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EQUIVARIANT WEDGE SUM CONSTRUCTION OF FINITE CONTRACTIBLE G-CW COMPLEXES WITH G-VECTOR BUNDLES

Dedicated to the memory of Professer Fuichi Uchida on his 60th birthday

MASAHARU MORIMOTO AND KRZYSZTOF PAWAŁOWSKI

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1. Introduction

Let G be a finite group and let B_GO denote the classifying space of real G-vector bundles. Let $\mathcal{P}(G)$ denote the set of all subgroups of G of prime power order. In [O2, p.587], Oliver defined a G-space B_G^*O which plays a role of the classifying space for the family of $\{\eta_P |\ P \in \mathcal{P}(G)\}$'s, where η_P is a real P-vector bundle, possessing natural G-compatibility. By definition, there is a canonical G-map $L_G : B_GO \to B_G^*O$.

Now suppose that the order of G is not a prime power. Let F be a finite CW-complex with base point $*$. We regard F as a G-space with trivial action. Let n_G be the Oliver number defined in [O1, p.155] (cf. [O2, Theorem 0.3] for a summary of computation of n_G). Thus, F can be realized as the G-fixed point set of G-action on a finite, contractible G-CW complex if and only if the Euler characteristic $\chi(F)$ is congruent to 1 modulo n_G. In case where F is a smooth manifold, there arise additional requirements for F so as to be realized as the G-fixed point set of a smooth G-action on a disk. By [O2] (in particular [O2, Theorem 0.1]), the requirements are interpreted into one condition that there exists a G-homotopically commutative diagram

$$\text{(DF)} \qquad \begin{array}{ccccc} F & & \xrightarrow{c(\eta)} & & B_GO \\ {\scriptstyle id_F}\big\downarrow & & & & \big\downarrow{\scriptstyle L_G} \\ F & \longrightarrow & \{*\} & \longrightarrow & B_G^*O \end{array}$$

where η is a real G-vector bundle over F such that

$$\eta^G \oplus \underline{\mathbb{R}^s} \cong T(F) \oplus \underline{\mathbb{R}^t}$$

for some nonnegative integers s, t, and $c(\eta)$ is the classifying map of η. Here $\underline{\mathbb{R}^s}$ denotes the product bundle over F with the fiber $\mathbb{R}^s$ on which G acts trivially. The main step of

Oliver's construction [O2] of a smooth G-action on a disk D with $D^G = F$ is to extend Diagram (DF) to a G-homotopically commutative diagram

$$
\text{(DX)} \qquad
\begin{array}{ccccc}
X & & \xrightarrow{\quad f_X \quad} & & B_G O \\
{\scriptstyle \varphi_X}\big\downarrow & & & & \big\downarrow{\scriptstyle L_G} \\
Y & \longrightarrow & \{*\} & \longrightarrow & B_G^* O
\end{array}
$$

such that X and Y are finite contractible G-CW complexes with the fixed point sets $X^G = Y^G = F$, $\varphi|_F = id_F$ and $f_X|_F = c(\eta)$. The G-map f_X can be regarded as the classifying map of a real G-vector bundle η_X over X. Applying the equivariant thickening theorem ([P, Theorems 2.4 and 3.1] or [O2, Theorem A.12]) to X and the G-vector bundle η_X derived from f_X, one can obtain a desired smooth G-action on D.

The purpose of this paper is to give refinements of Oliver's results in case where G is an Oliver group (i.e. in case of $n_G = 1$). That is, we shall construct Diagram (DX) with controlled K-fixed point sets X^K and Y^K for various subgroups K of G. More precisely to say with notation below, we shall control X^P and Y^P for all p-subgroups P (resp. $P \in \mathcal{P}_T(G)$) so that $\pi_1(D^P)$ are finite, abelian groups of order prime to p (resp. D^P are simply connected), and also control X^H, f_X^H and Y^H for $H \in \mathcal{L}(G)$ (hence $H \notin \mathcal{G}^1(G)$) so that the connected components D_i^H of D^H including F are without boundary in case where F is without boundary. These properties of D^P and D^H (cf. [M, (4.1.4), (4.1.4′) and (4.1.6)]) are very helpful to constructing various smooth G-actions on disks by equivariant surgery ([M, Theorem 4.1]) from the G-manifold D. Since the topic is concerned with details of equivariant surgery theory, we do not touch it further in the current article.

Let $\mathcal{S}(G)$ denote the set of all subgroups of G. For a prime p, let $\mathcal{P}_p(G)$ denote the set of all p-subgroups of G. The trivial group $\{1\}$ is contained in $\mathcal{P}_p(G)$ for any prime p. Let G^p denote the Dress subgroup of G of type p, namely it is the smallest normal subgroup of G whose index $[G : H]$ is a power of p. Clearly $G^p = G$ holds if p does not divide $|G|$. Let $\mathcal{L}^p(G)$ denote the set of all subgroups H of G such that H includes G^p. Set

$$\mathcal{L}(G) = \bigcup_p \mathcal{L}^p(G).$$

Furthermore, for a set T of primes dividing the order of G, we set $\mathcal{P}_T(G) = \bigcup_{p\in T} \mathcal{P}_p(G)$ and $\mathcal{L}^T(G) = \bigcup_{p\in T} \mathcal{L}^p(G)$. (As convention, we adopt $\mathcal{P}_\emptyset(G) = \{\{1\}\}$ and $\mathcal{L}^\emptyset(G) = \{G\}$.) If $\mathcal{A}$ is a set of subgroups of G then we define $\mathcal{A}_{>1}$ by

$$\mathcal{A}_{>1} = \mathcal{A} \smallsetminus \{\{1\}\}.$$

We shall use $\mathcal{G}_p^q$ and $\mathcal{G}^1$ in the sense of [O1]. That is, a finite group G belongs to $\mathcal{G}_p^q$ if and only if there exists a normal series $P \lhd H \lhd G$ such that P and G/H are a p-group and a

q-group respectively and H/P is cyclic. A finite group G is an Oliver group if and only if $G \notin \mathcal{G}_p^q$ for any primes p and q. Let $\mathcal{G}^1$ denote the union of all $\mathcal{G}_p^1$, where p runs over all primes. We set

$$\begin{aligned}
&\mathcal{G}_p^q(G) = \{H \mid H \le G,\ H \in \mathcal{G}_p^q\},\\
&\mathcal{G}^1(G) = \{H \mid H \le G,\ H \in \mathcal{G}^1\} \text{ and}\\
&\mathcal{G}^1(G)^T = \{H \in \mathcal{G}^1(G) \mid H \ge P \ \text{ for some } P \in \mathcal{P}_T(G)_{>1}\}.
\end{aligned}$$

In the following, an implication $A \supseteq B$ for CW complexes A and B should be understood that B is a subcomplex of A. Let X be a finite G-CW complex and $\mathcal{F}$ a set of subgroups of G. Define

$$\Pi(G, X; \mathcal{F}) := \coprod_{H \in \mathcal{F}} \pi_0(X^H).$$

For $\gamma \in \Pi(G, X; \mathcal{F})$, X_γ denotes the underlying space of γ. If $\mathcal{F}$ is invariant under conjugation by elements in G then $\Pi(G, X; \mathcal{F})$ inherits the canonical G-action. Define the map $\rho : \Pi(G, X; \mathcal{F}) \to \mathcal{F}$ by assigning each element $\gamma \in \pi_0(X^H)$ the subgroup H. For $\gamma, \delta \in \Pi(G, X; \mathcal{F})$, if $X_\gamma \subseteq X_\delta$ and $\rho(\gamma) > \rho(\delta)$ then we write $\gamma < \delta$.

In the first theorem, we control Y^P for all $P \in \mathcal{P}(G)$ and Y^H for all $H \notin \mathcal{G}^1(G)$.

Theorem 1.1. *Let G be an Oliver group and A a finite G-CW complex having the base point $*$ in $F := A^G$. Then there exists a finite G-CW complex Y including A and possessing the following properties:*

(C1) *Y is contractible.*

(C2) *$Y^G = F$.*

(C3) *For each $P \in \mathcal{P}(G)$, Y^P is 1-connected (i.e. connected and simply connected).*

(C4) *For each $H \in \mathcal{G}^1(G)$, Y^H is connected.*

(C5) *There is an order preserving G-map*

$$r : \Pi(G, Y; \mathcal{S}(G) \smallsetminus \mathcal{G}^1(G)) \to \Pi(G, A \amalg \{pt\}; \mathcal{S}(G) \smallsetminus \mathcal{G}^1(G))$$

with G_γ-homeomorphisms

$$k_\gamma : Y_\gamma \to A_{r(\gamma)} \text{ or } \{pt\}_{r(\gamma)},\ \gamma \in \Pi(G, Y; \mathcal{S}(G) \smallsetminus \mathcal{G}^1(G)),$$

such that $k_\gamma : Y_\gamma \to A_\gamma$ is the canonical map if γ is contained in the subset $\Pi(G, A; \mathcal{S}(G) \smallsetminus \mathcal{G}^1(G))$ of $\Pi(G, Y; \mathcal{S}(G) \smallsetminus \mathcal{G}^1(G))$; $k_{g\gamma} = g \circ k_\gamma \circ g^{-1}$ for all

$g \in G$ and $\gamma \in \Pi(G, Y; \mathcal{S}(G) \smallsetminus \mathcal{G}^1(G))$; and the square

$$\begin{array}{ccc} Y_\gamma & \xrightarrow{k_\gamma} & A_{r(\gamma)} \\ {\scriptstyle \text{inclusion}}\downarrow & & \downarrow{\scriptstyle \text{inclusion}} \\ Y_\delta & \xrightarrow{k_\delta} & A_{r(\delta)} \end{array}$$

commutes whenever $\gamma \le \delta \in \Pi(G, Y; \mathcal{S}(G) \smallsetminus \mathcal{G}^1(G))$.

The idea of the proof of Theorem 1.1 goes back to [LM]. In place of the equivariant connected sum that is used in [LM] for modification of smooth G-manifolds, we use in the current paper the equivariant wedge sum defined in Section 2 for modification of G-CW complexes. In case where G is an Oliver group, our equivariant wedge sum operation is effective to kill obstructions in the reduced projective class group $\widetilde{K}_0(\mathbb{Z}[G])$.

DEFINITION 1.2. We call a six-tuple $\boldsymbol{X} = (X, Y, f_X, g_Y, \varphi_X, h_X)$ a (B_GO, B_G^*O)-*system* if $\boldsymbol{X}$ consists of

finite G-CW complexes X and Y with base point, say $*$ ($\in X^G$, $\in Y^G$),
G-maps $f_X : X \to B_GO$ and $g_Y : Y \to B_G^*O$,
a base point preserving G-map $\varphi_X : X \to Y$, and
a G-homotopy $h_X : X \times I \to B_G^*O$ from $L_G \circ f_X$ to $g_Y \circ \varphi_X$.

If $\boldsymbol{X} = (X, Y, f_X, g_Y, \varphi_X, h_X)$ is a (B_GO, B_G^*O)-system then the associated diagram

$$\begin{array}{ccc} X & \xrightarrow{f_X} & B_GO \\ {\scriptstyle \varphi_X}\downarrow & & \downarrow{\scriptstyle L_G} \\ Y & \xrightarrow{g_Y} & B_G^*O \end{array}$$

G-homotopically commutes. Given another (B_GO, B_G^*O)-system $\boldsymbol{A} = (A, B, f_A, g_B, \varphi_A, h_A)$, we say that $\boldsymbol{A}$ *extends to* $\boldsymbol{X} = (X, Y, f_X, g_Y, \varphi_X, h_X)$ if $X \supseteq A$ and $Y \supseteq B$ as G-subcomplexes, and furthermore $f_X|_A = f_A$, $g_Y|_B = g_B$ and $h_X|_{A\times I} = h_A$.

A subset $\mathcal{F}$ of $\mathcal{G}^1(G)$ is said to be *upper closed* if K is contained in $\mathcal{F}$ whenever $K \in \mathcal{G}^1(G)$ and $K \ge H$ for some $H \in \mathcal{F}$.

In the two theorems below, we control X^H together with $f_X^H : X^H \to (B_GO)^H$ for all $H \notin \mathcal{G}^1(G)$ as well as X^P for all $P \in \mathcal{P}(G)$.

Theorem 1.3. *Let G be an Oliver group, T a set of primes dividing the order of G, $\mathcal{F}$ a conjugation invariant, upper closed subset of $\mathcal{G}^1(G)$ such that $\mathcal{F} \supseteq \mathcal{P}_T(G)_{>1}$, and $\boldsymbol{B} = (B, Y, f_B, g_Y, \varphi_B, h_B)$ a (B_GO, B_G^*O)-system such that Y is contractible, $Y^G \neq \emptyset$, $Y \supseteq B$, and $\varphi_B : B \to Y$ is the inclusion map. Assume*

(A1) *B^P is 1-connected for all $P \in \mathcal{P}_T(G)_{>1}$,*

(A2) *B^H is connected for all $H \in \mathcal{F}$,*

(A3) *For each $H \in \mathcal{S}(G) \smallsetminus \mathcal{G}^1(G)$, B^H coincides with Y^H,*

(A4) *Y^P is 1-connected for all $P \in \mathcal{P}(G)$, and*

(A5) *Y^H is connected for all $H \in \mathcal{G}^1(G)$.*

Then $\boldsymbol{B}$ extends to a (B_GO, B_G^*O)-system $\boldsymbol{X} = (X, Y, f_X, g_Y, \varphi_X, h_X)$ possessing the following properties:

(C1) *X is contractible.*

(C2) *All isotropy subgroups of $X \smallsetminus B$ are contained in $\mathcal{G}^1(G)$. Moreover, any cells of $X \smallsetminus B$ of isotropy type in $\mathcal{G}^1(G)^T$ has dimension ≥ 3.*

(C3) *For each $P \in \mathcal{P}_T(G)$, X^P is 1-connected.*

(C4) *For each prime p and each nontrivial p-subgroup P, X^P is connected and its fundamental group $\pi_1(X^P)$ is finite, abelian and of order prime to p.*

(C5) *For any $H \in \mathcal{F}$, X^H is connected.*

Our proof of the theorem is a slight modification of the proofs of [O2, Lemma 2.2 and Proposition 2.3]. The assumption (A2) is relevant to Condition [BM, (2.1.2)].

The main result of this paper is:

Theorem 1.4. *Let G be an Oliver group, T a set of primes dividing the order of G, $\mathcal{F}$ a conjugation invariant, upper closed subset of $\mathcal{G}^1(G)$ such that $\mathcal{F} \supseteq \mathcal{P}_T(G)_{>1}$, and $\boldsymbol{A} = (A, \{*\}, f_A, g_{\{*\}}, \mathrm{triv}, h_A)$ a (B_GO, B_G^*O)-system such that $A^G \ni *$. Assume*

(A1) *A^P is 1-connected for all $P \in \mathcal{P}_T(G)_{>1}$, and*

(A2) *A^H is connected for all $H \in \mathcal{F}$.*

Then $\boldsymbol{A}' = (A, A, f_A, g_{\{*\}} \circ c_A, id_A, h_A)$, where c_A is the map $A \to \{*\}$, extends to a (B_GO, B_G^*O)-system $\boldsymbol{X} = (X, Y, f_X, g_Y, \varphi_X, h_X)$ possessing the following properties:

(C1) *X and Y are finite contractible G-CW complexes including A.*

(C2) *$X^G = Y^G = F := A^G$, moreover*

$$\bigcup_{H \in \mathcal{S}(G) \smallsetminus \mathcal{G}^1(G)} X^H = \bigcup_{H \in \mathcal{S}(G) \smallsetminus \mathcal{G}^1(G)} Y^H \quad \text{(as G-CW complexes)}$$

and $\varphi_X|_{\bigcup_{H \in \mathcal{S}(G) \smallsetminus \mathcal{G}^1(G)} X^H}$ is the identity map.

(C3) *For each $P \in \mathcal{P}_T(G)$ ($P \in \mathcal{P}(G)$), X^P (resp. Y^P) is 1-connected.*

(C4) *For each prime p and each nontrivial p-subgroup P, X^P is connected and the fundamental group $\pi_1(X^P)$ is finite, abelian and of order prime to p.*

(C5) *For each $H \in \mathcal{F}$ (resp. $H \in \mathcal{G}^1(G)$), X^H (resp. Y^H) is connected.*

(C6) *There is an order preserving G-map*

$$r : \Pi(G, X; \mathcal{S}(G) \smallsetminus \mathcal{G}^1(G)) \to \Pi(G, A \amalg \{pt\}; \mathcal{S}(G) \smallsetminus \mathcal{G}^1(G))$$

with G_γ-homeomorphisms

$$k_\gamma : Y_\gamma \to A_{r(\gamma)} \text{ or } \{pt\}_{r(\gamma)},\ \gamma \in \Pi(G, Y; \mathcal{S}(G) \smallsetminus \mathcal{G}^1(G)),$$

such that $k_\gamma : Y_\gamma \to A_\gamma$ is the canonical map if γ is contained in the subset $\Pi(G, A; \mathcal{S}(G) \smallsetminus \mathcal{G}^1(G))$ of $\Pi(G, Y; \mathcal{S}(G) \smallsetminus \mathcal{G}^1(G))$; $k_{g\gamma} = g \circ k_\gamma \circ g^{-1}$ for all $g \in G$ and $\gamma \in \Pi(G, Y; \mathcal{S}(G) \smallsetminus \mathcal{G}^1(G))$; and the square

$$\begin{array}{ccc}
Y_\gamma & \xrightarrow{k_\gamma} & A_{r(\gamma)} \\
{\scriptstyle \text{inclusion}}\downarrow & & \downarrow{\scriptstyle \text{inclusion}} \\
Y_\delta & \xrightarrow{k_\delta} & A_{r(\delta)}
\end{array}$$

commutes whenever $\gamma \le \delta \in \Pi(G, Y; \mathcal{S}(G) \smallsetminus \mathcal{G}^1(G))$. Moreover, $f_X|_{X_\gamma} = f_A|_{A_{r(\gamma)}}$ when $r(\gamma)$ is contained in the subset $\Pi(G, A; \mathcal{S}(G) \smallsetminus \mathcal{G}^1(G))$, and $f_X(x) = f_A(*)$ for all $x \in X_\gamma$ when $r(\gamma)$ is contained in the subset $\Pi(G, pt; \mathcal{S}(G) \smallsetminus \mathcal{G}^1(G))$.

(C7) *$g_Y = g_{\{*\}} \circ c_Y$, where c_Y is the map $Y \to \{*\}$.*

This theorem is obtained from Theorems 1.1 and 1.3 with careful observation.

If we eliminate Assumptions (A2) and (A5) from Theorem 1.3 (resp. Assumption (A2) from Theorem 1.4) then we obtain similar conclusions in which Property (C5) of Theorem 1.3 (resp. Theorem 1.4) does not necessarily hold. The modified results are given as Theorems 5.1 and 5.2.

The organization of the paper is as follows. In Section 2, we explain what is our equivariant wedge sum. In Section 3, using the equivariant wedge sum, we prove Theorem 1.1. In Section 4, we prove Theorems 1.3 and 1.4. Section 5 is devoted to the modified versions of Theorems 1.3 and 1.4.

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2. Equivariant wedge sum operation

In the subsequent section, we use equivariant wedge sum operation in the following context.

Let X and Y be G-CW complexes, H be a subgroup of G, $x_H \in X$ a point such that $G_{x_H} = H$ and $* \in Y^G$. The equivariant wedge sum $X \vee_G (G \times_H Y)$ of X and $G \times_H Y$ with attaching data $(x_H, *)$ is the identification space

$$X \vee_G (G \times_H Y) = \{X \amalg (G \times_H Y)\}/\sim,$$

where $[g, *] \in G \times_H Y$ is identified with $gx_H \in X$ for each $[g] \in G/H$, obtained from the disjoint union of X and $G \times_H Y$. This wedge sum is said to be of *type* (H). Suppose $X^G \neq \emptyset$. In case where $Y = X$, let $(1 + G/H)X$ denote $X \vee_G (G \times_H X)$.

Let $H_1, \ldots, H_m$ be subgroups of G (possibly $H_i = H_j$) and $x_1, \ldots, x_m$ points in X such that $G_{x_i} = H_i$ for $1 \leq i \leq m$ and $Gx_i \cap Gx_j = \emptyset$ for $i \neq j$. Then we define the wedge sum $(1 + G/H_1 + \cdots + G/H_m)X$ with attaching data $\{(x_i, *) \mid 1 \leq i \leq m\}$ by

$$(1 + G/H_1 + \cdots + G/H_m)X = X \vee_G (G \times_{H_1} X) \vee_G \cdots \vee_G (G \times_{H_m} X).$$

If β is an element in the Burnside ring $\Omega(G)$ of the form

$$\beta = \sum_{i=1}^{m} [G/H_i],$$

then $(1 + \beta)X$ stands for the wedge sum $(1 + G/H_1 + \cdots + G/H_m)X$ with attaching data $\{(x_i, *) \mid 1 \leq i \leq m\}$.

Let X be a finite G-CW complex. Then $[X]$ stands for the element in the Burnside ring $\Omega(G)$ represented by X. It holds that

$$[(1 + G/H)X] - 1 = (1 + [G/H])([X] - 1) \quad \text{in } \Omega(G).$$

In general, we obtain $[(1 + \beta)X] - 1 = (1 + \beta)([X] - 1)$.

Let $f : X \to Z$ be a G-map such that $f(x_i) = f(*)$ for all $1 \leq i \leq m$. Then the G-map

$$(1 + G/H_1 + \cdots + G/H_m)f : (1 + G/H_1 + \cdots + G/H_m)X \to Z$$

is obtained canonically from $f, G \times_{H_1} f, \ldots, G \times_{H_m} f$.

3. Proof of Theorem 1.1

The proof of Theorem 1.1 contains supplementary remarks which are not necessary here but valid under the conditions (A1) and (A2):

(A1) A^P is 1-connected for all $P \in \mathcal{P}_T(G)_{>1}$, and

(A2) A^H is connected for all $H \in \mathcal{F}$.

These conditions are supposed in Theorems 1.3 and 1.4 and the remarks will be used in the proof of Theorem 1.4.

The proof of Theorem 1.1 is divided into the 5 steps below.

Step 1. Attaching 1-dimensional G-cells of isotropy type (H) in $\mathcal{G}^1(G)$ to A, we can obtain a finite G-CW complex A' such that A'^H is connected for all $H \in \mathcal{G}^1(G)$. (Note that if A satisfies (A2) above then we need only 1-dimensional G-cells of type (H) in $\mathcal{G}^1(G) \smallsetminus \mathcal{F}$. When A satisfies (A1, A2), so does A'.)

Step 2. Let β be an element in the Burnside ring $\Omega(G)$ such that $\chi_H(\beta) = 1$ for all $H \notin \mathcal{L}(G)$ and $\chi_G(\beta) = 0$ ([LM, Proposition 1.1 and Theorem 1.3]). Let

$$(-\beta)^{\%} = \sum_{(H)\neq(G)} a_H[G/H] \quad \in \Omega(G)$$

be an element such that

$$a_H > 0 \text{ and } (-\beta)^{\%} \equiv -\beta \mod 2|G||\widetilde{K}_0(\mathbb{Z}[G])| \cdot \Omega(G).$$

(Here $\widetilde{K}_0(\mathbb{Z}[G])$ is a finite group by [S, Proposition 9.1.])

Take a finite G-CW complex X_0 satisfying the following properties:

(1) X_0^G consists of exactly one point, say x_G.

(2) For each $H \in \mathcal{S}(G) \smallsetminus \mathcal{G}^1(G)$, $\dim X_0^H = 0$.

(3) For each $H \in \mathcal{G}^1(G)$, X_0^H is connected.

(4) For each $P \in \mathcal{P}(G)$, X_0^P is 1-connected.

(5) For each $H \lneqq G$,

$$|(G{\cdot}X_0^{=H})/G| \geq n_\beta := \max(\{a_H \mid (H) \neq (G)\}){\cdot}\{|(\mathcal{G}^1(G){\smallsetminus}\mathcal{P}(G))/G|{+}2\}.$$

Choose arbitrary points $x_{H,i} \in X_0^{=H}$ for all $(H) \lneqq (G)$ and $i = 1, \dots, n_\beta$. Set $X_1 = A' \vee X_0$. Then, X_1 clearly have the properties corresponding to (C2, C5) of Theorem 1.1. (When A satisfies (A1, A2), so does X_1.) .

Step 3. Let H be a maximal subgroup in $\mathcal{G}^1(G)$ such that $\chi(X^H) \neq 1$. Form the equivariant wedge sum $(1 + (-\beta)^{\%})X_1$ such that the points $[g, *]_i$ of $(G \times_H X_1)_i$, $(H) \neq (G)$, $[g] \in G/H$ and $1 \leq i \leq a_H$, are identified with the points $gx_{H,i}^{(1)}$ of X_0 with

$G_{x^{(1)}_{H,i}} = H$, respectively. Note that this wedge sum operation does not change the G-fixed point set and that $(1+(-\beta)^{\%})X_1$ has the properties corresponding to (C2, C5) of Theorem 1.1. (When A satisfies (A1, A2), so does $(1+(-\beta)^{\%})X_1$.) Also, it follows that

$$[(1+(-\beta)^{\%})\,X_1]-1=(1+(-\beta)^{\%})([X_1]-1) \quad \text{in } \Omega(G).$$

Suppose H is a maximal subgroup in $\mathcal{G}^1(G) \smallsetminus \mathcal{P}(G)$ such that $\chi(X_1^H) \neq 1$. Then

$$\begin{aligned}\chi(((1+(-\beta)^{\%})X_1)^H) &\equiv \chi(((1-\beta)X_1)^H) = (1-\chi_H(\beta))\chi(X_1^H) \\ &\equiv 0 \; mod\; |G|.\end{aligned}$$

Thus, we can form the equivariant wedge sum

$$X_2 = (1+(-\beta)^{\%})\,X_1 \vee_G (G/H \times S) \vee_G \cdots \vee_G (G/H \times S),$$

where $S = S^3$ or S^4, so that $\chi(X_2^H) = 1$. Here the points gx_{H,n_β} $(g \in G)$ are regarded as the base points of $(1+(-\beta)^{\%})\,X_1$ and the base point of $\{gH\} \times S$ is identified with gx_{H,n_β}. This X_2 has the properties corresponding to (C2, C5) of Theorem 1.1. (When A satisfies (A1, A2), so does X_2.)

Thus, we may suppose that X_2 satisfies $\chi(X_2^H) = 1$ for all $H \in \mathcal{G}^1(G) \smallsetminus \mathcal{P}(G)$. Note that all isotropy subgroups in $X_2 \smallsetminus (1+(-\beta)^{\%})\,X_1$ are contained in $\mathcal{G}^1(G)$.

Step 4. Suppose that $P \in \mathcal{P}(G)$ is a maximal subgroup among nontrivial p-groups such that X_2^P is not 1-connected. (When A satisfies (A1, A2), P is not contained in $\mathcal{P}_T(G)_{>1}$.) Attaching G-cells $G/P \times \mathrm{Int}(D^i)$'s with $i = 1, 2$ to X_2, we can construct a finite G-CW complexes X_3 such that X_3^P is 1-connected.

Thus we may assume that for all $P \in \mathcal{P}(G)_{>1}$, X_3^P is 1-connected. Note that all isotropy subgroups of cells in $X_3 \smallsetminus X_2$ are contained in $\mathcal{P}(G)$. (When A satisfies (A1, A2), they are contained in $\mathcal{P}(G) \smallsetminus \mathcal{P}_T(G)$.)

Now employing Oliver's method [O1], we can construct a finite G-CW complex X_4 including X_3 such that $X_4 \smallsetminus X_3$ consists of cells $G/P \times \mathrm{Int}(D^i)$'s with $P \in \mathcal{P}(G)_{>1}$ and $i \geq 3$ and X_4^P is $\mathbb{Z}_{(p)}$-acyclic. By construction above, X_4 have the properties corresponding to (C2, C4, C5) in Theorem 1.1.

Step 5. Set $d = \dim X_4$. We may suppose $d \geq 4$. We can attach free cells of dimension $\leq d$ so that X_4 is 1-connected and $\widetilde{H}_k(X_4;\mathbb{Z}) = 0$ for any $k < d$. Then $H_d(X_4;\mathbb{Z})$ is $\mathbb{Z}[G]$-projective.

Now perform the wedge sum operation on X_4 and obtain $X_5 = (1+(-\beta)^{\%})X_4$ such that the points $[g,*]_i$ of $(G \times_H X_4)_i$, $(H) \neq (G)$, $[g] \in G/H$ and $1 \leq i \leq a_H$, are identified with points $gx^{(4)}_{H,i}$ in

$$X_0 \smallsetminus \left(\bigcup_{(H),i} Gx^{(1)}_{H,i} \cup \bigcup_{(H)} Gx_{H,n_\beta}\right)$$

such that $G_{x^{(4)}_{H,i}} = H$, respectively. Thus, X_5 has the properties corresponding to (C2–5) in Theorem 1.1. By [S, Corollary 9.1], $\widetilde{K}_0(\mathbb{Z}[G])$ is a Green module over $K_0(\mathbb{Q}[G])$. Since $\mathrm{Res}^G_C(\beta) = 1$ in $\Omega(C)$ for any cyclic subgroup C of G, $(1-\beta)[\mathbb{Q}] = 0$ in $K_0(\mathbb{Q}[G])$. Thus,

$$[H_d(X_5;\mathbb{Z})] = 0 \in \widetilde{K}_0(\mathbb{Z}[G])$$

follows from

$$\begin{aligned}[H_d(X_5;\mathbb{Z})] &= (1+(-\beta)^{\%})[H_d(X_4;\mathbb{Z})] \\ &= \{(1-\beta)[\mathbb{Q}]\}[H_d(X_4;\mathbb{Z})].\end{aligned}$$

Thus, $H_d(X_5;\mathbb{Z})$ is stably $\mathbb{Z}[G]$-free.

By attaching free cells of dimension d to $* \in X_5$, we obtain

$$X_6 = X_5 \vee \Sigma^d(G^+) \vee \cdots \vee \Sigma^d(G^+)$$

such that $H_d(X_6;\mathbb{Z})$ is $\mathbb{Z}[G]$-free. Here $G^+ = G \amalg \{*\}$ and $\Sigma^d(G^+)$ denotes the d-fold reduced suspension of G^+. Now, by attaching free cells of dimension $d+1$, we produce a finite G-CW complex

$$Y = X_6 \cup (G \times D^{d+1}) \cup \cdots \cup (G \times D^{d+1})$$

such that Y is (nonequivariantly) contractible. Then Y satisfies (C1–5) in Theorem 1.1. □

4. Proofs of Theorems 1.3 and 1.4

This section is devoted to the proofs of Theorems 1.3 and 1.4. In order to improve Oliver's results, we employ again the equivariant wedge sum associated to elements of the Burnside ring.

Proof of Theorem 1.3. We can construct a finite G-CW complex

$$X_1 = B \vee \bigvee_{(H)\subseteq \mathcal{G}^1(G)\smallsetminus\mathcal{P}(G)} \{\Sigma^t((G/H)^+) \vee \cdots \vee \Sigma^t((G/H)^+)\}$$

with $t = 3$ or 4 such that $\chi(X_1^H) = \chi(Y^H)$ for all $H \in \mathcal{G}^1(G) \smallsetminus \mathcal{P}(G)$. Here $(G/H)^+ = G/H \amalg \{*\}$, $\Sigma^t((G/H)^+$ denotes the t-fold reduced suspension of $(G/H)^+$, and the wedge sum is taken at the base point $* \in A$. Clearly $\boldsymbol{B}$ is extendible to $\boldsymbol{X}_1 = (X_1, Y, f_{X_1}, g_Y, \varphi_{X_1}, h_{X_1})$.

Apply [O2, Lemma 2.2] together with its proof. (Recall Oliver's assertion in [O2, Proof of Lemma 2.2, Finite case] that $\pi_1(\alpha_1)$ is injective and $\mathrm{Ker}(\pi_1(\beta\alpha_1))$ is finite abelian of order prime to p.) By attachment of G-cells of isotropy type (P), where P runs over $\mathcal{P}(G)_{>1}$, extend $\boldsymbol{X}_1$ to a (B_GO, B_G^*O)-system $\boldsymbol{X}_2 = (X_2, Y, f_{X_2}, g_Y, \varphi_{X_2}, h_{X_2})$ such that X_2^P are mod p acyclic for all $P \in \mathcal{P}(G)_{>1}$, such that X_2^P are 1-connected for all $P \in \mathcal{P}_T(G)_{>1}$, and such that the fundamental groups $\pi_1(X_2^P)$ are finite, abelian and of order prime to p for all $P \in \mathcal{P}(G)_{>1} \smallsetminus \mathcal{P}_T(G)_{>1}$, where p is the prime dividing $|P|$. Here we can do in such a way that if a G-cell in $X_2 \smallsetminus X_1$ has isotropy type (H) in $\mathcal{P}_T(G)$ then the dimension of the cell is greater than or equal to 3.

Finally apply [O2, Proposition 2.3] and obtain a desired $\boldsymbol{X}$ from $\boldsymbol{X}_2$. □

Proof of Theorem 1.4. Let A be the G-CW complex given in Theorem 1.4. Apply the proof of Theorem 1.1 and let Y be the finite, contractible G-CW complex obtained in the proof of Theorem 1.1. Set

$$B = A \cup \bigcup_{H \in \mathcal{S}(G) \smallsetminus \mathcal{G}^1(G)} Y^H \cup \bigcup_{H \in \mathcal{F}} (Y^H)^{(2)},$$

where $(Y^H)^{(2)}$ is the 2-skeleton of Y^H. We shall see that $\boldsymbol{A}'$ is extendible to a (B_GO, B_G^*O)-system $\boldsymbol{B} = (B, Y, f_B, g_Y, \varphi_B, h_B)$. Firstly, define $\varphi_B : B \to Y$ to be the canonical inclusion. Secondly, define $g_Y : Y \to B_G^*O$ so as to satisfy (C7) in Theorem 1.4. The restrictions of G-maps $f_B : B \to B_GO$ and $h_B : B \times I \to B_G^*O$ to A are, of course, given as f_A and h_A. The restrictions of them to Y^H with $H \in \mathcal{S}(G) \smallsetminus \mathcal{G}^1(G)$ are given by

$$f_B(x) = \begin{cases} f_A|_{A_{r(\gamma)}}(x) & \text{when } x \in Y_\gamma \text{ such that } r(\gamma) \in \Pi(G, A; \mathcal{S}(G) \smallsetminus \mathcal{G}^1(G)), \\ f_A(*) & \text{when } x \in Y_\gamma \text{ such that } r(\gamma) \in \Pi(G, pt; \mathcal{S}(G) \smallsetminus \mathcal{G}^1(G)), \end{cases}$$

$$h_B(x,t) = \begin{cases} h_A|_{A_j^H \times I}(x,t) & \text{when } x \in Y_\gamma \\ & \quad \text{such that } r(\gamma) \in \Pi(G, A; \mathcal{S}(G) \smallsetminus \mathcal{G}^1(G)), t \in I, \\ h_A(*,t) & \text{when } x \in Y_\gamma \\ & \quad \text{such that } r(\gamma) \in \Pi(G, pt; \mathcal{S}(G) \smallsetminus \mathcal{G}^1(G)), t \in I, \end{cases}$$

where $\gamma \in \Pi(G, Y; \mathcal{S}(G) \smallsetminus \mathcal{G}^1(G))$ and $A_{r(\gamma)}$ is the connected component described in Theorem 1.1 (C5). It suffices to define the restrictions of f_B and h_B to $(Y^H)^{(2)}$, where $H \in \mathcal{F}$. By the construction of Y in the proof of Theorem 1.1, $(Y^H)^{(2)}$ is the wedge sum of $(A^H)^{(2)}$'s and $(X_0^H)^{(2)}$'s (see Proof of Theorem 1.1, Steps 1 and 2). Thus the desired maps can be constructed similarly.

Note that B^P is 1-connected for each $P \in \mathcal{P}_T(G)_{>1}$ and B^H is connected for every $H \in \mathcal{F}$. Now, apply Theorem 1.3 to this $\boldsymbol{B}$. Then we obtain a desired (B_GO, B_G^*O)-system $\boldsymbol{X} = (X, Y, f_X, g_Y, \varphi_X, h_X)$. □

5. Modifications of Theorems 1.3 and 1.4

The goal of the current section is to point out that Theorems 1.3 and 1.4 both have their analogies without their assumptions (A2). These new versions of both theorems we found useful in applications concerned with constructions of smooth G-actions on disks and spheres with specified G-fixed point sets. More precisely, we are able to modify Oliver's construction [O2] of smooth G-actions on disks so that to control the isotropy subgroups occurring around the G-fixed point sets. This, in turn, allows us to apply equivariant surgery as developed in [LMP], [LM], and [M] to convert the resulting smooth G-actions on disks into smooth G-actions on spheres without changing the G-fixed point sets.

Theorem 5.1. *Let G be an Oliver group, T a set of primes dividing the order of G and $\boldsymbol{B} = (B, Y, f_B, g_Y, \varphi_B, h_B)$ a (B_GO, B_G^*O)-system such that Y is contractible, $Y^G \neq \emptyset$, $Y \supseteq B$, and $\varphi_B : B \to Y$ is the inclusion map. Assume that*

(A1) *B^P is 1-connected for all $P \in \mathcal{P}_T(G)_{>1}$,*

(A2) *For each $H \in \mathcal{S}(G) \smallsetminus \mathcal{G}^1(G)$, B^H coincides with Y^H, and*

(A3) *Y^P is 1-connected for all $P \in \mathcal{P}(G)$.*

Then $\boldsymbol{B}$ extends to a (B_GO, B_G^*O)-system $\boldsymbol{X} = (X, Y, f_X, g_Y, \varphi_X, h_X)$ possessing the following properties:

(C1) *X is contractible.*

(C2) *All isotropy subgroups of $X \smallsetminus B$ are contained in $\mathcal{G}^1(G)$. Moreover, any cells of $X \smallsetminus B$ of isotropy type in $\mathcal{G}^1(G)^T$ has dimension ≥ 3.*

(C3) *For each $P \in \mathcal{P}_T(G)$, X^P is 1-connected.*

(C4) *For each prime p and each nontrivial p-subgroup P, X^P is connected and its fundamental group $\pi_1(X^P)$ is finite, abelian and of order prime to p.*

Now, we wish to point out that with some modifications of the arguments presented in the proof of Theorem 1.4, we obtain the following theorem.

Theorem 5.2. *Let G be an Oliver group, T a set of primes dividing the order of G, and $\boldsymbol{A} = (A, \{*\}, f_A, g_{\{*\}}, \mathrm{triv}, h_A)$ a (B_GO, B_G^*O)-system with a base point $* \in A^G$. Assume that*

(A1) *A^P is 1-connected for all $P \in \mathcal{P}_T(G)_{>1}$.*

Then $\boldsymbol{A}' = (A, A, f_A, g_{\{*\}} \circ c_A, id_A, h_A)$, where c_A is the map $A \to \{*\}$, extends to a $(B_G O, B_G^* O)$-system $\boldsymbol{X} = (X, Y, f_X, g_Y, \varphi_X, h_X)$ possessing the following properties:

(C1) *X and Y are finite contractible G-CW complexes including A.*

(C2) *$X^G = Y^G = F := A^G$, moreover*

$$\bigcup_{H \in \mathcal{S}(G) \smallsetminus \mathcal{G}^1(G)} X^H = \bigcup_{H \in \mathcal{S}(G) \smallsetminus \mathcal{G}^1(G)} Y^H \quad (\text{as } G\text{-CW complexes})$$

and $\varphi_X|_{\bigcup_{H \in \mathcal{S}(G) \smallsetminus \mathcal{G}^1(G)} X^H}$ is the identity map.

(C3) *For each $P \in \mathcal{P}_T(G)$ ($P \in \mathcal{P}(G)$), X^P (resp. Y^P) is 1-connected.*

(C4) *For each prime p and each nontrivial p-subgroup P, X^P is connected and the fundamental group $\pi_1(X^P)$ is finite, abelian and of order prime to p.*

(C5) *There is an order preserving G-map*

$$r : \Pi(G, X; \mathcal{S}(G) \smallsetminus \mathcal{G}^1(G)) \to \Pi(G, A \amalg \{pt\}; \mathcal{S}(G) \smallsetminus \mathcal{G}^1(G))$$

with G_γ-homeomorphisms

$$k_\gamma : X_\gamma \to A_{r(\gamma)} \text{ or } \{pt\}_{r(\gamma)}, \ \gamma \in \Pi(G, X; \mathcal{S}(G) \smallsetminus \mathcal{G}^1(G)),$$

such that $k_\gamma : X_\gamma \to A_\gamma$ is the canonical map if γ is contained in the subset $\Pi(G, A; \mathcal{S}(G) \smallsetminus \mathcal{G}^1(G))$ of $\Pi(G, X; \mathcal{S}(G) \smallsetminus \mathcal{G}^1(G))$; $k_{g\gamma} = g \circ k_\gamma \circ g^{-1}$ for all $g \in G$ and $\gamma \in \Pi(G, X; \mathcal{S}(G) \smallsetminus \mathcal{G}^1(G))$; and the square

$$\begin{tikzcd} X_\gamma \arrow[r, "k_\gamma"] \arrow[d, "\text{inclusion}"'] & A_{r(\gamma)} \arrow[d, "\text{inclusion}"] \\ X_\delta \arrow[r, "k_\delta"] & A_{r(\delta)} \end{tikzcd}$$

commutes whenever $\gamma \le \delta \in \Pi(G, X; \mathcal{S}(G) \smallsetminus \mathcal{G}^1(G))$. Moreover, $f_X|_{X_\gamma} = f_A|_{A_{r(\gamma)}} \circ k_\gamma$ when $r(\gamma)$ is contained in the subset $\Pi(G, A; \mathcal{S}(G) \smallsetminus \mathcal{G}^1(G))$, and $f_X(x) = f_A(*)$ for all $x \in X_\gamma$ when $r(\gamma)$ is contained in the subset $\Pi(G, pt; \mathcal{S}(G) \smallsetminus \mathcal{G}^1(G))$.

(C6) *$g_Y = g_{\{*\}} \circ c_Y$, where c_Y is the map $Y \to \{*\}$.*

REMARK. By Condition 5.2 (C2), it holds that

$$\Pi(G, X; \mathcal{S}(G) \smallsetminus \mathcal{G}^1(G)) = \Pi(G, Y; \mathcal{S}(G) \smallsetminus \mathcal{G}^1(G))$$

and $X_\gamma = Y_\gamma$ for all $\gamma \in \Pi(G, X; \mathcal{S}(G) \smallsetminus \mathcal{G}^1(G))$.

Proof of Theorem 5.2. First, in the proof of Theorem 1.1, Step 1 is replaced as follows.

Step 1′. By attachment of G-cells of isotropy type (P) in $\mathcal{P}(G)$ to A, one can obtain a finite G-CW complex A' such that ${A'}^P$ is connected for all $P \in \mathcal{P}(G)$.

With the replacement, the proof of Theorem 1.1 yields a finite G-CW complex X' such that $X' \supset A'$ and X' fulfills the conditions (C1), (C2), (C3), and (C5) stated in Theorem 1.1.

Second, in the proof of Theorem 1.4, B is replaced by

$$B' = A' \cup \bigcup_{H \in \mathcal{S}(G) \smallsetminus \mathcal{G}^1(G)} Y^H \cup \bigcup_{P \in \mathcal{P}(G)} (Y^P)^{(2)}$$

for Y as in Theorem 1.4, where $(Y^P)^{(2)}$ is the 2-skeleton of Y^P. Now, by the arguments presented in the proof of Theorem 1.4, we can obtain the desired $(B_G O, B_G^* O)$-system $\boldsymbol{X} = (X, Y, f_X, g_Y, \varphi_X, h_X)$, proving Theorem 5.2. □

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