Title	ON THE CONSTRUCTION OF DUALLY FLAT FINSLER METRICS
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Citation	Osaka Journal of Mathematics. 2015, 52(2), p. 377-391
Version Type	VoR
URL	https://doi.org/10.18910/57649
rights	
Note	

Huang, L., Liu, H. and Mo, X.
Osaka J. Math.
52 (2015), 377–391

ON THE CONSTRUCTION OF DUALLY FLAT FINSLER METRICS

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(Received April 15, 2013, revised November 14, 2013)

Abstract

In this paper, we give a new approach to find a dually flat Finsler metric. As its application, we produce many new spherically symmetric dually flat Finsler metrics by using known projective spherically symmetric Finsler metrics.

1. Introduction

A Finsler metric $F = F(x, y)$ on an open subset $\mathcal{U} \subset \mathbb{R}^n$ is *dually flat* if and only if it satisfies the following dually flat equations:

$$(F^2)_{x^i y^j} y^i - 2(F^2)_{x^j} = 0 \tag{1.1}$$

where $x = (x^1, \dots, x^n) \in \mathcal{U}$ and $y = y^j (\partial/\partial x^j)|_x \in T_x\mathcal{U}$. Such Finsler metrics arise from α-flat information structures on Riemann–Finsler manifolds [1, 14]. Recently the study of dually flat Finsler metrics has attracted a lot of attention [2, 3, 9, 14, 15, 16, 17].

In this paper, we give a new approach to find a dually flat Finsler metric. We establish the relation between the solutions of Hamel equations and ones of dually flat equations (1.1). Hamel equations are the following partial differential equations:

$$\Theta_{x^j y^i} y^j = \Theta_{x^i}, \tag{1.2}$$

where $\Theta\colon T\mathcal{U} \to \mathbb{R}$ and $\mathcal{U}$ is an open subset in $\mathbb{R}^n$. Formula (1.2) was first given by G. Hamel, in 1903, from the study of projectively flat Finsler metrics on an open subset $\mathcal{U} \subset \mathbb{R}^n$. By using (1.2), Finsler geometers manufacture projectively flat Finsler metrics.

To study and characterize projectively flat Finsler metrics on $\mathcal{U} \subset \mathbb{R}^n$ is the Hilbert's fourth problem in the smooth case. Funk metrics, Mo–Shen–Yang metrics, Bryant metrics with one parameter and Chern–Shen metrics are interesting projectively flat Finsler metrics [4, 11, 13]. In the words, their geodesics are straight lines. Furthermore, these

2000 Mathematics Subject Classification. 58E20, 53B40.

This work is supported by the National Natural Science Foundation of China 11371032, 11301283 and the Doctoral Program of Higher Education of China 20110001110069. The third author (Xiaohuan Mo) is the corresponding author.

notable Finsler metrics satisfy

$$F(Ax, Ay) = F(x, y) \tag{1.3}$$

for all $A \in O(n)$.

Very recently, Huang and Mo have constructed a lot of new projectively flat Finsler metrics satisfying (1.3) (see Proposition 4.2 below) [7]. When $\delta = 0$, these Finsler metrics, up to a scaling, were constructed in [6, Example 4.48].

According to [5], a Finsler metric $F = F(x, y)$ on an open subset $\mathcal{U} \subset \mathbb{R}^n$ is projectively flat if and only if it satisfies Hamel equations (1.2). It is natural to study the relation between PDE (1.1) and (1.2). Moreover, we wonder if the solutions of Hamel equations (1.2) produce the solutions of dually flat equations (1.1).

In this paper we are going to give a positive answer of this problem. We show that any solution of dually flat equations produces a solution of Hamel equations and vice versa (see Theorem 2.3). Using this correspondence, we are able to manufacture new dually flat Finsler metrics from known projectively flat Finsler metrics.

In the rest of this paper, we investigate how to construct the solutions of dually flat equations (1.1) from a projective spherically symmetric Finsler metric and seek conditions of producing Finsler metrics.

Recall that a Finsler metric $F = F(x, y)$ is called to be *spherically symmetric* if F satisfies (1.3) for all $A \in O(n)$, equivalently, the orthogonal group $O(n)$ act as isometries of F [9, 12, 19]. Huang–Mo proves that any spherically symmetric Finsler metric $F = F(x, y)$ can be expressed by [8]

$$F(x, y) = |y|\psi\left(|x|, \frac{\langle x, y\rangle}{|y|}\right).$$

Hence all spherically symmetric Finsler metrics are general (α, β)-metrics [18]. First, we give an explicit expression of the solution of dually flat equations (1.1) corresponding a projectively flat Finsler metric (see Proposition 3.1 below). Next, we produce many new spherically symmetric dually flat Finsler metric by using Huang–Mo metrics in Proposition 4.2. More precisely, we prove the following:

Theorem 1.1. *Let $f(\lambda)$ be a polynomial function defined by*

$$f(\lambda) = 1 + \delta\lambda + 2n \sum_{k=0}^{n-1} \frac{(-1)^k C_{n-1}^k \lambda^{2k+2}}{(2k+1)(2k+2)} \tag{1.4}$$

where

$$C_m^k = \frac{m(m-1)\cdots(m-k+1)}{k!}.$$

Suppose that $f(-1) < 0$. *Then the following Finsler metric on an open subset in* $\mathbb{R}^n \setminus \{0\}$

$$F = |y|\{|x|^{2n-1}[2n\lambda f(\lambda) + (1-\lambda^2) f'(\lambda)]\}^{1/2}$$

is dually flat where $\lambda = \langle x, y\rangle/(|x|\,|y|)$.

We have the following two interesting special cases:

(a) When $n = 1$, then

$$F = \frac{\sqrt{\delta\langle x, y\rangle^2 + 4|x|\,|y|\langle x, y\rangle + \delta|x|^2|y|^2\langle x, y\rangle^2}}{|x|^{1/2}}$$

is dually flat where $\delta > 2$.

(b) When $n = 2$, then

$$F = \frac{\sqrt{\delta|x|^3|y|^3 + 8|x|^2|y|^2\langle x, y\rangle + 3\delta|x|\,|y|\langle x, y\rangle^2 + (8/3)\langle x, y\rangle^3}}{|y|^{1/2}}$$

is dually flat where $\delta > 8/3$.

Finally we should point out that the notions of dual flat and projectively flat are not equivalent. For example, the following Finsler metric on $\mathbb{B}^n \subset \mathbb{R}^n$ is projectively flat [10],

$$F = \sqrt{\sqrt{A} + B}$$

where

$$A := \frac{|y|^4 + (|x|^2|y|^2 - \langle x, y\rangle^2)^2}{4(1 + |x|^4)^2}, \quad B := \frac{(1 + |x|^4)|x|^2|y|^2 + (1 - |x|^4)\langle x, y\rangle^2}{2(1 + |x|^4)^2},$$

but F is not dually flat. This fact follows from Cheng–Shen–Zhou's Proposition 2.6 in [2] (if a Finsler metric is dually flat and projectively flat, then it is of constant flag curvature) and the classification theorem of projective spherically symmetric Finsler metrics of constant flag curvature due to L. Zhou and Mo–Zhu [19, 12]. Very recently, C. Yu has constructed the following new dually flat Finsler metrics [17]

$$F(x, y) = (1 + |x|^2)^{1/4}|y| \pm (1 + |x|^2)^{-1/4}\langle x, y\rangle.$$

Based on the above arguments, we obtain F is not projectively flat.

2. Dually flat equations

In this section we are going to explore some nice properties of dually flat equations. In particular, we show any solution of Hamel equations produces a solution of dually flat equations (see Theorem 2.3 below).

Lemma 2.1. *Let $\mathcal{U}$ be an open subset in $\mathbb{R}^n$. Suppose that $F\colon T\mathcal{U} \to \mathbb{R}$ is a function which is positively homogeneous of degree one. Then F is a solution of dually flat equations* (1.1) *if and only if it satisfies the following equations*:

$$L_{x^i y^j} = L_{x^j y^i}, \tag{2.1}$$

where $L := F^2/2$.

Proof. A function ξ defined on $T\mathcal{U}$ can be expressed as $\xi(x^1, \dots, x^n; y^1, \dots, y^n)$. We use the following notation

$$\xi_0 := \frac{\partial \xi}{\partial x^i} y^i.$$

Then

$$(L_0)_{y^i} = (L_{x^j} y^j)_{y^i} = L_{x^j y^i} y^j + L_{x^i}. \tag{2.2}$$

Note that L is positively homogeneous of degree two. So does L_{x^i}, i.e. $L_{x^i}(x, \lambda y) = \lambda^2 L_{x^i}(x, y)$. It follows that

$$L_{x^i y^j} y^j = 2L_{x^i}. \tag{2.3}$$

First suppose that F satisfies (1.1). Combining (1.1) with (2.2), we get

$$2L_{x^i} = (L_0)_{y^i} - L_{x^i},$$

that is

$$(L_0)_{y^i} = 3L_{x^i}. \tag{2.4}$$

Differentiating (2.4) with respect to y^j, we obtain

$$L_{x^i y^j} = \frac{1}{3}(L_0)_{y^i y^j} = \frac{1}{3}(L_0)_{y^j y^i} = L_{x^j y^i}. \tag{2.5}$$

Thus we obtain (2.1).

Conversely, suppose that (2.1) holds. Together with (2.3) we have (1.1). $\square$

Lemma 2.2. *If $F\colon T\mathcal{U} \to \mathbb{R}$ is a solution of* (1.1) *where $\mathcal{U}$ is an open subset in $\mathbb{R}^n$, then there exists a function Θ such that*

$$\Theta_{x^i} = F F_{y^i}. \tag{2.6}$$

Proof. Let

$$p_i = \left(\frac{F^2}{2}\right)_{y^i} = L_{y^i}. \tag{2.7}$$

Using (2.1), one obtains

$$(p_j)_{x^i} = L_{y^j x^i} = L_{y^i x^j} = (p_i)_{x^j}. \tag{2.8}$$

Take fixed $x_0 \in \mathcal{U}$ and put

$$\Theta(x, y) = \int_{\overline{x_0 x}} p_1(u, y)\, du^1 + \cdots + p_n(u, y)\, du^n \tag{2.9}$$

where $u = x_0 + t(x - x_0)$ and $u = (u^1, \ldots, u^n)$. It follows that

$$du^j = (x^j - x_0^j)\, dt, \quad j = 1, \ldots, n$$

where $x = (x^1, \ldots, x^n)$ and $x_0 = (x_0^1, \ldots, x_0^n)$. Together with (2.9), we have

$$\Theta(x, y) = \int_0^1 [(x^1 - x_0^1) p_1(t(x - x_0) + x_0, y) + \cdots \\ + (x^n - x_0^n) p_n(t(x - x_0) + x_0, y)]\, dt.$$

It follows that Θ is differentiable with respect to y. Moreover we have

$$\begin{aligned}
\frac{\partial \Theta}{\partial x^i} &= \frac{\partial}{\partial x^i} \int_0^1 \sum_{j=1}^n (x^j - x_0^j) p_j(t(x - x_0) + x_0, y)\, dt \\
&= \int_0^1 \frac{\partial}{\partial x^i} \left[\sum_{j=1}^n (x^j - x_0^j) p_j(t(x - x_0) + x_0, y) \right] dt \\
&= \int_0^1 \sum_{j=1}^n \frac{\partial}{\partial x^i} [(x^j - x_0^j) p_j(t(x - x_0) + x_0, y)]\, dt \\
&= \int_0^1 \sum_{j=1}^n [\delta_i^j p_j(t(x - x_0) + x_0, y) + t(x^j - x_0^j)(p_j)_{u^i}(t(x - x_0) + x_0, y)]\, dt \\
&= \int_0^1 \left[p_i(t(x - x_0) + x_0, y) + t \sum_{j=1}^n (x^j - x_0^j)(p_j)_{u^i}(t(x - x_0) + x_0, y) \right] dt \\
&= \int_0^1 \frac{d}{dt}[t p_i(t(x - x_0) + x_0, y)]\, dt = t p_i(t(x - x_0) + x_0, y)\big|_0^1 = p_i(x, y)
\end{aligned}$$

where we have used (2.8). Then we complete the proof of the Lemma 2.2. □

Theorem 2.3. *Let $\mathcal{U}$ be an open subset in $\mathbb{R}^n$. Suppose that $F\colon T\mathcal{U} \to \mathbb{R}$ is a function which is positively homogeneous of degree one. Then $F = F(x,y)$ is a solution of* (1.1) *if and only if*

$$F^2 = \Theta_{x^i} y^i \tag{2.10}$$

where $\Theta\colon T\mathcal{U} \to \mathbb{R}$ *satisfies the following Hamel's equations*

$$\Theta_{x^j y^i} y^j = \Theta_{x^i}. \tag{2.11}$$

Proof. First suppose that F is a solution of (1.1). According to Lemma 2.2, there exists a function Θ such that (2.6) holds. Contracting (2.6) with y^i gives

$$\Theta_{x^i} y^i = F F_{y^i} y^i = F^2 = 2L. \tag{2.12}$$

Differentiating (2.12) with respect y^j, we obtain

$$2L_{y^j} = (\Theta_{x^i} y^i)_{y^j} = \Theta_{x^i y^j} y^i + \Theta_{x^j}. \tag{2.13}$$

Together with (2.6) yields (2.11).

Conversely, suppose that (2.10) holds, where $\Theta = \Theta(x, y)$ satisfies (2.11). Differentiating (2.10) with respect to x^j, we have

$$L_{x^j} = \left(\frac{F^2}{2}\right)_{x^j} = \frac{1}{2}\Theta_{x^i x^j} y^i.$$

It follows that

$$L_{x^j y^k} = \frac{1}{2}(I) + \frac{1}{2}\Theta_{x^k x^j} \tag{2.14}$$

where

$$(I) := \Theta_{x^i x^j y^k} y^i = \Theta_{x^i y^k x^j} y^i = (\Theta_{x^i y^k} y^i)_{x^j} = \Theta_{x^k x^j} \tag{2.15}$$

where we have used (2.11). Plugging (2.15) into (2.14) yields

$$L_{x^j y^k} = \Theta_{x^k x^j}.$$

Note that

$$\Theta_{x^j x^k} = \Theta_{x^k x^j}.$$

Hence we obtain (2.1). Combining this with Lemma 2.1 we obtain F is a solution of dually flat equations (1.1). $\square$

Theorem 2.3 tells us that there is a bijection between solutions Θ of projectively flat equations (i.e. Hamel equations) and solutions F of dually flat equations, which are positively homogeneous of degree one, given by (2.10) and (2.9).

By definition, a *Minkowski norm* on a vector space V is a nonnegative function $F\colon V \to [0, \infty)$ with the following properties:

(i) F is positively y-homogeneous of degree one, i.e., for any $y \in V$ and any $\lambda > 0$,

$$F(\lambda y) = \lambda F(y),$$

(ii) F is C^∞ on $V \setminus \{0\}$ and for any tangent vector $y \in V \setminus \{0\}$, the following bilinear symmetric form $\mathbf{g}_y(u, v)\colon V \times V \to R$ is positive definite,

$$\mathbf{g}_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}[F^2(y + su + tv)]_{s=t=0}.$$

Let M be a manifold. Let $TM = \bigcup_{x \in M} T_x M$ be the tangent bundle of M, where $T_x M$ is the tangent space at $x \in M$. We set $TM_o := TM \setminus \{0\}$ where $\{0\}$ stands for $\{(x, 0) \mid x \in M, 0 \in T_x M\}$. A *Finsler metric* on M is a function $F\colon TM \to [0, \infty)$ with the following properties

(a) F is C^∞ on TM_o;

(b) At each point $x \in M$, the restriction $F_x := F|_{T_x M}$ is a Minkowski norm on $T_x M$.

For instance, let $\phi = \phi(y)$ be a Minkowski norm on $\mathbb{R}^N$. Define

$$\Phi(x, y) := \phi(y), \quad y \in T_x\mathbb{R}^N \cong \mathbb{R}^N.$$

Then $\Phi = \Phi(x, y)$ is a Finsler metric. We call Φ the *Minkowski metric* on $\mathbb{R}^N$ [4, 11].

A Finsler metric $F = F(x, y)$ on an open subset $\mathcal{U} \subset \mathbb{R}^m$ is *dually flat* if and only if it satisfies (1.1) where $x = (x^1, \ldots, x^m) \in \mathcal{U}$ and $y = y^j(\partial/\partial x^j)|_x \in T_x\mathcal{U}$ [2].

3. Solutions from projectively flat metrics

In this section, we give an approach to manufacture solutions of (1.1) from projectively flat Finsler metrics in the form

$$\Phi(x, y) = |y|\left[\epsilon + \phi\left(|x|, \frac{\langle x, y\rangle}{|y|}\right)\right]$$

where $\epsilon > 0$.

Recall that a Finsler metric $F = F(x, y)$ on an open subset $\mathcal{U} \subset \mathbb{R}^n$ is said to be *projectively flat* if all geodesics are straight in $\mathcal{U}$.

Proposition 3.1. *Let $\Phi(x, y) := |y|[\epsilon + \phi(|x|, \langle x, y\rangle/|y|)]$ be a projectively flat Finsler metric on an open subset $\mathcal{U} \subset \mathbb{R}^n$. Then the following function on $T\mathcal{U}$*

$$F(x, y) = |y|\left[\psi\left(|x|, \frac{\langle x, y\rangle}{|y|}\right)\right]^{1/2} \tag{3.1}$$

is a solution of (1.1), *where ψ is given in* (3.7).

Proof. By Hamel Lemma (see (1.2) or [11]), Φ is projectively flat if and only if it satisfies $\Phi_{x^j y^i} y^j = \Phi_{x^i}$. Together with Theorem 2.3 we have

$$2L := F^2 = \Phi_{x^j} y^j \tag{3.2}$$

satisfies (1.1).

Now let us compute $\Phi_0 := \Phi_{x^i} y^i$ and F. Denote Φ by

$$\Phi = \Phi(r, s),$$

where

$$r = |x|, \quad s = \frac{\langle x, y\rangle}{|y|}. \tag{3.3}$$

By straightforward computations one obtains

$$\frac{\partial r}{\partial x^i} = \frac{x^i}{r}, \quad \frac{\partial s}{\partial x^i} = \frac{y^i}{|y|}. \tag{3.4}$$

It follows that

$$\begin{aligned} \Phi_{x^i} &= \frac{\partial}{\partial x^i}\left[|y|\phi\left(|x|, \frac{\langle x, y\rangle}{|y|}\right)\right] \\ &= |y|\left(\frac{\partial \phi}{\partial r}\frac{\partial r}{\partial x^i} + \frac{\partial \phi}{\partial s}\frac{\partial s}{\partial x^i}\right) = |y|\left(\frac{x^i}{r}\frac{\partial \phi}{\partial r} + \frac{y^i}{|y|}\frac{\partial \phi}{\partial s}\right). \end{aligned} \tag{3.5}$$

Contracting (3.5) with y^i yields

$$\Phi_{x^i} y^i = |y|^2 \psi\left(|x|, \frac{\langle x, y\rangle}{|y|}\right), \tag{3.6}$$

where we have used (3.2) and ψ is defined by

$$\psi(r, s) := \frac{\partial \phi}{\partial s} + \frac{s}{r}\frac{\partial \phi}{\partial r}. \tag{3.7}$$

From (3.2), (3.6) and (3.7), one obtains

$$\begin{aligned} F &= \sqrt{2L} \\ &= \left[|y|^2\left(\frac{\partial \phi}{\partial s} + \frac{s}{r}\frac{\partial \phi}{\partial r}\right)\right]^{1/2} = |y|\left[\psi\left(|x|, \frac{\langle x, y\rangle}{|y|}\right)\right]^{1/2} \end{aligned} \tag{3.8}$$

which completes the proof of Proposition 3.1. □

Taking $\phi(r, s) = \kappa + r^{\mu} f(s/r)$ in Proposition 3.1 where κ and μ are constants, we have

$$\frac{\partial\phi}{\partial s} = r^{\mu-1} f'(\lambda), \tag{3.9}$$

$$\frac{\partial\phi}{\partial r} = \mu r^{\mu-1} f(\lambda) - s r^{\mu-2} f'(\lambda), \tag{3.10}$$

where

$$\lambda = \frac{s}{r} = \frac{\langle x, y\rangle}{|x|\,|y|}. \tag{3.11}$$

Plugging (3.9) and (3.10) into (3.8) we obtain the following formula for F

$$\begin{aligned} F &= |y|\left(\frac{\partial\phi}{\partial s} + \lambda\frac{\partial\phi}{\partial r}\right)^{1/2} \\ &= |y|\{|x|^{\mu-1}[\mu\lambda f(\lambda) + (1-\lambda^2) f'(\lambda)]\}^{1/2}. \end{aligned}$$

Hence we obtain the following:

Corollary 3.2. *Let $\Phi(x, y) := |y|[\epsilon + |x|^{\mu} f(\langle x, y\rangle/(|x|\,|y|))]$ be a projectively flat Finsler metric on an open subset $\mathcal{U} \subset \mathbb{R}^n \setminus \{0\}$. Then the following function on $T\mathcal{U}$*

$$F(x, y) := |y|\{|x|^{\mu-1}[\mu\lambda f(\lambda) + (1-\lambda^2) f'(\lambda)]\}^{1/2}$$

is a solution of (1.1) *where $\lambda = \langle x, y\rangle/(|x|\,|y|)$.*

4. New dually flat Finsler metrics

In this section we are going to produce new dually flat Finsler metrics from a given projectively flat Finsler metric.

Lemma 4.1. *Let $\Phi(x, y) := |y|[\epsilon + |x|^{\mu} f(\langle x, y\rangle/(|x|\,|y|))]$ be a projectively flat Finsler metric on an open subset $\mathcal{U} \subset \mathbb{R}^n \setminus \{0\}$. Suppose that $f(-1) < 0$. Then*

$$F(x, y) := |y||x|^{(\mu-1)/2}[\mu\lambda f(\lambda) + (1-\lambda^2) f'(\lambda)]^{1/2}$$

is dually flat Finsler metric where $\mu > 0$.

Proof. In fact, F is expressed in the form

$$F = |y|\phi(r, s), \quad r = |x|, \quad s = \frac{\langle x, y\rangle}{|y|}$$

where

$$\phi = r^{(\mu-1)/2}\sqrt{\mu\lambda f(\lambda) + (1-\lambda^2) f'(\lambda)}, \tag{4.1}$$

and λ satisfies (3.11). Further, F satisfies (1.1) by Corollary 3.2. It is known that $F = |y|\phi(r,s)$ is a Finsler metric with $r < b_0$ if and only if ϕ is a positive function satisfying

$$\phi(s) - s\phi_s(s) > 0, \quad \phi(s) - s\phi_s(s) + (r^2 - s^2)\phi_{ss}(s) > 0, \quad |s| \le r < b_0$$

where $n \ge 3$ or

$$\phi(s) - s\phi_s(s) + (r^2 - s^2)\phi_{ss}(s) > 0, \quad |s| \le r < b_0$$

where $n = 2$ [18, Proposition 3.3]. Note that Φ is projectively flat. From [7], we have

$$(\lambda^2 - 1) f'' - \mu\lambda f' + \mu f = 0 \tag{4.2}$$

and

$$f''(\lambda) = \mu(1-\lambda^2)^{\mu/2-1}. \tag{4.3}$$

Differentiating (4.1) with respect to s, we obtain

$$\begin{aligned} 2\phi\phi_s &= r^{\mu-1}\frac{\partial\lambda}{\partial s}[\mu f + \mu\lambda f' - 2\lambda f' + (1+\lambda^2) f''] \\ &= r^{\mu-2}[\mu f + \mu\lambda f' - 2\lambda f' + \mu(f - \lambda f')] = 2r^{\mu-2}(\mu f - \lambda f') \end{aligned}$$

where we have used (3.11) and (4.2). It follows that

$$\phi_s = \frac{r^{\mu-2}(\mu f - \lambda f')}{\phi}. \tag{4.4}$$

Together with (4.1) and (3.11), we have

$$\phi - s\phi_s = \frac{1}{\phi}[\phi^2 - sr^{\mu-2}(\mu f - \lambda f')] = \frac{r^{\mu-1}}{\phi} f'. \tag{4.5}$$

Differentiating (4.4) with respect to s and using (4.4) one deduces

$$\phi_{ss} = \frac{(\mu-1)r^{\mu-3}}{\phi} f' - \frac{\lambda r^{\mu-3}}{\phi} f'' - \frac{r^{2\mu-4}}{\phi^3}(\mu f - \lambda f')^2.$$

Together with (3.11) and (4.2), we obtain

$$(r^2 - s^2)\phi_{ss} = \frac{r^{\mu-1}}{\phi}[\mu - (1-\lambda^2)] f' - \lambda\mu\frac{r^{\mu-1}}{\phi} f - \frac{r^{2\mu-2}}{\phi^3}(1-\lambda^2)(\mu f - \lambda f')^2.$$

Combining this with (4.5), we get

$$
\begin{aligned}
&\phi(s) - s\phi_s(s) + (r^2 - s^2)\phi_{ss}(s)\\
&= \frac{r^{\mu-1}}{\phi}[(\mu+\lambda^2)f' - \lambda\mu f] - \frac{r^{2\mu-2}}{\phi^3}(1-\lambda^2)(\mu f - \lambda f')^2\\
&= \frac{r^{2\mu-2}}{\phi^3}\times (I)
\end{aligned}
\tag{4.6}
$$

where

$$
\begin{aligned}
(I) &= [\mu\lambda f + (1-\lambda^2)f'][(\mu+\lambda^2)f' - \lambda\mu f]\\
&\quad - (1-\lambda^2)(\mu^2 f^2 - 2\mu\lambda f f' + \lambda^2 f'^2)\\
&= \mu[(1-\lambda^2)f'f' + (1+\mu)\lambda f f' - \mu f^2].
\end{aligned}
\tag{4.7}
$$

Plugging (4.7) into (4.6) yields

$$
\begin{aligned}
&\phi(s) - s\phi_s(s) + (r^2 - s^2)\phi_{ss}(s)\\
&= \frac{\mu r^{2\mu-2}}{\phi^3}[(1-\lambda^2)f'f' + \lambda(1+\mu)f f' - \mu f^2].
\end{aligned}
\tag{4.8}
$$

By (4.1), (4.5) and (4.8), $F = |y|\phi(r, s)$ is a Finsler metric if and only if

$$ f' > 0, \tag{4.9} $$

$$ g := \mu\lambda f + (1-\lambda^2)f' > 0, \tag{4.10} $$

$$ h := (1-\lambda^2)f'f' + \lambda(1+\mu)f f' - \mu f^2 > 0 \tag{4.11} $$

where $n \geq 3$ or, (4.10) and (4.11) hold when $n = 2$. By using (3.11) and Cauchy–Buniakowski inequality we are going to find conditions on f for (4.9), (4.10) and (4.11) to hold in $[-1, 1]$.

Note that $\mu > 0$. Together with (4.3) we get

$$ f''(\lambda) > 0 \tag{4.12} $$

where $\lambda \in (-1, 1)$. It follows that $f'(\lambda)$ is a monotonically increasing function on $[-1, 1]$. Thus

$$ f'(-1) > 0 \tag{4.13} $$

implies that (4.9) holds in $[-1, 1]$. Plugging (4.3) into (4.2) yields

$$ \mu(f - \lambda f') = (1-\lambda^2)f'' = (1-\lambda^2)\mu(1-\lambda^2)^{\mu/2-1} = \mu(1-\lambda^2)^{\mu/2}. $$

It follows that

$$f - \lambda f' = (1 - \lambda^2)^{\mu/2},$$

which immediately implies that

$$f(1) = f'(1), \quad f(-1) = -f'(-1). \tag{4.14}$$

This means that (4.9) holds in $[-1, 1]$ if $f(-1) < 0$.

Next, we are going to a find condition on f for (4.10) to hold in $[-1,1]$. By using (4.2) and (4.10), we have

$$g'(\lambda) = 2[\mu f(\lambda) - \lambda f'(\lambda)].$$

It follows that $g'(\lambda) = 0$ if and only if

$$\mu f(\lambda) = \lambda f'(\lambda). \tag{4.15}$$

Suppose that $\lambda_0 \in [-1, 1]$ such that $g'(\lambda_0) = 0$. Combining this with (4.15), we have

$$\mu f(\lambda_0) = \lambda_0 f'(\lambda_0). \tag{4.16}$$

Together with (4.10) we get

$$g(\lambda_0) = \mu\lambda_0 f(\lambda_0) + (1 - \lambda_0^2) f'(\lambda_0) = f'(\lambda_0). \tag{4.17}$$

On the other hand, from (4.10) and (4.14), we obtain

$$g(1) = \mu f'(1), \quad g(-1) = \mu f'(-1). \tag{4.18}$$

It is known that the minimum of g satisfies the following

$$\min_{\lambda\in[-1,1]} g(\lambda) = \min\{g(\lambda_0), g(\pm 1) \mid g'(\lambda_0) = 0\}.$$

It is easy to see that (4.10) holds for $\lambda \in [-1, 1]$ if and only if

$$\min_{\lambda\in[-1,1]} g(\lambda) > 0. \tag{4.19}$$

By (4.17) and (4.18), (4.19) holds if and only if

$$\min\{\mu f'(-1), f'(\lambda_0), \mu f'(1)\} > 0 \tag{4.20}$$

where $\lambda_0 \in [-1, 1]$ satisfying $g'(\lambda_0) = 0$. Note that $\mu > 0$ and f' is a monotonically increasing function. Together with the second equation of (4.14), we obtain that (4.20) holds if and only if $f(-1) < 0$.

Finally, we are going to find a condition on f for (4.11) to hold in $[-1, 1]$. Using (4.2) and (4.11) we get

$$\begin{aligned} h'(\lambda) &= (1-\mu) f'(f - \lambda f') + (1+\mu)\lambda f f'' + 2(1-\lambda^2) f' f'' \\ &= (1-\mu) f' \frac{(1-\lambda^2) f''}{\mu} + (1+\mu)\lambda f f'' + 2(1-\lambda^2) f' f'' \\ &= \frac{\mu+1}{\mu} f''[\mu\lambda f + (1-\lambda^2) f']. \end{aligned} \tag{4.21}$$

By (4.11) and the second equation of (4.14), we see that

$$h(-1) = -(1+\mu) f(-1) f'(-1) - \mu[f(-1)]^2 = [f(-1)]^2. \tag{4.22}$$

Suppose that $f(-1) < 0$. Together with (4.22) yields

$$h(-1) > 0. \tag{4.23}$$

Moreover, (4.10) holds where $\lambda \in [-1, 1]$. Combining this with (4.12) and (4.21), we have

$$h'(\lambda) > 0, \quad \lambda \in (-1, 1).$$

It follows that $h(\lambda)$ is a monotonically increasing function. Together with (4.23) ones obtain that (4.11) is true. □

In [7], authors gave an explicit construction of projectively flat spherically symmetric Finsler metric. Precisely, they have proved the following:

Proposition 4.2. *Let $f(\lambda)$ be a polynomial function defined by*

$$f(\lambda) = 1 + \delta\lambda + 2n \sum_{k=0}^{n-1} \frac{(-1)^k C_{n-1}^k \lambda^{2k+2}}{(2k+1)(2k+2)}. \tag{4.24}$$

Then the following Finsler metric on open subset in $\mathbb{R}^n \setminus \{0\}$

$$\Phi = |y| \left\{ \epsilon + |x|^{2n} f\left(\frac{\langle x, y\rangle}{|x|\,|y|} \right) \right\}$$

is projectively flat where $\epsilon > 0$.

Proof of Theorem 1.1. Combine Lemma 4.1 with Proposition 4.2. □

REMARK. We also obtain some other dually flat Finsler metrics by Proposition 4.5 and Theorem 1.1 in [7] and Lemma 4.1.

ACKNOWLEDGMENT. We thank the referee for pointing out the following interesting problem: study dimensions of moduli spaces of dually flat Finsler metrics and projectively flat Finsler metrics.

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