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Osaka University
6. Model $M(R, C, I)$

$M(R, C, I)$: A model of a geometry in which Axioms $R$, $C$ and $I$ alone hold besides Axiom $E$. (Notice that $I$ follows automatically from $E$, $R$ and $C$.)

The construction of $M(R, C, I)$ is quite different from those of other models, and its exposition here may be too long, but it seems to the authors appropriate to provide it with a full proof. It depends essentially upon Lemma below, and we will begin by introducing some definitions and auxiliary axioms needed in it.

Let $A$ be a finite number of linearly ordered points, in which congruence relations are supposed to hold among some of the segments, and let $P, Q, P'$ etc. denote points of $A$.

**Definition.** We write

$$PQ \approx Q'P' \quad \text{or} \quad Q'P' \approx PQ,$$

if and only if

$$PQ = Q'P' \quad \text{and} \quad Q'P' = PQ$$

at the same time.

**Axiom $E_u$:** If $PQ = Q'P'$ and $PQ = Q'P''$, then $P' = P''$.

**Axiom $C^+$ ($=\text{Axiom C}$)**

$$
\begin{align*}
&P < Q < R, \\
&R' < Q' < P', \\
&PQ = Q'P', \\
&QR = R'Q'
\end{align*}
\implies

PR = R'P'.

**Axiom $C^+$**

$$
\begin{align*}
&P < Q < R, \\
&R' < Q' < P', \\
&PQ \approx Q'P', \\
&QR \approx R'Q'
\end{align*}
\implies

\begin{align*}
&PR \approx R'P' \\
&QR \approx R'Q'.
\end{align*}

Axiom C" :
\[ P < R < Q, \]
\[ PQ = Q' P', \]
\[ RQ = Q' R' \]
\[ \Rightarrow PR = R' P'. \]

Axiom \( C' \) :
\[ P < R < Q, \]
\[ PQ \approx Q' P', \]
\[ RQ \approx Q' R' \]
\[ \Rightarrow PR \approx R' P', \]
\[ RQ \approx Q' R'. \]

The following is an important consequence of \( C' \), and will sometimes be denoted by \( e' \).

\[ e' : PQ = Q' P', \]
\[ Q' < P \Rightarrow PQ \approx Q' P', \]
\[ Q' P \approx P' Q. \]

Proof.

\[ Q' < P < Q, \]
\[ Q' Q \approx Q' Q, \]
\[ PQ \approx Q' P' \]
\[ \Rightarrow \quad \begin{cases} PQ \approx Q' P' , & \\
Q' P \approx P' Q. & \end{cases} \]

Definition. A segment \( PQ \) will be called elementary, if there is no point \( X \) with \( P < X < Q \).

Lemma. Let \( A_{n-1} = \{ A_1, A_2, \ldots, A_{n-1} \} \) be a finite number of points in some linear order such that they satisfy Axioms \( E_n, R, C^+, C^- \) and \( C^- \). Then, for a given elementary segment \( A_j A_j \) and a given point \( A_k \) such that the equality

\[ A_j A_j = A_k A_k \]

has no solution in \( A_i \subseteq A_{n-1} \), a new point \( A_n \) can be introduced, so that

\[ A_j A_j = A_k A_n \]

holds and the linearly ordered points \( A_n = \{ A_1, A_2, \ldots, A_{n-1}, A_n \} \) satisfy the same Axioms from \( E_n \) to \( C^- \).

Proof. Points as well as notations such as \( A, P, X, P' \) etc. will mean in this proof points of \( A_{n-1} \), except for \( A' \) which will be introduced below as a new point \( A_n \). If two segments are equal it is convenient to write the corresponding end points counterwise with and without dashes such as \( PQ = Q' P' \), since several axioms of the type of \( C \) are involved.

For the sake of simplicity, set \( A_i = A, A_j = B \) and \( A_k = B' \).
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Thus by assumption there is no point $X$ with $AB = B'X$

**DEFINITION OF THE NEW POINT $A'(=A_n)$ AND OF ORDERING.**

Let $A'$ be introduced as a new point such that $\{A_1, \ldots, A_{n-1}, A'\}$ satisfy the following linear ordering:

(i) $B' < A'$.
(ii) If $X < B'$, then $X < A'$ for any point $X \in \mathbb{A}_{n-1}$.
(iii) If $B' < X$, then $A' < X$ for any point $X \in \mathbb{A}_{n-1}$.

**DEFINITION OF THE BASIC EQUALITY.** The following is the basic congruence relation:

(i) $AB \approx B'A'$, i.e., $AB = B'A'$ and $B'A' = AB$ at the same time, if and only if there exist some $X$ and $X'$ such that $XB \approx B'X'$.

In particular, $AB \approx B'A'$ if $B' < A$, since $B'B \approx B'B$.

(ii) Otherwise $AB = B'A'$ but $B'A' \not\approx AB$, that is, $B'A' = AB$ is not defined.

**DEFINITION OF OTHER EQUALITIES.** Besides the above basic congruence relation we must define other new congruence relations in order to make the system of points $\mathbb{A}_n = \{A_1, \ldots, A_{n-1}, A_n\}$ satisfy all axioms from $E_1$ to $C^{-}$.

To insure Axiom $R$ we only need

**DEFINITION 0.** For any $X \in \mathbb{A}_{n-1}$: $A'X = A'X$ and $XA' =XA'$.

In the following are defined all the equalities between old segments and new ones with one end point $A'$. They are classified into four types according to the position of $A'$.

Some of them are redundant, such as $AB = B'A'$, $AA' = AA'$ and $A'A = A'A'$, but are included for the sake of completeness.

**DEFINITION 1.** $AP = P'A'$, if and only if

(i) $P = B$, $P' = B'$, i.e., $AB = B'A'$,

or (ii) $BP = P'B'$,

or (iii) $P = A'$, $P = A$, i.e., $AA' = AA'$. 
DEFINITION 2. \( P'A' = AP, \text{ if and only if} \)

(i) \( P' = B', P = B, \text{ i.e., } B'A' = AB, \)

or (ii) \( P' < A \) (or \( B'A' = AB \)) and \( P'B' = BP, \)

or (iii) \( P' = A, P = A', \text{ i.e., } AA' = AA'. \)

DEFINITION 3. \( PA = A'P', \text{ if and only if} \)

(i) \( PB = B'P', \)

or (ii) \( P = A', P' = A, \text{ i.e., } A'A = A'A. \)

DEFINITION 4. \( A'P' = PA, \text{ if and only if} \)

(i) \( B'P' \approx PB, \)

or (ii) \( P' = A, P = A', \text{ i.e., } A'A = A'A. \)

Having thus defined all congruence relations between old segments and new ones with one end point \( A' \), we are now going to verify Axioms \( E_u, C^+, C^- \) and \( C^0 \) one by one.

The verification will be done after a pattern: each equality under consideration is first classified according to its type, and then dealt with by Definitions 1, 2, 3 and 4 accordingly almost mechanically. Verbal explanations in detail will be omitted.

VERIFICATION OF \( E_u. \)

Type 1.

\[ AP = P'A', AP = P'X \Rightarrow X = A'. \]

Proof. According to Definition 1, we divide the proof into three cases.

Case (i). \( P = B, P' = B': AB = B'A'. \)

Then \( AB = B'X \) is impossible for \( X \in A_{n-1}. \)

Case (ii). \( BP = B'B'. \)

\[ A < B < P, AP = P'X, BP = P'B' \rightarrow AB = B'X, \]

which is impossible for any old point \( X \in A_{n-1}. \)

Case (iii). \( P = A', P' = A: AA' = AA'. \)

Then \( AA' = AX \) is impossible for any old point \( X \in A_{n-1}. \)

Type 2.

\[ P'A' = AP, P'A' = AX \Rightarrow X = P \]

Proof. Divide into three cases by Definition 2.

Case (i). \( P' = B', P = B: B'A' = AB. \)

Then \( B'A' = AX \) is only possible for \( X = B \) by Definition 2.

Case (ii). \( P' < A \) (or \( B'A' = AB \)) and \( P'B' = BP \) and \( P'B' = BX. \)

Then \( X = P \) by Axiom \( E_u \) applied to old congruence relations.

Type 3.

$PA=A'P', PA=A'X \Rightarrow X = P'$.

Proof. Divide into two cases by Definition 3.

Case (i). $PB=B'P'$. Then

$PB = B'P', PB = B'X \xrightarrow{(E_q)} X = P'$.

Case (ii). $P = A', P' = A$. Then

$A'A = A'A', A'A = A'X \Rightarrow X = A$ by Definition 3.

Type 4.

$A'P' = PA, A'P' = PX \Rightarrow X = A$.

Proof. Divide into two cases by Definition 4.

Case (i). $B'P' \approx PB$. Then

$B'P' \approx PB, A'P' = PX \Rightarrow X = A$ by Definition 4.


$A'A = A'A, A'A = A'X \Rightarrow X = A$ by Definition 4.

Verification of $C^+$.

To show that Axiom $C^+$ is satisfied for $A_n = \{A_1, \ldots, A_n, A'\}$ we consider six types of equalities.

Type 1.

$A < P < Q, \quad Q' < P' < A'$, 

$AP = P'A'$ \ (1) , 

$PQ = Q'P'$ \ (2) 

$\Rightarrow AQ = Q'A'$.

Proof. We divide the proof into three cases, according to (1); cf. Definition 1.

Case (i). $P = B, P' = B'$. Then from (2),

$BQ = Q'B' \xrightarrow{(\text{Def.1(ii)})} AQ = Q'A'$.

Case (ii). $BP = P'B'$.

$B < P < Q, \quad Q' < P' < B'$, 

$BP = P'B', PQ = Q'P' \xrightarrow{(C^-)} BQ = Q'B' \xrightarrow{(\text{Def.1(ii)})} AQ = Q'A'$.

Case (iii). $P = A', P' = A$. Then (2) becomes

$A'Q = Q'A$.

$\Rightarrow (2)'$

Divide into two subcases according to $(2)'$; cf. Definition 4.

Subcase (i). $B'Q \approx Q'B$. 

\[ Q' < B', \ B'Q \approx Q'B \quad \xrightarrow{(\varepsilon)} \quad BQ = Q'B' \quad \xrightarrow{\text{(Def. 1)}} \quad AQ = Q'A'. \]

Subcase (ii). \( Q = A, \ Q' = A' \). This is impossible, since \( A < Q \).

**Type 2.**
\[
\begin{align*}
Q' < P' < A', \ A < P < Q, \\
P'A' &= AP \quad \text{(1)}, \\
Q'P' &= PQ \quad \text{(2)}
\end{align*}
\]

Divide into three cases by (1); cf. Definition 2.

Case (i). \( P' = B', \ P = B; \ B'A' = AB \).
\[
B'A' = AB, \ Q'B' = BQ \quad \xrightarrow{\text{(Def. 2(ii))}} \quad Q'A' = AQ.
\]

Case (ii). \( P' < A \) (or \( B'A' = AB \)) and \( P'B' = BP \).
\[
P'B' = BP, \ Q'P' = PQ \quad \xrightarrow{\text{(C')}} \quad Q'B' = BQ \quad \xrightarrow{\text{(Def. 2(ii))}} \quad Q'A' = AQ.
\]

Case (iii). \( P' = A, \ P = A' \): \( A < A' \). Then (2) becomes
\[
Q'A = A'Q \quad \text{(2)'}
\]

We divide into two subcases according to (2)'; cf. Definition 3.

Subcase (i). \( Q'B = B'Q \).
Since \( A < A' \) and since \( AB \) and \( B'A' \) are elementary, either \( B < B' \) or \( B = B' \).

If \( B < B' \),
\[
\begin{align*}
Q'B &= B'Q, \\
BB &= BB' \quad \xrightarrow{(\varepsilon)} \quad Q'B = BQ \quad \xrightarrow{Q' < A} \quad Q'A = AQ.
\end{align*}
\]

If \( B = B' \),
\[
Q' < A, \ Q'B = BQ \quad \xrightarrow{\text{(Def. 2(ii))}} \quad Q'A = AQ.
\]

Subcase (ii). \( Q' = A', \ Q = A \).
This case is impossible, since \( Q' < A' \).

**Type 3.**
\[
\begin{align*}
P < Q < A, \ A' < Q' < P', \\
QA &= A'Q' \quad \text{(1)}, \\
PQ &= Q'P' \quad \text{(2)}
\end{align*}
\]
\[ \Rightarrow PA = A'P'. \]
We divide into two cases by (1); cf. Definition 3.

Case (i). \( QB = B'Q' \).

\[ QB = B'Q', \ PQ = Q'P' \quad \text{(C+)} \quad PB = B'P' \quad \text{(Def. 3)} \quad PA = A'P'. \]

Case (ii). \( Q = A', \ Q' = A : A' < A \). Then (2) becomes

\[ PA' = AP' \quad \text{(2')} \]

Divide into three subcases by (2)'; cf. Definition 2.

Subcase (i). \( P = B' \), \( P' = B \). Since \( B' < A' < A < B \),

\[ B'B = B'B \quad \text{(Def. 3)} \quad B'P = A'P, \ i.e., \ PA = A'P'. \]

Subcase (ii). \( P < A \) (or \( B'A = AB \)) and \( PB = B'P' \).

\[ PB = B'P', \ B'B = B'B \quad \text{(C+)} \quad PB = B'P' \quad \text{(Def. 3)} \quad PA = A'P'. \]

Subcase (iii). \( P = A \), \( P' = A' \). This case is impossible, since \( A' < P' \).

Type 4.

\[ A' < Q' < P', \ P < Q < A, \]
\[ A'Q' = QA \quad \text{(1)}, \quad A'P' = PA. \]
\[ Q'P' = PQ \quad \text{(2)} \]

Proof. Divide into two cases by (1); cf. Definition 4.

Case (i). \( B'Q' \approx QB \).

\[ B'Q' \approx QB, \ Q'P' = PQ \quad \text{(C+)} \quad B'P' \approx PB \quad \text{(Def. 4)} \quad A'P' = PA. \]

Case (ii). \( Q = A, \ Q' = A' : A' < A \). Then (2) becomes

\[ AP' = PA' \quad \text{(2')} \]

Divide into three subcases by (2)'; cf. Definition 1.

Subcase (i). \( P' = B, \ P = B' : AB = B'A' \). Then \( B' < B, \) since \( B' < A' = A < B \). Then

\[ B'B \approx B'B \quad \text{(Def. 4)} \quad A'B = B'A, \ i.e., \ A'P' = PA. \]

Subcase (ii).

\[ BP = PB', \ B'B \approx B'B \quad \text{(C+)} \quad B'P' \approx PB \quad \text{(Def. 4)} \quad A'P' = PA. \]

Subcase (iii). \( P = A, \ P' = A' \). Impossible, since \( A' < P' \).

Type 5.

\[ P < A' < Q, \ Q' < A' < P', \]
\[ PA = A'P' \quad \text{(1)}, \quad \Rightarrow P = Q'P'. \]
\[ AQ = Q'A' \quad \text{(2)} \]

Proof. Divide into two cases by (1); cf. Definition 3.
Case (i). $PB = B'P'$.

Divide into three subcases by (2); cf. Definition 1.

Subcase (i). $Q = B, Q' = B'$. Then

$$PB = B'P' \quad \text{gives} \quad PQ = Q'P'.$$

Subcase (ii). $BQ = Q'B'$.

$$PB = B'P', \quad BQ = Q'B' \quad (\text{C}^+) \quad PQ = Q'P'.$$

Subcase (iii). $Q = A', Q' = A$.

This case has been treated in Type 2.


Proved in Type 4.

**Type 6.**

$$Q' < A' < P', \quad P < A < Q,$$

$$Q'A' = AQ \quad (1), \quad A'P' = PA \quad (2) \quad \Rightarrow \quad Q'P' = PQ.$$

Proof. Divide into two cases by (2); cf. Definition 4.

Case (i). $B'P' \approx PB$.

Divide into three subcases by (1); cf. Definition 2.

Subcase (i). $Q' = B', Q = B$. Then

$$B'P' = PB \quad \text{gives} \quad Q'P' = PQ.$$

Subcase (ii). $Q' < A$ (or $B'A' = AB$) and $Q'B' = BQ$.

$$Q'B' = BQ, \quad B'P' = PB \quad (\text{C}^+) \quad Q'P' = PQ.$$

Subcase (iii). $Q' = A, Q = A'$.

This case has been proved in Type 1.


Has been proved in Type 3.

**Verification of $C^+$.**

**Type 1.**

$$A < P < Q, \quad Q' < P' < A', \quad AP \approx P'A' \quad (1), \quad PQ = Q'P' \quad (2) \quad \Rightarrow \quad PQ \approx Q'P', \quad AQ \approx Q'A'.$$

Proof. Divide into three cases by (1); cf. Definition 1.

Case (i). $P = B, P' = B'$. $AB \approx B'A'$.

$$AB \approx B'A' \quad (\text{Def}) \quad \exists \ X, X': XB \approx B'X'. \quad XB \approx B'X', \quad BQ = Q'B' \quad (C^+) \quad BQ \approx Q'B', \quad \text{i.e.,} \quad PQ \approx Q'P'.$$
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\[ PQ \cong Q'P', \ AB \cong B'A' \quad (\text{Def. 1.2}) \]

\[ AX \quad B=P \quad Q \]

\[ L \quad A \quad B=P \quad Q \quad L \quad Q' \quad P'=B' \quad A' \quad X' \]

Case (ii). \( P'<A \) (or \( B'A'=AB \)) and \( P'B' \cong BP \)

\[ \begin{aligned}
BP \cong P'B', \ PQ=Q'P' \quad (\text{Def. 3, 4}) \quad & \\
BQ \cong Q'B', \ \quad & Q' \prec A \quad (\text{or} \ B'A'=AB) \quad \rightarrow \quad \text{Def. 1.2} \quad AQ \cong Q'A'.
\end{aligned} \]

Case (iii). \( P=A', \ P'=A \).

\[ A'Q = Q'A \quad (2)' \]

Divide into two subcases by (2)'; cf. Definition 4.

Subcase (i). \( B'Q \cong Q'B \).

\[ B'Q \cong Q'B \quad (\text{Def. 3, 4}) \quad Q'A \cong Q'A', \ \text{i.e.,} \ Q'P' \cong PQ. \]

Since \( A<A' \), \( Q'<P'=A<B' \).

\[ Q'<B', \ B'Q \cong Q'B \quad (\text{Def. 3, 4}) \quad BQ \cong Q'B' \quad (\text{Def. 1, 2}) \quad AQ \cong Q'A'. \]

Subcase (ii). \( Q=A, \ Q'=A' \). Impossible, since \( A<Q \).

Type 2.

\[ Q'<P'<A', \ A<P<Q, \quad Q'P' \cong PQ \quad (1), \quad P'A' = AP \quad (2) \]

Divide into three cases by (2); cf. Definition 2.

Case (i). \( P'=B', \ P=B; \ B'A'=AB \).

\[ Q'B' \cong BQ, \ AB \cong B'A' \quad (\text{Def. 1, 2}) \quad AQ \cong Q'A'. \]

\[ B'A' \cong AB \quad \text{gives} \quad P'A' \cong AP. \]

Case (ii). \( P'<A \) (or \( B'A'=AB \)) and \( P'B'=BP \).

\[ \begin{aligned}
Q'P' \cong PQ, \ P'B'=BP \quad (\text{Def. 1, 2}) \quad & \\
Q' \cong BQ \quad & Q' \prec A \quad (\text{or} \ B'A'=AB), \ Q'B' \cong BQ \quad (\text{Def. 1, 2}) \quad AQ \cong Q'A'.
\end{aligned} \]

Case (iii). \( P'=A, \ P=A': \ A<A' \). Then (1) becomes

\[ Q'A \cong A'Q \quad (1)' \]
Divide into two subcases by (1)'; cf. Definition 3,4.

Subcase (i). \( Q'B \approx B'Q \).

Since \( A < A' \), \( Q' < P' = A < B \leq B' \).

\[
\begin{align*}
Q' < B', \ & Q'B \approx B'Q \ (\text{Def. 1, 2}) \\
& \quad \quad \rightarrow Q'B' \approx BQ' \ \ (\text{Def. 1, 2}) \\
& \quad \quad \rightarrow Q' < A \\
\end{align*}
\]

\[ AP \approx P'A' \] is evident.

Subcase (ii). \( Q' = A' \), \( Q = A \). Impossible, since \( Q' < A' \).

Type 3.

\[
\begin{align*}
& P < Q < A, \ A' < Q' < P', \\
& PQ \approx Q'P' \ (1), \\
& QA = A'Q' \ (2) \\
\Rightarrow & \quad \quad \{ PA \approx A'P', \\
& \qquad \quad QA \approx A'Q' \}.
\end{align*}
\]

Proof. Divide into two cases by (2); cf. Definition 3.

Case (i). \( Q'B = B'Q' \) \( (\text{Def. 3, 4}) \) \( PB \approx B'P' \) \( (\text{Def. 3, 4}) \) \( PA \approx A'P' \).

\[
\begin{align*}
PQ \approx Q'P' \\
\Rightarrow & \quad \quad \{ QB \approx B'Q' \} \ (\text{Def. 3, 4}) \\
& \quad \quad QA \approx A'Q'.
\end{align*}
\]

Case (ii). \( Q = A' \), \( Q' = A \): \( A' < A \). (1) becomes

\[ PA \approx A'P' \] \( (1)' \)

Divide into two subcases by (1)'; cf. Definition 1,2.

Subcase (i). \( P = B' \), \( P' = B \): \( B'A' \approx AB \).

\[ A'A \approx A'A \quad \text{gives} \quad QA \approx A'Q' \]

Since \( B' < A' < A < B, \)

\[ B'B \approx B'B \ (\text{Def. 3, 4}) \quad B'A \approx A'B, \quad \text{i.e.,} \quad PA \approx A'P'. \]

Subcase (ii). \( BP' \approx PB' \). Since \( P < A < B, \)

\[ BP' \approx PB', \ P < B \ (\text{Def. 3, 4}) \quad PB \approx B'P' \Rightarrow PA \approx A'P'. \]

Subcase (iii). \( P = A, \ P' = A' \). Impossible, since \( P < A \).

Type 4.

\[
\begin{align*}
& A' < Q' < P', \ P < Q < A, \\
& A'Q' \approx QA \quad (1), \\
& Q'P' = PQ \quad (2) \\
\Rightarrow & \quad \quad \{ A'P' \approx PA, \\
& \qquad \quad Q'P' \approx PQ \}.
\end{align*}
\]

Proof. Divide into two cases by (1); cf. Definition 4.

Case (i). \( B'Q' \approx QB \).

\[
\begin{align*}
B'Q' \approx QB', \ & Q'P' = PQ \ (\text{Def. 3, 4}) \\
& \quad \quad \rightarrow \{ PB \approx B'P' \Rightarrow A'P' \approx PA \}.
\end{align*}
\]

\[
\begin{align*}
B'Q' \approx QB, \ Q'P' = PQ \quad (\text{Def. 3, 4}) \\
& \quad \quad \rightarrow \{ Q'P' \approx PQ \}.
\end{align*}
\]
Case (ii). $Q'=A$, $Q=A'$: $A'<A$. (2) becomes

$$AP'=PA'$$

Divide into three subcases by (2)'; cf. Definition 1.

Subcase (i). $P'=B$, $P=B'$. Since $B'<A'<A<B$,

$$B'B\cong B'B \quad \Rightarrow \quad A'B\cong B'A', \text{ i.e., } A'P'\cong PA.$$ $$AB=B'A', \quad B'B\cong B'B \quad \Rightarrow \quad AB\cong B'A', \text{ i.e., } Q'P'\cong PQ.$$

Subcase (ii). $BP'=PB'$. Since $P<B'<A'<A<B$,

$$P<B, \quad BP'=PB' \quad \Rightarrow \quad \left\{ \begin{array}{l} PB\cong B'P' \quad \text{(Def. 3.4)} \\ BP'\cong PB' \\ P<A \end{array} \right\} \quad \Rightarrow \quad AP'\cong PA', \text{ i.e., } Q'P'\cong PQ.$$

Subcase (iii). $P'=A'$, $P=A$. Impossible, since $A'<P'$.

Type 5.

$$P<A<Q, \quad Q'<A'<P', \quad PA\cong A'P' \quad (1), \quad \Rightarrow \quad \left\{ \begin{array}{l} PQ\cong Q'P', \\ AQ\cong Q'A'. \end{array} \right\}$$

Proof. Divide into two cases by (1); cf. Definition 3.

Case (i). $B'P'\cong PB$.

Divide into three subcases by (2); cf. Definition 1.

Subcase (i). $Q=B$, $Q'=B'$.

$$B'P'\cong PB \quad \text{gives} \quad Q'P'\cong PQ.$$ $$B'P'\cong PB \quad \text{(Def.)} \quad \Rightarrow \quad AB\cong B'A', \text{ i.e., } AQ\cong Q'A'.$$

Subcase (ii). $BQ=Q'B'$.

$$BQ=Q'B', \quad PB\cong B'P' \quad \Rightarrow \quad P Q\cong Q'P'.$$ $$PB\cong B'P' \quad \text{(Def.)} \quad \Rightarrow \quad B' A'\cong AB \quad \Rightarrow \quad AQ\cong Q'A'.$$

Subcase (iii). $Q=A'$, $Q'=A$. Proved in Type 2.


Type 6.

$$Q'<A'<P', \quad P<A<Q,$$ $$Q'A'\cong AQ \quad (1), \quad \Rightarrow \quad \left\{ \begin{array}{l} Q'P'\cong PQ, \\ A'P'\cong PA. \end{array} \right\}$$

Proof. Divide into two cases by (2); cf. Definition 4.
Case (i). $B'P' \approx PB$.
Divide into three subcases by (1); cf. Definition 2.

Subcase (i). $Q' = B'$, $Q = B$: 

$B'P' \approx PB$ gives $Q'P' \approx PQ$.

$B'P' \approx PB \xrightarrow{(\text{Def. } 3, \text{4})} PA \approx A'P'$.

Subcase (ii). $Q' < A$ (or $B'A' = AB$) and $Q'B' \approx BQ$.

$B'P' \approx PB$, $Q'B' \approx BQ \xrightarrow{(C^-)} Q'P' \approx PQ$.

$B'P' \approx PB \xrightarrow{(\text{Def. } 3, \text{4})} PA \approx A'P'$.

Subcase (iii). $Q = A'$, $Q' = A$. Proved in Type 1.


Verification of $C^-$. 

Type 1.

\[
\begin{align*}
A &< P < Q, \\
AQ &\approx Q'A' \quad (1), \\
PQ &\approx Q'P' \quad (2)
\end{align*}
\]

Proof. Divide into three cases by (1); cf. Definition 1.

Case (i). $Q = B$, $Q' = B'$. Impossible, since $AB$ is elementary.

Case (ii). $BQ = Q'B'$.

\[
\begin{align*}
B &< P < Q, \\
BQ &\approx Q'B', \ PQ \approx Q'P' \quad (C^-) 
\end{align*}
\]

\[
\begin{array}{c}
L \\
A & B & P & Q \\
\hline
L \\
Q' & P' & B' & A'
\end{array}
\]

Case (iii). $Q = A'$, $Q' = A$: $A < A'$. (2) becomes

$PA' \approx AP'$

$\quad \text{(2)'}$

Divide into three subcases by (2)'; cf. Definition 2.

Subcase (i). $P = B'$, $P' = B$. Since $AB$ is elementary,

$A < B \leq B' < A'$.

If $B < B'$,

$BB' \approx BB' \xrightarrow{(\text{Def. } 1\text{(ii)})} AB' = BA'$, i.e., $AP = P'A'$.

If $B = B'$, $AB = B'A'$ gives $AP = P'A'$. 

---
Subcase (ii). $P < A$ (or $B'A' = AB$) and $PB' = BP'$.  
Since $A < P < A'$ and since $AB$ is elementary,  
$$B \leq P,$$
If $B < P$,
$$PB' = BP' \xrightarrow{(c^-)} BP = P'B \Rightarrow AP = P'A'.$$
If $B = P$, then $B' = P'$, so $AP = P'A'$.

Type 2.
$$Q' < P' < A', \quad Q'A' = AQ \quad (1), \quad \Rightarrow Q'P' = PQ.$$  
$$P'A' = AP \quad (2)$$
Proof. Divide into three cases by (2); cf. Definition 2.
Case (i). $P' = B'$, $P = B$: $B'A' = AB$.
Divide into three subcases by (1); cf. Definition 2.
Subcase (i). $Q' = B'$, $Q = B$. Impossible, since $Q' < P'$.
Subcase (ii). $Q' < A$ (or $B'A' = AB$) and $Q'B' = BQ$.
$$Q'B' = BQ \quad \text{ gives } \quad Q'P' = PQ.$$  
Subcase (iii). $Q = A'$, $Q' = A'$; $A < A'$.
If $B < B'$,
$$BB' = BB' \xrightarrow{\text{Def.}1} AB' = BA'$, \ i.e., \ Q'P' = PQ.$$  
If $B = B'$, $AB = B'A'$ gives $Q'P' = PQ$.
Case (ii). $P' < A$ (or $B'A' = AB$) and $P'B' = BP$.
Divide into three subcases by (1); cf. Definition 2.
Subcase (i). $Q' = B'$, $Q = B$. Impossible, since $Q' < P' < B'$.
Subcase (ii). $Q' < A$ (or $B'A' = AB$) and $Q'B' = BQ$.
$$Q' < P' < B', \quad Q'B' = BQ, \quad P'B' = BP \xrightarrow{(c^-)} Q'P' = PQ.$$  
Subcase (iii). $Q' = A$, $Q = A'$. Then $A < P' < A'$, since $Q' < P' < A'$.
If $B < P$,
$$B < P', \quad P'B' = BP \xrightarrow{(c^-)} BP' = PB \Rightarrow AP' = PA', \ i.e., \ Q'P' = PQ.$$  
If $B = P'$, then $B' = P$ and $AB = B'A'$ gives $Q'P' = PQ$.
Divide into three cases by (1); cf. Definition 2.
Subcase (i). $Q' = B'$, $Q = B$.
Impossible, since $Q' < P' < A'$ and since $B'A'$ is elementary.
Subcase (ii). $Q' < A$ (or $B'A' = AB$) and $Q'B' = BQ$.  

If \( B<B' \),

\[
B<B', \quad Q'B'=BQ \quad \xrightarrow{(C^-)} \quad Q'B=B'Q \quad \xrightarrow{(\text{Def.3})} \quad Q'A=A'Q, \quad \text{i.e.,} \quad Q'P'=PQ .
\]

If \( B=B' \),

\[
Q'B'=BQ \Rightarrow Q'A=A'Q, \quad \text{i.e.,} \quad Q'P'=PQ .
\]

Subcase (iii). \( Q'=A, \quad Q=A' \). Impossible, since \( Q'<P' \).

Type 3.

\[
\begin{align*}
P&<Q<A, \\
PA&=A'P' \quad (1), \\
QA&=A'Q' \quad (2)
\end{align*}
\]

Proof. Divide into two cases by (1); cf. Definition 3.

Case (i). \( PB=B'P' \).

Divide into two subcases by (2); cf. Definition 3.

Subcase (i). \( QB=B'Q' \).

\[
P<Q<B, \quad QB=B'Q', \quad PB=B'P' \quad \xrightarrow{(C^-)} \quad PQ=Q'P' .
\]

Subcase (ii). \( Q=A', \quad Q'=A: \quad A'<A \). Then \( P\leq B' \), since \( P<Q \) and since \( B'A' \) is elementary.

If \( P<B' \),

\[
P<B'<B \quad \xrightarrow{(C^-)} \quad PB'=BP' \quad \xrightarrow{(\text{Def.2(ii)})} \quad PA'=AP' , \quad \text{i.e.,} \quad PQ=Q'P' .
\]

If \( P=B' \), then \( B=P' \) and

\[
B'B=BB \quad \xrightarrow{(\text{Def.})} \quad B'A'=AB , \quad \text{i.e.,} \quad PQ=Q'P' .
\]

Case (ii). \( P=A', \quad P'=A: \quad A'<A \).

Divide into two subcases by (2); cf. Definition 3.

Subcase (i). \( QB=B'Q' \).

\[
B'<A'=P<Q \quad , \quad \xrightarrow{(\varepsilon^-)} \quad B'Q=Q'B \quad \xrightarrow{(\text{Def.4})} \quad A'Q=Q'A, \quad \text{i.e.,} \quad PQ=Q'P' .
\]

Subcase (ii). \( Q=A', \quad Q'=A \). Impossible, since \( P<Q \).

Type 4.

\[
\begin{align*}
A'&<Q'<P', \\
A'P'=PA \quad (1), \\
Q'P'=PQ \quad (2)
\end{align*}
\]

Proof. Divide into two cases by (1); cf. Definition 4.

Case (i). \( B'P'=PB \).
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\[ B'<A'<Q'<P' \quad B'P \approx PB, \quad Q'P' \approx PQ \quad \overset{(\text{Def. 4})}{\Rightarrow} B'Q' \approx QB \overset{(\text{Def. 4})}{\Rightarrow} A'Q' = QA. \]

Case (ii). \( P' = A, P = A' \); \( A' < A \). Then (2) becomes

\[ Q'A = A'Q. \] (2')

Divide into two subcases by (2)'; cf. Definition 3.

Subcase (i). \( Q'B = B'Q \).

\[ B' < A' < Q', \quad Q'B = B'Q \overset{(\text{Def. 4})}{\Rightarrow} B'Q' \approx QB \overset{(\text{Def. 4})}{\Rightarrow} A'Q' = QA. \]

Subcase (ii). \( Q' = A', \quad Q = A \). Impossible, since \( A' < Q' \).

Type 5.

\[
\begin{align*}
P < A < Q, \\
PQ = Q'P' \quad (1), \\
AQ = Q'A' \quad (2)
\end{align*}
\]

Proof. Divide into three cases by (2); cf. Definition 1.

Case (i). \( BQ = Q'B' \).

\[ P < B < Q \]

\[ BQ = Q'B', \quad PQ = Q'P' \overset{(\text{Def. 3})}{\Rightarrow} PB = B'P' \overset{(\text{Def. 3})}{\Rightarrow} PA = A'P'. \]

Case (ii). \( Q = B, Q' = B' \). Then

\[ PB = B'P' \Rightarrow PA = A'P'. \]

Case (iii). \( Q = A', Q' = A \). Proved in Type 2.

Type 6.

\[
\begin{align*}
Q' < A' < P', \\
Q'P' = PQ \quad (1), \\
A'P' = PA \quad (2)
\end{align*}
\]

Proof. Divide into two cases by (2); cf. Definition 4.

Case (i). \( B'P' \approx PB \).

If \( Q' < B' \)

\[ B'P' \approx PB, \quad Q'P' = PQ \overset{(\text{Def. 1})}{\Rightarrow} Q'B' = BQ, \]

\[ B'P' \approx PB \overset{(\text{Def. 2(ii)})}{\Rightarrow} B'A' = AB \]

If \( Q' = B' \), then \( Q = B \) and

\[ B'P' \approx PB \Rightarrow B'A' = AB, \quad \text{i.e.,} \quad Q'A' = AQ. \]

Case (ii). \( P' = A, P = A' \). Proved in Type 3.
Verification of $\tilde{C}^-$.

Type 1.

$A < P < Q$,

$A Q \approx Q' A'$ \hspace{1cm} (1) \Rightarrow \{ P Q \approx Q' P' , \newline P Q = Q' P' \}$ \hspace{1cm} (2)

$A P \approx P' A'$.

Proof. Divide into three cases by (1); cf. Definition 1.

Case (i). $Q = B$, $Q' = B'$.

Impossible, since $A < P < Q$ and since $A B$ is elementary.

Case (ii). $Q' < A$ (or $B' A' = A B$) and $B Q \approx Q' B'$.

Note that if $Q' < A$ then $Q' < B$ and

$B Q \approx Q' B' \quad (\varepsilon^{-1}) \Rightarrow Q' B \approx B' Q \Rightarrow B' A' = A B$.

Now $B \leq P$, since $A < P$.

If $B < P$,

$B Q \approx Q' B'$, $P Q = Q' P'$ \hspace{1cm} (\varepsilon^{-1}) \Rightarrow \{ B P \approx P' B' , \newline (B' A' = A B) \} \Rightarrow A P \approx P' A'$

$Q' P' \approx P Q$.

Evident, if $B = P$.

Case (iii). $Q = A'$, $Q' = A$: $A < A'$. Then (2) becomes

$P A' = A P'$ \hspace{1cm} (2)'$

Divide into three subcases by (2)'; cf. Definition 2.

Subcase (i). $P = B'$, $P' = B$: $B' A' = A B$.

If $B < B'$,

$B B' \approx B B'$, $B' A' = A B$ \hspace{1cm} (Def. 1, 2) \Rightarrow A B' \approx B' A$, i.e., $A P \approx P' A'$.

$B' A' \approx A B$ gives $P Q \approx Q' P'$.

If $B = B'$, evident.


If $B < P$,

$P B' = B P'$ \hspace{1cm} (\varepsilon^{-1}) \Rightarrow \{ P' B' \approx B P', B' A' = A B \Rightarrow A P \approx P' A' , \newline P B' \approx B P', B' A' = A B \Rightarrow A P \approx P' A'$, i.e. $Q' P' \approx P Q$.

If $B = P$, evident.


Type 2.

\[ Q' < P' < A' \]
\[ Q'A' \approx AQ \] (1)
\[ P'A' = AP \] (2)
\[ \Rightarrow \]
\[ P'A' \approx AP \]
\[ Q'P' \approx PQ \]

Proof. Divide into three cases by (2); cf. Definition 2.

Case (i). \( P' = B', P = B; B'A' = AB \).

Divide into three subcases by (1); cf. Definition 2.

Subcase (i). \( Q' = B', Q = B \). Impossible, since \( Q' < P' \).

Subcase (ii). \( Q'B' \approx BQ \).

\[ Q'B' \approx BQ \]
\[ \text{gives} \]
\[ Q'P' \approx PQ \]
\[ AB \approx B'A' \]
\[ \text{gives} \]
\[ AP \approx P'A' \]

Subcase (iii). \( Q' = A, Q = A' \). Evident.

Case (iii). \( P' < A \) (or \( B'A' = AB \)) and \( P'B' = BP \).

Divide into three subcases by (1); cf. Definition 2.

Subcase (i). \( Q' = B', Q = B \). Impossible, since \( Q' < P' \).

Subcase (ii). \( Q'B' \approx BQ \).

\[ Q'B' \approx BQ, P'B' = BP \]
\[ \xrightarrow{\text{(1)}} \]
\[ Q'P' \approx PQ \]
\[ P'B' \approx BP \]
\[ \Rightarrow \]
\[ AP \approx P'A' \]

Subcase (iii). \( Q' = A, Q = A' \). \( A < A' \).

If \( B < P' \).

\[ P'B' = BP \xrightarrow{\text{(1)}} \]
\[ BP \approx P'B', B'A' = AB \Rightarrow AP \approx P'A' \]
\[ PB' \approx BP', B'A' = AB \Rightarrow PA' \approx AP', \text{ i.e.,} \]
\[ P \approx Q'P' \]

If \( B = P' \), evident.

Case (iii). \( P' = A, P = A' \). \( A < A' \).

Divide into three subcases by (1); cf. Definition 2.

Subcase (i). \( Q' = B', Q = B \). Impossible, since \( B'A' \) is elementary.

Subcase (ii). \( Q' < A \) (or \( B'A' = AB \)) and \( Q'B' \approx BQ \).

Since \( Q' < P' = A < B \),

\[ Q'B' \approx BQ \]
\[ \Rightarrow \]
\[ Q'B' \approx Q'Q \Rightarrow Q'A \approx A'Q, \text{ i.e.,} \]
\[ Q'P' \approx PQ \]
\[ AA' \approx AA' \]
\[ \text{gives} \]
\[ AP \approx P'A' \]

Subcase (iii). \( Q' = A, Q = A' \). Impossible, since \( Q' < P' \).

Type 3.

\[ P < Q < A \]
\[ PA \approx A'P' \] (1)
\[ QA = A'Q' \] (2)
\[ \Rightarrow \]
\[ A'Q' \approx QA \]
\[ P \approx Q'P' \]
Proof. Divide into two cases by (1); cf. Definition 3.

Case (i). \( B'P' \approx PB \).

Divide into two subcases by (2); cf. Definition 3.

Subcase (i). \( QB = B'Q' \).

\[
\begin{align*}
P < Q &< B \\
PB \approx B'P', \; QB = B'Q' \end{align*}
\]

\[
\begin{align*}
\{(\; - \rangle \} & \quad \{ PQ \approx Q'P'. \\
QB \approx B'Q' \Rightarrow QA \approx A'Q'. \\
P < A \end{align*}
\]

Subcase (ii). \( Q = A', \; Q' = A \).

Since \( P \leq B' < A' \),

\[
\begin{align*}
B'P' \approx PB \quad \quad \rightarrow \quad \quad PB' \approx BP' \\
P < A \end{align*}
\]

\[
\begin{align*}
QB = B'Q' \rightarrow B'Q \approx Q'B \Rightarrow A'Q \approx Q'A, \; i.e., \; PQ \approx Q'P'. \\
( B' < Q ) \\
QB \approx B'Q' \Rightarrow QA \approx A'Q'. \\
B' < A = P < Q,
\end{align*}
\]

Subcase (ii). \( Q = A', \; Q' = A \). Impossible, since \( P < Q \).

Case (ii). \( P = A', \; P' = A; \; A' A \approx A' A \).

Divide into two subcases by (1); cf. Definition 3.

Subcase (i). \( QB = B'Q' \).

Since \( B' < A = P < Q \),

\[
\begin{align*}
QB = B'Q' \rightarrow B'Q \approx Q'B \Rightarrow A'Q \approx Q'A, \; i.e., \; PQ \approx Q'P'. \\
QB \approx B'Q' \quad ( B' < Q ) \\
\Rightarrow QA \approx A'Q'. \\
\end{align*}
\]

Subcase (ii). \( Q = A', \; Q' = A \). Impossible, since \( P < Q \).

Type 4.

\[
\begin{align*}
A' < Q' < P', \\
A'P' \approx PA \quad (1), \\
Q'P' = PQ \quad (2) \end{align*}
\]

\[
\begin{align*}
\rightarrow \{ PQ \approx Q'P'. \\
\Rightarrow A'Q' \approx QA. \\
\end{align*}
\]

Proof. Divide into two cases by (1); cf. Definition 4.

Case (i). \( B'P' \approx PB \).

\[
\begin{align*}
B' < Q' < P', \\
B'P' \approx PB, \; Q'P' = PQ \end{align*}
\]

\[
\begin{align*}
\rightarrow \{ PQ \approx Q'P'. \\
B'Q' \approx QB \Rightarrow A'Q' \approx QA. \\
\end{align*}
\]

Case (ii). \( P' = A, \; P = A'; \; A' < A \). (2) becomes

\[
Q' = A'Q
\]

Divide into two subcases by (2)'; cf. Definition 3.

Subcase (i). \( Q'B = B'Q \).

\[
\begin{align*}
B' < Q', \\
Q' = B'Q' \end{align*}
\]

\[
\begin{align*}
\rightarrow \{ B'Q' \approx QB \Rightarrow A'Q' \approx QA. \\
B'Q \approx Q'B \Rightarrow A'Q \approx Q'A, \; i.e., \; PQ \approx Q'P'. \\
\end{align*}
\]
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Subcase (ii). $Q' = A'$, $Q = A$. Impossible, since $Q' < P'$ and $A' < A$.

**Type 5.**

\[
\begin{align*}
P &< A < Q, \\
PQ &\approx Q'P' \quad \text{(1)} \implies \begin{cases} AQ \approx Q'A', \\
PA \approx A'P'. \end{cases} \\
AQ &\approx Q'A' \quad \text{(2)}
\end{align*}
\]

Proof. Divide into three cases by (2); cf. Definition 1.

Case (i). $Q = B$, $Q' = B'$. Then

\[
(1): \quad PB \approx B'P' \Rightarrow \begin{cases} PA \approx A'P', \\
AB \approx B'A', \text{ i.e., } AQ \approx Q'A'. \end{cases}
\]

Case (ii). $BQ = Q'B'$.

\[
P < B, \\
PQ \approx Q'P', \\
BQ \approx Q'B' \\
PB \approx B'P' \implies AB \approx B'A'
\]

Case (iii). $Q = A'$, $Q' = A$. Proved in Type 2.

**Type 6.**

\[
\begin{align*}
Q' &< A'< P', \\
Q'P' &\approx PQ, \quad \text{(1)} \implies \begin{cases} A'P' \approx PA, \\
Q'A' \approx AQ. \end{cases} \\
A'P' &\approx PA \quad \text{(2)}
\end{align*}
\]

Proof. Divide into two cases by (2); cf. Definition 4.

Case (i). $B'P' \approx PB$. Then

$Q' \leq B'$, since $Q' < A'$.

If $Q' < B'$, then

\[
\begin{align*}
Q'P' &\approx PQ \quad \text{(6)} \implies Q'B' \approx BQ \\
B'P' &\approx PB \implies AB \approx B'A' \\
PA &\approx A'P'
\end{align*}
\]

If $Q' = B'$, evident.


Thus the proof of Lemma is complete.

We are now in a position to construct a model $M(R, C, I)$ on the basis of Lemma.

First take all the triples of natural numbers $(i, j, k)$, make a numbering $N$ on them such that different triples $(i, j, k)$ and $(i', j', k')$ have different numbers $N(i, j, k) \neq N(i', j', k')$. 

Suppose a system $A_{n_i}$ of $n_i$ different points $A_1, A_2, \ldots, A_{n_i}$ has been already defined such that points are linearly ordered and that it satisfies Axioms $E_u, R, C^+, C^+, C^-$ and $C^-$. Call a triple of points $(A_i, A_j, A_k) \ (1 \leq i, j, k \leq n_i)$ with $A_i < A_j$ saturated if the equality

$$A_i A_j = A_k A_l$$

has a solution in $A_i \in A_{n_i}$, and insaturated if not, and let $(A_p, A_q, A_r)$ be the insaturated triple with the smallest $N(p, q, r)$.

For the sake of simplicity, set $A_p = A_m$, $A_q = A_{m+1}$, $A_r = A_{m+2}$, and choose points $P_{m-1}, P_{m-2}, \ldots, P_1$ of $A_{n_i}$ such that $A_p = P_m < P_{m-1} < \cdots < P_2 < P_1 = A_q$ and that the consecutive segments $P_1 P_2 P_3 P_4 \ldots P_1 P_2$ are all elementary.

If there is any saturated triple $(P_m, P_{m+1}, P_{m+2})$, let $(P_{s-1}, P_s, P_s')$ be such a one with the largest $s$. Then there must be a point $P_s' < P_{s-1}$ with

$$P_{s-1} P_s = P_s' P_{s-1}. \quad (1)$$

If there is no saturated triple, set $s=2$. Introduce then $m-s+1$ new points $P_x', P_{x+1}', \ldots, P_m'$ and define the linear ordering

$$P_{x-1}' < P_x' < \cdots < P_m' < P_1',$$

where either $P_{x-1}' P_1' (P_1' \in A_{n_i})$ is an elementary segment or $P_1'$ is to be regarded as the point at infinity, if there is no point $X \in A_{n_i}$ with $P_{x-1}' < X$.

Repeated applications of Lemma beginning with the successive introduction of basic congruence relations

$$P_s P_{s-1} = P_{s-1}' P_s', \quad P_{s-1} P_s = P_s' P_{s-1}', \quad (2)$$

lead us to a system of points $A_1, \ldots, A_{n_i}, P_s', \ldots, P_m'$ in a linear order, satisfying Axioms $E_u, R, C^+, C^+, C^-$ and $C^-$. Then we have from (1) and (2) on account of $C^+$

$$P_m P_1 = P_1' P_m'. \quad (3)$$

If we set
we have by (3)
\[ A_p A_q = A_r A_{n_i+1}, \]
and \((A_p, A_q, A_r)\) becomes a saturated triple in the system of points
\[ A_{n_i+1} = \{A_1, A_2, \ldots, A_{n_i}, \ldots, A_{n_i+1}\}. \]

Now let \(n\) be equal to 4 and let \(A_{n_1}\) be defined as a system of four points
\(A_1, A_2, A_3, A_4\) in a linear order
\[ A_1 < A_2 < A_3 < A_4 \]
with the following congruence relations:

i) \(A_i A_j = A_i A_j\) for all \(i, j = 1, \ldots, 4,\)
   provided \(A_i < A_j,\)

ii) \(A_1 A_2 = A_2 A_3\) but \(A_2 A_3 \neq A_1 A_4,\)

iii) \(A_2 A_4 = A_1 A_3,\)

and

iv) \(A_1 A_4 = A_2 A_3, A_1 A_2 = A_4 A_3, A_4 A_3 = A_1 A_2.\)

In \(A_{n_1}\) all Axioms \(E_u, R, C^+ (= C), C^+, C^-\) and \(C^-\) are seen to be fulfilled. Thus we see by induction that in each \(A_{n_i} (i = 1, 2, 3, \ldots)\) all Axioms from \(E_u\) to \(C^-\) are fulfilled, so that in particular Axioms \(E_u, R\) and \(C\) are satisfied in the system of points
\[ A = \bigcup_{i=1}^{\infty} A_{n_i}. \]

If \(A_p, A_q, A_r\) is any triple of points with \(A_p < A_q\) in \(A\), then there is by the way of introducing new points of \(A_{n_i+1}\) into each \(A_{n_i} (i = 1, 2, 3, \ldots)\) a natural number \(n_j\) such that the equality
\[ A_p A_q = A_r A_s \]
is satisfied by an \(A_s \in A_{n_j}.\) Thus Axiom E is satisfied in \(A.\)

Recalling the fact seen in the proof of Lemma that when the point \(A_n\) is added to the set \(A_{n-1}\) as a new point to obtain \(A_n\) the new congruence relations introduced with it are confined to those between some old segments and new ones having \(A_n\) as an end point, so we see that the relation \(A_2 A_3 = A_1 A_2\) in \(A_{n_1}\) remains true throughout all \(A_{n_i}.\) Thus in \(A:\)

→ S: Axiom S fails to be satisfied, for \(A_1 A_2 = A_3 A_2\) but
\(A_2 A_3 = A_1 A_2.\)

→ T: Axiom T fails to be satisfied, for \(A_1 A_2 = A_2 A_3,\)
\(A_2 A_3 = A_1 A_4\) but \(A_1 A_3 = A_1 A_4\) by Axiom \(E_u.\)

→ A: Axiom A fails to be satisfied, for if \(A\) holds, then by Theorem 11 (see Part I) Axiom S would hold good too.
Thus $A$ is the desired model $M(R, C, I)$ in which Axioms $R, C,$ and $I$ alone hold besides Axiom $E$.

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