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MAXIMAL TORI OF EXTRINSIC SYMMETRIC SPACES AND MERIDIANS

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Abstract

We give a different proof of a theorem of O. Loos [5] which characterizes maximal tori of extrinsically symmetric spaces. On the way we show some facts on certain symmetric subspaces, so called meridians, which previously have been known only using classification.

1. Introduction

Our aim in this paper is to give a geometric proof of the following theorem of Ottmar Loos [5]:

Theorem 1. *Let $X \subset \mathbb{R}^n$ be a compact extrinsically symmetric space. Then a maximal torus of X is rectangular, i.e. a Riemannian product of circles.*

Recall that a submanifold $X \subset \mathbb{R}^n$ is called *extrinsically symmetric* if it is fixed by the euclidean reflection s_x across the affine normal space $x + N_x X$ (where $N_x X = (T_x X)^{\perp}$) for every $x \in X$. Then s_x preserves the second fundamental form $\alpha\colon S^2(T_x X) \to N_x X$ and its derivative $\nabla\alpha : S^3(T_x X) \to N_x X$ which must be zero since s_x changes the sign on $S^3(T_x X)$ but not on $N_x X$. Clearly, an extrinsically symmetric submanifold with its induced Riemannian metric is a symmetric space. In particular it is an orbit of a subgroup G of the isometry group of $\mathbb{R}^n$. If X is compact, its center of mass is fixed by G. Choosing the origin at the center of mass, X is contained in a sphere around the origin, and $G \subset O_n$. It has been shown by Ferus ([4], also cf. [3]) that compact extrinsically symmetric submanifolds are particular orbits of the isotropy representation of other symmetric spaces. However we will not make use of this classification.

To prove Theorem 1, we will show the following alternative for any compact symmetric space X: Either X is a Riemannian product of euclidean spheres and flat tori or it contains a certain totally geodesic subspace, a so called *meridian* (Theorem 10). Further, a meridian M of a compact symmetric space X has the same rank as X (Theorem 5), and when X is extrinsically symmetric, then so is M (Corollary 6). Thus

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passing to meridians again and again, we will lower the dimension while preserving a maximal torus until we reach an extrinsic symmetric space which is a Riemannian product of spheres and flat tori. By Theorem 4, a maximal torus of such a space is rectangular which finishes the proof of Theorem 1.

2. Extrinsic reflective subspaces

We start with a simple principle how to reduce the dimension of an extrinsic symmetric space $X \subset \mathbb{R}^n$. An *extrinsic reflective subspace* of X is a connected component of the fixed point set of $r|_X$ where r is an isometry of $\mathbb{R}^n$ of order 2 (*extrinsic reflection*) with $r(X) = X$.

Theorem 2. *Let $X \subset \mathbb{R}^n$ be an extrinsic symmetric space. Then any extrinsic reflective subspace is again extrinsic symmetric.*

Proof. Let $Y \subset X$ be a connected component of the fixed point set of an extrinsic reflection $r \in O_n$. Let $V_\pm$ be the (± 1)-eigenspaces of r. Then $\mathbb{R}^n = V_+ \oplus V_-$, and Y is a connected component of $X \cap V_+$. Since $r|_X$ is an isometry, Y is totally geodesic in X. For any $y \in Y$, the symmetry s_y of X at y extends to the ambient space. Moreover, s_y commutes with r since y is fixed by r and hence $rs_yr = s_{ry} = s_y$. Thus s_y preserves the eigenspace V_+ of r and decomposes V_+ into eigenspaces, $V_+ = V_+^+ \oplus V_+^-$. Since $V_+^+ \subset V_+ \cap N_yX$ and $V_+^- \subset V_+ \cap T_yX$, we have equality in both inclusions and in particular $V_+^- = V_+ \cap T_yX = T_yY$. Thus $Y \subset V_+$ is extrinsic symmetric. □

3. Extrinsic symmetric products of tori and spheres

Theorem 3 ([4, Theorem 3])**.** *Let $F \subset \mathbb{R}^n$ be an extrinsic symmetric flat torus. Then F splits extrinsically as a product of round circles, $F = \mathbb{S}^1_{r_1} \times \cdots \times \mathbb{S}^1_{r_m} \subset \mathbb{R}^{2m} \subset \mathbb{R}^n$.*

Proof. We have an isometric immersion $f : \mathbb{R}^m \to \mathbb{R}^n$ with $f(\mathbb{R}^m) = F$, namely the universal covering of the torus F. Its partial derivatives f_i are parallel vector fields on F, and since the second fundamental form $\alpha : S^2TF \to NF$ is also parallel, the normal vectors $\alpha_{ij} = \alpha(f_i, f_j)$ are parallel, too. Thus the normal bundle is parallel and by Ricci equation, the corresponding shape operators $A_{\alpha_{ij}}$ commute with each other. Therefore they allow for a common parallel eigenspace decomposition $TF = E_1 \oplus \cdots \oplus E_r$. By compactness, each maximal integral leaf of the E_j is a sphere, but since F is flat, its dimension must be one. Thus F is a product of perpendicular circles $\mathbb{S}^1_{r_j} \subset \mathbb{R}^2_j \subset \mathbb{R}^n$. □

Theorem 4. *Let $X \subset \mathbb{R}^n$ be an extrinsic symmetric space which is intrinsically a Riemannian product*

$$(*) \qquad X = S_1 \times \cdots \times S_k \times F$$

where the S_i are round spheres and F is a flat torus. Then a maximal torus of X is rectangular.

Proof. We proceed by induction over k. For $k = 0$ the statement follows from Theorem 3. Now let $k \geq 1$. First we can split off every even-dimensional sphere factor. In fact, suppose that S_1 is even dimensional. Put $X' = S_2 \times \cdots \times S_k \times F$. Choose $p_1 \in S_1$ and let s_1 be the symmetry of S_1 at p_1. Then $\{p_1\} \times X'$ is a fixed component of the involution $s_1 \times \mathrm{id}$. Since S_1 is an even dimensional sphere, s_1 belongs to the transvection group of S_1 and $s_1 \times \mathrm{id}$ lies in the transvection group of X. Hence it extends to a reflection of the ambient space. Thus its fixed component $\{p_1\} \times X'$ is a reflective submanifold of X, and hence it is extrinsically symmetric in the fixed space of $s_1 \times \mathrm{id}$ (see Theorem 2). By induction hypothesis, X' has a rectangular maximal torus, and thus the same property follows for $X = S_1 \times X'$.

Now we may assume that every sphere factor is odd-dimensional. We choose great circles C_i in every S_i; then a maximal torus of X is $\hat{F} = C_1 \times \cdots \times C_k \times F$. Let r_i be the reflection across C_i in S_i. Since S_i is odd dimensional, r_i lies in the transvection group of S_i. Thus $r = r_1 \times \cdots \times r_k \times \mathrm{id}$ lies in the transvection group of X and has $\hat{F}$ as fixed set; since r extends to the ambient space, $\hat{F}$ is extrinsic symmetric (see Theorem 2) and we are done by Theorem 3 on extrinsic symmetric tori. □

REMARK. Though all factors of the product $(*)$ are extrinsic symmetric, we were not able to conclude that the splitting is extrinsic. In fact, for local products this is false as shown by the extrinsic symmetric space $(\mathbb{S}^p \times \mathbb{S}^q)/\pm\mathrm{id} = \mathbb{S}^p \otimes \mathbb{S}^q \subset \mathbb{R}^{p+1} \otimes \mathbb{R}^{q+1}$. However, for global products it follows from the classification of extrinsic symmetric spaces.

4. Polars and meridians

Recall from [1] that a *polar* of a point o in a symmetric space X is a positive dimensional connected component of the fixed set of the symmetry s_o of X while an isolated fixed point of s_o is called a *pole* of o. Polars and poles can also be characterized as certain orbits of K (the connected component of the isotropy group of o) through a fixed point of s_o. Elements of polars as well as poles are midpoints of closed geodesics starting and ending at o (see [1]). By definition a polar P is reflective, being a component of the fixed set of s_o. Through any $p \in P$ there is an orthogonal complementary reflective submanifold M which is the connected component through p of the fixed set of the involution $s_o s_p$ (note that s_o and s_p commute). This M is called a *meridian* of X (see [1]).

Theorem 5 ([1, Lemma 2.3], [7, Theorem 1.8])**.** *If X is a compact symmetric space of rank k, any meridian $M \subset X$ has the same rank k.*

Proof. Let $P \subset X$ be a polar corresponding to the base point $o \in X$. Consider a geodesic segment γ from o to $p \in P$. By the first variation formula, γ meets $P = Kp$

perpendicularly at p, since all $k\gamma$, $k \in K$, start at o and end in P and have the same length as γ. Therefore γ lies in M. It extends beyond p to a geodesic circle starting and ending at o with midpoint p. Let F be a maximal torus of X with $\gamma \subset F$. Then p is a pole or an element of a polar not only in X but also in F. Since a torus does not have polars, p is a pole for o in F. In other words, $s_p = s_o$ along F. Thus F belongs to the fixed component of $s_o s_p$ through p; this is the meridian M. □

If $X \subset \mathbb{R}^n$ happens to be extrinsic symmetric and $P, M \subset X$ denote a polar and a meridian of $o \in X$ through a common point $p \in X$, then s_o, s_p extend to commuting reflections on the ambient space $\mathbb{R}^n$. Thus we conclude immediately from Theorem 2:

Corollary 6. *A polar P and a meridian M in an extrinsically symmetric space X are extrinsically symmetric.*[1]

Any compact symmetric space X is finitely covered by a Riemannian product $\tilde{X} = Y \times F$ where Y is a simply connected symmetric space of compact type and F a torus. When there are no polars, the covering is trivial:

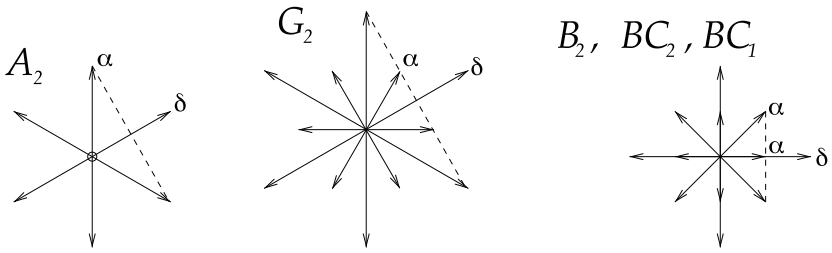
Lemma 7. *If X has no polars, then X itself is a Riemannian product of a simply connected symmetric space of compact type and possibly a torus.*

Proof. Let $\tilde{o}, \tilde{o}' \in \tilde{X}$ be two points in a fibre of the projection $\pi\colon \tilde{X} \to X$. Let $\tilde{\gamma}\colon [0,1] \to \tilde{X}$ be a shortest geodesic in $\tilde{X}$ joining $\tilde{o}$ to $\tilde{o}'$. Then $\gamma = \pi \circ \tilde{\gamma}$ is a closed geodesic starting and ending at $o = \pi(\tilde{o}) = \pi(\tilde{o}')$. Its midpoint $p = \gamma(1/2)$ is either a pole or an element of a polar of X, but the second case is excluded by our assumption. Let K be the connected isotropy subgroup at $\tilde{o}$ on $\tilde{X}$ (which is the same as that of X at o). Applying K to $\tilde{\gamma}$ we obtain a variation of shortest geodesics $\{k\tilde{\gamma}\colon k \in K\}$ from $\tilde{o}$ to $\tilde{o}'$. All $k\tilde{\gamma}$ pass through $\tilde{p} = \tilde{\gamma}(1/2)$ which is fixed by K since $p = \pi(\tilde{p})$ is a pole. Since $\tilde{\gamma}|_{[0,1)}$ has no conjugate points, being a shortest geodesics, we conclude that $\tilde{\gamma}$ is fixed by K. Hence the initial vector $\tilde{\gamma}'(0)$ is a fixed vector of K, thus tangent to the torus factor F. This shows that the covering map π can be nontrivial only on the torus factor. Since a symmetric space covered by a torus is a torus again, we have proved our claim. □

Lemma 8. *Let $\Sigma \subset V^*$ be an irreducible root system* (*not necessarily reduced, i.e. the BC-type is allowed*) *with $\Sigma \neq A_1$. Let $\delta \in \Sigma$ be any of the longest roots. Let*

[1]This theorem was mentioned already in [8], Lemma 3.1.

$H \in V$ with $H \perp \ker\delta$ and $\delta(H) = 2$. Then $\{\pm 1, \pm 2\} \subset \{\alpha(H)\colon \alpha \in \Sigma\} \subset \{0, \pm 1, \pm 2\}$.



Proof. Since only two roots δ, α are involved, it suffices to look at the simple root systems of dimension ≤ 2 where the statement is obvious from the figures. $\square$

Theorem 9. *Let X be an irreducible simply connected compact symmetric space without a polar. Then X is a round sphere.*

Proof. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of X at some point o; we view $\mathfrak{p}$ as the tangent space T_oX. Choose a maximal flat subspace $\mathfrak{a} \subset \mathfrak{p}$ and let $\Sigma \subset \mathfrak{a}^*$ be the corresponding root system. Let $\delta \in \Sigma$ be any of the longest roots and $H \in \mathfrak{a}$ with $H \perp \ker\delta$ and $\delta(H) = 2$. Then the geodesic $\gamma(t) = \exp(tH)o$ in X lies in the totally geodesic rank-one subspace $P_\delta \subset X$ with root system $\Sigma \cap \mathbb{R}\delta$, hence it is closed. Since X is simply connected, its cut locus and its first conjugate locus are the same, by a well known theorem of Crittenden [2]. The first conjugate point along γ can be easily computed from the curvature tensor of X: We have the orthogonal decomposition

$$\mathfrak{p} = \sum_{\alpha \in \Sigma \cup \{0\}} \mathfrak{p}_\alpha,$$

and for any $X_\alpha \in \mathfrak{p}_\alpha$,

$$R(X_\alpha, H)H = -\operatorname{ad}(H)^2 X_\alpha = \alpha(H)^2 X_\alpha.$$

Thus each X_α is an eigenvector of the Jacobi operator $R(\,\cdot\,, H)H$ with eigenvalue 4 (corresponding to X_δ) or 1 or 0, according to Lemma 8; for the root system A_1 we have $\alpha = \delta$ and the only eigenvalue is 4. Thus the first conjugate point along γ both in P_δ and in X occurs at $\pi/2$, the first zero of $f(t) = \sin(2t)$ solving $f'' + 4f = 0$. The midpoint $p = \gamma(t_o)$ of the simply closed geodesic γ in the rank-one symmetric space P_δ occurs not later than the first conjugate point, hence $t_o \leq \pi/2$ in our case (in fact $t_o = \pi/4$ when $P_\delta \cong \mathbb{RP}^k$ with curvature 4 while $t_o = \pi/2$ in all other cases). Therefore γ has period $2t_o \leq \pi$. Since the midpoint of a closed geodesic starting and ending at o is fixed by s_o, it is a pole or an element of a polar. In our case polars are excluded, so it must be a pole. Hence all Killing fields along γ vanishing at $o = \gamma(0)$ also vanish at $p = \gamma(t_o)$. But when Σ contains a root α with $\alpha(H) = 1$, then

$R(X_\alpha, H)H = X_\alpha$. Thus the Jacobi field J along γ with $J(0) = 0$ and $J'(0) = X_\alpha$ does not vanish before $t = \pi$. This is the Killing field corresponding to $Y_\alpha \in \mathfrak{k}$ where $Z_\alpha = X_\alpha + \sqrt{-1}Y_\alpha$ is the complex root vector, that is $[H, Z_\alpha] = \sqrt{-1}\alpha(H)Z_\alpha$ for all $H \in \mathfrak{a}$. Thus we have got a Killing field vanishing at o but not at $\gamma(t_o)$, a contradiction. Hence such a root α cannot exist, and by Lemma 8 the whole root system of X must be of type A_1. Consequently, X (being simply connected) is a sphere. □

Theorem 10. *Let X be a compact symmetric space without a polar. Then X is the Riemannian product of round spheres and tori.*

Proof. By Lemma 7, we have $X = Y \times F$ where F is a flat torus and Y a simply connected symmetric space of compact type. Thus Y splits into irreducible simply connected factors each of which is a round sphere by Theorem 9. □

By the classification of polars (see [1, 6]) this result is known, but here we have given a conceptional proof.

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