\approx BQ \xrightarrow{(\tilde{c}^-)} Q'B\approx B'Q \Rightarrow Q'A\approx A'Q, \text{ i.e., } Q'P'\approx PQ\,. \\ AA'\approx AA' \quad \text{gives} \quad AP\approx P'A'\,. \end{array}$$

Subcase (iii). $Q'=A$, $Q=A'$. Impossible, since $Q'<P'$.

Type 3.

$$\left.\begin{array}{ll} P<Q<A\,, & \\ PA\approx A'P' & (1)\,, \\ QA=A'Q' & (2) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} A'Q'\approx QA\,, \\ PQ\approx Q'P'\,. \end{array}\right.$$

Proof. Divide into two cases by (1); cf. Definition 3.
Case (i). $B'P' \approx PB$.
Divide into two subcases by (2); cf. Definition 3.
Subcase (i). $QB = B'Q'$.

$$\left.\begin{array}{l} P<Q<B \\ PB \approx B'P',\ QB = B'Q' \end{array}\right\} \xRightarrow{(\tilde{c}^-)} \left\{\begin{array}{l} PQ \approx Q'P' . \\ QB \approx B'Q' \Rightarrow QA \approx A'Q' . \end{array}\right.$$

Subcase (ii). $Q = A'$, $Q' = A$.
Since $P \leqq B' < A'$,

$$\left.\begin{array}{r} B'P' \approx PB \xRightarrow{(\tilde{c}^-)} PB' \approx BP' \\ P<A \end{array}\right\} \Rightarrow AP' \approx PA',\ \text{i.e.,}\ Q'P' \approx PQ .$$

$QA \approx A'Q'$ is evident.

Case (ii). $P = A'$, $P' = A$: $A'A \approx A'A$.
Divide into two subcases by (1); cf. Definition 3.
Subcase (i). $QB = B'Q'$.
Since $B' < A = P < Q$,

$$QB = B'Q' \xRightarrow{(\tilde{c}^-)} B'Q \approx Q'B \Rightarrow A'Q \approx Q'A,\ \text{i.e.,}\ PQ \approx Q'P' .$$

$$\left.\begin{array}{l} QB \approx B'Q' \\ (B'<Q) \end{array}\right\} \Rightarrow QA \approx A'Q' .$$

Subcase (ii). $Q = A'$, $Q' = A$. Impossible, since $P < Q$.

Type 4.

$$\left.\begin{array}{ll} A'<Q'<P' , & \\ A'P' \approx PA & (1) , \\ Q'P' = PQ & (2) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} PQ \approx Q'P' . \\ A'Q' \approx QA . \end{array}\right.$$

Proof. Divide into two cases by (1); cf. Definition 4.
Case (i). $B'P' \approx PB$.

$$\left.\begin{array}{l} B'<Q'<P' , \\ B'P' \approx PB,\ Q'P' = PQ \end{array}\right\} \xRightarrow{(\tilde{c}^-)} \left\{\begin{array}{l} PQ \approx Q'P' \\ B'Q' \approx QB \Rightarrow A'Q' \approx QA . \end{array}\right.$$

Case (ii). $P' = A$, $P = A'$: $A' < A$. (2) becomes

$$Q'A = A'Q \tag{2'}$$

Divide into two subcases by (2)′; cf. Definition 3.
Subcase (i). $Q'B = B'Q$.

$$\left.\begin{array}{l} B'<Q' , \\ Q'B = B'Q \end{array}\right\} \xRightarrow{(\tilde{c}^-)} \left\{\begin{array}{l} B'Q' \approx QB \Rightarrow A'Q' \approx QA . \\ B'Q \approx Q'B \Rightarrow A'Q \approx Q'A,\ \text{i.e.,}\ PQ \approx Q'P' . \end{array}\right.$$

Subcase (ii). $Q'=A'$, $Q=A$. Impossible, since $Q'<P'$ and $A'<A$.

Type 5.

$$\left.\begin{array}{ll} P<A<Q\,, & \\ PQ\approx Q'P' & (1)\,. \\ AQ=Q'A' & (2) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} AQ\approx Q'A'\,, \\ PA\approx A'P'\,. \end{array}\right.$$

Proof. Divide into three cases by (2); cf. Definition 1.
Case (i). $Q=B$, $Q'=B'$. Then

$$(1):\ PB\approx B'P' \Rightarrow \left\{\begin{array}{l} PA\approx A'P'\,. \\ AB\approx B'A',\ \text{i.e., } AQ\approx Q'A'\,. \end{array}\right.$$

Case (ii). $BQ=Q'B'$.

$$\begin{array}{lcl} \left.\begin{array}{l} P<B\,, \\ PQ\approx Q'P'\,, \\ BQ\approx Q'B' \end{array}\right\} & \xrightarrow{(\tilde{c}^-)} & \left\{\begin{array}{l} PB\approx B'P' \Rightarrow PA\approx A'P'\,. \\ BQ\approx Q'B' \end{array}\right. \\ PB\approx B'P' & \Longrightarrow & AB\approx B'A' \end{array} \quad \left.\begin{array}{l} BQ\approx Q'B' \\ AB\approx B'A' \end{array}\right\} \Rightarrow AQ\approx Q'A'\,.$$

Case (iii). $Q=A'$, $Q'=A$. Proved in Type 2.

Type 6.

$$\left.\begin{array}{ll} Q'<A'<P'\,, & \\ Q'P'\approx PQ, & (1)\,, \\ A'P'=PA & (2) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} A'P'\approx PA\,, \\ Q'A'\approx AQ\,. \end{array}\right.$$

Proof. Divide into two cases by (2); cf. Definition 4.
Case (i). $B'P'\approx PB$. Then

$$Q'\leqq B'\,,\quad \text{since}\quad Q'<A'\,.$$

If $Q'<B'$, then

$$\left.\begin{array}{l} \left.\begin{array}{l} Q'P'\approx PQ \\ B'P'\approx PB \end{array}\right\} \xrightarrow{(\tilde{c}^-)} Q'B'\approx BQ \\ B'P'\approx PB \quad \Rightarrow \left\{\begin{array}{l} AB\approx B'A' \\ PA\approx A'P' \end{array}\right. \end{array}\right\} \Rightarrow Q'A'\approx AQ\,.$$

If $Q'=B'$, evident.
Case (ii). $P'=A$, $P=A'$. Proved in Type 3.
Thus the proof of Lemma is complete.

We are now in a position to construct a model M(R, C, I) on the basis of Lemma.

First take all the triples of natural numbers (i, j, k), make a numbering N on them such that different triples (i, j, k) and (i', j', k') have different numbers $N(i, j, k)\neq N(i', j', k')$.

Suppose a system $\boldsymbol{A}_{n_i}$ of n_i different points $A_1, A_2, \cdots, A_{n_i}$ has been already defined such that points are linearly ordered and that it satisfies Axioms E_u, R, C^+, $\tilde{C}^+$, C^- and $\tilde{C}^-$. Call a triple of points (A_i, A_j, A_k) $(1 \leqq i, j, k \leqq n_i)$ with $A_i < A_j$ *saturated* if the equality

$$A_i A_j = A_k A_l$$

has a solution in $A_l \in \boldsymbol{A}_{n_i}$, and *insaturated* if not, and let (A_p, A_q, A_r) be the insaturated triple with the smallest $N(p, q, r)$.

For the sake of simplicity, set

$$A_p = P_m, \quad A_q = P_1, \quad A_r = P_1',$$

and choose points $P_{m-1}, P_{m-2}, \cdots, P_2$ of $\boldsymbol{A}_{n_i}$ such that

$$A_p = P_m < P_{m-1} < \cdots < P_2 < P_1 = A_q$$

and that the consecutive segments

$$P_m P_{m-1}, \ P_{m-1} P_{m-2}, \ \cdots, \ P_2 P_1$$

are all elementary.

If there is any saturated triple (P_n, P_1, P_1'), let (P_{s-1}, P_1, P_1') be such a one with the largest s. Then there must be a point $P_{s-1}' \in \boldsymbol{A}_{n_i}$ with

$$P_{s-1} P_1 = P_1' P_{s-1}'. \tag{1}$$

If there is no saturated triple, set $s=2$. Introduce then $m-s+1$ new points

$$P_s', \ P_{s+1}', \ \cdots, \ P_m'$$

and define the linear ordering

$$P_{s-1}' < P_s' < \cdots < P_m' < P'',$$

where either $P_{s-1}' P''$ $(P'' \in \boldsymbol{A}_{n_i})$ is an elementary segment or P'' is to be regarded as the point at infinity, if there is no point $X \in \boldsymbol{A}_{n_i}$ with $P_{s-1}' < X$.

Repeated applications of Lemma beginning with the successive introduction of basic congruence relations

$$\begin{aligned} P_s P_{s-1} &= P_{s-1}' P_s', \\ P_{s+1} P_s &= P_s' P_{s+1}', \\ &\ \vdots \\ P_m P_{m-1} &= P_{m-1}' P_m' \end{aligned} \tag{2}$$

lead us to a system of points $A_1, \cdots, A_{n_i}, P_s', \cdots, P_m'$ in a linear order, satisfying Axioms E_u, R, C^+, $\tilde{C}^+$, C^- and $\tilde{C}^-$. Then we have from (1) and (2) on account of C^+

$$P_m P_1 = P_1' P_m'. \tag{3}$$

If we set

$$P_s'=A_{n_i+1},\ P_{s+1}'=A_{n_i+2},\ \cdots,\ P_m'=A_{n_{i+1}},$$

we have by (3)

$$A_pA_q=A_rA_{n_{i+1}},$$

and (A_p, A_q, A_r) becomes a saturated triple in the system of points

$$\boldsymbol{A}_{n_{i+1}}=\{A_1, A_2, \cdots, A_{n_i}, \cdots, A_{n_{i+1}}\}.$$

Now let n_1 be equal to 4 and let $\boldsymbol{A}_{n_1}$ be defined as a system of four points A_1, A_2, A_3, A_4 in a linear order

$$A_1<A_4<A_2<A_3$$

with the following congruence relations:

i) $A_iA_j=A_iA_j$ for all $i, j=1, \cdots, 4$, provided $A_i<A_j$,

ii) $A_1A_2=A_2A_3$ but $A_2A_3 \neq A_1A_2$,

iii) $A_2A_3=A_1A_4$,

and

iv) $A_1A_4=A_2A_3,\ A_1A_2=A_4A_3,\ A_4A_3=A_1A_2$.

In $\boldsymbol{A}_{n_1}$ all Axioms E_u, R, C^+ ($=$C), $\tilde{C}^+$, C^- and $\tilde{C}^-$ are seen to be fulfillled. Thus we see by induction that in each $\boldsymbol{A}_{n_i}$ $(i=1, 2, 3, \cdots)$ all Axioms from E_u to $\tilde{C}^-$ are fulfilled, so that in particular Axioms E_u, R and C are satisfied in the system of points

$$\boldsymbol{A}=\bigcup_{i=1}^{\infty}\boldsymbol{A}_{n_i}.$$

If A_p, A_q, A_r is any triple of points with $A_p<A_q$ in $\boldsymbol{A}$, then there is by the way of introducing new points of $\boldsymbol{A}_{n_{i+1}}$ into each $\boldsymbol{A}_{n_i}$ $(i=1, 2, \cdots)$ a natural number n_j such that the equality

$$A_pA_q=A_rA_s$$

is satisfied by an $A_s \in \boldsymbol{A}_{n_j}$. Thus Axiom E is satisfied in $\boldsymbol{A}$.

Recalling the fact seen in the proof of Lemma that when the point A_n is added to the set $\boldsymbol{A}_{n-1}$ as a new point to obtain $\boldsymbol{A}_n$, the new congruence relations introduced with it are confined to those between some old segments and new ones having A_n as an end point, so we see that the relation $A_2A_3 \neq A_1A_2$ in $\boldsymbol{A}_{n_1}$ remains true throughout all $\boldsymbol{A}_{n_i}$. Thus in $\boldsymbol{A}$:

$\rightharpoondown$ S: Axiom S fails to be satisfied, for $A_1A_2=A_2A_3$ but $A_2A_3 \neq A_1A_2$.

$\rightharpoondown$ T: Axiom T fails to be satisfied, for $A_1A_2=A_2A_3$, $A_2A_3=A_1A_4$ but $A_1A_2 \neq A_1A_4$ by Axiom E_u.

$\rightharpoondown$ A: Axiom A fails to be satisfied, for if A holds, then by Theorem 11 (see Part I) Axiom S would hold good too.

Thus $\boldsymbol{A}$ is the desired model M(R, C, I) in which Axioms R, C, and I alone hold besides Axiom E.

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