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REMARKS ON "THE DORFMEISTER–NEHER THEOREM ON ISOPARAMETRIC HYPERSURFACES"

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Abstract

Sections 7 and 8 of "*The Dorfmeister–Neher theorem on isoparametric hypersurfaces*", (Osaka J. Math. **46**, 695–715) are the heart of the paper, but a lack of clear argument causes some questions, although the statement is true. The purpose of the present paper is to make it clear.

1. Dim $E = 2$ (§7 [2])

We follow the notation and the argument in [2]. First, we correct a typo in the last term of the displayed formula right above (35) of [2]: $(\Lambda_{63}^3)^2$ should be $(\Lambda_{63}^4)^2$.

We call a vector field $v(t)$ along L_6 parametrized by $p(t)$ *even* when $v(t+\pi) = v(t)$, and *odd* when $v(t+\pi) = -v(t)$. Note that E consists of $\nabla_{e_6}^k e_3(t)$, $k = 0, 1, \dots$ which are all odd or all even, and W consists of $\nabla_{e_6}^k \nabla_{e_3} e_6(t)$ of which evenness and oddness is the opposite of E, since $L(t+\pi) = -L(t)$.

Proposition 7.1 ([2]) dim $E = 2$ *does not occur at any point of* M_+.

Proof. $\dim E = 2$ implies $\dim W = 1$, and so W consists of even vectors ($\nabla_{e_3} e_6$ never vanish by Remark 5.3 of [2]). Thus E consists of odd vectors. For X_1, Z_1, X_2, Z_2 on p. 709, X_1 is parallel to $\nabla_{e_6} e_3$ at $p_0 = p(0)$ and $p(\pi)$, and so has opposite sign at $p(0)$ and $p(\pi)$. Note that $Z_1 \in W$ is a constant unit vector parallel to $\nabla_{e_3} e_6(t)$. Also, span$\{X_2, Z_2\}$ is parallel since this is the orthogonal complement of $E \oplus W$. Because $D_1(\pi) = D_5(0)$ and $D_2(\pi) = D_4(0)$ etc. hold, four cases occur;

$$(e_1 + e_5)(\pi) = (e_1 + e_5)(0) \quad \text{and} \quad (e_2 + e_4)(\pi) = (e_2 + e_4)(0),$$

$$(e_1 + e_5)(\pi) = (e_1 + e_5)(0) \quad \text{and} \quad (e_2 + e_4)(\pi) = -(e_2 + e_4)(0),$$

$$(e_1 + e_5)(\pi) = -(e_1 + e_5)(0) \quad \text{and} \quad (e_2 + e_4)(\pi) = (e_2 + e_4)(0),$$

$$(e_1 + e_5)(\pi) = -(e_1 + e_5)(0) \quad \text{and} \quad (e_2 + e_4)(\pi) = -(e_2 + e_4)(0).$$

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In the first case, $\alpha(\pi) = -\alpha(0)$ and $\beta(\pi) = -\beta(0)$ follow. Then X_2 becomes even and Z_2 becomes odd, which contradicts that span$\{X_2, Z_2\}$ is parallel. In the second case, $\alpha(\pi) = -\alpha(0)$ and $\beta(\pi) = \beta(0)$ hold, and so X_2 is odd, and Z_2 is even, again a contradiction. Other cases are similar. □

2. Dim $E = 3$ (§8 [2])

When $\dim E = 3$, $e_3(t)$ is an even vector, since E is parallel along L_6. Using Proposition 8.1 [2], we extend e_1, e_2, e_4, e_5 as follows: Taking the double cover $\tilde{c}(t)$ of $c(t)$, i.e., $t \in [0, 4\pi)$, if necessary, we choose a differentiable frame $e_i(t)$ as follows: First take $e_1(t)$, $e_2(t)$ continuously for $t \in [0, 4\pi)$. Then we define $e_5(t) = e_1(t + \pi)$ and $e_4(t) = e_2(t + \pi)$ for $t \in [0, 3\pi)$. Thus we have a differentiable frame $e_i(t)$ for $t \in [0, 3\pi)$, though we only need $t \in [0, 2\pi]$.

With respect to this frame, we can take a differentiable orthonormal frame of E and $E^\perp$ by

(1)
$$
\begin{aligned}
&e_3(t), \quad X_1 = \alpha(t)(e_1 + e_5)(t) + \beta(t)(e_2 + e_4)(t),\\
&X_2(t) = \frac{1}{\sqrt{\sigma(t)}}\left(\frac{\beta(t)}{\sqrt{3}}(e_1 - e_5)(t) - \sqrt{3}\alpha(t)(e_2 - e_4)(t)\right)
\end{aligned}
$$

and

(2)
$$
\begin{aligned}
&Z_1(t) = \frac{1}{\sqrt{\sigma(t)}}\left(\sqrt{3}\alpha(t)(e_1 - e_5)(t) + \frac{\beta(t)}{\sqrt{3}}(e_2 - e_4)(t)\right),\\
&Z_2(t) = \beta(t)(e_1 + e_5) - \alpha(t)(e_2 + e_4)(t),
\end{aligned}
$$

where $\alpha(t)$, $\beta(t)$, $\sigma(t)$ are differentiable for $t \in [0, 3\pi]$, satisfying

(3)
$$
\alpha^2(t) + \beta^2(t) = \frac{1}{2}, \quad \sigma(t) = 2\left(3\alpha^2(t) + \frac{\beta^2(t)}{3}\right).
$$

Note that $\sigma(t) = \sigma(t + \pi)$ holds, since $\sigma(t)$ is an eigenvalue of $T(t) = {}^tRR(t)$ (see (45) [2] and the statement after it).

Proposition 8.2 ([2]) *$\sigma(t)$ is constant and takes values $1/3$ or 3.*

REMARK. We need not distinguish the case $\sigma = 1$ in the proof.

Proof of Proposition 8.2 ([2]). From (3), the conclusion follows if we show $\alpha(t)\beta(t) \equiv 0$. Suppose $\alpha(t)\beta(t) \not\equiv 0$. By definition, we have

(4)
$$
e_1(\pi) = e_5(0), \quad e_2(\pi) = e_4(0).
$$

We must be careful for

$$e_5(\pi) = e_1(2\pi) = \epsilon_1 e_1(0), \quad e_4(\pi) = e_2(2\pi) = \epsilon_2 e_2(0),$$

where $\epsilon_i = \pm 1$. However, since e_3 is even and by (4), we obtain

$$\epsilon := \epsilon_1 = \epsilon_2.$$

CASE 1 $\epsilon = 1$. In this case, we have

$$\begin{aligned} X_1(\pi) &= \alpha(\pi)(e_1(\pi) + e_5(\pi)) + \beta(\pi)(e_2(\pi) + e_4(\pi)) \\ &= \alpha(\pi)(e_5(0) + e_1(0)) + \beta(\pi)(e_4(0) + e_2(0)), \end{aligned} \tag{5}$$

which belongs to E, and is orthogonal to $e_3(0)$ and $X_2(0)$. Thus we obtain

$$X_1(\pi) = \bar{\epsilon} X_1(0), \quad \text{namely,} \quad \alpha(\pi) = \bar{\epsilon}\alpha(0), \quad \beta(\pi) = \bar{\epsilon}\beta(0), \tag{6}$$

where $\bar{\epsilon} = \pm 1$. On the other hand, we have

$$\begin{aligned} X_2(\pi) &= \frac{1}{\sqrt{\sigma(\pi)}}\left(\frac{\beta(\pi)}{\sqrt{3}}(e_1(\pi) - e_5(\pi)) - \sqrt{3}\alpha(\pi)(e_2(\pi) - e_4(\pi))\right) \\ &= \frac{1}{\sqrt{\sigma(0)}}\left(\frac{\beta(\pi)}{\sqrt{3}}(e_5(0) - e_1(0)) - \sqrt{3}\alpha(\pi)(e_4(0) - e_2(0))\right), \end{aligned} \tag{7}$$

where we use $\sigma(\pi) = \sigma(0)$. Thus from (6), we obtain

$$X_2(\pi) = -\bar{\epsilon} X_2(0).$$

However, because E is parallel, X_1 and X_2 should be both even or both odd, a contradiction.

CASE 2 $\epsilon = -1$. In this case, we have

$$\begin{aligned} X_1(\pi) &= \alpha(\pi)(e_1(\pi) + e_5(\pi)) + \beta(\pi)(e_2(\pi) + e_4(\pi)) \\ &= \alpha(\pi)(e_5(0) - e_1(0)) + \beta(\pi)(e_4(0) - e_2(0)), \end{aligned} \tag{8}$$

which belongs to E, and is orthogonal to $e_3(0)$ and $X_1(0)$. Thus we obtain

$$X_1(\pi) = \bar{\epsilon} X_2(0), \quad \text{namely,} \quad \alpha(\pi) = -\bar{\epsilon}\frac{\beta(0)}{\sqrt{3\sigma(0)}}, \quad \text{and} \quad \beta(\pi) = \bar{\epsilon}\frac{\sqrt{3}\alpha(0)}{\sqrt{\sigma(0)}}, \tag{9}$$

for $\bar{\epsilon} = \pm 1$. On the other hand, we see that

$$\begin{aligned} X_2(\pi) &= \frac{1}{\sqrt{\sigma(\pi)}}\left(\frac{\beta(\pi)}{\sqrt{3}}(e_1(\pi) - e_5(\pi)) - \sqrt{3}\alpha(\pi)(e_2(\pi) - e_4(\pi))\right) \\ &= \frac{1}{\sqrt{\sigma(0)}}\left(\frac{\beta(\pi)}{\sqrt{3}}(e_5(0) + e_1(0)) - \sqrt{3}\alpha(\pi)(e_4(0) + e_2(0))\right) \end{aligned} \tag{10}$$

where we use $\sigma(\pi) = \sigma(0)$. Because it belongs to E and is orthogonal to $e_3(0)$ and $X_2(0)$, and further because $(X_1(0), X_2(0)) \mapsto (X_1(\pi), X_2(\pi))$ should be orientation preserving, we obtain,

$$(11)\quad X_2(\pi) = -\bar{\epsilon} X_1(0), \quad \text{namely,} \quad \frac{\beta(\pi)}{\sqrt{3\sigma(0)}} = -\bar{\epsilon}\alpha(0) \quad \text{and} \quad -\frac{\sqrt{3}\alpha(\pi)}{\sqrt{\sigma(0)}} = -\bar{\epsilon}\beta(0).$$

However, then (9) and (11) have no solution. □

These contradictions are caused by the assumption $\alpha(t)\beta(t) \not\equiv 0$. Thus $\alpha(t)\beta(t) \equiv 0$ follows. Now, by the argument in §9 [2], we obtain

Theorem 2.1 ([1], [2]) *Isoparametric hypersurfaces with* $(g, m) = (6, 1)$ *are homogeneous.*

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References

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