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TOWARDS A CRITERION FOR SLOPE STABILITY OF FANO MANIFOLDS ALONG DIVISORS

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Abstract

We give a simple criterion for slope stability of Fano manifolds X along divisors or smooth subvarieties. As an application, we show that X is slope stable along an ample effective divisor $D \subset X$ unless X is isomorphic to a projective space and D is a hyperplane section. We also give counterexamples to Aubin's conjecture on the relation between the anticanonical volume and the existence of a Kähler–Einstein metric. Finally, we consider the case that $\dim X = 3$; we give a complete answer for slope (semi)stability along divisors of Fano threefolds.

1. Introduction

Let X be a Fano manifold, that is, a smooth projective variety such that the anticanonical divisor $-K_X$ of X is ample. It has been conjectured that the K-polystability of $(X, -K_X)$ is equivalent to the existence of Kähler–Einstein metrics. However it is difficult to judge the K-(poly, semi)stability in general. In this article, we consider slope stability, which was introduced by Ross and Thomas (see [18]), that is weaker than K-stability but is easy to describe. For example, the case a Fano manifold is not slope (semi)stable along a smooth curve has been completely classified, see [6] and [8].

First, we give a simple criterion for slope stability of Fano manifolds along divisors (or smooth subvarieties).

Proposition 1.1 (see Proposition 3.2 for detail)**.** *For a Fano manifold X and a divisor $D \subset X$, X is slope stable* (*resp. slope semistable*) *along D if and only if $\xi(D) > 0$* (*resp.* ≥ 0)*, where*

$$\xi(D) = \mathrm{vol}_X(-K_X) + (\epsilon(D) - 1)\,\mathrm{vol}_X(K_X - \epsilon(D)D) - \int_0^{\epsilon(D)} \mathrm{vol}_X(-K_X - xD)\,dx$$

and $\epsilon(D)$ is the Seshadri constant of D with respect to $-K_X$.

As an application, we can investigate the case where $D \subset X$ is an ample divisor.

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Theorem 1.2 (= Theorem 4.6 (1))**.** *For a Fano manifold X and an ample divisor $D \subset X$, X is slope stable along D unless X is isomorphic to a projective space and D is a hyperplane section.*

We also construct the Fano manifolds which are not slope semistable along some divisors but have "small" anticanonical volumes, which are counterexamples to the following Conjecture 1.3 (cf. Remark 5.6) on the relation between the anticanonical volume of X and the existence of a Kähler–Einstein metric on X.

Conjecture 1.3 (see also Remark 5.6)**.** *Let X be a Fano n-fold. If the anticanonical volume* $\mathrm{vol}_X(-K_X)$ *is less than* $((n+1)^2/(2n))^n$, *then X admits Kähler–Einstein metrics.*

Counterexample 1.4 (= Corollary 5.5)**.** *For any* $n \geq 4$, *there exists a Fano n-fold X such that* $\mathrm{vol}_X(-K_X) = 2(3^n - 1)$ (*hence* $2(3^n-1) < ((n+1)^2/(2n))^n$ *holds if* $n \geq 5$) *but X does not admit Kähler–Einstein metrics.*

Finally, using the classification result [12], we give the complete answer for slope (semi)stability of Fano threefolds along divisors:

Theorem 1.5 (= Theorem 6.2)**.** *Let X be a smooth Fano threefold.*
(1) *X is slope semistable along any effective divisor but there exists a divisor $D \subset X$ such that X is not slope stable along D if and only if X is isomorphic to one of*:

$$\mathbb{P}^3, \quad \mathbb{P}^1 \times \mathbb{P}^2, \quad \mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O}(0,1) \oplus \mathcal{O}(1,0)),$$
$$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \quad \mathbb{P}^1 \times S_m \quad (1 \leq m \leq 7).$$

(2) *There exists a divisor $D \subset X$ such that X is not slope semistable along D if and only if X is isomorphic to one of*:

$$\mathrm{Bl}_{\mathrm{line}}\, \mathbb{Q}^3, \quad \mathrm{Bl}_{\mathrm{line}}\, \mathbb{P}^3, \quad \mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(1)), \quad \mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(2)),$$
$$\mathbb{P}^1 \times \mathbb{F}_1, \quad \mathbb{P}_{\mathbb{F}_1}(\mathcal{O} \oplus \mathcal{O}(e+f)), \quad \mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(1,1)).$$

Notation and terminology. We always consider over the complex number field $\mathbb{C}$. A *variety* means an irreducible and reduced scheme of finite type over $\mathrm{Spec}\,\mathbb{C}$. The theory of extremal contraction, we refer the readers to [9]. For a projective variety X, let $\mathrm{Eff}(X)$ (resp. $\mathrm{Nef}(X)$) be the effective (resp. nef) cone which is defined as the cone in $\mathrm{N}^1(X)$ spanned by the classes of effective (resp. nef) divisors on X. For a complete variety X, the Picard number of X is denoted by ρ_X. For a smooth projective variety X, let $\mathrm{NE}(X)$ be the cone in $\mathrm{N}_1(X)$ spanned by effective 1-cycles on X, and $\overline{\mathrm{NE}}(X)$ the closure of $\mathrm{NE}(X)$ in $\mathrm{N}_1(X)$. For a smooth projective variety X and a K_X-negative extremal ray $R \subset \overline{\mathrm{NE}}(X)$, we define the *length* $l(R)$ of R by

$$l(R) := \min\{(-K_X \cdot C) \mid C \text{ is a rational curve with } [C] \in R\},$$

and we define *a minimal rational curve of* R such that a rational curve $C \subset X$ with $[C] \in R$ and $(-K_X \cdot C) = l(R)$.

For an algebraic variety X and a closed subscheme $Y \subset X$, denotes corresponding ideal sheaf $\mathcal{I}_Y \subset \mathcal{O}_X$, and $\mathrm{Bl}_Y\colon \mathrm{Bl}_Y(X) \to X$ or $\mathrm{Bl}_{\mathcal{I}_Y}\colon \mathrm{Bl}_{\mathcal{I}_Y}(X) \to X$ denotes the blowing up of X along Y.

For algebraic varieties $X_1, \dots, X_k$, we write the projection $p_{1,\dots,t}\colon X_1 \times \cdots \times X_k \to X_1 \times \cdots \times X_t$.

We say X is a *Fano manifold* if X is a smooth projective variety whose anti-canonical divisor $-K_X$ is ample. We note that if X is a Fano manifold, then there is the canonical embedding $\mathrm{Pic}(X) \hookrightarrow \mathrm{N}^1(X)$. For a Fano manifold X, let the *Fano index* of X be

$$\max\{r \in \mathbb{Z}_{>0} \mid -K_X \sim rL \text{ for some Cartier divisor } L\}.$$

For a complete n-dimensional variety X and a nef Cartier divisor (or a nef invertible sheaf) D on X, the *volume* of D (denotes $\mathrm{vol}_X(D)$) means the self intersection number (D^n) of D.

The symbol $\mathbb{Q}^n$ denotes a smooth hyperquadric in $\mathbb{P}^{n+1}$. The symbol $\mathbb{F}_1$ denotes the Hirzebruch surface having the (-1)-curve $e \subset \mathbb{F}_1$, and let $f \subset \mathbb{F}_1$ be a fiber of $\mathbb{P}^1$-bundle $\mathbb{F}_1 \to \mathbb{P}^1$. The symbol S_m $(1 \le m \le 7)$ denotes a (smooth) del Pezzo surface S (Fano 2-fold) such that the anticanonical volume $\mathrm{vol}_S(-K_S)$ is equal to m.

2. Slope stabilities of polarized varieties

We recall slope stability of polarized varieties, which has been introduced by Ross and Thomas. See [18] in detail.

DEFINITION 2.1. Let (X, L) be a polarized variety of $\dim X = n$, let $Z \subset X$ be a closed subscheme, let $\sigma\colon \hat{X} \to X$ be the blowing up of X along Z and let $E \subset \hat{X}$ be the Cartier divisor defined by $\mathcal{O}_{\hat{X}}(-E) = \sigma^{-1}\mathcal{I}_Z \cdot \mathcal{O}_{\hat{X}}$.

- Let $\epsilon(\mathcal{I}_Z; (X, L))$ be the *Seshadri constant of Z with respect to L* (we often write $\epsilon(Z, X)$ or $\epsilon(Z)$ instead of $\epsilon(\mathcal{I}_Z; (X, L))$ for simplicity), which is defined as follows:

$$\epsilon(\mathcal{I}_Z; (X, L)) := \max\{c \in \mathbb{R}_{>0} \mid \sigma^*L - cE \text{ is nef on } \hat{X}\}.$$

- For $k, xk \in \mathbb{N}$ with $k \gg 0$, we can write

$$\chi(\hat{X}, \sigma^*(kL) - xkE) = a_0(x)k^n + a_1(x)k^{n-1} + \cdots + a_n(x),$$

where $a_i(x) \in \mathbb{Q}[x]$. Let $\mu_c(\mathcal{I}_Z, L)$ be the *slope of Z with respect to L and $c \in (0, \epsilon(Z)]$* (we often write $\mu_c(Z)$ instead of $\mu_c(\mathcal{I}_Z, L)$ for simplicity), which is defined as follows:

$$\mu_c(\mathcal{I}_Z, L) := \frac{\int_0^c (a_1(x) + a_0'(x)/2)\, dx}{\int_0^c a_0(x)\, dx}.$$

We also define *the slope of X with respect to L* as

$$\mu(X) = \mu(X, L) := \frac{a_1}{a_0},$$

where $a_i \in \mathbb{Q}$ are defined by $\chi(X, rL) = a_0 r^n + a_1 r^{n-1} + \cdots + a_n$.

DEFINITION 2.2 (slope (semi)stability). Let (X, L) (we often omit the polarization L) and $Z \subset X$ be as above.

(X, L) is *slope stable* (resp. *slope semistable*) *along* Z if:

- *slope semistability*:

$\mu_c(Z) \leq \mu(X)$ for all $c \in (0, \epsilon(Z)]$,

- *slope stability*:

$\mu_c(Z) < \mu(X)$ for all $c \in (0, \epsilon(Z))$, and also for $c = \epsilon(Z)$ if $\epsilon(Z) \in \mathbb{Q}$ and global sections of $L^k \otimes \mathcal{I}_Z^{k\epsilon(Z)}$ saturates for $k \gg 0$.

For a polarized variety (X, L) and a coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ with $\mathcal{O}_{\mathrm{Bl}_{\mathcal{I}}(X)}(-E) := \mathrm{Bl}_{\mathcal{I}}{}^{-1}\mathcal{I} \cdot \mathcal{O}_{\mathrm{Bl}_{\mathcal{I}}(X)}$, *global sections of $L \otimes \mathcal{I}$ saturates* if $\mathrm{Bl}_{\mathcal{I}}{}^* L(-E)$ is spanned by global sections of $L \otimes \mathcal{I}$. This condition is weaker than the condition such that $L \otimes \mathcal{I}$ is globally generated.

REMARK 2.3. If X is a Fano manifold, then we omit the polarization $(X, -K_X)$. More precisely, *slope stability of X along a closed subscheme $Z \subset X$* is nothing but slope stability of $(X, -K_X)$ along a closed subscheme $Z \subset X$.

The following is a fundamental result.

Theorem 2.4 ([5], [18])**.** *Let (X, L) be a polarized manifold. If (X, L) admits a Kähler metric with constant scalar curvature, then (X, L) is slope semistable along any closed subscheme $Z \subset X$.*

In particular, for a Fano manifold X, if X admits a Kähler–Einstein metric then X is slope semistable along any closed subscheme $Z \subset X$.

3. Slope stability of Fano manifolds along smooth subvarieties or divisors

In this section, we fix the notation.

NOTATION 3.1. We set that X is a Fano n-fold and $Z \subset X$ is a smooth subvariety of codimension $r \geq 2$ or an effective divisor (not necessary smooth). If Z is an effective divisor on X, we set $r := 1$. We set $\sigma := \mathrm{Bl}_Z \colon \hat{X} \to X$. Let $E \subset \hat{X}$ be the divisor which satisfies $\mathcal{O}_{\hat{X}}(-E) \simeq \sigma^{-1}\mathcal{I}_Z$ (i.e., if $r \geq 2$ then E is the exceptional divisor of σ, and if $r = 1$ then σ is the identity morphism and $E = Z$).

Under the notation, we consider slope stability of X along $Z \subset X$. We can show that

$$a_0(x) = \frac{1}{n!}\operatorname{vol}_{\hat{X}}(\sigma^*(-K_X) - xE),$$
$$a_1(x) = \frac{1}{2\cdot(n-1)!}(-K_{\hat{X}}\cdot(\sigma^*(-K_X) - xE)^{n-1}),$$
$$\mu(X) = \frac{n}{2}$$

by the weak Riemann–Roch formula (cf. [8]). Thus for $0 < c \le \epsilon(Z)$, we have

$$\begin{aligned}
&\mu_c(Z) < \mu(X)\\
&\Leftrightarrow \int_0^c (r-x)(E\cdot(\sigma^*(-K_X) - xE)^{n-1})\,dx > 0\\
&\Leftrightarrow r\operatorname{vol}_X(-K_X) + (c-r)\operatorname{vol}_{\hat{X}}(\sigma^*(-K_X) - cE)\\
&\qquad - \int_0^c \operatorname{vol}_{\hat{X}}(\sigma^*(-K_X) - xE)\,dx > 0.
\end{aligned}$$

We set

$$\begin{aligned}
\xi_c(Z) := r\operatorname{vol}_X(-K_X) + (c-r)\operatorname{vol}_{\hat{X}}(\sigma^*(-K_X) - cE)\\
- \int_0^c \operatorname{vol}_{\hat{X}}(\sigma^*(-K_X) - xE)\,dx.
\end{aligned}$$

Since $\sigma^*(-K_X) - cE$ is ample for any $0 < c < \epsilon(Z)$, we have

$$\begin{aligned}
&\frac{d}{dc}(\xi_c(Z))\\
&= n(r-c)(E\cdot(\sigma^*(-K_X) - cE)^{n-1})\begin{cases} > 0 & (\text{if } 0 < c < \min\{r, \epsilon(Z)\}),\\ < 0 & (\text{if } r < c < \epsilon(Z) \ (\text{if } \epsilon(Z) > r)).\end{cases}
\end{aligned}$$

Assume that X is not slope stable along Z. Then $\xi_c(Z) \le 0$ for some $0 < c \le \epsilon(Z)$. Hence $\epsilon(Z) > r$ and $\xi_c(Z) \ge \xi_{\epsilon(Z)}(Z)$ by the above argument. In particular, $\hat{X}$ is a Fano manifold and hence $\operatorname{Nef}(\hat{X})$ is a rational polyhedral cone spanned by semiample divisors (in particular, $\epsilon(Z) \in \mathbb{Q}_{>0}$ holds). Therefore we have the following result.

Proposition 3.2. *Let X be a Fano n-fold and $Z \subset X$ be a divisor or a smooth subvariety of codimension $r \ge 1$ (if Z is a divisor, then we set $r = 1$). Then X is slope* *stable* (*resp. slope semistable*) *along* Z *if and only if* $\xi(Z) > 0$ (*resp.* ≥ 0), *where*

$$\begin{aligned}\xi(Z) &:= \xi_{\epsilon(Z)}(Z)\\ &= r\,\mathrm{vol}_X(-K_X) + (\epsilon(Z)-r)\,\mathrm{vol}_{\hat{X}}(\sigma^*(-K_X)-\epsilon(Z)E)\\ &\quad - \int_0^{\epsilon(Z)} \mathrm{vol}_{\hat{X}}(\sigma^*(-K_X)-xE)\,dx\\ &= n\int_0^{\epsilon(Z)} (r-x)(E\cdot(\sigma^*(-K_X)-xE)^{n-1})\,dx.\end{aligned}$$

REMARK 3.3. For a Fano n-fold X and $Z \subset X$ a divisor or a smooth subvariety, the definition of slope stability of X along Z is equivalent to the definition in [8] and [6] by the above argument.

REMARK 3.4. If X is not slope stable along Z, then $\epsilon(Z) > r$ holds by the above argument. This result has been already known in [17, §8] (see also [8, Lemma 2.10]). In fact, Yuji Odaka pointed out to the author that if X is not slope stable (resp. not slope semistable) along Z then $\epsilon(Z) \geq r(n+1)/n$ (resp. $\epsilon(Z) > r(n+1)/n$) holds. See [16, Proposition 4.4] for detail.

Now, we show that slope stability of Fano manifolds along smooth subvarieties can reduce to slope stability of Fano manifolds along divisors.

Proposition 3.5. *Let* X *be a Fano* n*-fold and let* $Z \subset X$ *be a smooth subvariety of codimension* $r \geq 2$. *Let* $\sigma := \mathrm{Bl}_Z\colon \hat{X} \to X$ *and let* $E \subset \hat{X}$ *be the exceptional divisor of* σ. *If* X *is not slope stable along* Z, *then* $\hat{X}$ *itself is a Fano* n*-fold and* $\hat{X}$ *is not slope semistable along* E.

Proof. We have already seen that $\hat{X}$ is a Fano manifold. We note that

$$r < \epsilon(Z, X) = \epsilon(E, \hat{X}) + r - 1.$$

Hence we have

$$\begin{aligned}\frac{1}{n}\xi(E) &= \int_0^{\epsilon(E,\hat{X})} (1-x)(E\cdot(-K_{\hat{X}}-xE)^{n-1})\,dx\\ &= \int_0^{\epsilon(Z,X)-(r-1)} (1-x)(E\cdot(\sigma^*(-K_X)+(1-r-x)E)^{n-1})\,dx\\ &= \int_{r-1}^{\epsilon(Z,X)} (r-x)(E\cdot(\sigma^*(-K_X)-xE)^{n-1})\,dx\\ &= \frac{1}{n}\xi(Z) - \int_0^{r-1} (r-x)(E\cdot(\sigma^*(-K_X)-xE)^{n-1})\,dx\\ &< 0,\end{aligned}$$

since $\sigma^*(-K_X) - xE$ is ample for any $0 < x < r-1$ ($< \epsilon(Z, X)$). □

4. First properties

4.1. Convexity of the volume function. In this section, we consider slope stability of Fano manifolds in terms of the convexity of the volume function $\mathrm{vol}_{\hat{X}}(\sigma^*(-K_X) - xE)$ (under Notation 3.1).

Proposition 4.1. *We fix* Notation 3.1.
(1) *If $\mathcal{N}^\vee_{Z/X}$ (the dual of the normal bundle) is nef and $\epsilon(Z) \geq 2r$ holds, then X is not slope stable along Z.*
(2) *Furthermore, X is not slope semistable along Z if we assume the assumption in* (1) *and one of the following holds: $r \geq 2$, $\epsilon(Z) > 2r$ or $\mathcal{N}^\vee_{Z/X}$ is ample.*
(3) *If $r = 1$, Z^2 is a nonzero effective cycle and $\epsilon(Z) \leq 2$ holds, then X is slope stable along Z.*

We note that a vector bundle $\mathcal{E}$ on a projective variety Y is *nef* (resp. *ample*) if the corresponding tautological line bundle $\mathcal{O}_{\mathbb{P}_Y(\mathcal{E})}(1)$ on $\mathbb{P}_Y(\mathcal{E})$ is nef (resp. ample).

Proof of Propostion 4.1. We can assume $\epsilon(Z) > r$ by Remark 3.4. We write $\epsilon := \epsilon(Z)$ for simplicity. We define $f(x) := \mathrm{vol}_{\hat{X}}(\sigma^*(-K_X) - xE)$. Then we can write

$$\xi(Z) = rf(0) + (\epsilon - r)f(\epsilon) - \int_0^\epsilon f(x)\,dx.$$

We note that

$$\frac{df}{dx}(x) = -n(E \cdot (\sigma^*(-K_X) - xE)^{n-1}) < 0 \quad \text{(for any } 0 < x < \epsilon\text{)},$$
$$\frac{d^2 f}{dx^2}(x) = n(n-1)(E^2 \cdot (\sigma^*(-K_X) - xE)^{n-2}).$$

We recall that $\mathcal{O}_{\hat{X}}(-E)|_E \simeq \mathcal{O}_{\mathbb{P}_Y(\mathcal{E})}(1)$. Hence $f(x)$ is a convex upward (resp. strictly convex upward) and strictly monotone decreasing function over an interval $(0, \epsilon)$ if $\mathcal{N}^\vee_{Z/X}$ is nef (resp. ample). Then (1) and (2) follows immediately. The proof of (3) is same as those of (1) and (2). □

4.2. Product cases. We consider the case that a Fano manifold X can be decomposed into the product $X = X_1 \times X_2$. It is easy to show that both X_1 and X_2 are Fano manifolds, the vector space $\mathrm{N}^1(X)$ is naturally decomposed into $\mathrm{N}^1(X) = \mathrm{N}^1(X_1) \oplus \mathrm{N}^1(X_2)$ and the cones can be written as $\mathrm{Eff}(X) = \mathrm{Eff}(X_1) + \mathrm{Eff}(X_2)$, $\mathrm{Nef}(X) = \mathrm{Nef}(X_1) + \mathrm{Nef}(X_2)$ under the decomposition, respectively. We set $n_i := \dim X_i$ ($i = 1,2$).

Proposition 4.2. *Let $D_1 \subset X_1$ be a divisor on X_1. Then slope stability (resp. slope semistability) of X_1 along D_1 is equivalent to slope stability (resp. slope semistability) of X along $p_1^* D_1$.*

Proof. Let $\epsilon_1 := \epsilon(D_1, X_1)$. Then we know that $\epsilon_1 = \epsilon(p_1^* D_1, X)$. By the definition of $\xi(p_1^* D_1)$, we have

$$\begin{aligned}
\xi(p_1^* D_1) &= \mathrm{vol}_X(-K_X) + (\epsilon_1 - 1)\,\mathrm{vol}_X(-K_X - \epsilon_1 p_1^* D_1) \\
&\quad - \int_0^{\epsilon_1} \mathrm{vol}_X(-K_X - x p_1^* D_1)\,dx \\
&= \binom{n_1 + n_2}{n_1} \cdot \mathrm{vol}_{X_2}(-K_{X_2}) \Big\{ \mathrm{vol}_{X_1}(-K_{X_1}) + (\epsilon_1 - 1)\,\mathrm{vol}_{X_1}(-K_{X_1} - \epsilon_1 D_1) \\
&\qquad - \int_0^{\epsilon_1} \mathrm{vol}_{X_1}(-K_{X_1} - x D_1)\,dx \Big\} \\
&= \binom{n_1 + n_2}{n_1} \mathrm{vol}_{X_2}(-K_{X_2}) \cdot \xi(D_1).
\end{aligned}$$

Therefore the signs of $\xi(D_1)$ and $\xi(p_1^* D_1)$ are same. □

Proposition 4.3. *Let $D_i \subset X_i$ be divisors on X_i for $i = 1, 2$. If X_i is slope semistable along D_i for any $i = 1, 2$, then X is slope stable along $D := p_1^* D_1 + p_2^* D_2$.*

Proof. Let $\epsilon_i := \epsilon(D_i, X_i)$ for $i = 1, 2$ and let $\epsilon := \epsilon(D, X)$. We can show that $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. We can assume $\epsilon > 1$ by Remark 3.4. We note that

$$\frac{d}{dx}(\mathrm{vol}_{X_i}(-K_{X_i} - x D_i)) = -n_i (D_i \cdot (-K_{X_i} - x D_i)^{n_i - 1}) < 0 \quad (0 < x < \epsilon)$$

for any $i = 1, 2$. By the definition of $\xi(D)$, we have

$$\begin{aligned}
\xi(D) &= \int_0^1 (\mathrm{vol}_X(-K_X) - \mathrm{vol}_X(-K_X - x D))\,dx \\
&\quad - \int_1^{\epsilon} (\mathrm{vol}_X(-K_X - x D) - \mathrm{vol}_X(-K_X - \epsilon D))\,dx \\
&= \binom{n_1 + n_2}{n_1} \Big\{ \int_0^1 (\mathrm{vol}_{X_1}(-K_{X_1})\,\mathrm{vol}_{X_2}(-K_{X_2}) \\
&\qquad - \mathrm{vol}_{X_1}(-K_{X_1} - x D_1)\,\mathrm{vol}_{X_2}(-K_{X_2} - x D_2))\,dx \\
&\qquad - \int_1^{\epsilon} (\mathrm{vol}_{X_1}(-K_{X_1} - x D_1)\,\mathrm{vol}_{X_2}(-K_{X_2} - x D_2) \\
&\qquad - \mathrm{vol}_{X_1}(-K_{X_1} - \epsilon D_1)\,\mathrm{vol}_{X_2}(-K_{X_2} - \epsilon D_2))\,dx \Big\}
\end{aligned}$$

$$
\begin{aligned}
&= \binom{n_1+n_2}{n_1}\bigg\{\int_0^1 (\mathrm{vol}_{X_1}(-K_{X_1}) - \mathrm{vol}_{X_1}(-K_{X_1} - xD_1))\,\mathrm{vol}_{X_2}(-K_{X_2})\,dx \\
&\qquad + \int_0^1 \mathrm{vol}_{X_1}(-K_{X_1} - xD_1)(\mathrm{vol}_{X_2}(-K_{X_2}) - \mathrm{vol}_{X_2}(-K_{X_2} - xD_2))\,dx \\
&\qquad - \int_1^\epsilon (\mathrm{vol}_{X_1}(-K_{X_1} - xD_1) - \mathrm{vol}_{X_1}(-K_{X_1} - \epsilon D_1)) \\
&\qquad\qquad \times \mathrm{vol}_{X_2}(-K_{X_2} - xD_2)\,dx \\
&\qquad - \int_1^\epsilon \mathrm{vol}_{X_1}(-K_{X_1} - \epsilon D_1) \\
&\qquad\qquad \times (\mathrm{vol}_{X_2}(-K_{X_2} - xD_2) - \mathrm{vol}_{X_2}(-K_{X_2} - \epsilon D_2))\,dx\bigg\} \\
&> \binom{n_1+n_2}{n_1}\bigg\{\int_0^1 (\mathrm{vol}_{X_1}(-K_{X_1}) - \mathrm{vol}_{X_1}(-K_{X_1} - xD_1))\,\mathrm{vol}_{X_2}(-K_{X_2} - D_2)\,dx \\
&\qquad + \int_0^1 \mathrm{vol}_{X_1}(-K_{X_1} - D_1)(\mathrm{vol}_{X_2}(-K_{X_2}) - \mathrm{vol}_{X_2}(-K_{X_2} - xD_2))\,dx \\
&\qquad - \int_1^\epsilon (\mathrm{vol}_{X_1}(-K_{X_1} - xD_1) - \mathrm{vol}_{X_1}(-K_{X_1} - \epsilon D_1)) \\
&\qquad\qquad \times \mathrm{vol}_{X_2}(-K_{X_2} - D_2)\,dx \\
&\qquad - \int_1^\epsilon \mathrm{vol}_{X_1}(-K_{X_1} - D_1) \\
&\qquad\qquad \times (\mathrm{vol}_{X_2}(-K_{X_2} - xD_2) - \mathrm{vol}_{X_2}(-K_{X_2} - \epsilon D_2))\,dx\bigg\} \\
&= \binom{n_1+n_2}{n_1}\{\mathrm{vol}_{X_1}(-K_{X_1} - D_1)\cdot\xi_\epsilon(D_2) + \mathrm{vol}_{X_2}(-K_{X_2} - D_2)\cdot\xi_\epsilon(D_1)\} \\
&\geq \binom{n_1+n_2}{n_1}\{\mathrm{vol}_{X_1}(-K_{X_1} - D_1)\cdot\xi(D_2) + \mathrm{vol}_{X_2}(-K_{X_2} - D_2)\cdot\xi(D_1)\} \geq 0.
\end{aligned}
$$

Therefore X is slope stable along D. □

As a consequence of Propositions 4.2 and 4.3, we have the following result.

Corollary 4.4. *Let X be a Fano manifold which is the product of Fano manifolds $X = \prod_{i=1}^m X_i$. Then X is slope stable* (*resp. slope semistable*) *along any divisor if and only if X_i is slope stable* (*resp. slope semistable*) *along any divisor for any $1 \leq i \leq m$.*

4.3. Length of extremal rays. We show that if a Fano manifold X is not slope stable along a divisor, then there exists an extremal ray of the length ≥ 2.

Proposition 4.5. *Let X be a Fano manifold and $D \subset X$ be a divisor. Assume X is not slope stable along D. Then for any irreducible curve $C \subset X$, we have $(-K_X \cdot C) > (D \cdot C)$. In particular, there exists an extremal ray $R \subset \mathrm{NE}(X)$ such that $l(R) \geq 2$.*

Proof. We have $\epsilon(D) > 1$ by Remark 3.4. Hence we have

$$(D \cdot C) \leq (-K_X/\epsilon(D) \cdot C) < (-K_X \cdot C).$$

Since D is an effective divisor, there exists an extremal ray $R \subset \mathrm{NE}(X)$ with a minimal rational curve $[C] \in R$ such that $(D \cdot C) > 0$. Therefore we have $(-K_X \cdot C) \geq 2$ since $(-K_X \cdot C) > (D \cdot C)$ holds. □

4.4. Slope stability of Fano manifolds along nef divisors.

Theorem 4.6. *Let X be a Fano n-fold and $D \subset X$ be a divisor.*
(1) *If D is an ample divisor, then X is slope stable along D unless X is isomorphic to a projective space and D is a hyperplane section.*
(2) *If D is a nef divisor and $(D^i \cdot (-K_X - \epsilon(D)D)^{n-i}) = 0$ for any $1 \leq i \leq \epsilon(D) - 1$ (resp. $1 \leq i < \epsilon(D) - 1$), then X is slope stable (resp. slope semistable) along D.*

Proof. First, we consider the case that $-K_X$ and D are numerically proportional (i.e., there exists a positive rational number t such that $-K_X \equiv tD$). We note that $t \leq n+1$ and the equality holds if and only if $X \simeq \mathbb{P}^n$ and $D \in |\mathcal{O}_{\mathbb{P}^n}(1)|$ by [10]. In this case we have

$$\xi(D) = \mathrm{vol}_X(D)\left\{t^n - \int_0^t (t-x)^n\,dx\right\} = \mathrm{vol}_X(D)t^n\left(1 - \frac{t}{n+1}\right) \geq 0,$$

and equality holds if and only if $t = n+1$. Therefore we have proved the theorem for the case $-K_X$ and D are numerically proportional.

Now we consider the case that $-K_X$ and D are not numerically proportional. We can assume $\epsilon(D) > 1$ by Remark 3.4. Let $P \subset \mathrm{N}^1(X)$ be the 2-dimensional vector subspace spanned by $[-K_X]$ and $[D]$ (the classes of $-K_X$ and D in $\mathrm{N}^1(X)$). We take $[H_1], [H_2] \in P \cap \mathrm{Nef}(X)$ such that $P \cap \mathrm{Nef}(X) = \mathbb{R}_{\geq 0}[H_1] + \mathbb{R}_{\geq 0}[H_2]$. Then after interchanging H_1 and H_2, if necessary, we can write $-K_X \equiv p_1H_1 + p_2H_2$ and $D \equiv q_1H_1 + q_2H_2$, where $p_1, p_2, q_1 > 0$, $q_2 \geq 0$ and $1/\epsilon(D) = q_1/p_1 > q_2/p_2$ holds. We note that D is ample if and only if $q_2 > 0$. We also note that there exists extremal rays $R_1, R_2 \subset \mathrm{NE}(X)$ such that $(H_i \cdot R_j) = 0$ if and only if $i \neq j$ holds where $1 \leq i$, $j \leq 2$, since the class of $-K_X$ lives in the interior of $\mathrm{Nef}(X)$. We choose minimal rational curves C_1 and C_2 of R_1 and R_2, respectively. We have $(-K_X \cdot C_i) \leq n$ for any $i = 1, 2$ by [4]. If $q_i > 0$ then we have $p_i/q_i \leq n$ since $(-K_X \cdot C_i) \leq n$, $(D \cdot C_i) \geq 1$

and $p_i/q_i = (-K_X \cdot C_i)/(D \cdot C_i)$ hold. Then we can show that

$$\begin{aligned}
\xi(D) &= \mathrm{vol}_X(p_1H_1 + p_2H_2) + \left(\frac{p_1}{q_1} - 1\right)\mathrm{vol}_X\left(\left(p_2 - q_2\frac{p_1}{q_1}\right)H_2\right) \\
&\quad - \int_0^{p_1/q_1} \mathrm{vol}_X((p_1 - q_1x)H_1 + (p_2 - q_2x)H_2)\,dx \\
&= (H_2^n)p_2^n\left\{1 + \left(\frac{p_1}{q_1} - 1\right)\left(1 - \frac{q_2}{p_2}\frac{p_1}{q_1}\right)^n - \int_0^{p_1/q_1}\left(1 - \frac{q_2}{p_2}x\right)^n dx\right\} \\
&\quad + \sum_{i=1}^{n-1}(H_1^i \cdot H_2^{n-i})p_1^i p_2^{n-i}\binom{n}{i}\left\{1 - \int_0^{p_1/q_1}\left(1 - \frac{q_1}{p_1}x\right)^i\left(1 - \frac{q_2}{p_2}x\right)^{n-i} dx\right\} \\
&\quad + (H_1^n)p_1^n\left\{1 - \int_0^{p_1/q_1}\left(1 - \frac{q_1}{p_1}x\right)^n dx\right\}.
\end{aligned}$$

We denote the coefficient of $(H_1^i \cdot H_2^{n-i})$ by M_i. We claim that $(H_1^i \cdot H_2^{n-i}) \geq 0$ for any $0 \leq i \leq n$ and $(H_1^i \cdot H_2^{n-i}) > 0$ for some i since H_1 and H_2 are nef and $[-K_X] \in P$.

First, we consider the case D is ample. It is enough to show $M_i > 0$ for any i by the above claim. We have

$$\begin{aligned}
M_0 &= p_2^n\left\{1 - \frac{1}{n+1}\frac{p_2}{q_2} + \left(1 - \frac{q_2}{p_2}\frac{p_1}{q_1}\right)^n\left(\frac{p_1}{q_1} - 1 + \frac{1}{n+1}\left(\frac{p_2}{q_2} - \frac{p_1}{q_1}\right)\right)\right\} > 0, \\
M_n &= p_1^n\left(1 - \frac{1}{n+1}\frac{p_1}{q_1}\right) > 0, \\
M_i &> \binom{n}{i}p_1^i p_2^{n-i}\left\{1 - \int_0^{p_1/q_1}\left(1 - \frac{p_2}{q_2}x\right)^n dx\right\} \\
&= \binom{n}{i}p_1^i p_2^{n-i}\frac{p_2}{q_2(n+1)}\left\{(n+1)\frac{q_2}{p_2} - 1 + \left(1 - \frac{q_2}{p_2}\frac{p_1}{q_1}\right)^{n+1}\right\} > 0 \quad (0 < i < n).
\end{aligned}$$

Thus we have proved the theorem for the case D is ample.

Now, we consider the case that D is not ample. Since $q_2 = 0$, we have

$$\xi(D) = \sum_{i=1}^{n}(H_1^i \cdot H_2^{n-i})\binom{n}{i}p_1^i p_2^{n-i}\left(1 - \frac{1}{i+1}\frac{p_1}{q_1}\right).$$

Therefore we have $\xi(D) > 0$ (resp. ≥ 0) if $(H_1^i \cdot H_2^{n-i}) = 0$ for any $1 \leq i \leq p_1/q_1 - 1$ (resp. $1 \leq i < p_1/q_1 - 1$) by the same argument of the case D is ample. □

As an immediate corollary of Theorem 4.6, we get Odaka's result:

Corollary 4.7 (Odaka)**.** *Let X be a Fano manifold with the Picard number $\rho_X = 1$ and let $D \subset X$ be a divisor. Then X is slope stable along D unless X is isomorphic to a projective space and D is a hyperplane section.*

Proof. The divisor D is ample since $\rho_X = 1$. Hence the assertion is obvious from Theorem 4.6 (1). □

REMARK 4.8. There exists a Fano n-fold X and a nef effective divisor $D \subset X$ such that X is not slope semistable along D. For example, let X be the Fano manifold obtained by the blowing up of the n-dimensional projective space along a (reduced) point and D be the strict transform of a hyperplane passing through the center of the blowing up. Then D is a nef divisor and

$$\mathrm{vol}_X(-K_X - xD) = (n+1-x)^n - (n-1-x)^n$$

holds. Hence we have

$$\xi(D) = \frac{2(n-1)}{n+1}\{n \cdot 2^{n-1} - (n-1)^{n-1}\},$$

which takes a negative value if $n \geq 5$.

5. Examples and applications

5.1. Projective spaces. Let $Z \subset \mathbb{P}^n$ be a linear subspace of codimension $r \geq 1$. If $r = 1$, then

$$\xi(Z) = (n+1)^n - \int_0^{n+1} (n+1-x)^n \, dx = 0$$

(see also the proof of Theorem 4.6).

We consider the case $r \geq 2$. Let the blowing up of $\mathbb{P}^n$ along Z be $\sigma\colon \hat{X} \to \mathbb{P}^n$ and the exceptional divisor be E. We can show that $\epsilon(Z) = n+1$, $E \simeq \mathbb{P}^{n-r} \times \mathbb{P}^{r-1}$, $\mathcal{N}_{E/\hat{X}} \simeq \mathcal{O}_{\mathbb{P}^{n-r}\times\mathbb{P}^{r-1}}(1,-1)$ and $\mathcal{O}_{\hat{X}}(-K_{\hat{X}})|_E \simeq \mathcal{O}_{\mathbb{P}^{n-r}\times\mathbb{P}^{r-1}}(n-r+2, r-1)$. Hence

$$\begin{aligned}\xi(Z) &= n \int_0^{n+1} (r-x)\, \mathrm{vol}_{\mathbb{P}^{n-r}\times\mathbb{P}^{r-1}}(\mathcal{O}_{\mathbb{P}^{n-r}\times\mathbb{P}^{r-1}}(n+1-x, x))\, dx \\ &= n\binom{n-1}{r-1} \int_0^{n+1} (r-x)(n+1-x)^{n-r} x^{r-1}\, dx = 0\end{aligned}$$

by a simple calculation. Therefore, we have the following:

Proposition 5.1. *The projective space $\mathbb{P}^n$ is not slope stable but slope semistable along any linear subspace.*

In fact, it is well known that the n-dimensional projective space admits a Kähler–Einstein metrics; the Fubini–Study metric.

5.2. Surfaces. In Section 5.2, we consider the case such that the dimension is equal to two.

Proposition 5.2. *Let S be a del Pezzo surface, that is, S is a Fano manifold with* $\dim S = 2$.
(1) *S is slope semistable along any curve but there exists a curve $C \subset S$ such that S is not slope stable along C if and only if S is isomorphic to either $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$.*
(2) *There exists a curve $C \subset S$ such that S is not slope semistable along C if and only if S is isomorphic to $\mathbb{F}_1$.*

Proof. If $\mathrm{vol}_S(-K_S) \leq 7$, then we know that any extremal ray $R \subset \mathrm{NE}(S)$ satisfies that $l(R) = 1$. Hence S is slope stable along any curve by Proposition 4.5. If $S = \mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$, then the assertion (1) in Proposition 5.2 holds by Theorem 4.6 (1) and Corollary 4.4. If $S = \mathbb{F}_1$, then S is not slope semistable along $e \subset \mathbb{F}_1$ by Propositions 5.1 and 3.5. □

REMARK 5.3. In fact, Tian [21] proved that S does *not* admit Kähler–Einstein metrics if and only if S is isomorphic to $\mathbb{F}_1$ or S_7.

5.3. Non-slope-semistable examples. Let Z be a Fano $(n-1)$-fold of $\rho_Z = 1$ and the Fano index $t \geq 2$. Let $\mathcal{O}_Z(1)$ be the ample generator of $\mathrm{Pic}(Z)$. We note that $t \leq n$, see [10].

We set $X := \mathbb{P}_Z(\mathcal{O}_Z \oplus \mathcal{O}_Z(s)) \xrightarrow{\pi} Z$ with $t > s > 0$. We denote the section of π with $\mathcal{N}_{E/X} \simeq \mathcal{O}_Z(-s)$ by $E \subset X$. Then it is easy to show that X is a Fano n-fold which satisfies that

$$\mathrm{vol}_X(-K_X) = \frac{(t+s)^n - (t-s)^n}{s} \mathrm{vol}_Z(\mathcal{O}_Z(1))$$

and

$$\mathrm{NE}(X) = \mathbb{R}_{\geq 0}[f] + \mathbb{R}_{\geq 0}[e],$$

where f is a fiber of π and $e \subset E$ is an arbitrary irreducible curve in E. Then we can show that $\epsilon(E) = 2$. Hence we have the following result by Proposition 4.1 (2).

Proposition 5.4. *X is not slope semistable along E.*

As a corollary, we give the following counterexample.

Corollary 5.5 (counterexamples to Conjecture 1.3). *For any $n \geq 4$, there exists a Fano n-fold X such that*
(1) *the anticanonical volume of X is equal to $2(3^n - 1)$ (note that $2(3^n - 1) < ((n+1)^2/2n)^n$ if $n \geq 5$) and*

(2) *X does not admit Kähler–Einstein metrics.*

Proof. Let $\tau\colon Z \to \mathbb{P}^{n-1}$ be the double cover such that the branch locus $B \subset \mathbb{P}^{n-1}$ is a smooth divisor of degree $2(n-2)$. We note that Z is isomorphic to a weighted hypersurface of degree $2(n-2)$ in $\mathbb{P}(1^n, n-2)$. Let $\mathcal{O}_Z(1) := \tau^*\mathcal{O}_{\mathbb{P}^{n-1}}(1)$, then we have $\mathcal{O}_Z(-K_Z) \simeq \tau^*(\mathcal{O}_{\mathbb{P}^{n-1}}(-K_{\mathbb{P}^{n-1}}) \otimes \mathcal{O}_{\mathbb{P}^{n-1}}(2-n)) \simeq \mathcal{O}_Z(2)$. Hence Z is a Fano $(n-1)$-fold with $\rho_Z = 1$, the Fano index of Z is equal to 2 and $\mathrm{vol}_Z(\mathcal{O}_Z(1)) = 2$ holds.

Let $X := \mathbb{P}_Z(\mathcal{O}_Z \oplus \mathcal{O}_Z(1))$, then X is a Fano n-fold and $\mathrm{vol}_X(-K_X) = 2(3^n - 1)$. On the other hand, X is not slope semistable by Proposition 5.4. Thus X does not admit Kähler–Einstein metrics by Theorem 2.4. □

REMARK 5.6. In [1], Aubin reduced Conjecture 1.3 to [1, Inequality (4)]. However the inequality does not hold, as already pointed out by Yuji Sano, for example for $S_3 \times \mathbb{P}^1$.

REMARK 5.7. The above Fano manifolds, which are given by $X = \mathbb{P}_Z(\mathcal{O}_Z \oplus \mathcal{O}_Z(s))$ such that Z is a Fano $(n-1)$-fold of $\rho_Z = 1$ and the Fano index t which satisfies $t > s > 0$, are characterized by the smooth projective varieties which have an elementary birational K_X-negative extremal divisor-to-point contraction and have a $\mathbb{P}^1$-bundle structure. See [7, Remark 2.4 (a)] or [3, Lemma 3.6].

6. Threefold case

Throughout this section, let X be a Fano threefold which satisfies that $\xi(D) \leq 0$ for some divisor $D \subset X$. For the *type* of an extremal ray for smooth projective threefolds, we refer the readers to [14].

6.1. $\rho_X = 1$ case. This case has been shown in Theorem 4.6 (1) since any effective divisor is ample. We have $X \simeq \mathbb{P}^3$ and D is a hyperplane section. In this case, X is slope semistable along D.

6.2. $\rho_X = 2$ case. We set $\mathrm{NE}(X) = R_1 + R_2$ and we also set minimal rational curves $[l_1] \in R_1$ and $[l_2] \in R_2$. We denote the contractions $\phi_i := \mathrm{cont}_{R_i}\colon X \to Y_i$ and let $H_i \in \mathrm{Pic}(X)$ be the pullback of the ample generator of $\mathrm{Pic}(Y_i)$. We note that $\mathrm{Nef}(X) = \mathbb{R}_{\geq 0}[H_1] + \mathbb{R}_{\geq 0}[H_2]$. Then we have

- $\mathrm{Pic}(X) = \mathbb{Z}[H_1] \oplus \mathbb{Z}[H_2]$,
- $(H_1 \cdot l_2) = 1$, $(H_2 \cdot l_1) = 1$,
- $-K_X \sim l(R_2)H_1 + l(R_1)H_2$

by [14, Theorem 5.1].

First, we consider the case $l(R_1) = 3$ (i.e., ϕ_1 is a $\mathbb{P}^2$-bundle). Then X is either isomorphic to $\mathbb{P}^1 \times \mathbb{P}^2$ or $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(1))$.

(1) If $X \simeq \mathbb{P}^1 \times \mathbb{P}^2$, then X is not slope stable along some divisor but slope semistable along any divisor by Theorem 4.6 (1) and Corollary 4.4.

(2) If $X \simeq \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(1))$, then X is isomorphic to the blowing up of $\mathbb{P}^3$ along a line. Thus X is not slope semistable along the exceptional divisor by Propositions 5.1 and 3.5.

Hence we can assume $l(R_1) \leq 2$ and $l(R_2) \leq 2$. By Proposition 4.5, we can assume $l(R_1) = 2$ and $(D \cdot l_1) = 1$. Hence we can write $D \sim aH_1 + H_2$ $(a \in \mathbb{Z})$. Note that $a \leq 0$ by Theorem 4.6 (1).

Assume that $a = 0$. We set $b := l(R_2)$ (note that $b = 1$ or 2). Then we have $-K_X \sim bH_1 + 2H_2$ and $D \sim H_2$ hence $\epsilon(D) = 2$. However we have

$$\begin{aligned}\frac{1}{3}\xi(D) &= \int_0^2 (1-x)(H_2 \cdot (bH_1 + (2-x)H_2)^2)\, dx \\ &= \frac{4b}{3}(H_1 \cdot H_2^2) + \frac{4}{3}(H_2^3) \geq 0,\end{aligned}$$

and equality holds if and only if $(H_1 \cdot H_2^2) = 0$ and $(H_2^3) = 0$. In this case ϕ_2 is a del Pezzo fibration with $l(R_2) \leq 2$ and $l(R_1) = 2$. However there are no Fano threefolds satisfying these conditions by [14, Theorem 1.7]. Therefore $\xi(D)$ always takes a positive value; this leads to a contradiction.

As a consequence, we have $a < 0$. Since $D \sim aH_1 + H_2$ is effective, ϕ_2 is a divisorial contraction. Hence R_2 is of type E_1, E_2, E_3, E_4 or E_5.

(1) If R_2 is of type E_2, E_3, E_4 or E_5 (divisor-to-point type), X is either isomorphic to $\mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(1))$ or $\mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(2))$ by [14, Theorem 1.7]. These are not slope semistable along a divisor by Proposition 5.4.

(2) We consider the case that R_2 is of type E_1 (divisor to smooth curve). Let F be the exceptional divisor of ϕ_2 and t be the Fano index of Y_2. We have $F \sim -H_1 + (t-2)H_2$ since $-K_X \sim H_1 + 2H_2$ and $-K_X \sim tH_2 - F$. Since $\mathrm{Eff}(X) \cap (\mathbb{R}_{\geq 0}[-H_1] + \mathbb{R}_{\geq 0}[H_2]) = \mathbb{R}_{\geq 0}[F] + \mathbb{R}_{\geq 0}[H_2]$ and $D \sim aH_1 + H_2$ is an effective divisor, we have $t = 3$ (i.e., $Y_2 \simeq \mathbb{Q}^3$) and $a = -1$ (i.e., $D = F$) (hence $\epsilon(D) = 2$). Therefore X is isomorphic to either $\mathrm{Bl}_{\mathrm{conic}}\, \mathbb{Q}^3$ or $\mathrm{Bl}_{\mathrm{line}}\, \mathbb{Q}^3$ since $l(R_1) = 2$ (see [14, (5.3), (5.5)]).

- If $X \simeq \mathrm{Bl}_{\mathrm{conic}}\, \mathbb{Q}^3$, then it is easy to show that $F \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathcal{N}_{F/X} \simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2, -1)$ and $\mathcal{O}_X(-K_X)|_F \simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(4, 1)$. Hence we have

$$\frac{1}{3}\xi(F) = \int_0^2 (1-x)\, \mathrm{vol}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O}(4, 1) - x\mathcal{O}(2, -1))\, dx = \frac{8}{3} > 0,$$

this lead to a contradiction.

- If $X \simeq \mathrm{Bl}_{\mathrm{line}}\, \mathbb{Q}^3$, then it is easy to show that $F \simeq \mathbb{F}_1$ and $\mathcal{N}_{F/X} \simeq \mathcal{O}_{\mathbb{F}_1}(-e)$ and $-K_X|_F \simeq \mathcal{O}_{\mathbb{F}_1}(3f + e)$. Hence we have

$$\frac{1}{3}\xi(F) = \int_0^2 (1-x)\, \mathrm{vol}_{\mathbb{F}_1}(3f + e - x(-e))\, dx = -\frac{4}{3} < 0.$$

Therefore X is not slope semistable along F.

6.3. $\rho_X = 3$ case. By Proposition 4.5, there exists an extremal ray $R \subset \mathrm{NE}(X)$ with a minimal rational curve $[C_R] \in R$ such that $l(R) = 2$ and $(D.C_R) = 1$. Hence R is either of type E_2 or C_2.

(1) If R is of type E_2 (smooth point blowing up), then X is isomorphic to $\mathrm{Bl}_p(Y_d)$ (let $\phi := \mathrm{Bl}_p$) with $1 \le d \le 3$, where $\sigma\colon Y_d \to \mathbb{P}^3$ is the blowing up of $\mathbb{P}^3$ along B such that $H_0 \subset \mathbb{P}^3$ is a hyperplane, $B \subset H_0$ is a smooth curve of degree d, $H \subset Y_d$ is the strict transform of H_0 and satisfies $p \notin H$ (see [12, p. 160] or [2]).

Let E be the exceptional divisor of ϕ, let F be the exceptional divisor of σ and let F' be the strict transform of the locus of lines passing through p and B. We set $e \subset E$ and $h \subset H$ such that lines (both E and H are isomorphic to $\mathbb{P}^2$), $f \subset F$ be an exceptional curve of σ and $f' \subset F'$ be the strict transform of a line passing through p and a point in B. Then it is easy to show that

$$\begin{aligned} &\mathrm{NE}(X) = \mathbb{R}_{\ge 0}[e] + \mathbb{R}_{\ge 0}[h] + \mathbb{R}_{\ge 0}[f] + \mathbb{R}_{\ge 0}[f'],\\ &\mathrm{Pic}(X) = \mathbb{Z}[E] \oplus \mathbb{Z}[H] \oplus \mathbb{Z}[F],\\ &-K_X \sim -2E + 4H + 3F,\\ &F' \sim -dE + dH + (d-1)F. \end{aligned}$$

We can show that $E + F'$, $F + H$ and $(F + F')/d$ are nef. Therefore, for $[pE + qH + rF] \in \mathrm{Eff}(X)$ $(p, q, r \in \mathbb{R})$, we have

- $r = (pE + qH + rF \cdot (F + H)^2) \ge 0$,
- $p + r = (pE + qH + rF \cdot ((F + F')/d)^2) \ge 0$,
- $dq = (pE + qH + rF \cdot F + H \cdot E + F') \ge 0$,
- $(d-1)p + dq = (pE + qH + rF \cdot (F + F')/d \cdot E + F') \ge 0$.

Hence we have

$$\mathrm{Eff}(X) = \mathbb{R}_{\ge 0}[E] + \mathbb{R}_{\ge 0}[H] + \mathbb{R}_{\ge 0}[F] + \mathbb{R}_{\ge 0}[F'].$$

We write $D \sim pE + qH + rF$, where $p,q,r \in \mathbb{Z}$. Then we have $(D \cdot e) = 1$, $(D \cdot f) \le 0$, $(D \cdot f') \le 0$ and $(D \cdot h) < (-K_X \cdot h) = 4 - d$ by Proposition 4.5. Thus we have $p = -1$, $q = 1$, $r = 1$ since D is effective. Hence $D \sim -E + H + F$ and $\epsilon(D) = 2$. Therefore we have

$$\begin{aligned} \frac{1}{3}\xi(D) &= \int_0^2 (1-x)(-E + H + F \cdot ((x-2)E + (4-x)H + (3-x)F)^2)\,dx\\ &= \int_0^2 (1-x)(-4x + 12 - d)\,dx = \frac{8}{3} > 0; \end{aligned}$$

this leads to a contradiction.

(2) If R is of type C_2, then R induces a $\mathbb{P}^1$-bundle $\pi : X \to Z$. Since $\rho_X = 3$, Z is isomorphic to either $\mathbb{F}_1$ or $\mathbb{P}^1 \times \mathbb{P}^1$.

We claim that such Fano threefolds has been classified by Szurek and Wiśniewski [20]:

Claim 6.1. (i) *If* $Z \simeq \mathbb{F}_1$, X *is isomorphic to one of* $\mathbb{F}_1 \times_{\mathbb{P}^2} \mathbb{P}(T_{\mathbb{P}^2})$, $\mathbb{F}_1 \times \mathbb{P}^1$ *or* $\mathbb{F}_1 \times_{\mathbb{P}^2} \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1))$.
(ii) *If* $Z \simeq \mathbb{P}^1 \times \mathbb{P}^1$, X *is isomorphic to one of a smooth divisor of tridegree* $(1, 1, 1)$ *in* $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$, $\mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O}(0, 1) \oplus \mathcal{O}(1, 0))$, $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{F}_1 \times \mathbb{P}^1$ *or* $\mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(1, 1))$.

(I) Assume $X \simeq \mathbb{F}_1 \times_{\mathbb{P}^2} \mathbb{P}(T_{\mathbb{P}^2})$. Then we can show that $X \subset \mathbb{F}_1 \times \mathbb{P}^2$ is a smooth divisor with $X \in |\mathcal{O}_{\mathbb{F}_1 \times \mathbb{P}^2}(e + f, 1)|$. Let E, F, H be effective divisors on X correspond to $\mathcal{O}_X(e, 0)$, $\mathcal{O}_X(f, 0)$, $\mathcal{O}_X(0, 1)$, respectively. Then we can show that

$$\operatorname{Pic}(X) = \mathbb{Z}[E] \oplus \mathbb{Z}[F] \oplus \mathbb{Z}[H].$$

We can also show that there exists the structure of the blowing up $X \to \mathbb{P}^1 \times \mathbb{P}^2$ with the exceptional divisor $E' \sim H - E$. We note that F, H and $E + F$ are nef. Therefore, for $[pE + qF + rH] \in \operatorname{Eff}(X)$, we have
- $r = (pE + qF + rH \cdot (E + F)^2) \geq 0$,
- $q = (pE + qF + rH \cdot H^2) \geq 0$,
- $p + r = (pE + qF + rH \cdot F \cdot H) \geq 0$.

Hence we have

$$\operatorname{Eff}(X) = \mathbb{R}_{\geq 0}[E] + \mathbb{R}_{\geq 0}[F] + \mathbb{R}_{\geq 0}[E']$$

and it is easy to show that $-K_X \sim E + 2F + 2H$.

Let m be a fiber of π, let l be an exceptional curve of $X \to \mathbb{P}(T_{\mathbb{P}^2})$ and let l' be an exceptional curve of $X \to \mathbb{P}^1 \times \mathbb{P}^2$. Then it is easy to show that

$$\operatorname{NE}(X) = \mathbb{R}_{\geq 0}[m] + \mathbb{R}_{\geq 0}[l] + \mathbb{R}_{\geq 0}[l'].$$

We write $D \sim pE + qF + rE'$, where p, q, $r \in \mathbb{Z}$. Then we have $(D \cdot m) = 1$, $(D \cdot l) \leq 0$ and $(D \cdot l') \leq 0$ by Proposition 4.5. We also note that $p = 1$, $q = 0$, $r = 1$ (hence $D \sim H$) and $\epsilon(D) = 2$ since D is effective. Therefore we have

$$\begin{aligned}
\frac{1}{3}\xi(D) &= \int_0^2 (1 - x)(H \cdot (E + 2F + (2 - x)H)^2)\, dx \\
&= \int_0^2 (1 - x)(\mathcal{O}(0, 1) \cdot \mathcal{O}(e + 2f, 2 - x)^2 \cdot \mathcal{O}(e + f, 1))_{\mathbb{F}_1 \times \mathbb{P}^2}\, dx \\
&= \int_0^2 (1 - x)(11 - 4x)\, dx = \frac{8}{3} > 0;
\end{aligned}$$

this leads to a contradiction.

(II) Assume $X \simeq \mathbb{F}_1 \times \mathbb{P}^1$. Then X is not slope semistable along $p_1^* e$ by Propositions 5.2 and 4.2.

(III) Assume $X \simeq \mathbb{F}_1 \times_{\mathbb{P}^2} \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1))$. Let H be the section of π with normal bundle $\mathcal{N}_{H/X} \simeq \mathcal{O}_{\mathbb{F}_1}(-e - f)$, let E be the pullback of $e \subset \mathbb{F}_1$ with respect to π and let F be the pullback of $f \subset \mathbb{F}_1$ with respect to π. Then we can show that $-K_X \sim 4F + 3E + 2H$, $\epsilon(H) = 2$ and

$$\frac{1}{3}\xi(H) = \int_0^2 (1-x)(H \cdot (4F + 3E + (2-x)H)^2)\, dx$$
$$= \int_0^2 (1-x)(1+x)(3+x)\, dx = -4 < 0,$$

hence X is not slope semistable along H.

(IV) Assume $X \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^2}(1, 1, 1)|$. Let H_i $(1 \le i \le 3)$ be the restriction of $p_i^*\mathcal{O}(1)$ to X. Then we have

$$\mathrm{Pic}(X) = \mathbb{Z}[H_1] \oplus \mathbb{Z}[H_2] \oplus \mathbb{Z}[H_3]$$

by the theorem of Lefschetz. We can show that $-K_X \sim H_1 + H_2 + 2H_3$. We can also show that $p_{13}|_X \colon X \to \mathbb{P}^1 \times \mathbb{P}^2$ and $p_{23}|_X \colon X \to \mathbb{P}^1 \times \mathbb{P}^2$ are the blowing up along smooth curves with the exceptional divisors $F_{13} \sim H_1 - H_2 + H_3$ and $F_{23} \sim -H_1 + H_2 + H_3$, respectively.

We note that H_1, H_2 and H_3 are nef. Therefore, for $[a_1H_1 + a_2H_2 + a_3H_3] \in \mathrm{Eff}(X)$ $(a_1, a_2, a_3 \in \mathbb{R})$, we have

- $a_3 = (a_1H_1 + a_2H_2 + a_3H_3 \cdot H_1 \cdot H_2) \ge 0$,
- $a_1 + a_3 = (a_1H_1 + a_2H_2 + a_3H_3 \cdot H_2 \cdot H_3) \ge 0$,
- $a_2 + a_3 = (a_1H_1 + a_2H_2 + a_3H_3 \cdot H_1 \cdot H_3) \ge 0$,
- $a_1 + a_2 = (a_1H_1 + a_2H_2 + a_3H_3 \cdot H_3^2) \ge 0$.

Hence we have

$$\mathrm{Eff}(X) = \mathbb{R}_{\ge 0}[H_1] + \mathbb{R}_{\ge 0}[H_2] + \mathbb{R}_{\ge 0}[F_{13}] + \mathbb{R}_{\ge 0}[F_{23}].$$

Let l_3, l_2, l_1 be nontrivial irreducible fibers of $p_{12}|_X$, $p_{13}|_X$, $p_{23}|_X$, respectively. Then we can show that

$$\mathrm{NE}(X) = \mathbb{R}_{\ge 0}[l_1] + \mathbb{R}_{\ge 0}[l_2] + \mathbb{R}_{\ge 0}[l_3].$$

We write $D \sim a_1H_1 + a_2H_2 + a_3H_3$, where $a_1, a_2, a_3 \in \mathbb{Z}$. Then we have $(D \cdot l_1) \le 0$, $(D \cdot l_2) \le 0$ and $(D \cdot l_3) = 1$ by Proposition 4.5. We also know that $a_1 = 0$, $a_2 = 0$ and $a_3 = 1$ (hence $D \sim H_3$) and $\epsilon(D) = 2$ since D is effective. Hence we have

$$\frac{1}{3}\xi(D) = \int_0^2 (H_3 \cdot (H_1 + H_2 + (2-x)H_3)^2)\, dx$$
$$= \int_0^2 (1-x)(\mathcal{O}(0,0,1) \cdot \mathcal{O}(1,1,2-x)^2 \cdot \mathcal{O}(1,1,1))_{\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^2}\, dx$$
$$= \int_0^2 (1-x)(10-4x)\, dx = \frac{8}{3} > 0;$$

this leads to a contradiction.

(V) Assume $X \simeq \mathbb{P}_{\mathbb{P}^1\times\mathbb{P}^1}(\mathcal{O}(0,1)\oplus\mathcal{O}(1,0))$. Let E_1 and E_2 be the sections of π such that the normal bundles are $\mathcal{N}_{E_1/X} \simeq \mathcal{O}(-1,1)$ and $\mathcal{N}_{E_2/X} \simeq \mathcal{O}(1,-1)$, and $H_i := \pi^* p_i^*\mathcal{O}(1)$ $(i = 1, 2)$. Let $e_i \subset E_i$ be a fiber of the projection $p_j\colon E_i \simeq \mathbb{P}^1\times\mathbb{P}^1 \to \mathbb{P}^1$ (for $\{i, j\} = \{1, 2\}$) and f be a fiber of π. Then we can show that $-K_X \sim 3H_1 + H_2 + 2E_2$,

$$\begin{aligned}
\mathrm{NE}(X) &= \mathbb{R}_{\geq 0}[e_1] + \mathbb{R}_{\geq 0}[e_2] + \mathbb{R}_{\geq 0}[f],\\
\mathrm{Pic}(X) &= \mathbb{Z}[H_1]\oplus\mathbb{Z}[H_2]\oplus\mathbb{Z}[E_1],\\
\mathrm{Eff}(X) &= \mathbb{R}_{\geq 0}[H_1] + \mathbb{R}_{\geq 0}[H_2] + \mathbb{R}_{\geq 0}[E_1] + \mathbb{R}_{\geq 0}[E_2].
\end{aligned}$$

Hence we can show that $D \sim E_1$ or $D \sim E_2$ or $D \sim E_1 + H_1$ (in each case we have $\epsilon(D) = 2$).

If $D \sim E_1 + H_1$, then we have

$$\begin{aligned}
\frac{1}{3}\xi(D) &= \int_0^2 (1-x)(E_1 + H_1\cdot((3-x)H_1 + H_2 + (2-x)E_1)^2)\,dx\\
&= \int_0^2 (1-x)(2-x)(4-x)\,dx = \frac{8}{3} > 0;
\end{aligned}$$

this leads to a contradiction.

If $D \sim E_1$ (or E_2), then we have

$$\begin{aligned}
\frac{1}{3}\xi(D) &= \int_0^2 (1-x)(E_1\cdot(3H_1 + H_2 + (2-x)E_1)^2)\,dx\\
&= \int_0^2 2(1-x)(1+x)(3-x)\,dx = 0.
\end{aligned}$$

Hence X is slope semistable but not slope stable along E_1 (and also along E_2).

(VI) Assume $X \simeq \mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$. Then X is slope semistable along any divisor but is not slope stable along a fiber of p_1 by Theorem 4.6 (1) and Corollary 4.4.

(VII) Assume $X \simeq \mathbb{P}_{\mathbb{P}^1\times\mathbb{P}^1}(\mathcal{O}\oplus\mathcal{O}(1,1))$. Let E be the section of π with the normal bundle $\mathcal{N}_{E/X} \simeq \mathcal{O}(-1,-1)$. Then we have $\epsilon(E) = 2$. Therefore X is not slope semistable along E by Proposition 4.1 (2).

6.4. $\rho_X \geq 4$ case. There exists an extremal ray $R \subset \mathrm{NE}(X)$ of type C_2 by Proposition 4.5 and [12, p. 160]. We write its contraction $\pi\colon X \to S$. We know that S is a del Pezzo surface of $\rho_S \geq 3$. Hence we have $X \simeq \mathbb{P}^1\times S_m$ with $1 \leq m \leq 7$ by [15, Theorem 4.20] (see also [20]).

Hence X is slope semistable along any divisor but is not slope stable along some divisor by Proposition 5.2, Theorem 4.6 (1) and Corollary 4.4.

As a consequence, we have the following result:

Theorem 6.2. *Let X be a Fano threefold.*
(1) *X is slope semistable along any effective divisor but there exists a divisor $D \subset X$ such that X is not slope stable along D if and only if X is isomorphic to one of*:

$$\mathbb{P}^3, \quad \mathbb{P}^1 \times \mathbb{P}^2, \quad \mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O}(0, 1) \oplus \mathcal{O}(1, 0)),$$
$$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \quad \mathbb{P}^1 \times S_m \quad (1 \leq m \leq 7).$$

(2) *There exists a divisor $D \subset X$ such that X is not slope semistable along D if and only if X is isomorphic to one of*:

$$\mathrm{Bl}_{\mathrm{line}}\, \mathbb{Q}^3, \quad \mathrm{Bl}_{\mathrm{line}}\, \mathbb{P}^3, \quad \mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(1)), \quad \mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(2)),$$
$$\mathbb{P}^1 \times \mathbb{F}_1, \quad \mathbb{P}_{\mathbb{F}_1}(\mathcal{O} \oplus \mathcal{O}(e + f)), \quad \mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(1, 1)).$$

REMARK 6.3. By Theorem 6.2, [6, Theorem 1.1] and the result of Steffens [19, Theorem 3.1], there exists a Fano threefold X which is slope stable along all divisors and smooth subvarieties but has the unstable tangent bundle. For example, X is the blowing up of $\mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(1))$ along a line on the exceptional divisor ($\simeq \mathbb{P}^2$) of the blowing up $\mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(1)) \to \mathbb{P}^3$ (no. 29 in Table 3 in Mori and Mukai's list [12]). In fact, Mabuchi [11, Remark 2.5] observed that the above X does not admit Kähler–Einstein metrics.

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