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ON $H = 1/2$ SURFACES IN $\widetilde{PSL}_2(\mathbb{R}, \tau)$

CARLOS PEÑAFIEL

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Abstract

In this paper we prove that if $\Sigma$ is a properly embedded constant mean curvature $H = 1/2$ surface which is asymptotic to a horocylinder $C \subset \widetilde{PSL}_2(\mathbb{R}, \tau)$, in one side of $C$, such that the mean curvature vector of $\Sigma$ has the same direction as that of the $C$ at points of $\Sigma$ converging to $C$, then $\Sigma$ is a subset of $C$.

1. Introduction

In this paper we study complete constant mean curvature $H = 1/2$ surfaces immersed in $\widetilde{PSL}_2(\mathbb{R}, \tau)$. Recall that in [5] the authors generalized to $\mathbb{H}^2 \times \mathbb{R}$ the half-space theorem of Hoffman and Meeks which ensures that a properly immersed minimal surface in $\mathbb{R}^3$ that lies in a half-space must be a plane. The main theorem in [5] says that, if a properly embedded constant mean curvature $H = 1/2$ surface in $\mathbb{H}^2 \times \mathbb{R}$ which is asymptotic to a horocylinder $C$ and on one side of $C$; such that the mean curvature vector of the surface has the same direction as that of $C$ at points of the surface converging to $C$, then the surface is equal to $C$ (or a subset of $C$ if the surface has non-empty boundary).

We extend this result to the space $\widetilde{PSL}_2(\mathbb{R}, \tau)$. Remember that the space $\widetilde{PSL}_2(\mathbb{R}, \tau)$ is one of the eight Thurston’s geometries. Indeed it is well known there exists a classification due to W. Thurston of simply connected homogeneous 3-manifolds (see [8, Chapter eight]). Such a manifold has an isometry group of dimension 3, 4 or 6.

- When the manifold has 6-dimensional isometry group, we have the 3-dimensional space-forms: the Euclidean space $\mathbb{R}^3$, the Euclidean sphere $S^3(\kappa)$ (having sectional curvature $\kappa > 0$) and the hyperbolic space $\mathbb{H}^3(\kappa)$ (having sectional curvature $\kappa < 0$).
- When the manifold has 3-dimensional isometry group, we have the Lie group $Sol_3$.
- When the manifold has 4-dimensional isometry group (we label by $E(\kappa, \tau)$ these manifolds), there exists a Riemannian fibration over a 2-dimensional space form $M^2(\kappa)$.

The manifolds $E(\kappa, \tau)$ are classified, up to isometry, by the curvature $\kappa$ of the base surface and by the bundle curvature of the fibration $\tau$, where $\kappa$ and $\tau$ can be any real numbers satisfying $\kappa \neq 4\tau^2$. When $\tau = 0$ we have the metric product spaces $M^2(\kappa) \times \mathbb{R}$. When $\kappa = 0$ and $\tau \neq 0$ we have the 3-dimensional Heisenberg group. The

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space $\widetilde{PSL}_2(\mathbb{R}, \tau)$ is given when we consider $\tau \neq 0$ and $\kappa = -1$, that is $E(-1, \tau) = \widetilde{PSL}_2(\mathbb{R}, \tau)$.

We extend the aforementioned result to the space $\widetilde{PSL}_2(\mathbb{R}, \tau)$. In order to do that, note that, since exists a Riemannian submersion $P$ over the half-plane model for the 2-dimensional hyperbolic space $\mathbb{H}^2$, we call a horocylinder the inverse image $\pi^{-1}(h)$, where $h$ is a horocycle in $\mathbb{H}^2$. We also denote by $\partial$ the tangent field to the fibers on $\widetilde{PSL}_2(\mathbb{R}, \tau)$.

Let $C$ be a complete horocylinder in $\widetilde{PSL}_2(\mathbb{R}, \tau)$, we say that the surface $\Sigma$ is asymptotic to $C$ if $\Sigma$ contain an open subset $U \subset \Sigma$ (with $U \cap C = \emptyset$), such that, for each $\epsilon > 0$, there exists a compact set $K \subset U$, where the distance $d(p, C) < \epsilon$ for all $p \in (U - K)$, here $d(\cdot, \cdot)$ denotes the distance function in the space $\widetilde{PSL}_2(\mathbb{R}, \tau)$.

Following the same spirit as in [5], we show an analogous result in the space $\widetilde{PSL}_2(\mathbb{R}, \tau)$. More precisely, our main theorem is the following.

**Theorem 1.1.** Let $\Sigma$ be a properly embedded constant mean curvature $H = 1/2$ surface in $\widetilde{PSL}_2(\mathbb{R}, \tau)$. Suppose $\Sigma$ is asymptotic to a horocylinder $C$, and on one side of $C$. If the mean curvature vector of $\Sigma$ has the same direction as that of $C$ at points of $\Sigma$ converging to $C$, then $\Sigma$ is equal to $C$.

As a consequence of Theorem 1.1, we obtain (in the same sense as in [5]) the Theorem 1.2. Note that, the Theorem 1.2 is well known, see for instance [1] or [3, Corollary 4.6.3].

**Theorem 1.2.** Let $\Sigma$ be a complete immersed surface in $\widetilde{PSL}_2(\mathbb{R}, \tau)$ of constant mean curvature $H = 1/2$. If $\Sigma$ is transverse to the vertical Killing field $E_3 = \partial$, then $\Sigma$ is an entire vertical graph over $\mathbb{H}^2$.

Observe that the value $H = 1/2$ for constant mean curvature $H$ surfaces is special in the space $\widetilde{PSL}_2(\mathbb{R}, \tau)$. In fact, a constant mean curvature $H$ surface in the homogeneous space $E(\kappa, \tau)$ has critical constant mean curvature if the relation $H^2 = -\kappa/4$ holds. This terminology comes from the fact that it separates the case $H^2 > -\kappa/4$, in which compact constant mean curvature exists, from the case $H^2 < -\kappa/4$, in which no compact constant mean curvature can exists.

2. The space $\widetilde{PSL}_2(\mathbb{R}, \tau)$

The 3-dimensional space $\widetilde{PSL}_2(\mathbb{R}, \tau)$ is a complete homogeneous simply connected Riemannian manifold. Each such a manifold (depending on $\tau$) is the total space of a Riemannian submersion over the 2-dimensional hyperbolic space $\mathbb{H}^2$ (here the Gaussian
curvature of the hyperbolic space is \( \kappa = -1 \). The bundle curvature of the submersion is the number \( \tau \) such that \( \nabla_X E_3 = \tau X \times E_3 \) for any vector field \( X \) on \( \overline{PSL}_2(\mathbb{R}, \tau) \) (here \( \nabla \) denotes the Riemannian connection of \( \overline{PSL}_2(\mathbb{R}, \tau) \)). And each fiber is a complete geodesic tangent to a Killing field \( E_3 \). When \( \tau = 0 \), we obtain the space \( \overline{PSL}_2(\mathbb{R}, 0) \equiv \mathbb{H}^2 \times \mathbb{R} \).

From now on, we choice and fix a value for \( \tau \) different from zero. More precisely, the Riemannian manifold is \( (\overline{PSL}_2(\mathbb{R}, \tau), g) \), where \( \overline{PSL}_2(\mathbb{R}, \tau) \) is topologically \( \mathbb{H}^2 \times \mathbb{R} \) (\( \mathbb{R} \) the real line), that is \( \overline{PSL}_2(\mathbb{R}, \tau) = \{ (x, y, t) \in \mathbb{R}^3 : y > 0 \} \) endowed with the metric

\[
g = \lambda^2 (dx^2 + dy^2) + (-2\tau \lambda dx + dt)^2, \quad \lambda = \frac{1}{y}.
\]

There is a natural orthonormal frame \( \{ E_1, E_2, E_3 \} \) given by (in coordinates \( \{ \partial_x, \partial_y, \partial_t \} \))

\[
E_1 = \frac{\partial_x}{\lambda} + 2\tau \partial_t, \quad E_2 = \frac{\partial_y}{\lambda}, \quad E_3 = \partial_t.
\]

\( E_3 \) is the Killing field tangent to the fibers. The metric \( g \) induces a Riemannian connection \( \nabla \) given by

\[
\nabla_{E_1} E_1 = \frac{\lambda_y}{\lambda^2} E_2, \quad \nabla_{E_1} E_2 = \frac{\lambda_y}{\lambda^2} E_1 + \tau E_3, \quad \nabla_{E_1} E_3 = -\tau E_2,
\]

\[
\nabla_{E_2} E_1 = \frac{\lambda_x}{\lambda^2} E_2 - \tau E_3, \quad \nabla_{E_2} E_2 = -\frac{\lambda_x}{\lambda^2} E_1, \quad \nabla_{E_2} E_3 = \tau E_1,
\]

\[
\nabla_{E_3} E_1 = -\tau E_2, \quad \nabla_{E_3} E_2 = \tau E_1, \quad \nabla_{E_3} E_3 = 0.
\]

We also have

\[
[E_1, E_2] = \frac{\lambda_y}{\lambda^2} E_1 - \frac{\lambda_x}{\lambda^2} E_2 + 2\tau E_3, \quad [E_1, E_3] = 0, \quad [E_2, E_3] = 0.
\]

For more details see [6], [2], [8].

### 2.1. Graphs in \( \overline{PSL}_2(\mathbb{R}, \tau) \).

Now we give the definition of vertical and horizontal graphs in \( \overline{PSL}_2(\mathbb{R}, \tau) \).

#### 2.1.1. Vertical graph

A section of the Riemannian submersion

\[
\pi : \overline{PSL}_2(\mathbb{R}, \tau) \rightarrow \mathbb{H}^2
\]
is a map $s: \Omega \subset \mathbb{H}^2 \to \widetilde{PSL}_2(\mathbb{R}, \tau)$, where $\Omega$ is a domain, such that

$$\pi \circ s = id_{\mathbb{H}^2}|_{\Omega}$$

being $id_{\mathbb{H}^2}|_{\Omega}$ the identity map on $\mathbb{H}^2$ restrict to $\Omega$.

**Definition 2.1 (Vertical graph).** A vertical graph in $\widetilde{PSL}_2(\mathbb{R}, \tau)$ is the image of a section of the Riemannian submersion $\pi: \widetilde{PSL}_2(\mathbb{R}, \tau) \to \mathbb{H}^2$.

Given a domain $\Omega \subset \mathbb{H}^2$ we also denote by $\Omega$ its lift to $\mathbb{H}^2 \times \{0\}$, with this identification we have that the vertical graph $\Sigma(u)$ of $u \in C^0(\partial \Omega) \cap C^\infty(\Omega)$ is given by

$$\Sigma(u) = \{(x, y, u(x, y)) \in \widetilde{PSL}_2(\mathbb{R}, \tau); (x, y) \in \Omega\}.$$

If the vertical graph $\Sigma(u)$ has constant mean curvature $H$, then $u$ satisfies the following partial differential equation

$$(2.1) \quad L_H(u) := \text{div}_{\mathbb{H}^2}\left(\frac{\alpha}{W} e_1 + \frac{\beta}{W} e_2\right) - 2H = 0,$$

where $H$ is the mean curvature function with respect to the upward pointing normal vector and $W = \sqrt{1 + \alpha^2 + \beta^2}$,

- $\alpha = u_x / \lambda + 2\tau \lambda_x / \lambda^2$,
- $\beta = u_y / \lambda - 2\tau \lambda_y / \lambda^2$.

**2.1.2. Horizontal graph.** Following the ideas presented in [5], we consider a $C^2$-function $y = f(x, t)$, $f > 0$.

**Definition 2.2 (Horizontal graph).** We denote by $\Sigma_h(f) = \text{graph}(f)$, the horizontal graph of the function $f$, that is

$$\Sigma_h(f) = \{(x, f(x, t), t) \in \widetilde{PSL}_2(\mathbb{R}, \tau); (x, t) \in \text{Dom}(f)\}.$$

We denote by $N$ the natural normal vector to $\Sigma_h(f)$ (see equation (2.2)), and by $H$ the length of the mean curvature vector of $\Sigma_h(f)$ with respect to $N$. The mean curvature equation for horizontal graphs is given in the following lemma.

**Lemma 2.3.** Suppose that $H$ is the mean curvature function of $\Sigma_h(f)$. Then, the function $f$ satisfies the equation

$$\frac{2HW^3}{f^2} = (f^2 + f_t^2)f_{xx} - 2(f_x f_t - 2\tau f) f_{xt}$$

$$+ ((1 + 4\tau^2) + f_x^2)f_{tt} + f(1 + f_x^2) + 2\tau f_x f_t,$$
where $W = \sqrt{f^2 + f_t^2 + f^2(f_x + 2\tau f_t/f_t)^2}$. In particular the horocylinders $f(x, t) = \text{constant}$, has constant mean curvature.

Proof. The surface $\Sigma_h(f)$ is parameterized by $\varphi(x, t) = (x, f(x, t), t)$, so the adapted frame to $\Sigma_h(f)$ is given by

$$
\begin{align*}
\varphi_x &= \lambda(E_1 + f_x E_2 - 2\tau E_3), \\
\varphi_t &= \lambda f_t E_2 + E_3, \\
N &= \frac{-(f_x + 2\tau f_t)E_1 + E_2 - \lambda f_t E_3}{\sqrt{1 + (f_x + 2\tau f_t)^2 + \lambda^2 f_t^2}}.
\end{align*}
$$

(2.2)

where $N$ is the unit normal to $\Sigma_h(f)$, observe that $\langle N, \partial_t \rangle > 0$. Denoting by $g_{ij}$ and $b_{ij}$ the coefficients of the first and second fundamental form respectively we have that the function $H$ satisfies the equation

$$
2H = \frac{b_{11}g_{22} + b_{22}g_{11} - 2b_{12}g_{12}}{g_{11}g_{22} - g_{12}^2}.
$$

Since

$$
\begin{align*}
\tilde{\nabla}_{\varphi_t}\varphi_x &= -\lambda^2 f_x (2 + 4\tau^2)E_1 + [\lambda f_{xx} + \lambda^2((1 + 4\tau^2) - f_x)]E_2 + 2\tau \lambda^2 f_x E_3, \\
\tilde{\nabla}_{\varphi_t}\varphi_t &= [\tau \lambda f_x = \lambda^2 f_t (1 + 2\tau^2)]E_1 + [\lambda f_{xt} - \lambda^2 f_t f_t - \lambda \tau]E_2 + \lambda^2 \tau f_t E_3, \\
\tilde{\nabla}_{\varphi_t}\varphi_t &= 2\tau \lambda f_t E_1 + (\lambda f_{tt} - \lambda^2 f_t^2)E_2,
\end{align*}
$$

with

$$
\begin{align*}
b_{11} &= \lambda f_{xx} + \lambda^2(1 + 4\tau^2)f_t^2 + 2\tau \lambda^3(1 + 4\tau^2)f_x f_t + \lambda^2(1 + 4\tau^2), \\
b_{12} &= \lambda f_{xt} - \tau \lambda f_t^2 + 2\tau \lambda^3 \left(\frac{1}{2} + 2\tau^2\right)f_t^2 - \tau \lambda, \\
b_{22} &= \lambda f_{tt} - 2\tau \lambda f_t f_t - \lambda^2 f_t^2(1 + 4\tau^2),
\end{align*}
$$

and

$$
\begin{align*}
g_{11} &= \lambda^2[(1 + 4\tau^2) + f_t^2], \\
g_{12} &= \lambda^2 f_x f_t - 2\tau \lambda, \\
g_{22} &= 1 + \lambda^2 f_t^2,
\end{align*}
$$

a straightforward computation gives the result.

An interesting formula for the Laplacian is given in the next lemma.
Lemma 2.4. Considering $H = 1/2$, the function $f$ satisfies
\[
\Delta_{\Sigma(f)} f = \frac{f^2}{W} \left( 1 - \frac{f}{W} + \frac{ff_x + 2\tau f_t f_x}{W} \right),
\]
\[
\Delta_{\Sigma(f)} \left( \frac{1}{f} \right) = \frac{W - f}{fw} + \frac{f_t^2 + 2\tau (ff_x f_t + 2\tau f_t^2)}{W}.
\]

Proof. The proof follows from a hard computation by considering
\[
\Delta_{\Sigma(f)} = \frac{1}{\sqrt{g}} \sum_{ij} \partial_i (\sqrt{g} g^{ij} \partial_j),
\]
where $g$ is the determinant of the first fundamental form and $(g^{ij}) = (g_{ij})^{-1}$.

Observe that
\[
\Delta_s f = \frac{1}{\sqrt{g} W^3} \left[ f^2 ((f^2 + f_t^2) f_{xx} + 2(2\tau f - f_x f_t) f_{xt} + (f_t^2 + (1 + 4\tau^2)) f_{tt}) \\
+ (a^2 + f^3 f_x) f_x + (af_x - (1 + 4\tau^2) ff_t) f_t, \right]
\]
where $a = ff_x + 2\tau f_t$ and $W^2 = f^2 + f_t^2 + (ff_x + 2\tau f_t)^2$.

Remark 2.5. In the case $\tau \equiv 0$, that is, when the ambient space is $\mathbb{H}^2 \times \mathbb{R}$, it was proved in [5] that
\[
\Delta_{\Sigma(f)} f > 0,
\]
\[
\Delta_{\Sigma(f)} \left( \frac{1}{f} \right) > 0,
\]
which is surprising and plays an important role. Note that, we do not have this property when $\tau \neq 0$.

3. The main theorem

In order to prove the main theorem (Theorem 3.6), first we construct an $H = 1/2$ annulus. Which is an horizontal graph, this is the goal of the Proposition 3.2. Since we deal with horizontal graphs, the $H = 1/2$ mean curvature equation is given in the following lemma.

Lemma 3.1. Considering $H = 1/2$, the mean curvature equation for a horizontal graph is given by
\[
1 = \frac{f^2}{W^3} \left[ (f^2 + f_t^2) f_{xx} - 2(f_x f_t - 2\tau f) f_{xt} \\
+ ((1 + 4\tau^2) + f_x^2) f_{tt} + f(1 + f_t^2) + 2\tau f_t f_t, \right].
\]
which we can write in the form

\[(f^2 + f_1^2) f_{xx} - 2(f_x f_t - 2\tau f) f_{xt} + (f_x^2 + (1 + 4\tau^2)) f_{tt}\]

\[
(3.1) \quad - \left[ \frac{W}{f^2} + \frac{1}{W + f} \right] [(1 + 4\tau^2) f_t + 4\tau f f_x] f_t + \left[ 2\tau f_t - \frac{W^2}{W + f} f^2 \right] f_x = 0.
\]

Proof. Considering \( H = 1/2 \) in Lemma 2.4, we obtain

\[(f^2 + f_1^2) f_{xx} - 2(f_x f_t - 2\tau f) f_{xt} + (f_x^2 + (1 + 4\tau^2)) f_{tt}\]

\[= -f(1 + f_1^2) - 2\tau f_x f_t + \frac{W^3}{f^2},\]

which we can write in the form

\[(f^2 + f_1^2) f_{xx} - 2(f_x f_t - 2\tau f) f_{xt} + (f_x^2 + (1 + 4\tau^2)) f_{tt}\]

\[- \left[ \frac{W^2}{f^2(W + f)} + \frac{1}{f} \right] [(1 + 4\tau^2) f_t + 4\tau f f_x] f_t - \frac{W^2}{f^2(W + f)} f^2 f_x^2 + 2\tau f_x f_t = 0.
\]

After a straightforward computation, we obtain the equation (3.1). \(\square\)

3.1. \( H = 1/2 \) horizontal annuli. Consider the horocylinder \( C(1) \subset PSL_2(\mathbb{R}, \tau) \), given by

\[C(1) = \{(x, 1, t) \in PSL_2(\mathbb{R}, \tau)\}.
\]

Let \( R > 0 \) be a positive constant. We define the subset \( B_R \subset C(1) \) of the horocylinder, by

\[B_R = \{(x, 1, t) \in PSL_2(\mathbb{R}, \tau); x^2 + t^2 < R^2\}.
\]

Proposition 3.2 (\( H = 1/2 \) annuli). Let \( U \) be the annulus \( U = \bar{B}_{R_2} \setminus B_{R_1} \) with \( R_2 \geq 4R_1 \). Then for \( \epsilon > 0 \) sufficiently small (depending on \( R_1 \)), there exist constant mean curvature \( H = 1/2 \) horizontal graphs \( f^\pm \) and \( f^- \), satisfying equation (3.1) in \( U \) with Dirichlet boundary data \( f^\pm = 1 \pm \epsilon \) on \( \partial B_{R_1} \), \( f^- = 0 \) on \( \partial B_{R_2} \). Moreover \( f^\pm \) tends to \( 1 \pm \epsilon \) uniformly on compact subsets as \( R_2 \) tends to \( \infty \).

Remark 3.3. Note that the equation (3.1) implies that any solution \( f^\pm \) solving the Dirichlet problem of Proposition 3.2 satisfies \( 1 - \epsilon \leq f^- \leq 1 \) and \( 1 \leq f^+ \leq 1 + \epsilon \) on \( U \).

Proof. Let \( U = \bar{B}_{R_2} \setminus B_{R_1} \) be an annulus with \( R_2 \geq 4R_1 \) and fix

\[h = 1 \pm \frac{\epsilon}{\log(R_2/R_1)} \log \left( \frac{R_2}{r} \right),
\]
where \( r^2 = x^2 + t^2 \).

We define the weighted \( C^{2,\alpha} \) norm:

\[
|v|_{2,\alpha;U}^2 = \sup_X \{ |v(X)| + r(X)|Dv(X)| + r^2(X)|D^2v(X)| + r^{2+\alpha}(X)[D^2v]_a(X) \},
\]

where \( X = (x, t) \) and \([D^2v]_a(X)\) is the H"older coefficient of \( D^2v \) at \( X \).

We expect the solution \( f \) to be close to \( h \). Thus we consider the following definition.

**Definition 3.4.** We say \( f \) is an admissible solution of (3.1) if \( f \in \mathcal{A}_e \), where

\[
\mathcal{A}_e = \{ f \in C^{2,\alpha}(U), \ f = h \text{ on } \partial U : |f - h|_{2,\alpha;U}^2 \leq \epsilon \}.
\]

We note that \( \mathcal{A}_e \) is convex and compact subset of the Banach space \( \mathcal{B} = C^{2,\beta}(U) \), \( \beta < \alpha \). We will reformulate our existence problem as a fixed point of a continuous operator \( T : \mathcal{A}_e \to \mathcal{A}_e \).

We now define the operator \( w = Tf \) as follows: if \( f \in C^{2,\alpha}(U) \), we set \( Tf = w \), where \( w \) is the solution of the linear Dirichlet problem

\[
\begin{aligned}
L_f w &= a w_{xx} + 2b w_{xt} + c w_{tt} + d w_x + e w_t = 0, \quad \text{in } U; \\
w &= h, \quad \text{on } \partial U,
\end{aligned}
\]

where:

\[
a = f^2 + f_t^2, \\
b = 2 \tau f - f_x f_t, \\
c = f_t^2 + (1 + 4 \tau^2), \\
d = -\left[ \frac{W}{f^2} + \frac{1}{W + f} \right] [(1 + 4 \tau^2) f_t f_x + 4 \tau f f_x], \\
e = \left[ 2 \tau f_t - \frac{W^2}{W + f} f_x \right].
\]

**Proposition 3.5.** If \( \epsilon \) is sufficiently small, then \( Tf \in \mathcal{A}_e \) for every \( f \in \mathcal{A}_e \).

Proof. Set \( u = w - h \), then

(3.2) \[ L_f u = [(1 - f^2 - f_t^2) h_{xx} + 2 f_x f_t h_{xt} - f_t^2 h_{tt} - dh_x - eh_t] := F. \]

By the maximum principle [4, Theorem 3.1 (p. 32)], \( 1 \leq w \leq 1 + \epsilon \) (or \( 1 - \epsilon \leq w \leq 1 \)) so \( |u| \leq \epsilon \).

Applying Schauder interior or boundary estimates to \( L_f u = F \) in \( U \), we obtain (see [4, Theorem 6.6 (p. 98)], [4, Corollary 6.7 (p. 100)])

\[
|u|_{2,\alpha;U} \leq C(|u|_{0,\alpha;U} + |F|_{0,\alpha;U}).
\]
Observe that $|u| \leq \epsilon$ implies $|u|_{0;U} \leq \epsilon$. From equation (3.2) follows $|F|_{0;\alpha;U} \leq C \epsilon^{3/2}$. This implies

$$(3.3) \quad |u|_{2;\alpha;U} \leq C(|u|_{0;U} + |F|_{0;\alpha;U}) \leq C \epsilon.$$ 

Now, from [4, formula 4.17'(p. 60)], we obtain

$$|u|_{2;\alpha;U} \leq C \epsilon.$$ 

Since $u = w - h$, it follows that for $\epsilon$ small enough, $w \in A_\epsilon$, from Schauder estimates and for $R_2$ big enough $\epsilon$ depends only on $R_1$, thus the proposition is proved.

Applying the Schauder fixed point theorem to the operator $w = Tf$, we obtain a solution $f^+ \in A_\epsilon$ which satisfies equation (3.1).

Now we prove that $f^+$ converges to the horocylinder $C(1 + \epsilon)$ uniformly on compact subsets as $R_2$ tends to $+\infty$, the $f^-$ case is similar. Take $K$ a compact set in $U$. Now enlarge $U$ by making $R_2$ tend to infinity, this produces a family of functions $h$ (one for each such $R_2$). Note that the restriction of this sequences of functions to the fixed compact set $K$ converges uniformly to the value $1 + \epsilon$.

On the other hand, given $\rho > 0$ and some compact $K \subset (C(1) - B_{R_1})$, by the definition of $A_\epsilon$ and the existence part, there is some $R_2$ large enough and some $\epsilon_1$ small enough (depending only on $R_1$ and $\rho$, not on $R_2$ or $K$) such that for any $\epsilon < \epsilon_1$, the function $f$ associated to such $h$ is $\rho$-close to $1 + \epsilon$, that is, when $R_2$ tend to infinity the functions $f^+$ converges uniformly to $1 + \epsilon$.

3.2. The main theorem. Now we prove the main theorem.

**Theorem 3.6 (Main theorem).** Let $\Sigma$ be a properly embedded constant mean curvature $H = 1/2$ surface in $\overline{PSL}_2(\mathbb{R})$. Suppose $\Sigma$ is asymptotic to a horocylinder $C$, and one side of $C$. If the mean curvature vector of $\Sigma$ has the same direction as that of $C$ at points of $\Sigma$ converging to $C$, then $\Sigma$ is equal to $C$ (or a subset of $C$ if $\partial \Sigma \neq \emptyset$).

Proof. Assume that $\Sigma$ is not a subset of $C$. After an isometry, we can assume that, there is a sequence of points $p_i = (x_i, y_i, t_i) \in \Sigma$ with $y_i \to 1$. First, we suppose that $\Sigma$ is contained in the set $\{y > 1\}$, the other case is treated analogously. We denote by $C(\xi)$ the horocylinder in $\overline{PSL}_2(\mathbb{R})$ given by $\{y = \xi\}$. For $\epsilon > 0$ small we consider the slab $S^+$ bounded by $C(1)$ and $C(1 + \epsilon)$. Then by the maximum principle $\Sigma^+ = \Sigma \cap S^+$ has a non compact component with boundary $\partial \Sigma \subset C(1 + \epsilon)$.

Let $D(\xi, R)$ denote the disk in $C(\xi)$ defined by $D(\xi, R) = \{(x, \xi, t); x^2 + t^2 \leq R^2\}$.

By considering vertical translation, we can find a disk $D(1, 3R_1)$ such that:

$$(D(1, 3R_1) \times [1, 1 + \epsilon]) \cap \Sigma^+ = \emptyset.$$
By Theorem 3.2, for each $R \geq 4R_1$, there exist a horizontal graph $f_R^+$ defined on the annulus $U = \tilde{B}_{R_1} \setminus B_{R_1}$, this horizontal graph converge to $C(1 + \varepsilon)$, when $R$ goes to $+\infty$.

Now, consider $R$ large, such that the graph of $f_R^+$ (which we denote by $\Gamma^+$), satisfies $\Sigma^+ \cap \Gamma^+ \neq \emptyset$. By considering vertical translations and translations along the geodesic $\{x = 0, t = 0\}$, the translated surface of $\Gamma^+$ does not touch $\Sigma^+$, that is, there is a translated surface of $\Gamma^+$ (which we denote by $\Gamma_1^+$) such that $\Gamma_1^+$ and $\Sigma^+$ has an interior contact point. Since the mean curvature vectors are pointing up, this violates the maximum principle and $\Sigma^+$ cannot exist.

In the second case, we redo exactly the same argument exchanging the roles of $C(1 + \varepsilon)$ and $C(1 - \varepsilon)$.

4. The second theorem

In this section our second result concerns complete $H = 1/2$ surfaces in $\tilde{PSL}_2(\mathbb{R}, \tau)$ transverse to the vertical Killing field $E_3 = \partial_t$, we use Theorem 1.1 in order to prove such surfaces are entire graphs. This result was proved in a totally different way in [1] and [3].

Theorem 4.1. Let $\Sigma$ be a complete immersed surface in $\tilde{PSL}_2(\mathbb{R}, \tau)$ of constant mean curvature $H = 1/2$. If $\Sigma$ is transverse to $E_3$ then $\Sigma$ is an entire vertical graph over $\mathbb{H}^2$.

The proof of this theorem is analogous to this one in [5, Theorem 1.2] taking into account [7]. It was showed in [5, Theorem 1.2], that, there is $\varepsilon > 0$ and a horocylinder such that, a graph $G \subset \Sigma$ (over a domain in $\mathbb{H}^2 \times \{0\} \subset \tilde{PSL}_2(\mathbb{R}, \tau)$) is in the $\varepsilon$-tubular neighborhood of the cylinder. Since $G$ is proper the proof of the half-space theorem shows that this graph can not exist.

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