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ON $H = 1/2$ SURFACES IN $\widetilde{PSL}_2(\mathbb{R}, \tau)$

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Abstract

In this paper we prove that if Σ is a properly embedded constant mean curvature $H = 1/2$ surface which is asymptotic to a horocylinder $C \subset \widetilde{PSL}_2(\mathbb{R}, \tau)$, in one side of C, such that the mean curvature vector of Σ has the same direction as that of the C at points of Σ converging to C, then Σ is a subset of C.

1. Introduction

In this paper we study complete constant mean curvature $H = 1/2$ surfaces immersed in $\widetilde{PSL}_2(\mathbb{R}, \tau)$. Recall that in [5] the authors generalized to $\mathbb{H}^2 \times \mathbb{R}$ the half-space theorem of Hoffman and Meeks which ensures that a properly immersed minimal surface in $\mathbb{R}^3$ that lies in a half-space must be a plane. The main theorem in [5] says that, if a properly embedded constant mean curvature $H = 1/2$ surface in $\mathbb{H}^2 \times \mathbb{R}$ which is asymptotic to a horocylinder C and on one side of C; such that the mean curvature vector of the surface has the same direction as that of C at points of the surface converging to C, then the surface is equal to C (or a subset of C if the surface has non-empty boundary).

We extend this result to the space $\widetilde{PSL}_2(\mathbb{R}, \tau)$. Remember that the space $\widetilde{PSL}_2(\mathbb{R}, \tau)$ is one of the eight Thurston's geometries. Indeed it is well known there exists a classification due to W. Thurston of simply connected homogeneous 3-manifolds (see [8, Chapter eight]). Such a manifold has an isometry group of dimension 3, 4 or 6.

- When the manifold has 6-dimensional isometry group, we have the 3-dimensional space-forms: the Euclidean space $\mathbb{R}^3$, the Euclidean sphere $\mathbb{S}^3(\kappa)$ (having sectional curvature $\kappa > 0$) and the hyperbolic space $\mathbb{H}^3(\kappa)$ (having sectional curvature $\kappa < 0$).
- When the manifold has 3-dimensional isometry group, we have the Lie group Sol_3.
- When the manifold has 4-dimensional isometry group (we label by $E(\kappa, \tau)$ these manifolds), there exists a Riemannian fibration over a 2-dimensional space form $M^2(\kappa)$.

The manifolds $E(\kappa, \tau)$ are classified, up to isometry, by the curvature κ of the base surface and by the bundle curvature of the fibration τ, where κ and τ can be any real numbers satisfying $\kappa \neq 4\tau^2$. When $\tau = 0$ we have the metric product spaces $M^2(\kappa) \times \mathbb{R}$. When $\kappa = 0$ and $\tau \neq 0$ we have the 3-dimensional Heisenberg group. The

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space $\widetilde{PSL}_2(\mathbb{R}, \tau)$ is given when we consider $\tau \neq 0$ and $\kappa = -1$, that is $E(-1, \tau) = \widetilde{PSL}_2(\mathbb{R}, \tau)$.

We extend the aforementioned result to the space $\widetilde{PSL}_2(\mathbb{R}, \tau)$. In order to do that, note that, since exists a Riemannian submersion

$$\pi : \widetilde{PSL}_2(\mathbb{R}, \tau) \to \mathbb{H}^2$$

over the half-plane model for the 2-dimensional hyperbolic space $\mathbb{H}^2$, we call a horocylinder the inverse image $\pi^{-1}(\mathfrak{h})$, where $\mathfrak{h}$ is a horocycle in $\mathbb{H}^2$. We also denote by ∂_t the tangent field to the fibers on $\widetilde{PSL}_2(\mathbb{R}, \tau)$.

Let C be a complete horocylinder in $\widetilde{PSL}_2(\mathbb{R}, \tau)$, we say that the surface Σ is asymptotic to C if Σ contain a open subset $U \subset \Sigma$ (with $U \cap C = \emptyset$), such that, for each $\epsilon > 0$, there exists a compact set $K \subset U$, where the distance $d(p, C) < \epsilon$ for all $p \in (U - K)$, here $d(\cdot, \cdot)$ denotes the distance function in the space $\widetilde{PSL}_2(\mathbb{R}, \tau)$.

Following the same spirit as in [5], we show an analogous result in the space $\widetilde{PSL}_2(\mathbb{R}, \tau)$. More precisely, our main theorem is the following.

Theorem 1.1. *Let Σ be a properly embedded constant mean curvature $H = 1/2$ surface in $\widetilde{PSL}_2(\mathbb{R}, \tau)$. Suppose Σ is asymptotic to a horocylinder C, and on one side of C. If the mean curvature vector of Σ has the same direction as that of C at points of Σ converging to C, then Σ is equal to C.*

As a consequence of Theorem 1.1, we obtain (in the same sense as in [5]) the Theorem 1.2. Note that, the Theorem 1.2 is well known, see for instance [1] or [3, Corollary 4.6.3].

Theorem 1.2. *Let Σ be a complete immersed surface in $\widetilde{PSL}_2(\mathbb{R}, \tau)$ of constant mean curvature $H = 1/2$. If Σ is transverse to the vertical Killing field $E_3 = \partial_t$, then Σ is an entire vertical graph over $\mathbb{H}^2$.*

Observe that the value $H = 1/2$ for constant mean curvature H surfaces is special in the space $\widetilde{PSL}_2(\mathbb{R}, \tau)$. In fact, a constant mean curvature H surface in the homogeneous space $E(\kappa, \tau)$ has critical constant mean curvature if the relation $H^2 = -\kappa/4$ holds. This terminology comes from the fact that it separates the case $H^2 > -\kappa/4$, in which compact constant mean curvature exists, from the case $H^2 < -\kappa/4$, in which no compact constant mean curvature can exists.

2. The space $\widetilde{PSL}_2(\mathbb{R}, \tau)$

The 3-dimensional space $\widetilde{PSL}_2(\mathbb{R}, \tau)$ is a complete homogeneous simply connected Riemannian manifold. Each such a manifold (depending on τ) is the total space of a Riemannian submersion over the 2-dimensional hyperbolic space $\mathbb{H}^2$ (here the Gaussian curvature of the hyperbolic space is $\kappa = -1$). The bundle curvature of the submersion is the number τ such that $\bar\nabla_X E_3 = \tau X \times E_3$ for any vector field X on $\widetilde{PSL}_2(\mathbb{R},\tau)$ (here $\bar\nabla$ denotes the Riemannian connection of $\widetilde{PSL}_2(\mathbb{R},\tau)$). And each fiber is a complete geodesic tangent to a Killing field E_3. When $\tau = 0$, we obtain the space $\widetilde{PSL}_2(\mathbb{R},0) \equiv \mathbb{H}^2 \times \mathbb{R}$.

From now on, we choice and fix a value for τ different from zero. More precisely, the Riemannian manifold is $(\widetilde{PSL}_2(\mathbb{R},\tau), g)$, where $\widetilde{PSL}_2(\mathbb{R},\tau)$ is topologically $\mathbb{H}^2 \times \mathbb{R}$ ($\mathbb{R}$ the real line), that is

$$\widetilde{PSL}_2(\mathbb{R},\tau) = \{(x, y, t) \in \mathbb{R}^3;\ y > 0\}$$

endowed with the metric

$$g = \lambda^2(dx^2 + dy^2) + (-2\tau\lambda\, dx + dt)^2, \quad \lambda = \frac{1}{y}.$$

There is a natural orthonormal frame $\{E_1, E_2, E_3\}$ given by (in coordinates $\{\partial_x, \partial_y, \partial_t\}$)

$$E_1 = \frac{\partial_x}{\lambda} + 2\tau\,\partial_t, \quad E_2 = \frac{\partial_y}{\lambda}, \quad E_3 = \partial_t.$$

E_3 is the Killing field tangent to the fibers. The metric g induces a Riemannian connection $\bar\nabla$ given by

$$\begin{aligned}
&\bar\nabla_{E_1} E_1 = -\frac{\lambda_y}{\lambda^2} E_2, \quad \bar\nabla_{E_1} E_2 = \frac{\lambda_y}{\lambda^2} E_1 + \tau E_3, \quad \bar\nabla_{E_1} E_3 = -\tau E_2,\\
&\bar\nabla_{E_2} E_1 = \frac{\lambda_x}{\lambda^2} E_2 - \tau E_3, \quad \bar\nabla_{E_2} E_2 = -\frac{\lambda_x}{\lambda^2} E_1, \quad \bar\nabla_{E_2} E_3 = \tau E_1,\\
&\bar\nabla_{E_3} E_1 = -\tau E_2, \quad \bar\nabla_{E_3} E_2 = \tau E_1, \quad \bar\nabla_{E_3} E_3 = 0.
\end{aligned}$$

We also have

$$[E_1, E_2] = \frac{\lambda_y}{\lambda^2} E_1 - \frac{\lambda_x}{\lambda^2} E_2 + 2\tau E_3, \quad [E_1, E_3] = 0, \quad [E_2, E_3] = 0.$$

For more details see [6], [2], [8].

2.1. Graphs in $\widetilde{PSL}_2(\mathbb{R},\tau)$. Now we give the definition of vertical and horizontal graphs in $\widetilde{PSL}_2(\mathbb{R},\tau)$.

2.1.1. Vertical graph. A section of the Riemannian submersion

$$\pi : \widetilde{PSL}_2(\mathbb{R},\tau) \to \mathbb{H}^2$$

is a map $s\colon \Omega \subset \mathbb{H}^2 \to \widetilde{PSL}_2(\mathbb{R}, \tau)$, where Ω is a domain, such that

$$\pi \circ s = id_{\mathbb{H}^2}|_{\Omega}$$

being $id_{\mathbb{H}^2}|_{\Omega}$ the identity map on $\mathbb{H}^2$ restrict to Ω.

DEFINITION 2.1 (Vertical graph). A vertical graph in $\widetilde{PSL}_2(\mathbb{R}, \tau)$ is the image of a section of the Riemannian submersion $\pi\colon \widetilde{PSL}_2(\mathbb{R}, \tau) \to \mathbb{H}^2$.

Given a domain $\Omega \subset \mathbb{H}^2$ we also denote by Ω its lift to $\mathbb{H}^2 \times \{0\}$, with this identification we have that the vertical graph $\Sigma(u)$ of $u \in C^0(\partial\Omega) \cap C^\infty(\Omega)$ is given by

$$\Sigma(u) = \{(x, y, u(x, y)) \in \widetilde{PSL}_2(\mathbb{R}, \tau);\ (x, y) \in \Omega\}.$$

If the vertical graph $\Sigma(u)$ has constant mean curvature H, then u satisfies the following partial differential equation

$$L_H(u) := \operatorname{div}_{\mathbb{H}^2}\left(\frac{\alpha}{W}e_1 + \frac{\beta}{W}e_2\right) - 2H = 0, \tag{2.1}$$

where H is the mean curvature function with respect to the upward pointing normal vector and $W = \sqrt{1 + \alpha^2 + \beta^2}$,

- $\alpha = u_x/\lambda + 2\tau\lambda_y/\lambda^2$,
- $\beta = u_y/\lambda - 2\tau\lambda_x/\lambda^2$.

2.1.2. Horizontal graph. Following the ideas presented in [5], we consider a C^2-function $y = f(x, t)$, $f > 0$.

DEFINITION 2.2 (Horizontal graph). We denote by $\Sigma_h(f) = graph(f)$, the horizontal graph of the function f, that is

$$\Sigma_h(f) = \{(x, f(x, t), t) \in \widetilde{PSL}_2(\mathbb{R}, \tau);\ (x, t) \in \mathfrak{Dom}(f)\}.$$

We denote by N the natural normal vector to $\Sigma_h(f)$ (see equation (2.2)), and by H the length of the mean curvature vector of $\Sigma_h(f)$ with respect to N. The mean curvature equation for horizontal graphs is given in the following lemma.

Lemma 2.3. *Suppose that H is the mean curvature function of $\Sigma_h(f)$. Then, the function f satisfies the equation*

$$\begin{aligned}\frac{2HW^3}{f^2} &= (f^2 + f_t^2) f_{xx} - 2(f_x f_t - 2\tau f) f_{xt}\\ &\quad + ((1 + 4\tau^2) + f_x^2) f_{tt} + f(1 + f_x^2) + 2\tau f_x f_t,\end{aligned}$$

where $W = \sqrt{f^2 + f_t^2 + f^2(f_x + 2\tau f_t/f)^2}$. *In particular the horocylinders* $f(x,t) =$ *constant, has constant mean curvature.*

Proof. The surface $\Sigma_h(f)$ is parameterized by $\varphi(x,t) = (x, f(x,t), t)$, so the adapted frame to $\Sigma_h(f)$ is given by

$$\begin{aligned}\varphi_x &= \lambda(E_1 + f_x E_2 - 2\tau E_3),\\ \varphi_t &= \lambda f_t E_2 + E_3,\\ N &= \frac{-(f_x + 2\tau\lambda f_t)E_1 + E_2 - \lambda f_t E_3}{\sqrt{1 + (f_x + 2\tau\lambda f_t)^2 + \lambda^2 f_t^2}},\end{aligned} \tag{2.2}$$

where N is the unit normal to $\Sigma_h(f)$, observe that $\langle N, \partial_y\rangle > 0$. Denoting by g_{ij} and b_{ij} the coefficients of the first and second fundamental form respectively we have that the function H satisfies the equation

$$2H = \frac{b_{11}g_{22} + b_{22}g_{11} - 2b_{12}g_{12}}{g_{11}g_{22} - g_{12}^2}.$$

Since

$$\begin{aligned}\bar\nabla_{\varphi_x}\varphi_x &= -\lambda^2 f_x(2 + 4\tau^2)E_1 + [\lambda f_{xx} + \lambda^2((1 + 4\tau^2) - f_x)]E_2 + 2\tau\lambda^2 f_x E_3,\\ \bar\nabla_{\varphi_t}\varphi_x &= [\tau\lambda f_x = \lambda^2 f_t(1 + 2\tau^2)]E_1 + [\lambda f_{xt} - \lambda^2 f_x f_t - \lambda\tau]E_2 + \lambda^2\tau f_t E_3,\\ \bar\nabla_{\varphi_t}\varphi_t &= 2\tau\lambda f_t E_1 + (\lambda f_{tt} - \lambda^2 f_t^2)E_2,\end{aligned}$$

with

$$\begin{aligned}b_{11} &= \lambda f_{xx} + \lambda^2(1 + 4\tau^2) f_x^2 + 2\tau\lambda^3(1 + 4\tau^2) f_x f_t + \lambda^2(1 + 4\tau^2),\\ b_{12} &= \lambda f_{xt} - \tau\lambda f_x^2 + 2\tau\lambda^3\left(\frac{1}{2} + 2\tau^2\right) f_t^2 - \tau\lambda,\\ b_{22} &= \lambda f_{tt} - 2\tau\lambda f_x f_t - \lambda^2 f_t^2(1 + 4\tau^2),\end{aligned}$$

and

$$\begin{aligned}g_{11} &= \lambda^2[(1 + 4\tau^2) + f_x^2],\\ g_{12} &= \lambda^2 f_x f_t - 2\tau\lambda,\\ g_{22} &= 1 + \lambda^2 f_t^2,\end{aligned}$$

a straightforward computation gives the result. $\square$

An interesting formula for the Laplacian is given in the next lemma.

Lemma 2.4. *Considering $H = 1/2$, the function f satisfies*

$$\Delta_{\Sigma_h(f)} f = \frac{f^2}{W}\left(1 - \frac{f}{W} + \frac{f f_x^2 + 2\tau f_t f_x}{W}\right),$$
$$\Delta_{\Sigma_h(f)}\left(\frac{1}{f}\right) = \frac{W - f}{fW} + \frac{f_t^2 + 2\tau(f f_x f_t + 2\tau f_t^2)}{W}.$$

Proof. The proof follows from a hard computation by considering

$$\Delta_{\Sigma_h(f)} = \frac{1}{\sqrt{g}} \sum_{ij} \partial_{x_i}(\sqrt{g} g^{ij} \partial_{x_j}),$$

where g is the determinant of the first fundamental form and $(g^{ij}) = (g_{ij})^{-1}$.

Observe that

$$\Delta_s f = \frac{1}{\sqrt{g}W^3}[f^2[(f^2 + f_t^2) f_{xx} + 2(2\tau f - f_x f_t) f_{xt} + (f_x^2 + (1 + 4\tau^2)) f_{tt}]$$
$$+ (a^3 + f^3 f_x) f_x + (a f_x - (1 + 4\tau^2) f f_t) f_t],$$

where $a = f f_x + 2\tau f_t$ and $W^2 = f^2 + f_t^2 + (f f_x + 2\tau f_t)^2$. □

REMARK 2.5. In the case $\tau \equiv 0$, that is, when the ambient space is $\mathbb{H}^2 \times \mathbb{R}$, it was proved in [5] that

$$\Delta_{\Sigma_h(f)} f > 0,$$
$$\Delta_{\Sigma_h(f)}\left(\frac{1}{f}\right) > 0,$$

which is surprising and plays an important role. Note that, we do not have this property when $\tau \neq 0$.

3. The main theorem

In order to prove the main theorem (Theorem 3.6), first we construct an $H = 1/2$ annulus. Which is an horizontal graph, this is the goal of the Proposition 3.2. Since we deal with horizontal graphs, the $H = 1/2$ mean curvature equation is given in the following lemma.

Lemma 3.1. *Considering $H = 1/2$, the mean curvature equation for a horizontal graph is given by*

$$1 = \frac{f^2}{W^3}[(f^2 + f_t^2) f_{xx} - 2(f_x f_t - 2\tau f) f_{xt}$$
$$+ ((1 + 4\tau^2) + f_x^2) f_{tt} + f(1 + f_x^2) + 2\tau f_x f_t],$$

which we can write in the form

$$(3.1)\quad \begin{aligned}&(f^2+f_t^2)f_{xx}-2(f_xf_t-2\tau f)f_{xt}+(f_x^2+(1+4\tau^2))f_{tt}\\&-\left[\frac{W}{f^2}+\frac{1}{W+f}\right][(1+4\tau^2)f_t+4\tau ff_x]f_t+\left[2\tau f_t-\frac{W^2}{W+f}f_x\right]f_x=0.\end{aligned}$$

Proof. Considering $H\equiv 1/2$ in Lemma 2.4, we obtain

$$\begin{aligned}&(f^2+f_t^2)f_{xx}-2(f_xf_t-2\tau f)f_{xt}+(f_x^2+(1+4\tau^2))f_{tt}\\&=-f(1+f_x^2)-2\tau f_xf_t+\frac{W^3}{f^2},\end{aligned}$$

which we can write in the form

$$\begin{aligned}&(f^2+f_t^2)f_{xx}-2(f_xf_t-2\tau f)f_{xt}+(f_x^2+(1+4\tau^2))f_{tt}\\&-\left[\frac{W^2}{f^2(W+f)}+\frac{1}{f}\right][(1+4\tau^2)f_t+4\tau ff_x]f_t-\frac{W^2}{f^2(W+f)}f^2f_x^2+2\tau f_xf_t=0.\end{aligned}$$

After a straightforward computation, we obtain the equation (3.1). □

3.1. *H* = 1/2 horizontal annuli. Consider the horocylinder $C(1)\subset\widetilde{PSL}_2(\mathbb{R},\tau)$, given by

$$C(1)=\{(x,1,t)\in\widetilde{PSL}_2(\mathbb{R},\tau)\}.$$

Let $R>0$ be a positive constant. We define the subset $B_R\subset C(1)$ of the horocylinder, by

$$B_R=\{(x,1,t)\in\widetilde{PSL}_2(\mathbb{R},\tau);\ x^2+t^2<R^2\}.$$

Proposition 3.2 ($H=1/2$ annuli)**.** *Let U be the annulus $U=\bar{B}_{R_2}\setminus B_{R_1}$ with $R_2\geq 4R_1$. Then for $\epsilon>0$ sufficiently small (depending on R_1), there exist constant mean curvature $H=1/2$ horizontal graphs f^+ and f^-, satisfying equation* (3.1) *in U with Dirichlet boundary data $f^\pm=1\pm\epsilon$ on ∂B_{R_1}, $f^\pm=1$ on ∂B_{R_2}. Moreover $f^\pm$ tends to $1\pm\epsilon$ uniformly on compact subsets as R_2 tends to ∞.*

REMARK 3.3. Note that the equation (3.1) implies that any solution $f^\pm$ solving the Dirichlet problem of Proposition 3.2 satisfies $1-\epsilon\leq f^-\leq 1$ and $1\leq f^+\leq 1+\epsilon$ on U.

Proof. Let $U=\bar{B}_{R_2}\setminus B_{R_1}$ be an annulus with $R_2\geq 4R_1$ and fix

$$h=1\pm\frac{\epsilon}{\log(R_2/R_1)}\log\left(\frac{R_2}{r}\right),$$

where $r^2 = x^2 + t^2$.

We define the weighted $C^{2,\alpha}$ norm:

$$|v|^*_{2,\alpha;U} = \sup_X\{|v(X)| + r(X)|Dv(X)| + r^2(X)|D^2_v(X)| + r^{2+\alpha}(X)[D^2v]_\alpha(X)\},$$

where $X = (x, t)$ and $[D^2v]_\alpha(X)$ is the $H\ddot{o}lder$ coefficient of D^2v at X.

We expect the solution f to be close to h. Thus we consider the following definition.

DEFINITION 3.4. We say f is an admissible solution of (3.1) if $f \in \mathcal{A}_\epsilon$, where

$$\mathcal{A}_\epsilon = \{f \in C^{2,\alpha}(U),\ f = h \text{ on } \partial U\colon |f - h|^*_{2,\alpha;U} \le \sqrt{\epsilon}\}.$$

We note that $\mathcal{A}_\epsilon$ is convex and compact subset of the Banach space $\mathfrak{B} = C^{2,\beta}(U)$, $\beta < \alpha$. We will reformulate our existence problem as a fixed point of a continuous operator $T\colon \mathcal{A}_\epsilon \to \mathcal{A}_\epsilon$.

We now define the operator $w = Tf$ as follows: if $f \in C^{2,\alpha}(U)$, we set $Tf = w$, where w is the solution of the linear Dirichlet problem

$$\begin{cases} L_f w := aw_{xx} + 2bw_{xt} + cw_{tt} + dw_x + ew_t = 0, & \text{in} \quad U; \\ w = h, & \text{on} \quad \partial U, \end{cases}$$

where:

$$\begin{aligned} a &= f^2 + f_x^2, \\ b &= 2\tau f - f_x f_t, \\ c &= f_x^2 + (1 + 4\tau^2), \\ d &= -\left[\frac{W}{f^2} + \frac{1}{W + f}\right][(1 + 4\tau^2) f_t + 4\tau f f_x], \\ e &= \left[2\tau f_t - \frac{W^2}{W + f} f_x\right]. \end{aligned}$$

Proposition 3.5. *If ϵ is sufficiently small, then $Tf \in \mathcal{A}_\epsilon$ for every $f \in \mathcal{A}_\epsilon$.*

Proof. Set $u = w - h$, then

$$L_f u = [(1 - f^2 - f_t^2)h_{xx} + 2 f_x f_t h_{xt} - f_x^2 h_{tt} - dh_x - eh_t] := F. \tag{3.2}$$

By the maximum principle [4, Theorem 3.1 (p. 32)], $1 \le w \le 1 + \epsilon$ (or $1 - \epsilon \le w \le 1$) so $|u| \le \epsilon$.

Applying Schauder interior or boundary estimates to $L_f u = F$ in U, we obtain (see [4, Theorem 6.6 (p. 98)], [4, Corollary 6.7 (p. 100)])

$$|u|_{2,\alpha;U} \le C(|u|_{0;U} + |F|_{0,\alpha;U}).$$

Observe that $|u| \le \epsilon$ implies $|u|_{0;U} \le \epsilon$. From equation (3.2) follows $|F|_{0,\alpha;U} \le C\epsilon^{3/2}$. This implies

$$|u|_{2,\alpha;U} \le C(|u|_{0;U} + |F|_{0,\alpha;U}) \le C\epsilon. \tag{3.3}$$

Now, from [4, formula 4.17'(p. 60)], we obtain

$$|u|^*_{2,\alpha;U} \le C\epsilon.$$

Since $u = w - h$, it follows that for ϵ small enough, $w \in \mathcal{A}_\epsilon$, from Schauder estimates and for R_2 big enough ϵ depends only on R_1, thus the proposition is proved. □

Applying the Schauder fixed point theorem to the operator $w = Tf$, we obtain a solution $f^\pm \in \mathcal{A}_\epsilon$ which satisfies equation (3.1).

Now we prove that f^+ converges to the horocylinder $C(1+\epsilon)$ uniformly on compact subsets as R_2 tends to $+\infty$, the f^- case is similar. Take K a compact set in U. Now enlarge U by making R_2 tend to infinity, this produces a family of functions h (one for each such R_2). Note that the restriction of this sequences of functions to the fixed compact set K converges uniformly to the value $1 + \epsilon$.

On the other hand, given $\rho > 0$ and some compact $K \subset (C(1) - B_{R_1})$, by the definition of $\mathcal{A}_\epsilon$ and the existence part, there is some R_2 large enough and some ϵ_1 small enough (depending only on R_1 and ρ, not on R_2 or K) such that for any $\epsilon < \epsilon_1$, the function f associated to such h is ρ-close to $1+\epsilon$, that is, when R_2 tend to infinity the functions f^+ converges uniformly to $1 + \epsilon$. □

3.2. The main theorem.

Now we prove the main theorem.

Theorem 3.6 (Main theorem)**.** *Let Σ be a properly embedded constant mean curvature $H = 1/2$ surface in $\widetilde{PSL}_2(\mathbb{R}, \tau)$. Suppose Σ is asymptotic to a horocylinder C, and one side of C. If the mean curvature vector of Σ has the same direction as that of C at points of Σ converging to C, then Σ is equal to C (or a subset of C if $\partial\Sigma \neq \emptyset$).*

Proof. Assume that Σ is not a subset of C. After an isometry, we can assume that, there is a sequence of points $p_i = (x_i, y_i, t_i) \in \Sigma$ with $y_i \to 1$. First, we suppose that Σ is contained in the set $\{y > 1\}$, the other case is treated analogously. We denote by $C(\xi)$ the horocylinder in $\widetilde{PSL}_2(\mathbb{R}, \tau)$ given by $\{y = \xi\}$. For $\epsilon > 0$ small we consider the slab S^+ bounded by $C(1)$ and $C(1 + \epsilon)$. Then by the maximum principle $\Sigma^+ = \Sigma \cap S^+$ has a non compact component with boundary $\partial\Sigma \subset C(1 + \epsilon)$.

Let $D(\xi, R)$ denote the disk in $C(\xi)$ defined by $D(\xi, R) = \{(x, \xi, t);\ x^2 + t^2 \le R^2\}$. By considering vertical translation, we can find a disk $D(1, 3R_1)$ such that:

$$(D(1, 3R_1) \times [1, 1 + \epsilon]) \cap \Sigma^+ = \emptyset.$$

By Theorem 3.2, for each $R \geq 4R_1$, there exist a horizontal graph f_R^+ defined on the annulus $U = \bar{B}_{R_2} \setminus B_{R_1}$, this horizontal graph converge to $C(1+\epsilon)$, when R goes to $+\infty$.

Now, consider R large, such that the graph of f_R^+ (which we denote by Γ^+), satisfies $\Sigma^+ \cap \Gamma^+ \neq \emptyset$. By considering vertical translations and translations along the geodesic $\{x = 0, t = 0\}$, the translated surface of Γ^+ does not touch Σ^+, that is, there is a translated surface of Γ^+ (which we denote by Γ_1^+) such that Γ_1^+ and Σ^+ has an interior contact point. Since the mean curvature vectors are pointing up, this violates the maximum principle and Σ^+ cannot exist.

In the second case, we redo exactly the same argument exchanging the roles of $C(1+\epsilon)$ and $C(1-\epsilon)$. $\square$

4. The second theorem

In this section our second result concerns complete $H = 1/2$ surfaces in $\widetilde{PSL}_2(\mathbb{R},\tau)$ transverse to the vertical Killing field $E_3 = \partial_t$, we use Theorem 1.1 in order to prove such surfaces are entire graphs. This result was proved in a totally different way in [1] and [3].

Theorem 4.1. *Let Σ be a complete immersed surface in $\widetilde{PSL}_2(\mathbb{R},\tau)$ of constant mean curvature $H = 1/2$. If Σ is transverse to E_3 then Σ is an entire vertical graph over $\mathbb{H}^2$.*

The proof of this theorem is analogous to this one in [5, Theorem 1.2] taking into account [7]. It was showed in [5, Theorem 1.2], that, there is $\epsilon > 0$ and a horocylinder such that, a graph $G \subset \Sigma$ (over a domain in $\mathbb{H}^2 \times \{0\} \subset \widetilde{PSL}_2(\mathbb{R},\tau)$) is in the ϵ-tubular neighborhood of the cylinder. Since G is proper the proof of the half-space theorem shows that this graph can not exist.

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