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ON CONGRUENT AXIOMS IN LINEARLY ORDERED SPACES, II¹⁾

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6. Model $M(R, C, I)$

$M(R, C, I)$: *A model of a geometry in which Axioms R, C and I alone hold besides Axiom E. (Notice that I follows automatically from E, R and C.)*

The construction of $M(R, C, I)$ is quite different from those of other models, and its exposition here may be too long, but it seems to the authors appropriate to provide it with a full proof. It depends essentially upon Lemma below, and we will begin by introducing some definitions and auxiliary axioms needed in it.

Let A be a finite number of linearly ordered points, in which congruence relations are supposed to hold among some of the segments, and let P, Q, P' etc. denote points of A .

DEFINITION. We write

$$PQ \approx Q'P' \quad \text{or} \quad Q'P' \approx PQ,$$

if and only if

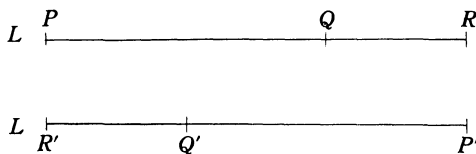
$$PQ = Q'P' \quad \text{and} \quad Q'P' = PQ$$

at the same time.

Axiom E_u : *If $PQ = Q'P'$ and $PQ = Q'P''$, then $P' = P''$.*

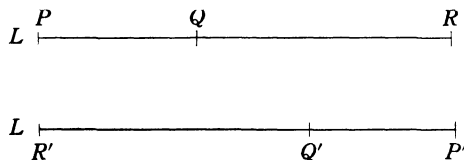
Axiom C^+ (=Axiom C)

$$\left. \begin{array}{l} P < Q < R, \\ R' < Q' < P', \\ PQ = Q'P', \\ QR = R'Q' \end{array} \right\} \Rightarrow PR = R'P'.$$



Axiom \tilde{C}^+

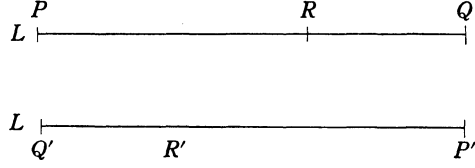
$$\left. \begin{array}{l} P < Q < R, \\ R' < Q' < P', \\ PQ \approx Q'P', \\ QR = R'Q' \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} PR \approx R'P' \\ QR \approx R'Q' \end{array} \right.$$



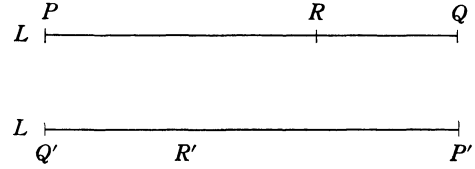
1) Continuation of Part I, this Journal, vol. 3 (1966), 269-292. Referred to as Part I.

Axiom C⁻

$$\left. \begin{array}{l} P < R < Q, \\ PQ = Q'P', \\ RQ = Q'R' \end{array} \right\} \Rightarrow PR = R'P'.$$

**Axiom C⁻:**

$$\left. \begin{array}{l} P < R < Q, \\ PQ \approx Q'P', \\ RQ = Q'R' \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} PR \approx R'P', \\ RQ \approx Q'R'. \end{array} \right.$$

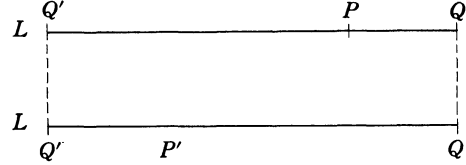


The following is an important consequence of \tilde{C}^- , and will sometimes be denoted by \tilde{c}^- .

$$\tilde{c}^-: PQ = Q'P', Q' < P \Rightarrow PQ \approx Q'P', Q'P \approx P'Q.$$

Proof.

$$\left. \begin{array}{l} Q' < P < Q \\ Q'Q \approx Q'Q \\ PQ = Q'P' \end{array} \right\} \xrightarrow{(\tilde{C}^-)} \left\{ \begin{array}{l} PQ \approx Q'P', \\ Q'P \approx P'Q. \end{array} \right.$$



DEFINITION. A segment PQ will be called *elementary*, if there is no point X with $P < X < Q$.

Lemma. Let $A_{n-1} = \{A_\lambda | \lambda = 1, 2, \dots, n-1\}$ be a finite number of points in some linear order such that they satisfy Axioms E_u , R , C^+ , \tilde{C}^+ , C^- and \tilde{C}^- . Then, for a given elementary segment $A_i A_j$ and a given point A_k such that the equality

$$A_i A_j = A_k A_l$$

has no solution in $A_l \in A_{n-1}$, a new point A_n can be introduced, so that

$$A_i A_j = A_k A_n$$

holds and the linearly ordered points $A_n = \{A_1, A_2, \dots, A_{n-1}, A_n\}$ satisfy the same Axioms from E_u to \tilde{C}^- .

Proof. Points as well as notations such as A , P , X , P' etc. will mean in this proof points of A_{n-1} except for A' which will be introduced below as a new point A_n . If two segments are equal it is convenient to write the corresponding end points counterwise with and without dashes such as $PQ = Q'P'$, since several axioms of the type of C are involved.

For the sake of simplicity, set $A_i = A$, $A_j = B$ and $A_k = B'$.

Thus by assumption there is no point X with

$$AB = B'X$$

DEFINITION OF THE NEW POINT $A' (=A_n)$ AND OF ORDERING.

Let A' be introduced as a new point such that $\{A_1, \dots, A_{n-1}, A'\}$ satisfy the following linear ordering:

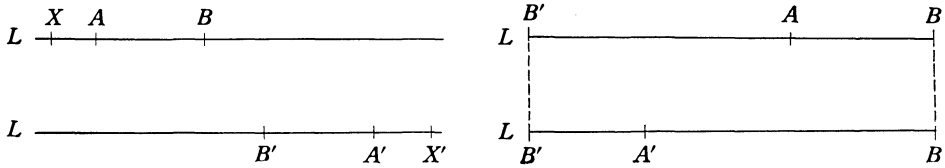
- (i) $B' < A'$.
- (ii) If $X < B'$, then $X < A'$ for any point $X \in A_{n-1}$.
- (iii) If $B' < X$, then $A' < X$ for any point $X \in A_{n-1}$.

DEFINITION OF THE BASIC EQUALITY. The following is the basic congruence relation:

- (i) $AB \approx B'A'$,

i.e. $AB = B'A'$ and $B'A' = AB$ at the same time, if and only if there exist some X and X' such that $XB \approx B'X'$.

In particular, $AB \approx B'A'$ if $B' < A$, since $B'B \approx B'B$.



- (ii) Otherwise

$$AB = B'A' \text{ but } B'A' \neq AB,$$

that is, $B'A' = AB$ is not defined.

DEFINITION OF OTHER EQUALITIES. Besides the above basic congruence relation we must define other new congruence relations in order to make the system of points $A_n = \{A_1, \dots, A_{n-1}, A_n\}$ satisfy all axioms from E_n to \tilde{C}^- .

To insure Axiom R we only need

DEFINITION 0. For any $X \in A_{n-1}$: $A'X = A'X$ and $XA' = XA'$.

In the following are defined all the equalities between old segments and new ones with one end point A' . They are classified into four types according to the position of A' .

Some of them are redundant, such as $AB = B'A'$, $AA' = AA'$ and $A'A = A'A$, but are included for the sake of completeness.

DEFINITION 1. $AP = P'A'$, if and only if

- (i) $P = B, P' = B'$, i.e., $AB = B'A'$,
- or (ii) $BP = P'B'$,
- or (iii) $P = A', P' = A$, i.e., $AA' = AA'$.

DEFINITION 2. $P'A' = AP$, if and only if

- (i) $P' = B'$, $P = B$, i.e., $B'A' = AB$,
- or (ii) $P' < A$ (or $B'A' = AB$) and $P'B' = BP$,
- or (iii) $P' = A$, $P = A'$, i.e., $AA' = AA'$.

DEFINITION 3. $PA = A'P'$, if and only if

- (i) $PB = B'P'$,
- or (ii) $P = A'$, $P' = A$, i.e., $A'A = A'A$.

DEFINITION 4. $A'P' = PA$, if and only if

- (i) $B'P' \approx PB$,
- or (ii) $P' = A$, $P = A'$, i.e., $A'A = A'A$.

Having thus defined all congruence relations between old segments and new ones with one end point A' , we are now going to verify Axioms E_μ , C^+ , \check{C}^+ , C^- and \check{C}^- one by one.

The verification will be done after a pattern: each equality under consideration is first classified according to its type, and then dealt with by Definitions 1, 2, 3 and 4 accordingly almost mechanically. Verbal explanations in detail will be omitted.

VERIFICATION OF E_μ .

Type 1.

$$AP = P'A', AP = P'X \Rightarrow X = A'.$$

Proof. According to Definition 1, we divide the proof into three cases.

Case (i). $P = B$, $P' = B'$: $AB = B'A'$.

Then $AB = B'X$ is impossible for $X \in A_{n-1}$.

Case (ii). $BP = P'B'$.

$$A < B < P, AP = P'X, BP = P'B' \xrightarrow{(C^-)} AB = B'X,$$

which is impossible for any old point $X \in A_{n-1}$.

Case (iii). $P = A'$, $P' = A$: $AA' = AA'$.

Then $AA' = AX$ is impossible for any old point $X \in A_{n-1}$.

Type 2.

$$P'A' = AP, P'A' = AX \Rightarrow X = P$$

Proof. Divide into three cases by Definition 2.

Case (i). $P' = B'$, $P = B$: $B'A' = AB$.

Then $B'A' = AX$ is only possible for $X = B$ by Definition 2.

Case (ii). $P' < A$ (or $B'A' = AB$) and $P'B' = BP$ and $P'B' = BX$.

Then $X = P$ by Axiom E_μ applied to old congruence relations.

Case (iii). $P'=A, P=A': AA'=AA',$
 $AA'=AX.$ Then $X=A'$ by Definition 2.

Type 3.

$$PA=A'P', PA=A'X \Rightarrow X=P'.$$

Proof. Divide into two cases by Definition 3.

Case (i). $PB=B'P'.$ Then

$$PB=B'P', PB=B'X \xrightarrow{(E_n)} X=P'.$$

Case (ii). $P=A', P'=A.$ Then

$$A'A=A'A, A'A=A'X \Rightarrow X=A \quad \text{by Definition 3.}$$

Type 4.

$$A'P'=PA, A'P'=PX \Rightarrow X=A.$$

Proof. Divide into two cases by Definition 4.

Case (i). $B'P' \approx PB.$ Then

$$B'P' \approx PB, A'P'=PX \Rightarrow X=A \quad \text{by Definition 4.}$$

Case (ii). $P'=A, P=A'.$

$$A'A=A'A, A'A=A'X \Rightarrow X=A \quad \text{by Definition 4.}$$

VERIFICATION OF $C^+.$

To show that Axiom C^+ is satisfied for $A_n = \{A_1, \dots, A_{n-1}, A'\}$ we consider six types of equalities.

Type 1.

$$\left. \begin{array}{l} A < P < Q, \quad Q' < P' < A', \\ AP = P'A' \quad (1) \quad , \\ PQ = Q'P' \quad (2) \end{array} \right\} \Rightarrow AQ = Q'A'.$$

Proof. We divide the proof into three cases, according to (1); cf. Definition 1.

Case (i). $P=B, P'=B'.$ Then from (2),

$$BQ = Q'B' \xrightarrow{(\text{Def. 1(ii)})} AQ = Q'A'.$$

Case (ii). $BP = P'B'.$

$$\left. \begin{array}{l} B < P < Q, \quad Q' < P' < B', \\ BP = P'B', \quad PQ = Q'P' \end{array} \right\} \xrightarrow{(C^+)} BQ = Q'B' \xrightarrow{(\text{Def. 1(ii)})} AQ = Q'A'.$$

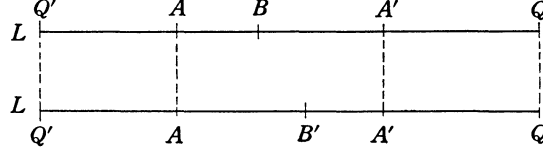
Case (iii). $P=A', P'=A.$ Then (2) becomes

$$A'Q = Q'A. \quad (2)'$$

Divide into two subcases according to (2)'; cf. Definition 4.

Subcase (i). $B'Q \approx Q'B,$

$$Q' < B', B'Q \approx Q'B \xrightarrow{(\tilde{c}^-)} BQ = Q'B' \xrightarrow{(\text{Def. 1})} AQ = Q'A'.$$



Subcase (ii). $Q = A, Q' = A'$. This is impossible, since $A < Q$.

Type 2.

$$\left. \begin{array}{l} Q' < P' < A', \quad A < P < Q, \\ P'A' = AP \quad (1) \\ Q'P' = PQ \quad (2) \end{array} \right\} \Rightarrow Q'A' = AQ.$$

Divide into three cases by (1); cf. Definition 2.

Case (i). $P' = B', P = B: B'A' = AB$.

$$B'A' = AB, Q'B' = BQ \xrightarrow{(\text{Def. 2(ii)})} Q'A' = AQ.$$

Case (ii). $P' < A$ (or $B'A' = AB$) and $P'B' = BP$.

$$\left. \begin{array}{l} P'B' = BP, Q'P' = PQ \xrightarrow{(C^+)} Q'B' = BQ \\ P' < A \text{ (or } B'A' = AB) \end{array} \right\} \xrightarrow{(\text{Def. 2(ii)})} Q'A' = AQ.$$

Case (iii). $P' = A, P = A': A < A'$. Then (2) becomes

$$Q'A = A'Q \quad (2)'$$

We divide into two subcases according to (2)'; cf. Definition 3.

Subcase (i). $Q'B = B'Q$.

Since $A < A'$ and since AB and $B'A'$ are elementary, either $B < B'$ or $B = B'$.

If $B < B'$,

$$\left. \begin{array}{l} Q'B = B'Q \\ BB' = BB' \end{array} \right\} \xrightarrow{(C^+)} \left. \begin{array}{l} Q'B' = BQ \\ Q' < A \end{array} \right\} \Rightarrow Q'A' = AQ.$$

If $B = B'$,

$$Q' < A, Q'B' = BQ \xrightarrow{(\text{Def. 2(ii)})} Q'A' = AQ.$$

Subcase (ii). $Q' = A', Q = A$.

This case is impossible, since $Q' < A'$.

Type 3.

$$\left. \begin{array}{l} P < Q < A, \quad A' < Q' < P', \\ QA = A'Q' \quad (1) \\ PQ = Q'P' \quad (2) \end{array} \right\} \Rightarrow PA = A'P'.$$

We divide into two cases by (1); cf. Definition 3.

Case (i). $QB=B'Q'$.

$$QB=B'Q', PQ=Q'P' \xrightarrow{(C^+)} PB=B'P' \xrightarrow{(Def. 3)} PA=A'P'.$$

Case (ii). $Q=A', Q'=A: A'<A$. Then (2) becomes

$$PA'=AP' \quad (2)'$$

Divide into three subcases by (2)'; cf. Definition 2.

Subcase (i). $P=B', P'=B$. Since $B'<A'<A<B$,

$$B'B=B'B \xrightarrow{(Def. 3)} B'A=A'B, \text{ i.e., } PA=A'P'.$$

Subcase (ii). $P<A$ (or $B'A'=AB$) and $PB'=BP'$.

$$PB'=BP', B'B=B'B \xrightarrow{(C^+)} PB=B'P' \xrightarrow{(Def. 3)} PA=A'P'.$$

Subcase (iii). $P=A, P'=A'$. This case is impossible, since $A'<P'$.

Type 4.

$$\left. \begin{array}{l} A'<Q'<P', P<Q<A, \\ A'Q'=QA \quad (1), \\ Q'P'=PQ \quad (2) \end{array} \right\} A'P'=PA.$$

Proof. Divide into two cases by (1); cf. Definition 4.

Case (i). $B'Q' \approx QB$.

$$B'Q' \approx QB, Q'P'=PQ \xrightarrow{(\tilde{C}^+)} B'P' \approx PB \xrightarrow{(Def. 4)} A'P'=PA.$$

Case (ii). $Q'=A, Q=A': A'<A$. Then (2) becomes

$$AP'=PA' \quad (2)'$$

Divide into three subcases by (2)'; cf. Definition 1.

Subcase (i). $P'=B, P=B': AB=B'A'$. Then

$B'<B$, since $B'<A'<A<B$. Then

$$B'B \approx B'B \xrightarrow{(Def. 4)} A'B=B'A, \text{ i.e., } A'P'=PA.$$

Subcase (ii).

$$BP'=PB', B'B \approx B'B \xrightarrow{(\tilde{C}^+)} B'P' \approx PB \xrightarrow{(Def. 4)} A'P'=PA.$$

Subcase (iii). $P'=A', P=A$. Impossible, since $A'<P'$.

Type 5.

$$\left. \begin{array}{l} P<A<Q, Q'<A'<P', \\ PA=A'P' \quad (1), \\ AQ=Q'A' \quad (2) \end{array} \right\} \Rightarrow PQ=Q'P'.$$

Proof. Divide into two cases by (1); cf. Definition 3.

Case (i). $PB=B'P'$.

Divide into three subcases by (2); cf. Definition 1.

Subcase (i). $Q=B, Q'=B'$. Then

$$PB=B'P' \text{ gives } PQ=Q'P'.$$

Subcase (ii). $BQ=Q'B'$.

$$PB=B'P', BQ=Q'B' \xrightarrow{(C^+)} PQ=Q'P'.$$

Subcase (iii). $Q=A', Q'=A$.

This case has been treated in Type 2.

Case (ii). $P=A', P'=A$.

Proved in Type 4.

Type 6.

$$\left. \begin{array}{l} Q' < A' < P', P < A < Q, \\ Q'A' = AQ \quad (1), \\ A'P' = PA \quad (2) \end{array} \right\} \Rightarrow Q'P' = PQ.$$

Proof. Divide into two cases by (2); cf. Definition 4.

Case (i). $B'P' \approx PB$.

Divide into three subcases by (1); cf. Definition 2.

Subcase (i). $Q'=B', Q=B$. Then

$$B'P' = PB \text{ gives } Q'P' = PQ$$

Subcase (ii). $Q' < A$ (or $B'A' = AB$) and $Q'B' = BQ$.

$$Q'B' = BQ, B'P' = PB \xrightarrow{(C^+)} Q'P' = PQ.$$

Subcase (iii). $Q'=A, Q=A'$.

This case has been proved in Type 1.

Case (ii). $P'=A, P=A'$.

Has been proved in Type 3.

VERIFICATION OF \tilde{C}^+ .

Type 1.

$$\left. \begin{array}{l} A < P < Q, Q' < P' < A', \\ AP \approx P'A' \quad (1), \\ PQ = Q'P' \quad (2) \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} PQ \approx Q'P', \\ AQ \approx Q'A'. \end{array} \right.$$

Proof. Divide into three cases by (1); cf. Definition 1.

Case (i). $P=B, P'=B'$: $AB \approx B'A'$.

$$AB \approx B'A' \xrightarrow{(\text{Def.})} \exists X, X': XB \approx B'X'.$$

$$XB \approx B'X', BQ = Q'B' \xrightarrow{(\tilde{C}^+)} BQ \approx Q'B', \text{ i.e., } PQ \approx Q'P'.$$

$$PQ \approx Q'P', AB \approx B'A', \stackrel{(\text{Def. 1.2})}{\implies} AQ \approx Q'A'$$

$$L \quad \begin{array}{c} X \quad A \quad B=P \quad Q \\ | \quad | \quad | \quad | \end{array}$$

$$L \quad \begin{array}{c} Q' \quad P'=B' \quad A' \quad X' \\ | \quad | \quad | \quad | \end{array}$$

Case (ii). $P' < A$ (or $B'A' = AB$) and $P'B' \approx BP$

$$\left. \begin{array}{l} BP \approx P'B', PQ = Q'P' \stackrel{(\tilde{C}^+)}{\implies} BQ \approx Q'B', \\ Q' < A \text{ (or } B'A' = AB) \end{array} \right\} \stackrel{(\text{Def. 1.2})}{\implies} AQ \approx Q'A'.$$

Case (iii). $P = A', P' = A$.

$$A'Q = Q'A'. \quad (2)'$$

Divide into two subcases by (2)'; cf. Definition 4.

Subcase (i). $B'Q \approx Q'B$.

$$B'Q \approx Q'B \stackrel{(\text{Def. 3,4})}{\implies} Q'A \approx A'Q, \text{ i.e., } Q'P' \approx PQ.$$

Since $A < A', Q' < P' = A < B'$.

$$\left. \begin{array}{l} Q' < B', B'Q \approx Q'B \stackrel{(\tilde{C}^-)}{\implies} BQ \approx Q'B', \\ Q' < A \end{array} \right\} \stackrel{(\text{Def. 1.2})}{\implies} AQ \approx Q'A'.$$

Subcase (ii). $Q = A, Q' = A'$. Impossible, since $A < Q$.

Type 2.

$$\left. \begin{array}{l} Q' < P' < A', A < P < Q, \\ Q'P' \approx PQ \quad (1), \\ P'A' = AP \quad (2) \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} Q'A' \approx AQ, \\ P'A' \approx AP. \end{array} \right.$$

Proof. Divide into three cases by (2); cf. Definition 2.

Case (i). $P' = B', P = B: B'A' = AB$.

$$\begin{array}{l} Q'B' \approx BQ, AB \approx B'A' \stackrel{(\text{Def. 1.2})}{\implies} AQ \approx Q'A'. \\ B'A' \approx AB \text{ gives } P'A' \approx AP. \end{array}$$

Case (ii). $P' < A$ (or $B'A' = AB$) and $P'B' = BP$.

$$Q'P' \approx PQ, P'B' = BP \stackrel{(\tilde{C}^+)}{\implies} \left\{ \begin{array}{l} Q'B' \approx BQ \\ P'B' \approx BP \Rightarrow P'A' \approx AP. \end{array} \right.$$

$$Q' < A \text{ (or } B'A' = AB), Q'B' \approx BQ \stackrel{(\text{Def. 1.2})}{\implies} AQ \approx Q'A'.$$

Case (iii). $P' = A, P = A': A < A'$. Then (1) becomes

$$Q'A \approx A'Q \quad (1)'$$

Divide into two subcases by (1)'; cf. Definition 3,4.

Subcase (i). $Q'B \approx B'Q$.

Since $A < A'$, $Q' < P' = A < B \leq B'$.

$$\left. \begin{array}{l} Q' < B', Q'B \approx B'Q \xrightarrow{(\tilde{c}^-)} Q'B' \approx BQ \\ Q' < A \end{array} \right\} \xrightarrow{(\text{Def. 1, 2})} Q'A' \approx AQ.$$

$AP \approx P'A'$ is evident.

Subcase (ii). $Q' = A'$, $Q = A$. Impossible, since $Q' < A'$.

Type 3.

$$\left. \begin{array}{l} P < Q < A, A' < Q' < P', \\ PQ \approx Q'P' \quad (1), \\ QA = A'Q' \quad (2) \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} PA \approx A'P', \\ QA \approx A'Q'. \end{array} \right.$$

Proof. Divide into two cases by (2); cf. Definition 3.

$$\text{Case (i). } \left. \begin{array}{l} QB = B'Q' \\ PQ \approx Q'P' \end{array} \right\} \xrightarrow{(\tilde{c}^+)} \left\{ \begin{array}{l} PB \approx B'P' \xrightarrow{(\text{Def. 3, 4})} PA \approx A'P'. \\ QB \approx B'Q' \xrightarrow{(\text{Def. 3, 4})} QA \approx A'Q'. \end{array} \right.$$

Case (ii). $Q = A'$, $Q' = A$: $A' < A$. (1) becomes

$$PA' \approx AP' \quad (1)'$$

Divide into two subcases by (1)'; cf. Definition 1,2.

Subcase (i). $P = B'$, $P' = B$: $B'A' \approx AB$.

$$A'A \approx A'A \text{ gives } QA \approx A'Q'$$

Since $B' < A' < A < B$,

$$B'B \approx B'B \xrightarrow{(\text{Def. 3, 4})} B'A \approx A'B, \text{ i.e., } PA \approx A'P'.$$

Subcase (ii). $BP' \approx PB'$. Since $P < A < B$,

$$BP' \approx PB', P < B \xrightarrow{(\tilde{c}^-)} PB \approx B'P' \Rightarrow PA \approx A'P'.$$

Subcase (iii). $P = A$, $P' = A'$. Impossible, since $P < A$.

Type 4.

$$\left. \begin{array}{l} A' < Q' < P', P < Q < A, \\ A'Q' \approx QA \quad (1), \\ Q'P' = PQ \quad (2) \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} A'P' \approx PA, \\ Q'P' \approx PQ. \end{array} \right.$$

Proof. Divide into two cases by (1); cf. Definition 4.

Case (i). $B'Q' \approx QB$.

$$B'Q' \approx QB, Q'P' = PQ \xrightarrow{(\tilde{c}^+)} \left\{ \begin{array}{l} PB \approx B'P' \Rightarrow A'P' \approx PA. \\ Q'P' \approx PQ. \end{array} \right.$$

Case (ii). $Q'=A, Q=A': A'<A$. (2) becomes

$$AP'=PA' \quad (2)'$$

Divide into three subcases by (2)'; cf. Definition 1.

Subcase (i). $P'=B, P=B'$. Since $B'<A'<A<B$,

$$B'B \approx B'B \xrightarrow{(\text{Def. 3, 4})} A'B \approx B'A, \text{ i.e., } A'P' \approx PA'.$$

$$AB=B'A', B'B \approx B'B \xrightarrow{(\text{Def.})} AB \approx B'A', \text{ i.e., } Q'P' \approx PQ.$$

Subcase (ii). $BP'=PB'$. Since $P<B'<A'<A<B$,

$$P<B, BP'=PB' \xrightarrow{(\tilde{c}^-)} \left\{ \begin{array}{l} PB \approx B'P' \xrightarrow{(\text{Def. 3, 4})} A'P' \approx PA' \\ BP' \approx PB' \\ P<A \end{array} \right\} \xrightarrow{(\text{Def. 1, 2})} AP' \approx PA', \text{ i.e., } Q'P' \approx PQ.$$

Subcase (iii). $P'=A', P=A$. Impossible, since $A'<P'$.

Type 5.

$$\left. \begin{array}{l} P<A<Q, Q'<A'<P', \\ PA \approx A'P' \quad (1), \\ AQ=Q'A' \quad (2) \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} PQ \approx Q'P', \\ AQ \approx Q'A'. \end{array} \right.$$

Proof. Divide into two cases by (1); cf. Definition 3.

Case (i). $B'P' \approx PB$.

Divide into three subcases by (2); cf. Definition 1.

Subcase (i). $Q=B, Q'=B'$.

$$B'P' \approx PB \text{ gives } Q'P' \approx PQ.$$

$$B'P' \approx PB \xrightarrow{(\text{Def.})} AB \approx B'A', \text{ i.e., } AQ=Q'A'.$$

Subcase (ii). $BQ=Q'B'$.

$$BQ=Q'B', PB \approx B'P' \xrightarrow{(\tilde{C}^+)} PQ \approx Q'P'.$$

$$\left. \begin{array}{l} PB \approx B'P' \xrightarrow{(\text{Def.})} B'A' \approx AB \\ BQ \approx Q'B' \end{array} \right\} \xrightarrow{(\text{Def. 1, 2})} AQ \approx Q'A'.$$

Subcase (iii). $Q=A', Q'=A$. Proved in Type 2.

Case (ii). $P=A', P'=A$. Proved in Type 4.

Type 6.

$$\left. \begin{array}{l} Q'<A'<P', P<A<Q, \\ Q'A' \approx AQ \quad (1), \\ A'P'=PA \quad (2) \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} Q'P' \approx PQ, \\ A'P' \approx PA. \end{array} \right.$$

Proof. Divide into two cases by (2); cf. Definition 4.

Case (i). $B'P' \approx PB$.

Divide into three subcases by (1); cf. Definition 2.

Subcase (i). $Q' = B'$, $Q = B$:

$$B'P' \approx PB \text{ gives } Q'P' \approx PQ.$$

$$B'P' \approx PB \xrightarrow{\text{(Def. 3, 4)}} PA \approx A'P'.$$

Subcase (ii). $Q' < A$ (or $B'A' = AB$) and $Q'B' \approx BQ$.

$$B'P' \approx PB, Q'B' \approx BQ \xrightarrow{(C^+)} Q'P' \approx PQ.$$

$$B'P' \approx PB \xrightarrow{\text{(Def. 3, 4)}} PA \approx A'P'.$$

Subcase (iii). $Q = A'$, $Q' = A$. Proved in Type 1.

Case (ii). $P' = A$, $P = A'$. Proved in Type 3.

VERIFICATION OF C^- .

Type 1.

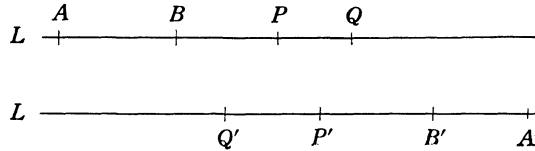
$$\left. \begin{array}{l} A < P < Q, \\ AQ = Q'A' \quad (1), \\ PQ = Q'P' \quad (2) \end{array} \right\} \Rightarrow AP = P'A'.$$

Proof. Divide into three cases by (1); cf. Definition 1.

Case (i). $Q = B$, $Q' = B'$. Impossible, since AB is elementary.

Case (ii). $BQ = Q'B'$.

$$\left. \begin{array}{l} B < P < Q, \\ BQ = Q'B', PQ = Q'P' \end{array} \right\} \xrightarrow{(C^-)} BP = P'B' \Rightarrow AP = P'A'.$$



Case (iii). $Q = A'$, $Q' = A$: $A < A'$. (2) becomes

$$PA' = AP' \quad (2)'$$

Divide into three subcases by (2)'; cf. Definition 2.

Subcase (i). $P = B'$, $P' = B$. Since AB is elementary,

$$A < B \leq B' < A'.$$

If $B < B'$,

$$BB' = BB' \xrightarrow{\text{(Def. 1(ii))}} AB' = BA', \text{ i.e., } AP = P'A'.$$

If $B = B'$, $AB = B'A'$ gives $AP = P'A'$.

Subcase (ii). $P < A$ (or $B'A' = AB$) and $PB' = BP'$.
 Since $A < P < A'$ and since AB is elementary,

$$B \leq P,$$

If $B < P$,

$$PB' = BP' \xrightarrow{(\tilde{c}^-)} BP = P'B' \Rightarrow AP = P'A'.$$

If $B = P$, then $B' = P'$, so $AP = P'A'$.

Subcase (iii). $P = A$, $P' = A'$. Impossible, since $A < P$.

Type 2.

$$\left. \begin{array}{l} Q' < P' < A', \\ Q'A' = AQ \quad (1), \\ P'A' = AP \quad (2) \end{array} \right\} \Rightarrow Q'P' = PQ.$$

Proof. Divide into three cases by (2); cf. Definition 2.

Case (i). $P' = B'$, $P = B$: $B'A' = AB$.

Divide into three subcases by (1); cf. Definition 2.

Subcase (i). $Q' = B'$, $Q = B$. Impossible, since $Q' < P'$.

Subcase (ii). $Q' < A$ (or $B'A' = AB$) and $Q'B' = BQ$.

$$Q'B' = BQ \text{ gives } Q'P' = PQ$$

Subcase (iii). $Q = A'$, $Q' = A$; $A < A'$.

If $B < B'$,

$$BB' = BB' \xrightarrow{(\text{Def. 1})} AB' = BA', \text{ i.e., } Q'P' = PQ.$$

If $B = B'$, $AB = B'A'$ gives $Q'P' = PQ$.

Case (ii). $P' < A$ (or $B'A' = AB$) and $P'B' = BP$.

Divide into three subcases by (1); cf. Definition 2.

Subcase (i). $Q' = B'$, $Q = B$. Impossible, since $Q' < P' < B'$.

Subcase (ii). $Q' < A$ (or $B'A' = AB$) and $Q'B' = BQ$.

$$Q' < P' < B', Q'B' = BQ, P'B' = BP \xrightarrow{(\tilde{c}^-)} Q'P' = PQ.$$

Subcase (iii). $Q' = A$, $Q = A'$. Then $A < P' < A'$, since $Q' < P' < A'$.

If $B < P'$,

$$B < P', P'B' = BP \xrightarrow{(\tilde{c}^-)} BP' = PB' \Rightarrow AP' = PA', \text{ i.e., } Q'P' = PQ.$$

If $B = P'$, then $B' = P$ and $AB = B'A'$ gives $Q'P' = PQ$.

Case (iii). $P' = A$, $P = A'$: $AA' = AA'$.

Divide into three cases by (1); cf. Definition 2.

Subcase (i). $Q' = B'$, $Q = B$.

Impossible, since $Q' < P' < A'$ and since $B'A'$ is elementary.

Subcase (ii). $Q' < A$ (or $B'A' = AB$) and $Q'B' = BQ$.

If $B < B'$,

$$B < B', Q'B' = BQ \xrightarrow{(C^-)} Q'B = B'Q \xrightarrow{(\text{Def. 3})} Q'A = A'Q, \text{ i.e., } Q'P' = PQ.$$

If $B = B'$,

$$Q'B' = BQ \Rightarrow Q'A = A'Q, \text{ i.e., } Q'P' = PQ.$$

Subcase (iii). $Q' = A, Q = A'$. Impossible, since $Q' < P'$.

Type 3.

$$\left. \begin{array}{l} P < Q < A, \\ PA = A'P' \quad (1), \\ QA = A'Q' \quad (2) \end{array} \right\} \Rightarrow PQ = Q'P'.$$

Proof. Divide into two cases by (1); cf. Definition 3.

Case (i). $PB = B'P'$.

Divide into two subcases by (2); cf. Definition 3.

Subcase (i). $QB = B'Q'$.

$$P < Q < B, QB = B'Q', PB = B'P' \xrightarrow{(C^-)} PQ = Q'P'.$$

Subcase (ii). $Q = A', Q' = A: A' < A$. Then $P \leq B'$, since $P < Q$ and since $B'A'$ is elementary.

If $P < B'$,

$$\left. \begin{array}{l} P < B' < B \\ PB = B'P', B'B = B'B \end{array} \right\} \xrightarrow{(C^-)} PB' = BP' \xrightarrow{(\text{Def. 2(ii)})} PA' = AP', \text{ i.e., } PQ = Q'P'.$$

If $P = B'$, then $B = P'$ and

$$B'B \approx B'B \xrightarrow{(\text{Def.})} B'A' = AB, \text{ i.e., } PQ = Q'P'.$$

Case (ii). $P = A', P' = A: A' < A$.

Divide into two subcases by (2); cf. Definition 3.

Subcase (i). $QB = B'Q'$.

$$\left. \begin{array}{l} B' < A' = P < Q, \\ QB = B'Q' \end{array} \right\} \xrightarrow{(\tilde{C}^-)} B'Q \approx Q'B \xrightarrow{(\text{Def. 4})} A'Q = Q'A, \text{ i.e., } PQ = Q'P'.$$

Subcase (ii). $Q = A', Q' = A$. Impossible, since $P < Q$.

Type 4.

$$\left. \begin{array}{l} A' < Q' < P', \\ A'P' = PA \quad (1), \\ Q'P' = PQ \quad (2) \end{array} \right\} \Rightarrow A'Q' = QA.$$

Proof. Divide into two cases by (1); cf. Definition 4.

Case (i). $B'P' \approx PB$.

$$\left. \begin{array}{l} B' < A' < Q' < P' \\ B'P' \approx PB, Q'P' = PQ \end{array} \right\} \xrightarrow{(\tilde{c}^-)} B'Q' \approx QB \xrightarrow{(\text{Def. 4})} A'Q' = QA.$$

Case (ii). $P' = A, P = A': A' < A$. Then (2) becomes

$$Q'A = A'Q. \quad (2)'$$

Divide into two subcases by (2)'; cf. Definition 3.

Subcase (i). $Q'B = B'Q$.

$$B' < A' < Q', Q'B = B'Q \xrightarrow{(\tilde{c}^-)} B'Q' \approx QB \xrightarrow{(\text{Def. 4})} A'Q' = QA.$$

Subcase (ii). $Q' = A', Q = A$. Impossible, since $A' < Q'$.

Type 5.

$$\left. \begin{array}{l} P < A < Q, \\ PQ = Q'P' \quad (1), \\ AQ = Q'A' \quad (2) \end{array} \right\} \Rightarrow PA = A'P'.$$

Proof. Divide into three cases by (2); cf. Definition 1.

Case (i). $BQ = Q'B'$.

$$\left. \begin{array}{l} P < B < Q \\ BQ = Q'B', PQ = Q'P' \end{array} \right\} \xrightarrow{(\text{C}^-)} PB = B'P' \xrightarrow{(\text{Def. 3})} PA = A'P'.$$

Case (ii). $Q = B, Q' = B'$. Then

$$PB = B'P' \Rightarrow PA = A'P'.$$

Case (iii). $Q = A', Q' = A$. Proved in Type 2.

Type 6.

$$\left. \begin{array}{l} Q' < A' < P', \\ Q'P' = PQ \quad (1), \\ A'P' = PA \quad (2) \end{array} \right\} \Rightarrow Q'A' = AQ.$$

Proof. Divide into two cases by (2); cf. Definition 4.

Case (i). $B'P' \approx PB$.

If $Q' < B'$

$$\left. \begin{array}{l} B'P' \approx PB, Q'P' = PQ \xrightarrow{(\text{C}^-)} Q'B' = BQ, \\ B'P' \approx PB \xrightarrow{(\text{Def.})} B'A' = AB \end{array} \right\} \xrightarrow{(\text{Def. 2(ii)})} Q'A' = AQ.$$

If $Q' = B'$, then $Q = B$ and

$$B'P' \approx PB \Rightarrow B'A' = AB, \text{ i.e., } Q'A' = AQ.$$

Case (ii). $P' = A, P = A'$. Proved in Type 3.

VERIFICATION OF \tilde{C}^- .

Type 1.

$$\left. \begin{array}{l} A < P < Q, \\ AQ \approx Q'A' \quad (1), \\ PQ = Q'P' \quad (2) \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} PQ \approx Q'P', \\ AP \approx P'A'. \end{array} \right.$$

Proof. Divide into three cases by (1); cf. Definition 1.

Case (i). $Q = B, Q' = B'$.

Impossible, since $A < P < Q$ and since AB is elementary.

Case (ii). $Q' < A$ (or $B'A' = AB$) and $BQ \approx Q'B'$.

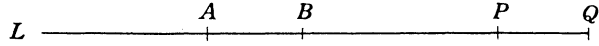
Note that if $Q' < A$ then $Q' < B$ and

$$BQ \approx Q'B' \xrightarrow{(\tilde{c}^-)} Q'B \approx B'Q \Rightarrow B'A' = AB.$$

Now $B \leq P$, since $A < P$.

If $B < P$,

$$BQ \approx Q'B', PQ = Q'P' \xrightarrow{(\tilde{c}^-)} \left\{ \begin{array}{l} BP \approx P'B', \\ (B'A' = AB) \\ Q'P' \approx PQ. \end{array} \right\} \Rightarrow AP \approx P'A'$$



Evident, if $B = P$.



Case (iii). $Q = A', Q' = A: A < A'$. Then (2) becomes

$$PA' = AP' \quad (2)'$$

Divide into three subcases by (2)'; cf. Definition 2.

Subcase (i). $P = B', P' = B: B'A' = AB$.

If $B < B'$,

$$\begin{aligned} BB' \approx BB', B'A' = AB &\xrightarrow{(\text{Def. 1, 2})} AB' \approx B'A, \text{ i.e., } AP \approx P'A'. \\ B'A' \approx AB &\text{ gives } PQ \approx Q'P'. \end{aligned}$$

If $B = B'$, evident.

Subcase (ii). $B'A' = AB$ and $PB' = BP'$.

If $B < P$,

$$PB' = BP' \xrightarrow{(\tilde{c}^-)} \left\{ \begin{array}{l} P'B' \approx BP, B'A' = AB \Rightarrow AP \approx P'A'. \\ PB' \approx BP', B'A' = AB \Rightarrow AP' \approx PA', \text{ i.e. } Q'P' \approx PQ. \end{array} \right.$$

If $B = P$, evident.

Subcase (iii). $P = A, P' = A'$. Impossible, since $A < P$.

Type 2.

$$\left. \begin{array}{l} Q' < P' < A', \\ Q'A' \approx AQ \quad (1), \\ P'A' = AP \quad (2) \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} P'A' \approx AP, \\ Q'P' \approx PQ. \end{array} \right.$$

Proof. Divide into three cases by (2); cf. Definition 2.

Case (i). $P' = B', P = B: B'A' = AB$.

Divide into three subcases by (1); cf. Definition 2.

Subcase (i). $Q' = B', Q = B$. Impossible, since $Q' < P'$.

Subcase (ii). $Q'B' \approx BQ$.

$$Q'B' \approx BQ \text{ gives } Q'P' \approx PQ.$$

$$AB \approx B'A' \text{ gives } AP \approx P'A'.$$

Subcase (iii). $Q' = A, Q = A'$. Evident.

Case (iii). $P' < A$ (or $B'A' = AB$) and $P'B' = BP$.

Divide into three subcases by (1); cf. Definition 2.

Subcase (i). $Q' = B', Q = B$. Impossible, since $Q' < P'$.

Subcase (ii). $Q'B' \approx BQ$.

$$\left. \begin{array}{l} Q' < P' \\ Q'B' \approx BQ, P'B' = BP \end{array} \right\} \xrightarrow{(\tilde{c}^-)} \left\{ \begin{array}{l} Q'P' \approx PQ \\ P'B' \approx BP \end{array} \right\} \Rightarrow AP \approx P'A'.$$

$P' < A \text{ (or } B'A' = AB)$

Subcase (iii). $Q' = A, Q = A': A < A'$.

If $B < P'$.

$$P'B' = BP \xrightarrow{(\tilde{c}^-)} \left\{ \begin{array}{l} BP \approx P'B', B'A' = AB \Rightarrow AP \approx P'A'. \\ PB' \approx BP', B'A' = AB \Rightarrow PA' \approx AP', \text{ i.e., } PQ \approx Q'P'. \end{array} \right.$$

If $B = P'$, evident.

Case (iii). $P' = A, P = A': A < A'$.

Divide into three subcases by (1); cf. Definition 2.

Subcase (i). $Q' = B', Q = B$. Impossible, since $B'A'$ is elementary.

Subcase (ii). $Q' < A$ (or $B'A' = AB$) and $Q'B' \approx BQ$.

Since $Q' < P' = A < B$,

$$Q'B' \approx BQ \xrightarrow{(\tilde{c}^-)} Q'B \approx B'Q \Rightarrow Q'A \approx A'Q, \text{ i.e., } Q'P' \approx PQ.$$

$$AA' \approx AA' \text{ gives } AP \approx P'A'.$$

Subcase (iii). $Q' = A, Q = A'$. Impossible, since $Q' < P'$.

Type 3.

$$\left. \begin{array}{l} P < Q < A, \\ PA \approx A'P' \quad (1), \\ QA = A'Q' \quad (2) \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} A'Q' \approx QA, \\ PQ \approx Q'P'. \end{array} \right.$$

Proof. Divide into two cases by (1); cf. Definition 3.

Case (i). $B'P' \approx PB$.

Divide into two subcases by (2); cf. Definition 3.

Subcase (i). $QB = B'Q'$.

$$\left. \begin{array}{l} P < Q < B \\ PB \approx B'P', QB = B'Q' \end{array} \right\} \xrightarrow{(\tilde{c}^-)} \left\{ \begin{array}{l} PQ \approx Q'P' \\ QB \approx B'Q' \Rightarrow QA \approx A'Q' \end{array} \right.$$

Subcase (ii). $Q = A', Q' = A$.

Since $P \leq B' < A'$,

$$\left. \begin{array}{l} B'P' \approx PB \xrightarrow{(\tilde{c}^-)} PB' \approx BP' \\ P < A \end{array} \right\} \Rightarrow AP' \approx PA', \text{ i.e., } Q'P' \approx PQ.$$

$QA \approx A'Q'$ is evident.

Case (ii). $P = A', P' = A$: $A'A \approx A'A$.

Divide into two subcases by (1); cf. Definition 3.

Subcase (i). $QB = B'Q'$.

Since $B' < A = P < Q$,

$$\left. \begin{array}{l} QB = B'Q' \xrightarrow{(\tilde{c}^-)} B'Q \approx Q'B \Rightarrow A'Q \approx Q'A, \text{ i.e., } PQ \approx Q'P' \\ QB \approx B'Q' \\ (B' < Q) \end{array} \right\} \Rightarrow QA \approx A'Q'.$$

Subcase (ii). $Q = A', Q' = A$. Impossible, since $P < Q$.

Type 4.

$$\left. \begin{array}{l} A' < Q' < P', \\ A'P' \approx PA \quad (1), \\ Q'P' = PQ \quad (2) \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} PQ \approx Q'P' \\ A'Q' \approx QA \end{array} \right.$$

Proof. Divide into two cases by (1); cf. Definition 4.

Case (i). $B'P' \approx PB$.

$$\left. \begin{array}{l} B' < Q' < P', \\ B'P' \approx PB, Q'P' = PQ \end{array} \right\} \xrightarrow{(\tilde{c}^-)} \left\{ \begin{array}{l} PQ \approx Q'P' \\ B'Q' \approx QB \Rightarrow A'Q' \approx QA \end{array} \right.$$

Case (ii). $P' = A, P = A'$: $A' < A$. (2) becomes

$$Q'A = A'Q \quad (2)'$$

Divide into two subcases by (2)'; cf. Definition 3.

Subcase (i). $Q'B = B'Q$.

$$\left. \begin{array}{l} B' < Q', \\ Q'B = B'Q \end{array} \right\} \xrightarrow{(\tilde{c}^-)} \left\{ \begin{array}{l} B'Q' \approx QB \Rightarrow A'Q' \approx QA \\ B'Q \approx Q'B \Rightarrow A'Q \approx Q'A, \text{ i.e., } PQ \approx Q'P' \end{array} \right.$$

Subcase (ii). $Q' = A'$, $Q = A$. Impossible, since $Q' < P'$ and $A' < A$.

Type 5.

$$\left. \begin{array}{l} P < A < Q, \\ PQ \approx Q'P' \quad (1), \\ AQ = Q'A' \quad (2) \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} AQ \approx Q'A', \\ PA \approx A'P'. \end{array} \right.$$

Proof. Divide into three cases by (2); cf. Definition 1.

Case (i). $Q = B$, $Q' = B'$. Then

$$(1): PB \approx B'P' \Rightarrow \left\{ \begin{array}{l} PA \approx A'P', \\ AB \approx B'A', \text{ i.e., } AQ \approx Q'A'. \end{array} \right.$$

Case (ii). $BQ = Q'B'$.

$$\left. \begin{array}{l} P < B, \\ PQ \approx Q'P', \\ BQ \approx Q'B' \end{array} \right\} \xrightarrow{(\tilde{c}^-)} \left\{ \begin{array}{l} PB \approx B'P' \Rightarrow PA \approx A'P', \\ BQ \approx Q'B' \end{array} \right\} \Rightarrow AQ \approx Q'A'.$$

$$PB \approx B'P' \implies AB \approx B'A'$$

Case (iii). $Q = A'$, $Q' = A$. Proved in Type 2.

Type 6.

$$\left. \begin{array}{l} Q' < A' < P', \\ Q'P' \approx PQ, \quad (1), \\ A'P' = PA \quad (2) \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} A'P' \approx PA, \\ Q'A' \approx AQ. \end{array} \right.$$

Proof. Divide into two cases by (2); cf. Definition 4.

Case (i). $B'P' \approx PB$. Then

$$Q' \leq B', \text{ since } Q' < A'.$$

If $Q' < B'$, then

$$\left. \begin{array}{l} Q'P' \approx PQ \\ B'P' \approx PB \end{array} \right\} \xrightarrow{(\tilde{c}^-)} \left\{ \begin{array}{l} Q'B' \approx BQ \\ B'P' \approx PB \end{array} \right\} \Rightarrow Q'A' \approx AQ.$$

$$B'P' \approx PB \Rightarrow \left\{ \begin{array}{l} AB \approx B'A' \\ PA \approx A'P' \end{array} \right.$$

If $Q' = B'$, evident.

Case (ii). $P' = A$, $P = A'$. Proved in Type 3.

Thus the proof of Lemma is complete.

We are now in a position to construct a model $M(R, C, I)$ on the basis of Lemma.

First take all the triples of natural numbers (i, j, k) , make a numbering N on them such that different triples (i, j, k) and (i', j', k') have different numbers $N(i, j, k) \neq N(i', j', k')$.

Suppose a system A_{n_i} of n_i different points A_1, A_2, \dots, A_{n_i} has been already defined such that points are linearly ordered and that it satisfies Axioms E_w , R, C^+ , \tilde{C}^+ , C^- and \tilde{C}^- . Call a triple of points (A_i, A_j, A_k) ($1 \leq i, j, k \leq n_i$) with $A_i < A_j$ *saturated* if the equality

$$A_i A_j = A_k A_l$$

has a solution in $A_l \in A_{n_i}$, and *insaturated* if not, and let (A_p, A_q, A_r) be the insaturated triple with the smallest $N(p, q, r)$.

For the sake of simplicity, set

$$A_p = P_m, \quad A_q = P_1, \quad A_r = P_1',$$

and choose points $P_{m-1}, P_{m-2}, \dots, P_2$ of A_{n_i} such that

$$A_p = P_m < P_{m-1} < \dots < P_2 < P_1 = A_q$$

and that the consecutive segments

$$P_m P_{m-1}, P_{m-1} P_{m-2}, \dots, P_2 P_1$$

are all elementary.

If there is any saturated triple (P_m, P_1, P_1') , let (P_{s-1}, P_1, P_1') be such a one with the largest s . Then there must be a point $P_{s-1}' \in A_{n_i}$ with

$$P_{s-1} P_1 = P_1' P_{s-1}'. \quad (1)$$

If there is no saturated triple, set $s=2$. Introduce then $m-s+1$ new points

$$P_s', P_{s+1}', \dots, P_m'$$

and define the linear ordering

$$P_{s-1}' < P_s' < \dots < P_m' < P'',$$

where either $P_{s-1}' P''$ ($P'' \in A_{n_i}$) is an elementary segment or P'' is to be regarded as the point at infinity, if there is no point $X \in A_{n_i}$ with $P_{s-1}' < X$.

Repeated applications of Lemma beginning with the successive introduction of basic congruence relations

$$\begin{aligned} P_s P_{s-1} &= P_{s-1}' P_s', \\ P_{s+1} P_s &= P_s' P_{s+1}', \\ &\vdots \\ P_m P_{m-1} &= P_{m-1}' P_m' \end{aligned} \quad (2)$$

lead us to a system of points $A_1, \dots, A_{n_i}, P_s', \dots, P_m'$ in a linear order, satisfying Axioms E_w , R, C^+ , \tilde{C}^+ , C^- and \tilde{C}^- . Then we have from (1) and (2) on account of C^+

$$P_m P_1 = P_1' P_m'. \quad (3)$$

If we set

$$P'_s = A_{n_i+1}, P'_{s+1} = A_{n_i+2}, \dots, P'_m = A_{n_i+1},$$

we have by (3)

$$A_p A_q = A_r A_{n_i+1},$$

and (A_p, A_q, A_r) becomes a saturated triple in the system of points

$$\mathbf{A}_{n_i+1} = \{A_1, A_2, \dots, A_{n_i}, \dots, A_{n_i+1}\}.$$

Now let n_1 be equal to 4 and let \mathbf{A}_{n_1} be defined as a system of four points A_1, A_2, A_3, A_4 in a linear order

$$A_1 < A_4 < A_2 < A_3$$

with the following congruence relations:

- i) $A_i A_j = A_i A_j$ for all $i, j = 1, \dots, 4$,
provided $A_i < A_j$,
- ii) $A_1 A_2 = A_2 A_3$ but $A_2 A_3 \neq A_1 A_2$,
- iii) $A_2 A_3 = A_1 A_4$,

and

$$\text{iv) } A_1 A_4 = A_2 A_3, A_1 A_2 = A_4 A_3, A_4 A_3 = A_1 A_2.$$

In \mathbf{A}_{n_1} all Axioms $E_u, R, C^+ (=C), \tilde{C}^+, C^-$ and \tilde{C}^- are seen to be fulfilled. Thus we see by induction that in each \mathbf{A}_{n_i} ($i=1, 2, 3, \dots$) all Axioms from E_u to \tilde{C}^- are fulfilled, so that in particular Axioms E_u, R and C are satisfied in the system of points

$$\mathbf{A} = \bigcup_{i=1}^{\infty} \mathbf{A}_{n_i}.$$

If A_p, A_q, A_r is any triple of points with $A_p < A_q$ in \mathbf{A} , then there is by the way of introducing new points of \mathbf{A}_{n_i+1} into each \mathbf{A}_{n_i} ($i=1, 2, \dots$) a natural number n_j such that the equality

$$A_p A_q = A_r A_s$$

is satisfied by an $A_s \in \mathbf{A}_{n_j}$. Thus Axiom E is satisfied in \mathbf{A} .

Recalling the fact seen in the proof of Lemma that when the point A_n is added to the set \mathbf{A}_{n-1} as a new point to obtain \mathbf{A}_n , the new congruence relations introduced with it are confined to those between some old segments and new ones having A_n as an end point, so we see that the relation $A_2 A_3 \neq A_1 A_2$ in \mathbf{A}_{n_1} remains true throughout all \mathbf{A}_{n_i} . Thus in \mathbf{A} :

- S: Axiom S fails to be satisfied, for $A_1 A_2 = A_2 A_3$ but $A_2 A_3 \neq A_1 A_2$.
- T: Axiom T fails to be satisfied, for $A_1 A_2 = A_2 A_3$, $A_2 A_3 = A_1 A_4$ but $A_1 A_2 \neq A_1 A_4$ by Axiom E_u .
- A: Axiom A fails to be satisfied, for if A holds, then by Theorem 11 (see Part I) Axiom S would hold good too.

Thus \mathcal{A} is the desired model $M(R, C, I)$ in which Axioms R, C, and I alone hold besides Axiom E.

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