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SIMPLICAL COMPLEXES ASSOCIATED TO QUIVERS ARISING FROM FINITE GROUPS

Dedicated to Professor Hiroyoshi Yamaki on the occasion of his 75th birthday

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Abstract

In this paper, we will introduce a simplicial complex $\mathrm{T}_Q(\mathcal{H})$ defined by a quiver Q and a family $\mathcal{H}$ of paths in Q. We call $\mathrm{T}_Q(\mathcal{H})$ a path complex of $\mathcal{H}$ in Q. Let G be a finite group, and denote by $\mathrm{Sgp}(G)$ and $\mathrm{Coset}(G)$ respectively the totality of subgroups of G, and that of left cosets $gL \in G/L$ of subgroups L of G. We will particularly focus on quivers Q_G and Q_{CG} obtained naturally from posets $\mathrm{Sgp}(G)$ and $\mathrm{Coset}(G)$ ordered by the inclusion-relation. Then various properties of path complexes associated to Q_G and Q_{CG} will be studied.

Contents

1. Introduction 162
2. Preliminaries 163
3. Some variations of Q 164
4. Quivers from groups 166
 4.1. Subgroup quivers Q_G. 166
 4.2. Coset quivers Q_{CG}. 167
5. Homology R-modules associated to Q 169
 5.1. Families of paths and R-complexes. 169
 5.2. Homology R-modules. 173
6. Simplicial complexes associated to paths 175
 6.1. Path complexes $\mathrm{T}_Q(\mathcal{H})$. 176
 6.2. Subgroup complexes $\mathrm{T}_{Q_G^{\mathrm{ud}}}(\mathcal{D})$. 180
 6.3. Coset complexes $\mathrm{T}_{Q_{CG}^{\mathrm{ud}}}(\mathcal{H})$ and G-simplicial maps. 182
7. Some properties of subgroup and coset complexes 184
 7.1. Ranges of paths in $\tilde{\mathfrak{G}}(\Delta)$. 184
 7.2. The Euler characteristic of $\mathrm{T}_{Q_{CG}^{\mathrm{ud}}}(\tilde{\mathfrak{G}}(\Delta))$. 191
 7.3. The top homology of $\mathrm{T}_{Q_{CG}^{\mathrm{ud}}}(\tilde{\mathfrak{G}}(\Delta))$ is trivial. 196
 7.4. A relation with coset geometries. 197
 7.5. Connectedness. 199
 7.6. Preimages under $\varphi_{G,\Delta}$. 202
References 203

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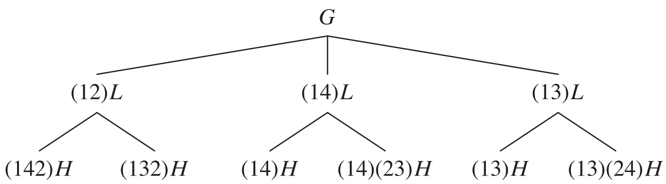
1. Introduction

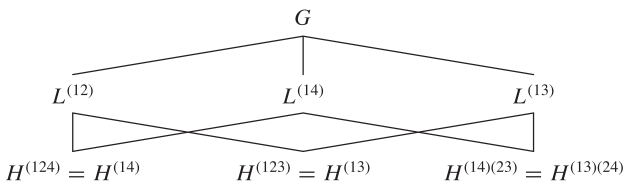
Let G be a finite group. Denote by $\mathrm{Sgp}(G)$ the totality of subgroups of G. The structure of the subgroup lattice $(\mathrm{Sgp}(G), \leq)$, where $\leq$ is the inclusion-relation, is quite important for investigating the group G itself. In more general, for a family $\mathcal{D} \subseteq \mathrm{Sgp}(G)$ of subgroups, we are interested in the structure of a poset $(\mathcal{D}, \leq)$. This tells us that how certain subgroups of G are piled up, or related each other. On the other hand, denote by $\mathrm{Coset}(G)$ the totality of left cosets $gL \in G/L$ of subgroups L of G which is regarded as a poset with respect to the inclusion-relation $\subseteq$. Let φ: $\mathrm{Coset}(G) \to \mathrm{Sgp}(G)$ be a surjective G-map defined by $\varphi(gL) := L^{g^{-1}}$ for all cosets $gL \in \mathrm{Coset}(G)$. Then the subgroup lattice $(\mathrm{Sgp}(G), \leq)$ is contained in $(\mathrm{Coset}(G), \subseteq)$. Furthermore, $(\mathrm{Sgp}(G), \leq)$ is realized by gluing, via φ, some cosets in $(\mathrm{Coset}(G), \subseteq)$. For example, we consider a sequence

$$G := \mathrm{Sym}(\{1, 2, 3, 4\}) > L := \mathrm{Sym}(\{2, 3, 4\}) > H := \mathrm{Sym}(\{3, 4\})$$

of subgroups where $\mathrm{Sym}(X)$ is the symmetric group on a set X. The following is a part of $(\mathrm{Coset}(G), \subseteq)$.



Then identifying some cosets via φ, we have the following which is a part of $(\mathrm{Sgp}(G), \leq)$.



From this reason, our main interests are $(\mathrm{Sgp}(G), \leq)$ and $(\mathrm{Coset}(G), \subseteq)$, and also their subposets. In order to examine those posets in more detail, we consider a quiver $Q_{\mathcal{P}}$ associated to a poset $(\mathcal{P}, \leq)$ whose vertex set is $\mathcal{P}$ and an arrow $(a \to b)$ for $a, b \in \mathcal{P}$ is defined precisely when $a > b$. Then we obtain quivers Q_G and Q_{CG} from posets $\mathrm{Sgp}(G)$ and $\mathrm{Coset}(G)$ respectively. Furthermore, we introduce a simplicial complex $\mathrm{T}_Q(\mathcal{H})$ defined by a quiver Q and a family $\mathcal{H}$ of paths in Q. We call $\mathrm{T}_Q(\mathcal{H})$ a path complex of $\mathcal{H}$ in Q. In this paper, we study various properties of path complexes associated to Q_G and Q_{CG}. At the same time, a general theory by using arbitrary quivers instead of Q_G and Q_{CG} is also developed.

The paper is organized as follows: In Section 2, we recall the basic definitions on quivers Q. In Section 3, some variations of Q are defined. In particular, the extended quiver Q^{ud} of Q is our fundamental object in this paper, and the closure $\overline{Q}$ of Q is important for defining homology of Q. In Section 4, we consider quivers Q_G and Q_{CG} appeared in the above. In Section 5, we establish three kinds of homologies of Q by using families $\mathsf{P}(\overline{Q})$, $\mathsf{P}(\overline{Q})^{\mathrm{pr}}$, $E\overline{Q}$ of paths in the closure $\overline{Q}$ of Q. A family $E\overline{Q}$ also provides a simplicial complex which reflects the original quiver Q. In Section 6, we introduce a path complex $\mathrm{T}_Q(\mathcal{H})$ mentioned above, and develop some general theory on $\mathrm{T}_Q(\mathcal{H})$. Moreover, we deal with those complexes associated to the extended quivers Q_G^{ud} and Q_{CG}^{ud} which we call subgroup complexes and coset complexes respectively. In Section 7, some other properties of our subgroup and coset complexes are investigated.

2. Preliminaries

In this section, we recall some definitions related to quivers, and establish our notations which will be used later. Throughout this paper, let R be a commutative ring with the identity element. For a set X, denote by $\mathrm{Sym}(X)$ the symmetric group on X. For maps $f\colon X \to Y$ and $g\colon Y \to Z$, the composition map $g \circ f$ is read from right to left, namely $(g \circ f)(x) := g(f(x)) \in Z$ for any $x \in X$.

DEFINITION 2.1. A quiver Q is a quadruple

$$Q = (Q_0, Q_1, (s\colon Q_1 \to Q_0), (r\colon Q_1 \to Q_0))$$

where Q_0 $(\neq \emptyset)$ and Q_1 are sets, and their elements are called vertices and arrows of Q respectively. Furthermore s and r are maps from Q_1 to Q_0. For an arrow $\alpha \in Q_1$, if $s(\alpha) = a$ and $r(\alpha) = b$ then denote by $a \xrightarrow{\alpha} b$ or $\alpha = (a \to b)$. Elements $s(\alpha)$ and $r(\alpha)$ are called the start and range of α respectively.

DEFINITION 2.2. Let $Q = (Q_0, Q_1, s, r)$ be a quiver.
(1) A path Δ in Q is either a sequence $(\alpha_1, \alpha_2, \dots, \alpha_k)$ $(k \geq 1)$ of arrows $\alpha_i = (a_{i-1} \to a_i) \in Q_1$ satisfying $r(\alpha_i) = s(\alpha_{i+1})$ for $(1 \leq i \leq k-1)$, or the symbol e_a for $a \in Q_0$ which is called the trivial path. In this case, we also write

$$\Delta = (a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} a_2 \to \cdots \to a_{k-1} \xrightarrow{\alpha_k} a_k)$$

or

$$e_a = (a).$$

Note that we identify a vertex a with e_a. Denote by $\mathsf{P}(Q)$ and $\mathsf{P}(Q)^{\mathrm{non}}$ respectively the totality of paths in Q, and that of non-trivial paths in Q.
(2) For a non-trivial path $\Delta = (\alpha_1, \alpha_2, \dots, \alpha_k) \in \mathsf{P}(Q)$, define $s(\Delta) := s(\alpha_1)$ and $r(\Delta) := r(\alpha_k)$. Furthermore, define $s(e_a) := a$ and $r(e_a) := a$ for $a \in Q_0$.

(3) For a non-trivial path $\Delta = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathsf{P}(Q)$, denote by $l(\Delta)$ the length k of Δ. We set $l(e_a) := 0$ for $a \in Q_0$. The notation $\mathsf{P}(Q)_i$ $(i \geq 0)$ stands for the totality of paths of length i, namely $\mathsf{P}(Q)_i := \{\Delta \in \mathsf{P}(Q) \mid l(\Delta) = i\}$. In particular, $\mathsf{P}(Q)_0 = \{(a) \mid a \in Q_0\}$ is the totality of trivial paths in Q.
(4) For $a, b \in Q_0$ and a subset $\mathcal{H} \subseteq \mathsf{P}(Q)$, denote by $\mathcal{H}_{a \Rightarrow b}$ the totality of paths $\Delta \in \mathcal{H}$ with $s(\Delta) = a$ and $r(\Delta) = b$.
(5) The path algebra $R[Q]$ of Q over R is the R-free module with all paths in Q as basis, and a multiplication on $R[Q]$ is defined by extending bilinearly the composition

$$\Delta_1 \Delta_2 := \begin{cases} (\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_m) & \text{if} \quad r(\alpha_k) = s(\beta_1), \\ 0 & \text{otherwise} \end{cases}$$

of paths $\Delta_1 = (\alpha_1, \ldots, \alpha_k)$, $\Delta_2 = (\beta_1, \ldots, \beta_m) \in \mathsf{P}(Q)$. Then $R[Q]$ is an associative R-algebra.

DEFINITION 2.3. For a set X, denote by $\mathsf{Mod}(X)_R$ the R-free module with basis X. Under this notation, we have that $R[Q] = \mathsf{Mod}(\mathsf{P}(Q))_R$ as R-modules.

DEFINITION 2.4 (Proper paths). Let $Q = (Q_0, Q_1, s, r)$ be a quiver, and $\Delta = (a_0 \to \cdots \to a_k) \in \mathsf{P}(Q)$ $(k \geq 0)$ be a path in Q.
(1) Denote by $\mathsf{Ob}(\Delta) := \{a_0, \ldots, a_k\} \subseteq Q_0$ the set of vertices of Q which make Δ.
(2) Δ is proper if $a_i \neq a_j$ for all distinct i, j $(0 \leq i, j \leq k)$. Note that Δ is proper if and only if $|\mathsf{Ob}(\Delta)| = k + 1$. In particular, a trivial path is always proper.
(3) For a subset $\mathcal{H} \subseteq \mathsf{P}(Q)$, denote by $\mathcal{H}^{\mathrm{pr}}$ the totality of proper paths in $\mathcal{H}$. For example, the notation $\mathsf{P}(Q)_i^{\mathrm{pr}}$ $(i \geq 0)$ means $(\mathsf{P}(Q)_i)^{\mathrm{pr}} = \mathsf{P}(Q)_i \cap \mathsf{P}(Q)^{\mathrm{pr}}$.

DEFINITION 2.5 (G-quivers). Let $Q = (Q_0, Q_1, s, r)$ be a quiver, and G be a group. We call Q a G-quiver if the following conditions hold:
(1) G acts on the sets Q_0 and Q_1, that is, there exist group homomorphisms $f_0 \colon G \to \mathrm{Sym}(Q_0)$ and $f_1 \colon G \to \mathrm{Sym}(Q_1)$. For $g \in G$, $a \in Q_0$, and $\alpha \in Q_1$, denote by $g \cdot a := f_0(g)(a)$ and $g \cdot \alpha := f_1(g)(\alpha)$.
(2) For $g \in G$ and $\alpha \in Q_1$, we have that $s(g \cdot \alpha) = g \cdot s(\alpha)$ and $r(g \cdot \alpha) = g \cdot r(\alpha)$. In other words, if $\alpha = (a \to b)$ then $g \cdot \alpha = (g \cdot a \to g \cdot b)$.
Note that for vertices $a, b \in Q_0$, if $g \cdot a = b$ for some $g \in G$ then we write $a \sim_G b$.

REMARK 2.6. Let $Q = (Q_0, Q_1, s, r)$ be a G-quiver. Then G acts on both $\mathsf{P}(Q)$ and $\mathsf{P}(Q)^{\mathrm{pr}}$ in such a way that $g \cdot \Delta := ((g \cdot \alpha_1), \ldots, (g \cdot \alpha_k))$ for $\Delta = (\alpha_1, \ldots, \alpha_k) \in \mathsf{P}(Q)$ and $g \in G$.

3. Some variations of Q

In this section, we introduce some variations of a quiver Q. In particular, the extended quiver Q^{ud} of Q is a fundamental object in this paper, and the closure $\overline{Q}$ of Q will play an important role in Section 5 where a homology of Q is defined.

DEFINITION 3.1 (Extended quivers; cf. Definition 3.9 and Remark 3.10 in [2]). Let $Q = (Q_0, Q_1, s, r)$ be a quiver. For each arrow $\alpha = (a \to b) \in Q_1$, we define the symbol ${}^t\alpha$. Set $Q_1^{\mathrm{opp}} := \{{}^t\alpha \mid \alpha \in Q_1\}$ and $Q_1^{\mathrm{ud}} := Q_1 \cup Q_1^{\mathrm{opp}}$. Then

$$Q^{\mathrm{ud}} := (Q_0, Q_1^{\mathrm{ud}}, (s\colon Q_1^{\mathrm{ud}} \to Q_0), (r\colon Q_1^{\mathrm{ud}} \to Q_0))$$

forms a quiver where s and r are extended on Q_1^{ud} as $s({}^t\alpha) := r(\alpha) = b$ and $r({}^t\alpha) := s(\alpha) = a$ for $\alpha = (a \to b) \in Q_1$. Thus ${}^t\alpha = (b \to a)$. We call ${}^t\alpha$ the opposite arrow of α. Note that $\mathsf{P}(Q) \subseteq \mathsf{P}(Q^{\mathrm{ud}})$.

REMARK 3.2. If Q is a G-quiver then, for $\alpha = (a \to b) \in Q_1$ and $g \in G$, we define,

$$g \cdot ({}^t\alpha) := {}^t(g \cdot \alpha) = {}^t(g \cdot a \to g \cdot b) = (g \cdot b \to g \cdot a).$$

This makes Q^{ud} a G-quiver.

NOTATION 3.3 (Up-down paths). For arrows $\alpha, \beta \in Q_1$ such that $r(\alpha) = r(\beta)$, we just write $\Delta = (\alpha, \beta)$ for a path $\Delta = (\alpha, {}^t\beta)$ in Q^{ud} where $r(\alpha) = r(\beta) = s({}^t\beta)$. Similarly, for arrows $\alpha, \beta \in Q_1$ such that $s(\alpha) = s(\beta)$, the notation $\Delta = (\alpha, \beta)$ indicates a path $\Delta = ({}^t\alpha, \beta)$ in Q^{ud} where $r({}^t\alpha) = s(\alpha) = s(\beta)$. For example, for arrows $\alpha_1 = (a \to b)$, $\alpha_2 = (c \to b)$, $\alpha_3 = (d \to c)$, $\alpha_4 = (d \to e)$ in Q_1, the notation

$$\Delta = (a \xrightarrow{\alpha_1} b \xleftarrow{\alpha_2} c \xleftarrow{\alpha_3} d \xrightarrow{\alpha_4} e)$$

implies a path

$$\Delta = (a \xrightarrow{\alpha_1} b \xrightarrow{{}^t\alpha_2} c \xrightarrow{{}^t\alpha_3} d \xrightarrow{\alpha_4} e)$$

in Q^{ud}. So any path $\Delta \in \mathsf{P}(Q^{\mathrm{ud}})$ in Q^{ud} can be expressed as $\Delta = (a_0 \overset{\alpha_1}{-} a_1 \overset{\alpha_2}{-} a_2 - \cdots - a_{k-1} \overset{\alpha_k}{-} a_k)$ for some $\alpha_i \in Q_1$ $(i = 1, \dots, k)$ where $-$ means $\to$ or $\leftarrow$. Throughout this paper, we frequently use this way of writing for paths without using opposite arrows.

DEFINITION 3.4 (Restrictions). Let $Q = (Q_0, Q_1, s, r)$ be a quiver. For a subset $A \subseteq Q_1$, we set

$$(Q_A)_0 := \{s(\alpha) \mid \alpha \in A\} \cup \{r(\alpha) \mid \alpha \in A\} \subseteq Q_0.$$

Denote the restrictions $s|_A\colon A \to (Q_A)_0$ and $r|_A\colon A \to (Q_A)_0$ by just the same notations s and r. Then $Q_A := ((Q_A)_0, A, s, r)$ forms a quiver which we call the restriction of Q to A.

DEFINITION 3.5 (Closures). Let $Q = (Q_0, Q_1, s, r)$ be a quiver. The maps $s, r\colon Q_1 \to Q_0$ can be extended as maps

$$s\colon \mathsf{P}(Q)^{\mathrm{non}} \to Q_0 \quad \text{by} \quad \Delta \mapsto s(\Delta),$$
$$r\colon \mathsf{P}(Q)^{\mathrm{non}} \to Q_0 \quad \text{by} \quad \Delta \mapsto r(\Delta).$$

Then $\overline{Q} := (Q_0, \mathsf{P}(Q)^{\mathrm{non}}, s, r)$ forms a quiver which we call the closure of Q.

EXAMPLE 3.6. Let Q be a quiver defined as follows:

$$a \overset{\alpha}{\to} b \overset{\beta}{\to} c.$$

This yields that $\mathsf{P}(Q)_0 = Q_0 = \{a, b, c\}$, $\mathsf{P}(Q)_1 = \{(a \overset{\alpha}{\to} b), (b \overset{\beta}{\to} c)\}$, and $\mathsf{P}(Q)_2 = \{(a \overset{\alpha}{\to} b \overset{\beta}{\to} c)\}$. Then by the definition of the closure of Q, we have that

$$\mathsf{P}(\overline{Q})_0 = \{a, b, c\}, \quad \mathsf{P}(\overline{Q})_1 = \{\alpha, \beta, \Delta_1 = (a \to c)\}, \quad \mathsf{P}(\overline{Q})_2 = \{(a \overset{\alpha}{\to} b \overset{\beta}{\to} c)\}$$

where Δ_1 comes from $\mathsf{P}(Q)_2$. Similarly

$$\mathsf{P}(\overline{\overline{Q}})_0 = \{a, b, c\}, \quad \mathsf{P}(\overline{\overline{Q}})_1 = \{\alpha, \beta, \Delta_1, \Delta_2 = (a \to c)\}, \quad \mathsf{P}(\overline{\overline{Q}})_2 = \{(a \overset{\alpha}{\to} b \overset{\beta}{\to} c)\}$$

where Δ_2 comes from $\mathsf{P}(\overline{Q})_2$. Now set $Q^0 := Q$ and $Q^k := \overline{Q^{k-1}}$ $(k \geq 1)$. Then we have that

$$\mathsf{P}(Q^k)_0 = \{a, b, c\}, \quad \mathsf{P}(Q^k)_1 = \{\alpha, \beta, \Delta_1, \dots, \Delta_k\}, \quad \mathsf{P}(Q^k)_2 = \{(a \overset{\alpha}{\to} b \overset{\beta}{\to} c)\}$$

where Δ_i $(1 \leq i \leq k)$ is an arrow from a to c. In Section 5.2, we will calculate a homology of Q^k $(k \geq 0)$.

4. Quivers from groups

Let G be a group. In this section, we introduce quivers Q_G and Q_{CG} associated to subgroups of G, and to left cosets of subgroups of G. Later in Sections 6 and 7, they will be investigated in more detail.

4.1. Subgroup quivers Q_G.

First of all, we establish a quiver associated to a poset (partially ordered set) in general.

DEFINITION 4.1. Let $(\mathcal{P}, \leq)$ be a poset. For elements $a, b \in \mathcal{P}$, we define an arrow $(a \to b)$ precisely when $a > b$. Put $(Q_{\mathcal{P}})_0 := \mathcal{P}$ and $(Q_{\mathcal{P}})_1 := \{(a \to b) \mid a, b \in \mathcal{P},\ a > b\}$. Then denote by

$$Q_{\mathcal{P}} := ((Q_{\mathcal{P}})_0, (Q_{\mathcal{P}})_1, (s\colon (Q_{\mathcal{P}})_1 \to (Q_{\mathcal{P}})_0), (r\colon (Q_{\mathcal{P}})_1 \to (Q_{\mathcal{P}})_0))$$

a quiver where $s(\alpha) := a$ and $r(\alpha) := b$ for $\alpha = (a \to b) \in (Q_{\mathcal{P}})_1$.

REMARK 4.2. Suppose that $(\mathcal{P}, \leq)$ is a G-poset, namely, G acts on the set $\mathcal{P}$, and the action of G preserves the ordering $\leq$. Then it is clear that $Q_{\mathcal{P}}$ becomes a G-quiver.

DEFINITION 4.3. Let G be a group, and let $\mathrm{Sgp}(G)$ be the totality of subgroups of G including the whole group G and the trivial subgroup $\{e\}$. This can be viewed as a poset together with the inclusion-relation $\leq$. Denote by

$$Q_G := Q_{(\mathrm{Sgp}(G),\leq)}$$

a quiver associated to a poset $(\mathrm{Sgp}(G), \leq)$ (see Definition 4.1). We call Q_G a subgroup quiver of G.

REMARK 4.4. In this paper, the extended quiver Q_G^{ud} of Q_G (see Definition 3.1) is our interest rather than just Q_G itself. Indeed, a path $\Delta = (L_0 \to \cdots \to L_k)$ in Q_G is simply an inclusion-chain $(L_0 > \cdots > L_k)$ of subgroups of G. However, in Q_G^{ud}, inclusion-chains $(L_0 > \cdots > L_k = M_1 < \cdots < M_t)$, $(L_0 < \cdots < L_k = M_1 > \cdots > M_t)$, and their every combinations are considered as paths. Thus Q_G^{ud} has much more information of the subgroup lattice $(\mathrm{Sgp}(G), \leq)$.

4.2. Coset quivers Q_{CG}.

DEFINITION 4.5. Let G be a group, and let $\mathrm{Coset}(G) := \bigcup_{L\in \mathrm{Sgp}(G)} G/L$ be the totality of left cosets of L in G for all subgroups $L \in \mathrm{Sgp}(G)$. We regard $\mathrm{Coset}(G)$ as a poset together with the inclusion-relation $\subseteq$. Denote by

$$Q_{CG} := Q_{(\mathrm{Coset}(G),\subseteq)}$$

a quiver associated to a poset $(\mathrm{Coset}(G), \subseteq)$ (see Definition 4.1). We call Q_{CG} a coset quiver of G. As in Remark 4.4, the extended quiver Q_{CG}^{ud} of Q_{CG} is an object for consideration rather than just Q_{CG}.

REMARK 4.6 (G-quivers Q_G and Q_{CG}). $(\mathrm{Sgp}(G), \leq)$ is a G-poset together with G-conjugate action, that is, for $g \in G$ and $L \in \mathrm{Sgp}(G)$, $g \cdot L := gLg^{-1} = L^{g^{-1}}$ is a member of $\mathrm{Sgp}(G)$. Furthermore G acts on $\mathrm{Coset}(G)$ by the left multiplication $x \cdot gL := x(gL) = (xg)L$ for $x \in G$ and $gL \in \mathrm{Coset}(G)$. This makes $(\mathrm{Coset}(G), \subseteq)$ a G-poset. Thus by Remark 4.2, associated quivers Q_G and Q_{CG} are both G-quivers, and so are Q_G^{ud} and Q_{CG}^{ud} by Remark 3.2. In particular, $\mathsf{P}(Q_G^{\mathrm{ud}})$ and $\mathsf{P}(Q_{CG}^{\mathrm{ud}})$ are G-invariant by Remark 2.6.

On the other hand, for a coset $gL \in G/L$ and $x \in G$, we have that $(gL)x = (gx)(x^{-1}Lx) = (gx)L^x$. This implies that G acts on $\mathrm{Coset}(G)$ by the right multiplication.

Now, we introduce paths in Q_{CG}^{ud} obtained from paths in Q_G^{ud}.

DEFINITION 4.7. For a path $\Delta = (L_0 - \cdots - L_k) \in \mathsf{P}(Q_G^{\mathrm{ud}})$ in Q_G^{ud} and a subset $\mathcal{D} \subseteq \mathsf{P}(Q_G^{\mathrm{ud}})$, set

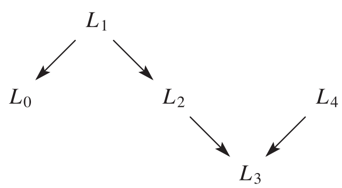
$$\tilde{\mathfrak{G}}(\Delta) := \{(A_0 - \cdots - A_k) \in \mathsf{P}(Q_{CG}^{\mathrm{ud}}) \mid A_j \in G/L_j \ (0 \le j \le k)\},$$
$$\tilde{\mathfrak{G}}(\mathcal{D}) := \bigcup_{\Delta \in \mathcal{D}} \tilde{\mathfrak{G}}(\Delta) \subseteq \mathsf{P}(Q_{CG}^{\mathrm{ud}}).$$

In particular, we have that $\tilde{\mathfrak{G}}(\{\Delta\}) = \tilde{\mathfrak{G}}(\Delta)$.

REMARK 4.8. Let $H < K \le G$ be subgroups of G. Suppose that G is finite.
(1) For each coset gH in G/H, there exists an unique coset A in G/K containing gH, that is, $A = gK$.
(2) For each coset gK in G/K, there are exactly $t := |K : H|$ cosets in G/H contained in gK. Indeed, if $K = a_1H \cup \cdots \cup a_tH$ is a decomposition into left cosets of H in K then $\{ga_1H, \ldots, ga_tH\}$ is the set of all required cosets.
(3) For subgroups $L_1, L_2 \le G$, suppose that $aL_1 \subseteq bL_2$ for $a, b \in G$. Then $b^{-1}a \in (b^{-1}a)L_1 \subseteq L_2$ and $L_1 = (b^{-1}a)^{-1}(b^{-1}a)L_1 \subseteq (b^{-1}a)^{-1}L_2 = L_2$. Thus $L_1 \le L_2$. In particular, if $aL_1 = bL_2$ then $L_1 = L_2$, and this implies that $G/L_1 \cap G/L_2 = \emptyset$ if $L_1 \ne L_2$. It follows that if $\Delta \in \mathsf{P}(Q_G^{\mathrm{ud}})$ is proper then so is any path in $\tilde{\mathfrak{G}}(\Delta)$.

From the observations in Remark 4.8 (1) and (2), paths in $\tilde{\mathfrak{G}}(\Delta)$ can be described according to up-down information of Δ. We demonstrate this fact in the next example.

EXAMPLE 4.9. Suppose that G is finite. Let $\Delta = (L_0 \leftarrow L_1 \rightarrow L_2 \rightarrow L_3 \leftarrow L_4) \in \mathsf{P}(Q_G^{\mathrm{ud}})$ be a path in Q_G^{ud} which is drawn as follows:



In other words, Δ is an inclusion-chain $(L_0 < L_1 > L_2 > L_3 < L_4)$ of subgroups L_i of G. In this case, any paths $(A_0 \leftarrow A_1 \rightarrow A_2 \rightarrow A_3 \leftarrow A_4)$ in $\tilde{\mathfrak{G}}(\Delta)$ are described as follows: First, any coset gL_0 in G/L_0 can be taken as A_0. A coset A_1 in G/L_1 must contain $A_0 = gL_0$. So it is uniquely determined as $A_1 = gL_1$. Since a coset A_2 in G/L_2 is contained in $A_1 = gL_1$, it is one of $ga_1L_2, \ldots, ga_tL_2$ where $L_1 = a_1L_2 \cup \cdots \cup a_tL_2$ is a decomposition into left cosets of L_2 in L_1. By the same way, for each $A_2 = ga_iL_2$, there are exactly $|L_2 : L_3|$ cosets A_3 in G/L_3 contained in A_2. Finally, for each such coset $A_3 = hL_3$ in G/L_3, A_4 is uniquely determined as $A_4 = hL_4$. Therefore the number of paths in $\tilde{\mathfrak{G}}(\Delta)$ is

$$|G : L_0| \times 1 \times |L_1 : L_2| \times |L_2 : L_3| \times 1.$$

Refer to Proposition 7.4 for a general result on the number of paths in $\tilde{\mathfrak{G}}(\Delta)$.

REMARK 4.10 (G-invariant $\tilde{\mathfrak{G}}(\mathcal{D})$). Recall that Q_G^{ud} and Q_{CG}^{ud} are both G-quivers, and that $\mathsf{P}(Q_G^{\mathrm{ud}})$ and $\mathsf{P}(Q_{CG}^{\mathrm{ud}})$ are preserved by G-conjugate action and the left multiplication respectively (see Remark 4.6). Let $\Delta = (L_0 - \cdots - L_k) \in \mathsf{P}(Q_G^{\mathrm{ud}})$. Then $\tilde{\mathfrak{G}}(\Delta)$ is G-invariant under the left multiplication, that is, for a path $\Gamma = (g_0 L_0 - \cdots - g_k L_k) \in \tilde{\mathfrak{G}}(\Delta)$ and $x \in G$, a path $x \cdot \Gamma := (xg_0 L_0 - \cdots - xg_k L_k)$ is in $\tilde{\mathfrak{G}}(\Delta)$. In particular, so is $\tilde{\mathfrak{G}}(\mathcal{D})$ for any subset $\mathcal{D} \subseteq \mathsf{P}(Q_G^{\mathrm{ud}})$.

On the other hand, for a coset $gL \in G/L$ and $x \in G$, we have that $(gL)x = (gx)(x^{-1}Lx) = (gx)L^x$. This tells us that if $\mathcal{D}$ is G-invariant, then so is $\tilde{\mathfrak{G}}(\mathcal{D})$ under the right multiplication.

Lemma 4.11 (Semi-regularity)**.** *Let $\Delta = (L_0 - \cdots - L_k) \in \mathsf{P}(Q_G^{\mathrm{ud}})$ be a path in Q_G^{ud}, and let G be a finite group. Suppose that there exist i, j $(0 \le i \ne j \le k)$ such that* $\gcd(|L_i|, |L_j|) = 1$. *Then the action of G on $\tilde{\mathfrak{G}}(\Delta)$ is semi-regular. In particular, $|G|$ divides* $|\tilde{\mathfrak{G}}(\Delta)|$.

Proof. For any $\Gamma = (x_0 L_0 - \cdots - x_k L_k) \in \tilde{\mathfrak{G}}(\Delta)$, let S be the stabilizer in G of Γ, that is, $S = \{g \in G \mid g \cdot \Gamma = \Gamma\} = \bigcap_{u=0}^{k} (L_u)^{x_u^{-1}} \le (L_i)^{x_i^{-1}} \cap (L_j)^{x_j^{-1}}$. Since $\gcd(|L_i|, |L_j|) = 1$ by our assumption, we have that $S = \{e\}$. □

5. Homology *R*-modules associated to *Q*

Let $Q = (Q_0, Q_1, s, r)$ be a quiver, and let $\overline{Q} = (Q_0, \mathsf{P}(Q)^{\mathrm{non}}, s, r)$ be the closure of Q (see Definition 3.5). The path algebra $R[\overline{Q}] = \mathsf{Mod}(\mathsf{P}(\overline{Q}))_R$ of $\overline{Q}$ over R can be regarded as an R-complex together with a certain R-endomorphism ∂ of $R[\overline{Q}]$. Then we consider subcomplexes $\mathsf{Mod}(\mathsf{P}(\overline{Q})^{\mathrm{pr}})_R$ and $\mathsf{Mod}(E\overline{Q})_R$ of $R[\overline{Q}]$ corresponding to families $\mathsf{P}(\overline{Q})^{\mathrm{pr}}$ and $E\overline{Q}$ in $\mathsf{P}(\overline{Q})$, so that three kinds of homologies associated to Q are defined. Furthermore, we see that $E\overline{Q}$ provides a simplicial complex $K_{E\overline{Q}}$ which reflects the original quiver Q. For homological algebras, we refer to [1, Chapter IV].

5.1. Families of paths and *R*-complexes.

Recall that the set $\mathsf{P}(\overline{Q})$ of paths in $\overline{Q}$ is described as follows:

$$\mathsf{P}(\overline{Q}) = \{(x_0 \xrightarrow{\Delta_1} x_1 \to \cdots \to x_{k-1} \xrightarrow{\Delta_k} x_k) \mid k \ge 0,\ \Delta_i \in \mathsf{P}(Q)^{\mathrm{non}}\}.$$

Note that a sequence $(\Delta_1, \dots, \Delta_k)$ of paths $\Delta_i \in \mathsf{P}(Q)^{\mathrm{non}}$ is a member of $\mathsf{P}(\overline{Q})$ if and only if the product $\Delta_1 \cdots \Delta_k$ in the path algebra $R[Q]$ of Q is non-zero. The path algebra $R[\overline{Q}]$ of $\overline{Q}$ over R is a positive graded R-free module (cf. [1, p. 58])

$$R[\overline{Q}] = \mathsf{Mod}(\mathsf{P}(\overline{Q}))_R = \bigoplus_{n \ge 0} C_n(\overline{Q})$$

where $C_n(\overline{Q}) := \mathsf{Mod}(\mathsf{P}(\overline{Q})_n)_R$ is the R-free module with all paths in $\overline{Q}$ of length n as basis. In particular, $\mathsf{P}(\overline{Q})_0$ is the set of all trivial paths $e_x = (x)$ ($x \in Q_0$) in $\overline{Q}$, so we have $C_0(\overline{Q}) = \mathsf{Mod}((x) \mid x \in Q_0)_R$.

Let $\partial\colon R[\overline{Q}] \to R[\overline{Q}]$ be a map defined by, for $(\Delta_1, \ldots, \Delta_n) \in \mathsf{P}(\overline{Q})_n$ $(n \geq 2)$,

$$\partial(\Delta_1, \ldots, \Delta_n) := (\Delta_2, \ldots, \Delta_n) + \sum_{i=1}^{n-1}(-1)^i(\Delta_1, \ldots, (\Delta_i\Delta_{i+1}), \ldots, \Delta_n) + (-1)^n(\Delta_1, \ldots, \Delta_{n-1}).$$

In other words,

$$\begin{aligned}
&\partial(x_0 \xrightarrow{\Delta_1} x_1 \to \cdots \to x_{n-1} \xrightarrow{\Delta_n} x_n)\\
&:= (x_1 \xrightarrow{\Delta_2} x_2 \to \cdots \to x_{n-1} \xrightarrow{\Delta_n} x_n)\\
&\quad + \sum_{i=1}^{n-1}(-1)^i(x_0 \xrightarrow{\Delta_1} \cdots \to x_{i-1} \xrightarrow{\Delta_i\Delta_{i+1}} x_{i+1} \to \cdots \xrightarrow{\Delta_n} x_n)\\
&\quad + (-1)^n(x_0 \xrightarrow{\Delta_1} x_1 \to \cdots \to x_{n-2} \xrightarrow{\Delta_{n-1}} x_{n-1}).
\end{aligned}$$

Furthermore, for $(x_0 \xrightarrow{\Delta} x_1) \in \mathsf{P}(\overline{Q})_1$ and $(x) \in \mathsf{P}(\overline{Q})_0$, we set $\partial(x_0 \xrightarrow{\Delta} x_1) := (x_1) - (x_0) \in C_0(\overline{Q})$ and $\partial(x) := 0$.

Lemma 5.1. *The map $\partial\colon R[\overline{Q}] \to R[\overline{Q}]$ is an R-endomorphism of $R[\overline{Q}]$ such that $\partial \circ \partial = 0$ and $\partial(C_n(\overline{Q})) \subseteq C_{n-1}(\overline{Q})$ for $n \geq 0$ where $C_{-1}(\overline{Q}) := \{0\}$. This yields that a pair $(R[\overline{Q}], \partial)$ is an R-complex.*

Proof. Straightforward. □

Next we introduce two subcomplexes of $(R[\overline{Q}], \partial)$.

DEFINITION 5.2. Take two families from $\mathsf{P}(\overline{Q})$ as follows:

$$\mathsf{P}(\overline{Q})^{\mathrm{pr}} = \{(x_0 \xrightarrow{\Delta_1} x_1 \to \cdots \to x_{k-1} \xrightarrow{\Delta_k} x_k) \in \mathsf{P}(\overline{Q}) \mid x_i \neq x_j \text{ if } i \neq j\},$$

$$E\overline{Q} := \{(x_0 \xrightarrow{\Delta_1} x_1 \to \cdots \to x_{k-1} \xrightarrow{\Delta_k} x_k) \in \mathsf{P}(\overline{Q}) \mid \Delta_1 \cdots \Delta_k \in \mathsf{P}(Q)^{\mathrm{pr}}\} \subseteq \mathsf{P}(\overline{Q})^{\mathrm{pr}}.$$

Then the R-free modules

$$\mathsf{Mod}(\mathsf{P}(\overline{Q})^{\mathrm{pr}})_R = \bigoplus_{n\geq 0} D_n(\overline{Q}) \quad \text{where} \quad D_n(\overline{Q}) := \mathsf{Mod}(\mathsf{P}(\overline{Q})^{\mathrm{pr}}_n)_R,$$

$$\mathsf{Mod}(E\overline{Q})_R = \bigoplus_{n\geq 0} E_n(\overline{Q}) \quad \text{where} \quad E_n(\overline{Q}) := \mathsf{Mod}(E\overline{Q}_n)_R$$

are positive graded R-modules. Here $E\overline{Q}_n$ is the set of all paths $\Delta \in E\overline{Q}$ of length n. Let $X = \mathsf{P}(\overline{Q})^{\mathrm{pr}}$ or $X = E\overline{Q}$. Then by the definitions of ∂ and X, it is easy to see that $\partial(\mathsf{Mod}(X)_R) \subseteq \mathsf{Mod}(X)_R$, and that ∂ maps paths of length n to those of length $n-1$. Denote the restriction $\partial_{\mathsf{Mod}(X)_R} : \mathsf{Mod}(X)_R \to \mathsf{Mod}(X)_R$ by just the same notation ∂. Then a pair $(\mathsf{Mod}(X)_R, \partial)$ is a subcomplex of $(R[\overline{Q}], \partial)$.

REMARK 5.3. Paths in $E\overline{Q}$ provide a geometric information of the original quiver Q. Indeed, as in the next, we gather sets $\mathsf{Ob}(\sigma)$ for all $\sigma \in E\overline{Q}$ which are thought of forgetting arrows of σ. Then it will be shown in Lemma 5.5 that such collection $K_{E\overline{Q}}$ forms a simplicial complex.

DEFINITION 5.4. Denote by $K_{E\overline{Q}}$ a collection of sets $\mathsf{Ob}(\sigma)$ for all $\sigma \in E\overline{Q}$, that is,

$$K_{E\overline{Q}} := \{\mathsf{Ob}(\sigma) \subseteq Q_0 \mid \sigma \in E\overline{Q}\}.$$

Lemma 5.5. *A pair $(Q_0, K_{E\overline{Q}})$ forms a simplicial complex.*

Proof. Take any path $\sigma = (x_0 \xrightarrow{\Delta_1} x_1 \to \cdots \to x_{k-1} \xrightarrow{\Delta_k} x_k) \in E\overline{Q}$. Then $\mathsf{Ob}(\sigma) = \{x_0, \ldots, x_k\}$. For a non-empty subset $\{x_{i_0}, \ldots, x_{i_m}\} \subseteq \mathsf{Ob}(\sigma)$ with $i_0 < \cdots < i_m$, we define a path

$$\Gamma_{s+1} := \Delta_{i_s+1}\Delta_{i_s+2}\Delta_{i_s+3}\cdots\Delta_{i_{s+1}} \in \mathsf{P}(Q)^{\mathrm{non}} \quad (0 \le s \le m-1).$$

Then $\Gamma_{s+1} = (x_{i_s} \xrightarrow{\Gamma_{s+1}} x_{i_{s+1}}) \in \mathsf{P}(\overline{Q})_1$. Since $\Delta_1 \cdots \Delta_k \in \mathsf{P}(Q)^{\mathrm{pr}}$, we have that $\Gamma_1 \cdots \Gamma_m \in \mathsf{P}(Q)^{\mathrm{pr}}$ and

$$(\Gamma_1, \Gamma_2, \ldots, \Gamma_m) = (x_{i_0} \xrightarrow{\Gamma_1} x_{i_1} \to \cdots \to x_{i_{m-1}} \xrightarrow{\Gamma_m} x_{i_m}) \in E\overline{Q}.$$

Thus $\{x_{i_0}, \ldots, x_{i_m}\} \in K_{E\overline{Q}}$. This completes the proof. □

REMARK 5.6. (1) It might be $\mathsf{Ob}(\sigma) = \mathsf{Ob}(\tau)$ for distinct paths $\sigma, \tau \in E\overline{Q}$. This is caused by ignoring arrows of $\sigma \in E\overline{Q}$ when we get $\mathsf{Ob}(\sigma) \in K_{E\overline{Q}}$. From this reason, we will consider in Lemma 5.8 an R-homomorphism ε. This map reflects such difference between $E\overline{Q}$ and $K_{E\overline{Q}}$.
(2) Define a simplicial complex $\mathrm{T}_Q(\mathsf{P}(Q)^{\mathrm{pr}})$ whose vertex set is

$$\bigcup_{\Delta \in \mathsf{P}(Q)^{\mathrm{pr}}} \mathsf{Ob}(\Delta) \quad (= Q_0),$$

and the totality

$$\bigcup_{\Delta \in \mathsf{P}(Q)^{\mathrm{pr}}} (2^{\mathsf{Ob}(\Delta)} \setminus \{\emptyset\})$$

of all non-empty subsets of $\mathsf{Ob}(\Delta)$ for all $\Delta \in \mathsf{P}(Q)^{\mathrm{pr}}$ forms the set of simplices. This complex depends only on a subset $\mathsf{P}(Q)^{\mathrm{pr}}$ of the set $\mathsf{P}(Q)$ of all paths in Q. We will investigate such complexes in general later in Section 6. It is clear from the definitions that $\mathrm{T}_Q(\mathsf{P}(Q)^{\mathrm{pr}})$ coincides with $(Q_0, K_{E\overline{Q}})$ as simplicial complexes.
(3) For proper paths $\Delta_1, \Delta_2 \in \mathsf{P}(Q)^{\mathrm{pr}}$, we define a pre-ordering $\Delta_1 \ll \Delta_2$ precisely when $\mathsf{Ob}(\Delta_1) \subseteq \mathsf{Ob}(\Delta_2)$. Let $\mathsf{P}(Q)^{\mathrm{pr}}_{\max}$ be the totality of all maximal paths in $\mathsf{P}(Q)^{\mathrm{pr}}$ with respect to $\ll$. Then a complex $\mathrm{T}_Q(\mathsf{P}(Q)^{\mathrm{pr}})$ is the same as a complex $\mathrm{T}_Q(\mathsf{P}(Q)^{\mathrm{pr}}_{\max})$ whose sets of vertices and simplices are respectively Q_0 and the totality of all non-empty subsets of $\mathsf{Ob}(\Delta)$ for all $\Delta \in \mathsf{P}(Q)^{\mathrm{pr}}_{\max}$. This implies that $\mathrm{T}_Q(\mathsf{P}(Q)^{\mathrm{pr}})$ can be realized by using fewer paths than those in $\mathsf{P}(Q)^{\mathrm{pr}}$.

DEFINITION 5.7. For a simplex $X = \{x_0, \ldots, x_n\} \in K_{E\overline{Q}}$ of dimension n and a total ordering on X, denote by $\langle X\rangle = \langle x_0, \ldots, x_n\rangle$ an oriented simplex. This means that, for $X = \{x_{i_0}, \ldots, x_{i_n}\} = \{x_{j_0}, \ldots, x_{j_n}\}$, if these two orderings differ by an even permutation then $\langle x_{i_0},\ldots,x_{i_n}\rangle = \langle x_{j_0},\ldots,x_{j_n}\rangle$, and otherwise we understand $\langle x_{i_0},\ldots,x_{i_n}\rangle = -\langle x_{j_0}, \ldots, x_{j_n}\rangle$ or $-\langle x_{i_0}, \ldots, x_{i_n}\rangle = \langle x_{j_0}, \ldots, x_{j_n}\rangle$. Denote by $K^{\mathrm{or}}_{E\overline{Q}} := \{\langle X\rangle \mid X \in K_{E\overline{Q}}\}$ the totality of oriented simplices of $K_{E\overline{Q}}$. Then the R-free module

$$\mathsf{Mod}\big(K^{\mathrm{or}}_{E\overline{Q}}\big)_R = \bigoplus_{n\geq 0} K_n(\overline{Q})$$

is a positive graded R-module where $K_n(\overline{Q})$ is the R-free module with all n-dimensional oriented simplices in $K^{\mathrm{or}}_{E\overline{Q}}$ as basis. Let $\delta\colon \mathsf{Mod}\big(K^{\mathrm{or}}_{E\overline{Q}}\big)_R \to \mathsf{Mod}\big(K^{\mathrm{or}}_{E\overline{Q}}\big)_R$ be an R-endomorphism defined by

$$\delta(\langle x_0, x_1, \ldots, x_n\rangle) := \sum_{i=0}^{n} (-1)^i \langle x_0, \ldots, \hat{x}_i, \ldots, x_n\rangle$$

for $\langle x_0, x_1, \ldots, x_n\rangle \in K^{\mathrm{or}}_{E\overline{Q}}$ with $n \geq 1$, and by extending by linearity. Here $\hat{x}_i$ means delete the vertex x_i. Furthermore, we set $\delta(\langle x_0\rangle) := 0$ for $x_0 \in Q_0$. Then it is shown that a pair $\big(\mathsf{Mod}\big(K^{\mathrm{or}}_{E\overline{Q}}\big)_R, \delta\big)$ is an R-complex.

As mentioned in Remark 5.6, the following map ε is defined by forgetting arrows of paths in $E\overline{Q}$.

Lemma 5.8. *A surjective R-homomorphism*

$$\varepsilon\colon \mathsf{Mod}(E\overline{Q})_R \to \mathsf{Mod}\big(K^{\mathrm{or}}_{E\overline{Q}}\big)_R$$

defined by $\sigma \mapsto \langle \mathsf{Ob}(\sigma)\rangle$ for $\sigma \in E\overline{Q}$ is a map between R-complexes $(\mathsf{Mod}(E\overline{Q})_R, \partial)$ and $\big(\mathsf{Mod}\big(K^{\mathrm{or}}_{E\overline{Q}}\big)_R, \delta\big)$, that is, conditions $\varepsilon(E_n(\overline{Q})) \subseteq K_n(\overline{Q})$ and $\delta \circ \varepsilon = \varepsilon \circ \partial$ are satisfied.

Proof. Straightforward. □

REMARK 5.9. Summarizing the procedure for constructing R-complexes and simplicial complexes, we have the following:

$$\begin{array}{lccccccc}
R\text{-complexes: } & \mathsf{Mod}(\mathsf{P}(\overline{Q}))_R & \supseteq & \mathsf{Mod}(\mathsf{P}(\overline{Q})^{\mathrm{pr}})_R & \supseteq & \mathsf{Mod}(E\overline{Q})_R & \xrightarrow{\varepsilon} & \mathsf{Mod}\big(K^{\mathrm{or}}_{E\overline{Q}}\big)_R \\
 & \Uparrow & & \Uparrow & & \Uparrow & & \\
\text{Paths: } & \mathsf{P}(\overline{Q}) & \supseteq & \mathsf{P}(\overline{Q})^{\mathrm{pr}} & \supseteq & E\overline{Q} & & \\
 & & & & & \downarrow \mathsf{Ob} & & \\
\text{Sim. complexes: } & & & & & (Q_0, K_{E\overline{Q}}) = \mathrm{T}_Q(\mathsf{P}(Q)^{\mathrm{pr}}) & &
\end{array}$$

Note that $\mathrm{T}_Q(\mathsf{P}(Q)^{\mathrm{pr}}) = \mathrm{T}_Q(\mathsf{P}(Q)^{\mathrm{pr}}_{\max})$ (see Remark 5.6).

5.2. Homology *R*-modules.

DEFINITION 5.10. For R-complexes defined in Section 5.1, we use the following notations for their homology R-modules:

$$H(Q, R) := H(\mathsf{Mod}(\mathsf{P}(\overline{Q}))_R, \partial), \quad H(Q, R)^{\mathrm{pr}} := H(\mathsf{Mod}(\mathsf{P}(\overline{Q})^{\mathrm{pr}})_R, \partial),$$
$$H(E\overline{Q}, R) := H(\mathsf{Mod}(E\overline{Q})_R, \partial), \quad H(K_{E\overline{Q}}, R) := H\big(\mathsf{Mod}\big(K^{\mathrm{or}}_{E\overline{Q}}\big)_R, \delta\big).$$

Recall that $Z_n(\overline{Q}) := C_n(\overline{Q}) \cap \mathrm{Ker}\partial$, $B_n(\overline{Q}) := C_n(\overline{Q}) \cap \mathrm{Im}\partial \le Z_n(\overline{Q})$, $H_n(\overline{Q}) := Z_n(\overline{Q})/B_n(\overline{Q})$. Then we have positive graded R-modules

$$\mathrm{Ker}\partial = \bigoplus_{n\ge 0} Z_n(\overline{Q}), \quad \mathrm{Im}\partial = \bigoplus_{n\ge 0} B_n(\overline{Q}),$$
$$H(\mathsf{Mod}(\mathsf{P}(\overline{Q}))_R, \partial) := \mathrm{Ker}\partial/\mathrm{Im}\partial \cong \bigoplus_{n\ge 0} H_n(\overline{Q}).$$

The other homology R-modules are similarly defined.

EXAMPLE 5.11. Let Q be a quiver defined as follows:

$$a \xrightarrow{\alpha} b \xrightarrow{\beta} c.$$

Set $Q^0 := Q$ and $Q^k := \overline{Q^{k-1}}$ ($k \ge 1$). Then paths in a quiver Q^k ($k \ge 0$) are described in Example 3.6, and a $\mathbb{Z}$-complex $\mathbb{Z}[\overline{Q^k}] = \mathsf{Mod}(\mathsf{P}(\overline{Q^k}))_{\mathbb{Z}}$ where $\overline{Q^k} = Q^{k+1}$ is as follows:

$$\{0\} \to \mathsf{Mod}((a \xrightarrow{\alpha} b \xrightarrow{\beta} c))_{\mathbb{Z}} \xrightarrow{\partial_2} \mathsf{Mod}(\alpha, \beta, \Delta_1, \ldots, \Delta_{k+1})_{\mathbb{Z}} \xrightarrow{\partial_1} \mathsf{Mod}(a, b, c)_{\mathbb{Z}} \to \{0\}.$$

Each Δ_i $(1 \le i \le k+1)$ is an arrow from a to c, and we may assume that $\partial_2((a \overset{\alpha}{\to} b \overset{\beta}{\to} c)) = \beta - \Delta_1 + \alpha$. Then it is straightforward to calculate that

$$\begin{aligned}\mathrm{Ker}\partial_1 &= \mathsf{Mod}(\alpha + \beta - \Delta_1,\, \Delta_i - \Delta_1 \ (2 \le i \le k+1))_{\mathbb{Z}} \\ &\geq \mathsf{Mod}(\alpha + \beta - \Delta_1)_{\mathbb{Z}} = \mathrm{Im}\partial_2.\end{aligned}$$

It follows that

$$H_n(Q^k, \mathbb{Z}) = H_n(\mathsf{Mod}(\mathsf{P}(\overline{Q^k}))_{\mathbb{Z}}, \partial) \cong \begin{cases} \mathbb{Z} & n = 0, \\ \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \quad (k \text{ times}) & n = 1, \\ \{0\} & n = 2. \end{cases}$$

Furthermore, since $\mathsf{P}(\overline{Q^k}) = \mathsf{P}(\overline{Q^k})^{\mathrm{pr}} = E\overline{Q^k}$ in this case, we have that $H_n(Q^k, \mathbb{Z}) = H_n(Q^k, \mathbb{Z})^{\mathrm{pr}} = H_n(E\overline{Q^k}, \mathbb{Z})$ for all $n \ge 0$. On the other hand, by the definition, $K_{E\overline{Q^k}} = \{\mathsf{Ob}(\sigma) \mid \sigma \in E\overline{Q^k}\} = 2^{\{a,b,c\}} \setminus \{\emptyset\}$. So a complex $K_{E\overline{Q^k}}$ is contractible. This implies that $H_0\big(K_{E\overline{Q^k}}, \mathbb{Z}\big) \cong \mathbb{Z}$ and $H_n\big(K_{E\overline{Q^k}}, \mathbb{Z}\big) = \{0\}$ for all $n \ge 1$.

REMARK 5.12 (Restricting to arrows). Let $Q = (Q_0, Q_1, s, r)$ be a quiver, and let $Q^{\mathrm{ud}} = (Q_0, Q_1^{\mathrm{ud}}, s, r)$ be the extended quiver of Q in Definition 3.1. For a subset $A \subseteq Q_1^{\mathrm{ud}}$ of arrows, we focus on the restriction

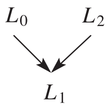
$$Q_A^{\mathrm{ud}} := ((Q_A^{\mathrm{ud}})_0, A, s, r)$$

to A in Definition 3.4 where $(Q_A^{\mathrm{ud}})_0 := \{s(\alpha), r(\alpha) \mid \alpha \in A\}$. This quiver Q_A^{ud} allows us to investigate paths in Q^{ud} constructed by arrows in A. Then applying Q_A^{ud} to Remark 5.9 on complexes, we have that

$$\Big((Q_A^{\mathrm{ud}})_0,\, K_{E\overline{Q_A^{\mathrm{ud}}}}\Big) = \mathrm{T}_{Q_A^{\mathrm{ud}}}(\mathsf{P}(Q_A^{\mathrm{ud}})^{\mathrm{pr}}) = \mathrm{T}_{Q_A^{\mathrm{ud}}}(\mathcal{H})$$

where $\mathcal{H} := \mathsf{P}(Q_A^{\mathrm{ud}})^{\mathrm{pr}}_{\max} \subseteq \mathsf{P}(Q^{\mathrm{ud}})^{\mathrm{pr}} \subseteq \mathsf{P}(Q^{\mathrm{ud}})$. Thus $H(K_{E\overline{Q_A^{\mathrm{ud}}}}, R) = H(\mathrm{T}_{Q_A^{\mathrm{ud}}}(\mathcal{H}), R)$. This homology R-module should contain much information on paths in Q^{ud} obtained from arrows in A.

EXAMPLE 5.13. Let $Q_{CG} = ((Q_{CG})_0, (Q_{CG})_1, s, r)$ be a coset quiver in Definition 4.5, and Q_{CG}^{ud} be the extended quiver of Q_{CG}. Let $\Delta \in \mathsf{P}(Q_G^{\mathrm{ud}})^{\mathrm{pr}}$ be a proper path in Q_G^{ud} of the form



This means that $L_0 > L_1$, $L_1 < L_2$, and $L_0 \neq L_2$. We take a subset $A := A^{(1)} \cup A^{(2)} \subseteq (Q^{\mathrm{ud}}_{CG})_1$ of arrows in Q^{ud}_{CG} as follows:

$$A^{(1)} := \{(aL_0 \to bL_1) \in (Q^{\mathrm{ud}}_{CG})_1 \mid a, b \in G\},$$
$$A^{(2)} := \{(cL_1 \leftarrow dL_2) \in (Q^{\mathrm{ud}}_{CG})_1 \mid c, d \in G\}.$$

Then the restriction $S := (Q^{\mathrm{ud}}_{CG})_A$ to A and the set $\mathsf{P}(S)$ of paths are given as follows:

$$S = (Q^{\mathrm{ud}}_{CG})_A = (G/L_0 \cup G/L_1 \cup G/L_2, A, s, r),$$
$$\mathsf{P}(S)_0 = (\text{trivial paths}) = G/L_0 \cup G/L_1 \cup G/L_2,$$
$$\mathsf{P}(S)_1 = A = A^{(1)} \cup A^{(2)},$$
$$\mathsf{P}(S)_2 = \{(aL_0 \xrightarrow{\alpha} bL_1 \xleftarrow{\beta} cL_2) \mid \alpha \in A^{(1)}, \beta \in A^{(2)}\}.$$

Since Δ is proper, so is any path in $\mathsf{P}(S)$, namely $\mathsf{P}(S)^{\mathrm{pr}} = \mathsf{P}(S)$ (cf. Remark 4.8 (3)). Furthermore $\mathsf{P}(S)^{\mathrm{pr}}_{\max} = \mathsf{P}(S)_2 = \tilde{\mathfrak{G}}(\Delta)$ in our notation in Definition 4.7. Thus, by Remark 5.9, $(S_0, K_{E\bar{S}}) = \mathrm{T}_S(\tilde{\mathfrak{G}}(\Delta))$, and so $H(K_{E\bar{S}}, R) = H(\mathrm{T}_S(\tilde{\mathfrak{G}}(\Delta)), R)$. In Section 7.3, we will study the top homology of this kind of a simplicial complex. As mentioned in Remark 5.12, we may say that this homology R-module contains much information on particular paths in

$$\tilde{\mathfrak{G}}(\Delta) = \{(x_0L_0 \to x_1L_1 \leftarrow x_2L_2) \mid x_iL_i \in G/L_i \ (0 \le i \le 2)\} \subseteq \mathsf{P}(Q^{\mathrm{ud}}_{CG})^{\mathrm{pr}}.$$

Finally we note that the closure $\bar{S}$ of S is as

$$\bar{S} := \overline{(Q^{\mathrm{ud}}_{CG})_A} = (G/L_0 \cup G/L_1 \cup G/L_2, \mathsf{P}(S)^{\mathrm{non}}, s, r),$$

and the set $\mathsf{P}(\bar{S})$ of paths is described as follows:

$$\mathsf{P}(\bar{S})_0 = (\text{trivial paths}) = G/L_0 \cup G/L_1 \cup G/L_2,$$
$$\mathsf{P}(\bar{S})_1 = \mathsf{P}(S)^{\mathrm{non}} = \mathsf{P}(S)_1 \cup \{(aL_0 \xrightarrow{\Delta} cL_2) \mid \Delta \in \mathsf{P}(S)_2\},$$
$$\mathsf{P}(\bar{S})_2 = \{(aL_0 \xrightarrow{\alpha} bL_1 \xleftarrow{\beta} cL_2) \mid \alpha \in A^{(1)}, \beta \in A^{(2)}\}.$$

This shows that $\mathsf{P}(\bar{S}) = \mathsf{P}(\bar{S})^{\mathrm{pr}} = E\bar{S}$.

6. Simplicial complexes associated to paths

In this section, we introduce a simplicial complex $\mathrm{T}_Q(\mathcal{H})$ defined by a quiver Q and a set $\mathcal{H}$ of paths in Q, which we call a path complex. First, we develop some general theory on $\mathrm{T}_Q(\mathcal{H})$. Next we apply a subgroup quiver Q^{ud}_G to $\mathrm{T}_Q(\mathcal{H})$. This is a natural generalization of the usual subgroup complex of G. The contractibility of such complexes is studied. Moreover, we adapt a coset quiver Q_{CG}^{ud} to $\mathrm{T}_Q(\mathcal{H})$, and examine a G-simplicial map between path complexes in Q_{CG}^{ud} and Q_G^{ud}. Further properties of path complexes will be devoted in Section 7. Throughout this section, let G be a finite group.

Now, we recall the geometric realization of a simplicial complex $K = (V(K), S(K))$ where $V(K)$ and $S(K)$ are the sets of vertices and simplices respectively. Let $\mathbb{E}^{V(K)}$ be the set of all maps $v = (v_x)_{x \in V(K)}$ from $V(K)$ to $\mathbb{R}$ such that $v_x \neq 0$ for finitely many values of $x \in V(K)$. This is called a generalized Euclidean space with topology given by the metric $|v - w| := \max\{|v_x - w_x| \mid x \in V(K)\}$. We identify a vertex $x \in V(K)$ with a map in $\mathbb{E}^{V(K)}$ whose value is 1 on x and 0 on all other elements of $V(K)$. Then $V(K)$ forms a basis of $\mathbb{E}^{V(K)}$.

DEFINITION 6.1 (cf. pp. 142, 197 in [4]). Let $K = (V(K), S(K))$ be a simplicial complex. For a simplex $\sigma = \{x_0, \dots, x_n\} \in S(K)$, define an Euclidean closed n-simplex

$$[\sigma] = [x_0, x_1, x_2, \dots, x_n] := \left\{ \sum_{i=0}^{n} t_i x_i \;\middle|\; \sum_{i=0}^{n} t_i = 1,\ 0 \leq t_i \leq 1 \right\} \subseteq \mathbb{E}^{V(K)}.$$

An Euclidean open n-simplex (σ) is defined by the set of all elements $\sum_{i=0}^{n} t_i x_i$ in $[\sigma]$ such that $t_i > 0$ for all $0 \leq i \leq n$. Set

$$[K] := \bigcup_{\sigma \in S(K)} [\sigma] \subseteq \mathbb{E}^{V(K)}$$

which is viewed as a topological space by topology coherent with a collection $\{[\sigma] \mid \sigma \in K\}$ of subspaces $[\sigma]$ of $\mathbb{E}^{V(K)}$. A space $[K]$ is called the geometric realization of K. We say that K is contractible if so is $[K]$, and that simplicial complexes K and T are homotopy equivalent if so are $[K]$ and $[T]$.

6.1. Path complexes $\mathbf{T}_{\boldsymbol{Q}}(\boldsymbol{\mathcal{H}})$. Although the notion of $\mathrm{T}_Q(\mathcal{H})$ already appeared in Remark 5.6, we formulate its definition here.

DEFINITION 6.2. Let Q be a quiver, and $\mathcal{H} \subseteq \mathsf{P}(Q)$ be a subset of paths in Q. Define a simplicial complex $\mathrm{T}_Q(\mathcal{H})$ whose vertex set is

$$\bigcup_{\Delta \in \mathcal{H}} \mathsf{Ob}(\Delta) \subseteq Q_0,$$

and the totality

$$\bigcup_{\Delta \in \mathcal{H}} (2^{\mathsf{Ob}(\Delta)} \setminus \{\emptyset\})$$

of all non-empty subsets of $\mathsf{Ob}(\Delta)$ for all $\Delta \in \mathcal{H}$ forms the set of simplices. We call

$\mathrm{T}_Q(\mathcal{H})$ a path complex of $\mathcal{H}$ in Q. The dimension of $\mathrm{T}_Q(\mathcal{H})$ is $\max\{|\mathsf{Ob}(\Delta)|-1 \mid \Delta \in \mathcal{H}\}$. Note that $|\mathsf{Ob}(\Delta)|-1 \leq l(\Delta)$ in general, but if Δ is proper then the equality holds.

REMARK 6.3. Let Q be a G-quiver. For a G-invariant subset $\mathcal{H} \subseteq \mathsf{P}(Q)$, a complex $\mathrm{T}_Q(\mathcal{H})$ becomes a G-simplicial complex, namely, the sets of vertices and simplices of $\mathrm{T}_Q(\mathcal{H})$ are preserved by the action of G. So various G-simplicial complexes can be obtained from G-quivers Q_G^{ud} and Q_{CG}^{ud} (see Remark 4.6).

Here we study conditions of the contractibility of $\mathrm{T}_Q(\mathcal{H})$. But before doing this, we recall some technique from poset topology. Let $(\mathcal{P}, \leq)$ be a poset. Denote by $\mathsf{O}(\mathcal{P}) = \mathsf{O}(\mathcal{P}, \leq)$ the order complex of $\mathcal{P}$, which is a simplicial complex defined by all inclusion-chains $(x_0 < \cdots < x_k)$, where $x_i \in \mathcal{P}$, as simplices. Let K be a simplicial complex. Denote by $\mathsf{sd}(K)$ the poset of all simplices in K ordered by the inclusion-relation. This is called the barycentric subdivision of K. It is worth mentioning that $[K]$ and $[\mathsf{O}(\mathsf{sd}(K))]$ are homeomorphic each other as geometric realizations (topological spaces) (see [3, (1.3)]).

DEFINITION 6.4 ((1.5) in [3]). We say that a poset $(\mathcal{P}, \leq)$ is conically contractible if there exist a poset map $f : \mathcal{P} \to \mathcal{P}$ and an element $x_0 \in \mathcal{P}$ such that $x \leq f(x)$ and $f(x) \geq x_0$ for all $x \in \mathcal{P}$. Recall that a poset map f is defined by the property that $x \leq y$ $(x, y \in \mathcal{P})$ implies $f(x) \leq f(y)$.

Lemma 6.5 ((1.5) in [3]). *If a poset $(\mathcal{P}, \leq)$ is conically contractible then the order complex $\mathsf{O}(\mathcal{P})$ is contractible.*

Lemma 6.6 (Proposition 6.1 in [6]). *Let $(\mathcal{P}, \leq)$ be a poset. For $x \in \mathcal{P}$, set*

$$\mathcal{P}_{<x} := \{y \in \mathcal{P} \mid y < x\}$$

and

$$\mathcal{P}_{>x} := \{y \in \mathcal{P} \mid y > x\}.$$

If $\mathsf{O}(\mathcal{P}_{<x})$ or $\mathsf{O}(\mathcal{P}_{>x})$ is contractible then $\mathsf{O}(\mathcal{P})$ and $\mathsf{O}(\mathcal{P} \setminus \{x\})$ are homotopy equivalent.

Proposition 6.7. *Let Q be a quiver, and $\mathcal{H} \subseteq \mathsf{P}(Q)$ be a subset of paths in Q. Suppose that*

$$\bigcap_{\Delta \in \mathcal{H}} \mathsf{Ob}(\Delta) \neq \emptyset.$$

Then $\mathrm{T}_Q(\mathcal{H})$ is contractible.

Proof. Let $\mathcal{P} := \mathsf{sd}(\mathrm{T}_Q(\mathcal{H}))$ be the barycentric subdivision of $\mathrm{T}_Q(\mathcal{H})$. It is enough to show that $\mathcal{P}$ is conically contractible by Lemma 6.5. Take any element $a \in$

$\bigcap_{\Delta \in \mathcal{H}} \mathsf{Ob}(\Delta)$. Then $\{a\} \in \mathcal{P}$. Furthermore, for any $\sigma \in \mathcal{P}$, there exists $\Gamma \in \mathcal{H}$ such that $\sigma \subseteq \mathsf{Ob}(\Gamma)$ by the definition of simplices. Since $a \in \mathsf{Ob}(\Gamma)$ and $\sigma \cup \{a\} \subseteq \mathsf{Ob}(\Gamma)$, we have that $\sigma \cup \{a\} \in \mathcal{P}$. This yields that a map

$$f : \mathcal{P} \to \mathcal{P}$$

defined by

$$\sigma \mapsto \sigma \cup \{a\}$$

is a poset map such that $\sigma \subseteq f(\sigma)$ and $f(\sigma) \supseteq \{a\}$ for any $\sigma \in \mathcal{P}$. Therefore $\mathcal{P}$ is conically contractible, and this completes the proof. □

DEFINITION 6.8 (Trees; cf. page 101 in [5]). Let $Q = (Q_0, Q_1, s, r)$ be a quiver with no loops, that is, $s(\alpha) \neq r(\alpha)$ for all $\alpha \in Q_1$. Thus $\mathsf{Ob}(\alpha)$ is a two-points set for all $\alpha \in Q_1$.
(1) Denote by $V(Q_{\leq 1}) := Q_0$ and $S(Q_{\leq 1}) := \{\mathsf{Ob}(\alpha) \mid \alpha \in Q_1\} \cup Q_0$. Then a pair

$$Q_{\leq 1} := (V(Q_{\leq 1}), S(Q_{\leq 1}))$$

forms a simplicial complex of dimension less than or equal to 1.
(2) Q is a tree if the geometric realization $[Q_{\leq 1}]$ of $Q_{\leq 1}$ is an arcwise connected, and $[Q_{\leq 1}] \setminus (\mathsf{Ob}(\alpha))$ is disconnected for each 1-simplex $\mathsf{Ob}(\alpha)$ ($\alpha \in Q_1$) (see Definition 6.1 for notations). Note that if Q is a tree then $\mathsf{P}(Q) = \mathsf{P}(Q)^{\mathrm{pr}}$.

DEFINITION 6.9 (End-vertices; cf. page 101 in [5]). Let $Q = (Q_0, Q_1, s, r)$ be a quiver. A vertex $x \in Q_0$ is an end-vertex in Q if there exists a unique arrow $\gamma_x \in Q_1$ such that $s(\gamma_x) \neq r(\gamma_x)$, and that $s(\gamma_x) = x$ or $r(\gamma_x) = x$.

DEFINITION 6.10. Let $Q = (Q_0, Q_1, s, r)$ be a quiver. For $x \in Q_0$, set $Q(x)_0 := Q_0 \setminus \{x\}$, and $Q(x)_1 := \{\beta \in Q_1 \mid s(\beta) \neq x \text{ and } r(\beta) \neq x\}$. Then

$$Q(x) := (Q(x)_0, Q(x)_1, s|_{Q(x)_1}, r|_{Q(x)_1})$$

forms a quiver. In particular, if x is an end-vertex in Q then we have that $Q(x)_1 = Q_1 \setminus \{\gamma_x\}$.

Lemma 6.11. *Let $Q = (Q_0, Q_1, s, r)$ be a quiver. For an end-vertex $x \in Q_0$, set $Q' := Q(x)$,*

$$\mathcal{P} := \mathsf{sd}(T_Q(\mathsf{P}(Q))),$$

and

$$\mathcal{P}' := \mathsf{sd}(T_{Q'}(\mathsf{P}(Q'))).$$

(1) $\mathsf{O}(\mathcal{P}_{\supset\{x\}})$ *is contractible, and thus* $\mathsf{O}(\mathcal{P})$ *and* $\mathsf{O}(\mathcal{P}\setminus\{x\})$ *are homotopy equivalent.*
(2) *For any poset* $\mathcal{Q}$ *such that* $\mathcal{P}'\subseteq\mathcal{Q}\subseteq(\mathcal{P}\setminus\{x\})$, *and for any minimal element* σ *in* $(\mathcal{Q}\setminus\mathcal{P}')$, *we have that* $\mathsf{O}(\mathcal{Q})$ *and* $\mathsf{O}(\mathcal{Q}\setminus\{\sigma\})$ *are homotopy equivalent. In particular, suppose that* Q *is finite, that is,* Q_0 *and* Q_1 *are both finite sets. Then, repeating this process, we conclude that* $\mathsf{O}(\mathcal{P}\setminus\{x\})$ *and* $\mathsf{O}(\mathcal{P}')$ *are homotopy equivalent.*
(3) *If* Q *is finite then* $T_Q(\mathsf{P}(Q))$ *and* $T_{Q'}(\mathsf{P}(Q'))$ *are homotopy equivalent.*

Proof. By the definition of an end-vertex $x\in Q_0$, there exists a unique arrow $\gamma_x\in Q_1$ such that $s(\gamma_x)\neq r(\gamma_x)$, and that $s(\gamma_x)=x$ or $r(\gamma_x)=x$. Let $\mathsf{Ob}(\gamma_x)=\{x,z\}$ $(x\neq z)$.

(1) For any $\tau\in\mathcal{P}_{\supset\{x\}}$, there exists a path $\Delta\in\mathsf{P}(Q)$ such that $\{x\}\subset\tau\subseteq\mathsf{Ob}(\Delta)$. So Δ must be of the form

$$\Delta=(x\xrightarrow{\gamma_x}z=:z_0\xrightarrow{\alpha_1}z_1\to\cdots\to z_{k-1}\xrightarrow{\alpha_k}z_k)$$

or

$$\Delta=(z_0\xrightarrow{\alpha_1}z_2\to\cdots\to z_{k-1}\xrightarrow{\alpha_k}z_k:=z\xrightarrow{\gamma_x}x).$$

Note that $\{x,z\}\in\mathcal{P}_{\supset\{x\}}$. Since $\tau\cup\{x,z\}\subseteq\mathsf{Ob}(\Delta)$, we have that $\tau\cup\{x,z\}\in\mathcal{P}_{\supset\{x\}}$. This yields that a map

$$f:\mathcal{P}_{\supset\{x\}}\to\mathcal{P}_{\supset\{x\}}$$

defined by

$$\tau\mapsto\tau\cup\{x,z\}$$

is a poset map such that $\tau\subseteq f(\tau)$ and $f(\tau)\supseteq\{x,z\}$ for any $\tau\in\mathcal{P}_{\supset\{x\}}$. Thus $\mathcal{P}_{\supset\{x\}}$ is conically contractible. The results follow from Lemmas 6.5 and 6.6.

(2) Since $\sigma\notin\mathcal{P}'$, we have that $x\in\sigma$. Furthermore, since $\sigma\in\mathcal{Q}\subseteq(\mathcal{P}\setminus\{x\})$, we have that $\sigma\in\mathcal{P}_{\supset\{x\}}$. Then there exists a path $\Delta\in\mathsf{P}(Q)$ as in the proof of (1) such that $\sigma\subseteq\mathsf{Ob}(\Delta)$. Set

$$\Delta':=(z_0\xrightarrow{\alpha_1}z_1\to\cdots\to z_{k-1}\xrightarrow{\alpha_k}z_k)\in\mathsf{P}(Q')\quad(k\geq 0).$$

Since $\emptyset\neq(\sigma\setminus\{x\})\subseteq\mathsf{Ob}(\Delta')$, we have that $(\sigma\setminus\{x\})\in\mathcal{P}'_{\subset\sigma}\subseteq\mathcal{Q}_{\subset\sigma}$. Take any $\tau\in\mathcal{Q}_{\subset\sigma}$. If $\tau\notin\mathcal{P}'$ then $\tau\in(\mathcal{Q}\setminus\mathcal{P}')$ with $\tau\subset\sigma$. This contradicts the minimality of σ. Thus $\tau\in\mathcal{P}'$, so that $x\notin\tau$. It follows that $\tau=(\tau\setminus\{x\})\subseteq(\sigma\setminus\{x\})$. This implies that $\mathcal{Q}_{\subset\sigma}$ possesses the maximum element $(\sigma\setminus\{x\})$, and thus $\mathsf{O}(\mathcal{Q}_{\subset\sigma})$ is contractible. Then by Lemma 6.6, $\mathsf{O}(\mathcal{Q})$ and $\mathsf{O}(\mathcal{Q}\setminus\{\sigma\})$ are homotopy equivalent.

(3) The result follows from (1) and (2) above. □

REMARK 6.12. (1) Put $\mathrm{T}:=\mathrm{T}_Q(\mathsf{P}(Q))$ and $\mathrm{T}':=\mathrm{T}_{Q'}(\mathsf{P}(Q'))$. We mention that the result in Lemma 6.11 (3) can be also proved according to the definition of a homotopy equivalence. Namely, we are able to construct continuous maps $f:\mathrm{T}\to\mathrm{T}'$ and

$g\colon \mathrm{T}' \to \mathrm{T}$ such that $f \circ g$ is homotopic to $\mathrm{Id}_{\mathrm{T}'}$, and $g \circ f$ is homotopic to Id_{T} where Id_X is the identity map on a set X.
(2) Under the situation of Lemma 6.11, suppose that Q is finite. Then by the same way, we can prove that $\mathrm{T}_Q(\mathsf{P}(Q)^{\mathrm{pr}})$ and $\mathrm{T}_{Q'}(\mathsf{P}(Q')^{\mathrm{pr}})$ are homotopy equivalent. Thus applying Remark 5.9, we have a $\mathbb{Z}$-module isomorphism as follows:

$$H(K_{E\overline{Q}}, \mathbb{Z}) = H(\mathrm{T}_Q(\mathsf{P}(Q)^{\mathrm{pr}}), \mathbb{Z}) \cong H(\mathrm{T}_{Q'}(\mathsf{P}(Q')^{\mathrm{pr}}), \mathbb{Z}) = H(K_{E\overline{Q'}}, \mathbb{Z}).$$

Lemma 6.13 (cf. pp. 101–102 in [5])**.** *Let $Q = (Q_0, Q_1, s, r)$ be a finite quiver with no loops. Suppose that Q is a tree such that $|Q_0| \geq 2$. Then there exists an end-vertex $x \in Q_0$ in Q. Furthermore, $Q(x)$ is again a tree.*

Combining Lemma 6.13 with our Lemma 6.11, we have the following.

Proposition 6.14. *Let $Q = (Q_0, Q_1, s, r)$ be a finite quiver with no loops. Suppose that Q is a tree. Then $\mathsf{P}(Q) = \mathsf{P}(Q)^{\mathrm{pr}}$, and $\mathrm{T}_Q(\mathsf{P}(Q)) = \mathrm{T}_Q(\mathsf{P}(Q)^{\mathrm{pr}})$ is contractible.*

REMARK 6.15 (Homology of trees). Let $Q = (Q_0, Q_1, s, r)$ be a finite quiver with no loops. Suppose that Q is a tree. Then since $\mathsf{P}(Q) = \mathsf{P}(Q)^{\mathrm{pr}}$, we have that $\mathsf{P}(\overline{Q}) = \mathsf{P}(\overline{Q})^{\mathrm{pr}} = E\overline{Q}$. It follows that $H(Q, \mathbb{Z}) = H(Q, \mathbb{Z})^{\mathrm{pr}} = H(E\overline{Q}, \mathbb{Z})$ (see Definition 5.10). Furthermore, because of $\mathsf{P}(Q) = \mathsf{P}(Q)^{\mathrm{pr}}$, we have that $H(E\overline{Q}, \mathbb{Z}) = H(K_{E\overline{Q}}, \mathbb{Z}) = H(\mathrm{T}_Q(\mathsf{P}(Q)^{\mathrm{pr}}))$ (see Remark 5.9). By Proposition 6.14, $\mathrm{T}_Q(\mathsf{P}(Q)^{\mathrm{pr}})$ is contractible, so that $H_0(Q, \mathbb{Z}) \cong \mathbb{Z}$ and $H_n(Q, \mathbb{Z}) = \{0\}$ for all $n \geq 1$.

6.2. Subgroup complexes $\mathbf{T}_{Q_G^{\mathrm{ud}}}(\mathcal{D})$. Let Q_G be a subgroup quiver of G in Definition 4.3, and let Q_G^{ud} be the extended quiver of Q_G in Definition 3.1. In this section, for a subset $\mathcal{D} \subseteq \mathsf{P}(Q_G^{\mathrm{ud}})$, we deal with a path complex $\mathrm{T}_{Q_G^{\mathrm{ud}}}(\mathcal{D})$ of $\mathcal{D}$ in Q_G^{ud} which we call a subgroup complex of G.

REMARK 6.16. Let $\mathcal{D} \subseteq \mathsf{P}(Q_G)$ ($\subseteq \mathsf{P}(Q_G^{\mathrm{ud}})$) be a subset of paths in Q_G. Then since $\mathcal{D}$ is a family of inclusion-chains ($H_0 > \cdots > H_k$) for some subgroups $H_i \leq G$, a complex $\mathrm{T}_{Q_G}(\mathcal{D})$ is nothing else but just the usual subgroup complex of G (see [3] for example). Therefore $\mathrm{T}_{Q_G^{\mathrm{ud}}}(\mathcal{D})$ for $\mathcal{D} \subseteq \mathsf{P}(Q_G^{\mathrm{ud}})$ can be thought of a natural generalization of the usual.

DEFINITION 6.17. For a subset $\mathcal{X} \subseteq \mathrm{Sgp}(G) = (Q_G)_0 = (Q_G^{\mathrm{ud}})_0$ of vertices, put

$$\mathsf{P}(Q_G) \cap \mathcal{X} := \{\Delta \in \mathsf{P}(Q_G) \mid \mathsf{Ob}(\Delta) \subseteq \mathcal{X}\},$$
$$\mathsf{P}(Q_G^{\mathrm{ud}}) \cap \mathcal{X} := \{\Delta \in \mathsf{P}(Q_G^{\mathrm{ud}}) \mid \mathsf{Ob}(\Delta) \subseteq \mathcal{X}\}.$$

We denote by $\mathrm{T}_{Q_G}(\mathcal{X})$, $\mathrm{T}_{Q_G^{\mathrm{ud}}}(\mathcal{X})$ respectively complexes $\mathrm{T}_{Q_G}(\mathsf{P}(Q_G) \cap \mathcal{X})$, $\mathrm{T}_{Q_G^{\mathrm{ud}}}(\mathsf{P}(Q_G^{\mathrm{ud}}) \cap \mathcal{X})$.

DEFINITION 6.18. Let G be a finite group, and p be a prime divisor of the order of G. Denote by $\mathcal{S}_p(G)$ the set of all non-trivial p-subgroups of G.

The following result is well-known.

Proposition 6.19 (Lemma 2.2 in [3]). *If $O_p(G) \neq \{e\}$ then $\mathrm{T}_{Q_G}(\mathcal{S}_p(G))$ is contractible where $O_p(G)$ is the largest normal p-subgroup of G.*

The converse of the statement of Proposition 6.19 is known as Quillen's conjecture. The next result is an extended version of Proposition 6.19.

Proposition 6.20. *If $O_p(G) \neq \{e\}$ then $\mathrm{T}_{Q_G^{\mathrm{ud}}}(\mathcal{S}_p(G))$ is contractible.*

Proof. Let $\mathcal{P} := \mathsf{sd}(\mathrm{T}_{Q_G^{\mathrm{ud}}}(\mathcal{S}_p(G)))$ be the barycentric subdivision of $\mathrm{T}_{Q_G^{\mathrm{ud}}}(\mathcal{S}_p(G))$, and $N := O_p(G)$. It is enough to show that $\mathcal{P}$ is conically contractible by Lemma 6.5. First we note that $\{N\} \in \mathcal{P}$. Furthermore, for any $\sigma \in \mathcal{P}$, there exists $\Gamma = (L_0 - \cdots - L_k) \in \mathsf{P}(Q_G^{\mathrm{ud}}) \cap \mathcal{S}_p(G)$ such that $\sigma \subseteq \mathsf{Ob}(\Gamma)$ by the definition of simplices. We set

$$\Gamma' := \begin{cases} (L_0 - \cdots - L_k = N) = \Gamma & \text{if} \quad L_k = N, \\ (L_0 - \cdots - L_k - N) & \text{if} \quad L_k > N \quad \text{or} \quad L_k < N, \\ (L_0 - \cdots - L_k \leftarrow L_k N \rightarrow N) & \text{if} \quad L_k \not\geq N \quad \text{and} \quad L_k \not\leq N. \end{cases}$$

Then $\Gamma' \in \mathsf{P}(Q_G^{\mathrm{ud}}) \cap \mathcal{S}_p(G)$. Since $\sigma \cup \{N\} \subseteq \mathsf{Ob}(\Gamma) \cup \{N\} \subseteq \mathsf{Ob}(\Gamma')$, we have that $\sigma \cup \{N\} \in \mathcal{P}$. This yields that a map

$$f : \mathcal{P} \to \mathcal{P}$$

defined by

$$\sigma \mapsto \sigma \cup \{N\}$$

is a poset map such that $\sigma \subseteq f(\sigma)$ and $f(\sigma) \supseteq \{N\}$ for any $\sigma \in \mathcal{P}$. Therefore $\mathcal{P}$ is conically contractible. The proof is complete. □

We give one more result on the contractibility in the next.

Proposition 6.21. $\mathrm{T}_{Q_G}(\mathsf{P}(Q_G))$, $\mathrm{T}_{Q_G^{\mathrm{ud}}}(\mathsf{P}(Q_G^{\mathrm{ud}}))$, *and* $\mathrm{T}_{Q_G^{\mathrm{ud}}}(\mathsf{P}(Q_G^{\mathrm{ud}})^{\mathrm{pr}})$ *are all contractible.*

Proof. Let $\mathcal{P} := \mathsf{sd}\big(\mathrm{T}_{Q_G^{\mathrm{ud}}}(\mathsf{P}(Q_G^{\mathrm{ud}}))\big)$ be the barycentric subdivision of $\mathrm{T}_{Q_G^{\mathrm{ud}}}(\mathsf{P}(Q_G^{\mathrm{ud}}))$. It is enough to show that $\mathcal{P}$ is conically contractible by Lemma 6.5. First we note that $\{G\} \in \mathcal{P}$. Furthermore, for any $\sigma \in \mathcal{P}$, there exists $\Gamma = (L_0 - \cdots - L_k) \in \mathsf{P}(Q_G^{\mathrm{ud}})$ such that $\sigma \subseteq \mathsf{Ob}(\Gamma)$ by the definition of simplices. We set

$$\Gamma' := \begin{cases} \Gamma & \text{if} \quad L_i = G \quad \text{for some} \quad 0 \leq i \leq k, \\ (L_0 - \cdots - L_k \leftarrow G) & \text{if} L_i \neq G \quad \text{for any} \quad 0 \leq i \leq k. \end{cases}$$

Then $\Gamma' \in \mathsf{P}(Q_G^{\mathrm{ud}})$, and since $\sigma \cup \{G\} \subseteq \mathsf{Ob}(\Gamma) \cup \{G\} = \mathsf{Ob}(\Gamma')$, we have that $\sigma \cup \{G\} \in \mathcal{P}$. This yields that a map

$$f : \mathcal{P} \to \mathcal{P}$$

defined by

$$\sigma \mapsto \sigma \cup \{G\}$$

is a poset map such that $\sigma \subseteq f(\sigma)$ and $f(\sigma) \supseteq \{G\}$ for any $\sigma \in \mathcal{P}$. Therefore $\mathcal{P}$ is conically contractible. The proof is complete. The same argument can be applied to $\mathrm{T}_{Q_G}(\mathsf{P}(Q_G))$ and $\mathrm{T}_{Q_G^{\mathrm{ud}}}(\mathsf{P}(Q_G^{\mathrm{ud}})^{\mathrm{pr}})$. □

6.3. Coset complexes $\mathrm{T}_{Q_{CG}^{\mathrm{ud}}}(\mathcal{H})$ and G-simplicial maps. Let Q_{CG} be a coset quiver of G in Definition 4.5, and let Q_{CG}^{ud} be the extended quiver of Q_{CG} in Definition 3.1. For a subset $\mathcal{H} \subseteq \mathsf{P}(Q_{CG}^{\mathrm{ud}})$, a path complex $\mathrm{T}_{Q_{CG}^{\mathrm{ud}}}(\mathcal{H})$ of $\mathcal{H}$ in Q_{CG}^{ud} is called a coset complex of G. In this section, we focus on a subset $\tilde{\mathfrak{G}}(\mathcal{D}) \subseteq \mathsf{P}(Q_{CG}^{\mathrm{ud}})$ of paths in Q_{CG}^{ud} obtained from $\mathcal{D} \subseteq \mathsf{P}(Q_G^{\mathrm{ud}})$ (see Definition 4.7), and deal with a coset complex $\mathrm{T}_{Q_{CG}^{\mathrm{ud}}}(\tilde{\mathfrak{G}}(\mathcal{D}))$.

Recall that Q_G^{ud} and Q_{CG}^{ud} are both G-quivers, and that $\mathsf{P}(Q_G^{\mathrm{ud}})$ and $\mathsf{P}(Q_{CG}^{\mathrm{ud}})$ are preserved by G-conjugate action and the left multiplication respectively (see Remark 4.6). Since $\tilde{\mathfrak{G}}(\mathcal{D})$ is G-invariant for any $\mathcal{D} \subseteq \mathsf{P}(Q_G^{\mathrm{ud}})$ (see Remark 4.10), we have that $\mathrm{T}_{Q_{CG}^{\mathrm{ud}}}(\tilde{\mathfrak{G}}(\mathcal{D}))$ is a G-simplicial complex (see Remark 6.3). Then we will introduce a G-simplicial map $\varphi_{G,\mathcal{D}}$ between G-simplicial complexes $\mathrm{T}_{Q_{CG}^{\mathrm{ud}}}(\tilde{\mathfrak{G}}(\mathcal{D}))$ and $\mathrm{T}_{Q_G^{\mathrm{ud}}}(\mathsf{P}(Q_G^{\mathrm{ud}}))$. This G-map is suggested in Introduction.

REMARK 6.22. Let $\mathcal{D} \subseteq \mathsf{P}(Q_G^{\mathrm{ud}})^{\mathrm{pr}}$ be a subset of proper paths in Q_G^{ud}. Since a path $\Delta \in \mathcal{D}$ is proper, so is any path in $\tilde{\mathfrak{G}}(\Delta)$ (see Remark 4.8 (3)). Thus $|\mathsf{Ob}(\Gamma)| - 1 = l(\Gamma) = l(\Delta) = |\mathsf{Ob}(\Delta)| - 1$ for all $\Gamma \in \tilde{\mathfrak{G}}(\Delta)$. It follows that complexes $\mathrm{T}_{Q_G^{\mathrm{ud}}}(\mathcal{D})$ and $\mathrm{T}_{Q_{CG}^{\mathrm{ud}}}(\tilde{\mathfrak{G}}(\mathcal{D}))$ have the same dimensions.

In order to define our simplicial map, we first prepare a covering $\mathcal{C}_\Delta$ of a path $\Delta \in \mathsf{P}(Q^{\mathrm{ud}})$ in general.

DEFINITION 6.23. Let Q be a G-quiver, and $\Delta = (a_0 - a_1 - \cdots - a_k) \in \mathsf{P}(Q^{\mathrm{ud}})$ be a path in Q^{ud}.
(1) For $0 \leq i \leq j \leq k$, define $\Delta_{[i,j]} := (a_i - a_{i+1} - \cdots - a_j) \in \mathsf{P}(Q^{\mathrm{ud}})$ which we call an interval of Δ. For intervals $\Delta_{[i,j]}$ and $\Delta_{[s,t]}$ of Δ, define an ordering $\Delta_{[i,j]} \preceq \Delta_{[s,t]}$ precisely when $s \leq i \leq j \leq t$.

(2) Put

$$\mathcal{C}_\Delta := \left\{ \Gamma\colon \text{interval of } \Delta \;\middle|\; \begin{array}{l} \Gamma \text{ is maximal w.r.t. } \preceq \text{ among intervals} \\ \Lambda \text{ of } \Delta \text{ such that } a \not\sim_G b \text{ for all distinct} \\ a, b \in \mathsf{Ob}(\Lambda) \end{array} \right\} \subseteq \mathsf{P}(Q^{\mathrm{ud}}).$$

We call $\mathcal{C}_\Delta$ a covering of Δ. Then $\mathcal{C}_\Delta$ has the following properties.

(a) $\mathsf{Ob}(\Delta) = \bigcup_{\Gamma \in \mathcal{C}_\Delta} \mathsf{Ob}(\Gamma)$.

(b) Any $\Gamma \in \mathcal{C}_\Delta$ is proper, so that $\mathcal{C}_\Delta \subseteq \mathsf{P}(Q^{\mathrm{ud}})^{\mathrm{pr}}$.

(c) If $a \not\sim_G b$ for all distinct $a, b \in \mathsf{Ob}(\Delta)$ then $\mathcal{C}_\Delta = \{\Delta\}$.

DEFINITION 6.24. Let $\Delta = (L_0 - \cdots - L_k) \in \mathsf{P}(Q_G^{\mathrm{ud}})$ be a path in Q_G^{ud}, and let $\mathcal{C}_\Delta \subseteq \mathsf{P}(Q_G^{\mathrm{ud}})$ be a covering of Δ (see Definition 6.23). Recall that the vertex sets V_1 and V_2 of simplicial complexes $\mathrm{T}_{Q_{CG}^{\mathrm{ud}}}(\tilde{\mathfrak{G}}(\mathcal{C}_\Delta))$ and $\mathrm{T}_{Q_G^{\mathrm{ud}}}(\mathsf{P}(Q_G^{\mathrm{ud}}))$ are respectively

$$V_1 = \bigcup_{\Gamma \in \tilde{\mathfrak{G}}(\mathcal{C}_\Delta)} \mathsf{Ob}(\Gamma) = \bigcup_{i=0}^{k} G/L_i$$

and

$$V_2 = \mathrm{Sgp}(G).$$

Denote by $\varphi_{G,\Delta}\colon V_1 \to V_2$ a map defined by $\varphi_{G,\Delta}(gL) := L^{g^{-1}} = gLg^{-1}$ for all cosets $gL \in V_1$.

Proposition 6.25. *Under the above situation, $\varphi_{G,\Delta}$ induces a G-simplicial map*

$$\varphi_{G,\Delta}\colon \mathrm{T}_{Q_{CG}^{\mathrm{ud}}}(\tilde{\mathfrak{G}}(\mathcal{C}_\Delta)) \to \mathrm{T}_{Q_G^{\mathrm{ud}}}(\mathsf{P}(Q_G^{\mathrm{ud}})).$$

Furthermore $\varphi_{G,\Delta}$ preserves the dimensions of simplices.

Proof. Take any q-simplex $\sigma = \{g_{i_0}L_{i_0}, \ldots, g_{i_q}L_{i_q}\}$ of $\mathrm{T}_{Q_{CG}^{\mathrm{ud}}}(\tilde{\mathfrak{G}}(\mathcal{C}_\Delta))$. Then by the definition of simplices, for a certain interval $\Delta_{[s,t]} = (L_s - L_{s+1} - \cdots - L_t) \in \mathcal{C}_\Delta$ $(0 \le s \le t \le k)$, there exists a path

$$\Gamma = (g_sL_s - g_{s+1}L_{s+1} - \cdots - g_tL_t) \in \tilde{\mathfrak{G}}(\Delta_{[s,t]}) \subseteq \tilde{\mathfrak{G}}(\mathcal{C}_\Delta)$$

such that $\sigma \subseteq \mathsf{Ob}(\Gamma) = \{g_sL_s, g_{s+1}L_{s+1}, \ldots, g_tL_t\}$. Suppose that $g_jL_j < g_{j+1}L_{j+1}$ for some $s \le j \le t-1$. Then since $L_jg_j^{-1} < L_{j+1}g_{j+1}^{-1}$, we have that $g_jL_jg_j^{-1} < g_{j+1}L_{j+1}g_{j+1}^{-1}$. Similarly $g_jL_j > g_{j+1}L_{j+1}$ forces $g_jL_jg_j^{-1} > g_{j+1}L_{j+1}g_{j+1}^{-1}$. Thus we obtain a path

$$\Lambda := \left(L_s^{g_s^{-1}} - L_{s+1}^{g_{s+1}^{-1}} - \cdots - L_t^{g_t^{-1}}\right) \in \mathsf{P}(Q_G^{\mathrm{ud}}).$$

Therefore

$$\varphi_{G,\Delta}(\sigma) := \{\varphi_{G,\Delta}(g_{i_0}L_{i_0}), \ldots, \varphi_{G,\Delta}(g_{i_q}L_{i_q})\} = \left\{L_{i_0}^{g_{i_0}^{-1}}, \ldots, L_{i_q}^{g_{i_q}^{-1}}\right\}$$

is a subset of $\mathsf{Ob}(\Lambda)$, this is, a simplex of $\mathrm{T}_{Q_G^{\mathrm{ud}}}(\mathsf{P}(Q_G^{\mathrm{ud}}))$. This shows that $\varphi_{G,\Delta}$ is a simplicial map. Furthermore, by Definition 6.23 of a covering $\mathcal{C}_\Delta$, $g_iL_ig_i^{-1} \neq g_jL_jg_j^{-1}$ for any $s \leq i \neq j \leq t$. Thus $\varphi_{G,\Delta}$ preserves the dimensions of simplices. Finally we will show that $\varphi_{G,\Delta}$ commutes with the G-action. However it is clear from the fact that the G-action on Q_G^{ud} and Q_{CG}^{ud} are respectively defined by G-conjugation $x \cdot L := L^{x^{-1}}$ for $x \in G$ and $L \leq G$, and defined by the left multiplication $x \cdot gL := (xg)L$ for $x \in G$ and $gL \in G/L$ (see Remark 4.6). The proof is complete. □

In Section 7.6, we will describe the preimage under $\varphi_{G,\Delta}$.

REMARK 6.26. Let $\Delta = (L_0 - \cdots - L_k) \in \mathsf{P}(Q_G^{\mathrm{ud}})$ be a path in Q_G^{ud}.
(1) Suppose that $L_i \not\sim_G L_j$ for any $0 \leq i \neq j \leq k$ then $\mathcal{C}_\Delta = \{\Delta\}$ (see Definition 6.23). Thus in this case, we have a G-simplicial map

$$\varphi_{G,\Delta}\colon \mathrm{T}_{Q_{CG}^{\mathrm{ud}}}(\tilde{\mathfrak{G}}(\Delta)) \to \mathrm{T}_{Q_G^{\mathrm{ud}}}(\mathsf{P}(Q_G^{\mathrm{ud}})).$$

(2) Let $\mathcal{D} \subseteq \mathsf{P}(Q_G^{\mathrm{ud}})$ be a subset of paths in Q_G^{ud}. Suppose that $L \not\sim_G L'$ for all distinct $L, L' \in \mathsf{Ob}(\Gamma)$ for any $\Gamma \in \mathcal{D}$. Then a G-simplicial map

$$\varphi_{G,\mathcal{D}}\colon \mathrm{T}_{Q_{CG}^{\mathrm{ud}}}(\tilde{\mathfrak{G}}(\mathcal{D})) \to \mathrm{T}_{Q_G^{\mathrm{ud}}}(\mathsf{P}(Q_G^{\mathrm{ud}})),$$

can be also defined by the same way as in the case of Proposition 6.25.

7. Some properties of subgroup and coset complexes

In this section, we provide some properties of subgroup and coset complexes introduced in Section 6. In particular, the Euler characteristic and the top homology of a coset complex are calculated. Furthermore, we show that the automorphism group of a coset geometry is realized as the intersection of those of certain coset complexes. The connectedness of subgroup and coset complexes is also examined. Finally, we describe the preimage of a G-simplicial map defined in Proposition 6.25. Throughout this section, let G be a finite group, and let Q_G and Q_{CG} be respectively a subgroup quiver and a coset quiver of G defined in Section 4.

7.1. Ranges of paths in $\tilde{\mathfrak{G}}(\Delta)$. Let Δ be a path in Q_G^{ud}. First of all, we explicitly describe ranges $r(\Gamma)$ of paths $\Gamma \in \tilde{\mathfrak{G}}(\Delta) \subseteq \mathsf{P}(Q_{CG}^{\mathrm{ud}})$ although the proofs are somewhat tedious (see Definition 4.7 for $\tilde{\mathfrak{G}}(\Delta)$). The results in this section will be used later in various places.

NOTATION 7.1. For a path $\Delta = (L_0 - \cdots - L_k) \in \mathsf{P}(Q_G^{\mathrm{ud}})$ in Q_G^{ud} and $g_0L_0 \in G/L_0$, denote by

$$\tilde{\mathfrak{G}}(\Delta)_{g_0L_0} := \{\Gamma \in \tilde{\mathfrak{G}}(\Delta) \mid s(\Gamma) = g_0L_0\}$$

the set of all paths in $\tilde{\mathfrak{G}}(\Delta) \subseteq \mathsf{P}(Q_{CG}^{\mathrm{ud}})$ with start g_0L_0. The notation $r(\tilde{\mathfrak{G}}(\Delta)_{g_0L_0})$ implies a subset $\{r(\Gamma) \mid \Gamma \in \tilde{\mathfrak{G}}(\Delta)_{g_0L_0}\}$ of G/L_k.

Lemma 7.2. *Let $\Delta = (L_0 - \cdots - L_k) \in \mathsf{P}(Q_G^{\mathrm{ud}})$ be a path in Q_G^{ud}.*
(1) $|\tilde{\mathfrak{G}}(\Delta)_{xL_0}| = |\tilde{\mathfrak{G}}(\Delta)_{yL_0}|$ *for any $xL_0, yL_0 \in G/L_0$.*
(2) $|\tilde{\mathfrak{G}}(\Delta)| = |G : L_0| \times |\tilde{\mathfrak{G}}(\Delta)_{xL_0}|$ *for any $xL_0 \in G/L_0$.*
(3) *For any sequence $(0 =: i_0 < i_1 < \cdots < i_{q-1} < i_q := k)$ of indices, we have that*

$$|\tilde{\mathfrak{G}}(\Delta)| = |G \colon L_0| \times \prod_{s=0}^{q-1} \frac{|\tilde{\mathfrak{G}}(\Delta_{[i_s,i_{s+1}]})|}{|G : L_{i_s}|}$$

where $\Delta_{[i_s,i_{s+1}]}$ is an interval of Δ (see Definition 6.23).

Proof. (1) A set $\tilde{\mathfrak{G}}(\Delta)$ is G-invariant under the left multiplication (see Remark 4.10). Then, for any cosets $xL_0, yL_0 \in G/L_0$, a map $f \colon \tilde{\mathfrak{G}}(\Delta)_{xL_0} \to \tilde{\mathfrak{G}}(\Delta)_{yL_0}$ defined by $\Gamma \mapsto yx^{-1} \cdot \Gamma$ is bijective. Thus the number $|\tilde{\mathfrak{G}}(\Delta)_{xL_0}|$ is independent of a choice of xL_0.

(2) The intersection of $\tilde{\mathfrak{G}}(\Delta)_{xL_0}$ and $\tilde{\mathfrak{G}}(\Delta)_{yL_0}$ is empty if and only if $xL_0 \neq yL_0$. Thus the result follows from the previous (1).

(3) Take any path $\Gamma = (A_0 \overset{\alpha_1}{-} A_1 \overset{\alpha_2}{-} A_2 - \cdots - A_{k-1} \overset{\alpha_k}{-} A_k) \in \tilde{\mathfrak{G}}(\Delta)$ where $A_i \in G/L_i$ $(0 \leq i \leq k)$. Then we divide Γ according to a partition $(0 = i_0 < i_1 < \cdots < i_{q-1} < i_q = k)$ of indices. In other words, we identify Γ with a path in the closure $\overline{Q_{CG}^{\mathrm{ud}}}$ (see Definition 3.5) as follows:

$$(A_{i_0} \xrightarrow{\Gamma_1} A_{i_1} \xrightarrow{\Gamma_2} A_{i_2} \to \cdots \to A_{i_{q-1}} \xrightarrow{\Gamma_q} A_{i_q})$$

where $\Gamma_{s+1} = (\alpha_{i_s+1}, \alpha_{i_s+2}, \alpha_{i_s+3}, \ldots, \alpha_{i_{s+1}}) \in \tilde{\mathfrak{G}}(\Delta_{[i_s,i_{s+1}]})$ $(0 \leq s \leq q-1)$. Any coset $gL_0 \in G/L_0$ can be taken as $A_{i_0} = A_0$. Furthermore

$$\Gamma_1 \in \tilde{\mathfrak{G}}(\Delta_{[i_0,i_1]})_{A_{i_0}}$$

and

$$\Gamma_{s+1} \in \tilde{\mathfrak{G}}(\Delta_{[i_s,i_{s+1}]})_{r(\Gamma_s)} \quad (1 \leq s \leq q-1).$$

Therefore, using the previous results, we have that

$$|\tilde{\mathfrak{G}}(\Delta)| = |G : L_0| \times \prod_{s=0}^{q-1} |\tilde{\mathfrak{G}}(\Delta_{[i_s,i_{s+1}]})_{L_{i_s}}| = |G : L_0| \times \prod_{s=0}^{q-1} \frac{|\tilde{\mathfrak{G}}(\Delta_{[i_s,i_{s+1}]})|}{|G : L_{i_s}|}.$$

The proof is complete. □

NOTATION 7.3. For a path $\Delta = (L_0 - \cdots - L_k) \in \mathsf{P}(Q_G^{\mathrm{ud}})$ in Q_G^{ud}, let $I(\Delta) := \{0, 1, \ldots, k\}$ and

$$I'(\Delta)^{\max} := \{u \in I(\Delta) \mid L_{u-1} < L_u > L_{u+1}\},$$

$$I'(\Delta)^{\min} := \{v \in I(\Delta) \mid L_{v-1} > L_v < L_{v+1}\},$$

$$I(\Delta)^{\max} := \begin{cases} I'(\Delta)^{\max} & \text{if} \quad L_0 < L_1 \quad \text{and} \quad L_{k-1} > L_k, \\ I'(\Delta)^{\max} \cup \{0\} & \text{if} \quad L_0 > L_1 \quad \text{and} \quad L_{k-1} > L_k, \\ I'(\Delta)^{\max} \cup \{k\} & \text{if} \quad L_0 < L_1 \quad \text{and} \quad L_{k-1} < L_k, \\ I'(\Delta)^{\max} \cup \{0, k\} & \text{if} \quad L_0 > L_1 \quad \text{and} \quad L_{k-1} < L_k, \end{cases}$$

$$I(\Delta)^{\min} := \begin{cases} I'(\Delta)^{\min} & \text{if} \quad L_0 > L_1 \quad \text{and} \quad L_{k-1} < L_k, \\ I'(\Delta)^{\min} \cup \{0\} & \text{if} \quad L_0 < L_1 \quad \text{and} \quad L_{k-1} < L_k, \\ I'(\Delta)^{\min} \cup \{k\} & \text{if} \quad L_0 > L_1 \quad \text{and} \quad L_{k-1} > L_k, \\ I'(\Delta)^{\min} \cup \{0, k\} & \text{if} \quad L_0 < L_1 \quad \text{and} \quad L_{k-1} > L_k. \end{cases}$$

Another expression of $|\tilde{\mathfrak{G}}(\Delta)|$ different from that in Lemma 7.2 is given in the next.

Proposition 7.4 (cf. Lemma 4.10 in [2]). *For a path* $\Delta = (L_0 - \cdots - L_k) \in \mathsf{P}(Q_G^{\mathrm{ud}})$ *in* Q_G^{ud}, *we have that*

$$(*) \qquad |\tilde{\mathfrak{G}}(\Delta)| = |G : L_0| \times \frac{\prod_{u \in I(\Delta)^{\max}} |L_u|}{\prod_{v \in I(\Delta)^{\min}} |L_v|} \times m_\Delta$$

where

$$m_\Delta := \begin{cases} 1 & \textit{if} \quad 0 \in I(\Delta)^{\max} \quad \textit{and} \quad k \in I(\Delta)^{\min}, \\ \dfrac{1}{|L_k|} & \textit{if} \quad 0 \in I(\Delta)^{\max} \quad \textit{and} \quad k \in I(\Delta)^{\max}, \\ |L_0| & \textit{if} \quad 0 \in I(\Delta)^{\min} \quad \textit{and} \quad k \in I(\Delta)^{\min}, \\ \dfrac{|L_0|}{|L_k|} & \textit{if} \quad 0 \in I(\Delta)^{\min} \quad \textit{and} \quad k \in I(\Delta)^{\max}. \end{cases}$$

Proof. We proceed by induction on the length k of Δ. Suppose that $k = 1$, that is, $\Delta = (L_0 - L_1)$. If $L_0 < L_1$ then $I(\Delta)^{\max} = \{1\}$ and $I(\Delta)^{\min} = \{0\}$, so that $m_\Delta = |L_0|/|L_1|$. For each $g_0 L_0 \in G/L_0$, there exists an unique coset $A \in G/L_1$ containing $g_0 L_0$, that is, $A = g_0 L_1$. Thus

$$|\tilde{\mathfrak{G}}(\Delta)| = |G : L_0| \times 1 = |G : L_0| \times \frac{|L_1|}{|L_0|} \times m_\Delta.$$

On the other hand, if $L_0 > L_1$ then $I(\Delta)^{\max} = \{0\}$ and $I(\Delta)^{\min} = \{1\}$, so that $m_\Delta = 1$. Let $L_0/L_1 = \{z_1L_1, \dots, z_sL_1\}$. For each $g_0L_0 \in G/L_0$, there are exactly $s = |L_0 : L_1|$ cosets $g_0z_iL_1 \in G/L_1$ $(1 \le i \le s)$ contained in g_0L_0. Thus

$$|\tilde{\mathfrak{G}}(\Delta)| = |G : L_0| \times |L_0 : L_1| = |G : L_0| \times \frac{|L_0|}{|L_1|} \times m_\Delta.$$

Suppose next that $k \ge 1$. Take any $\Delta_1 := (L_k - L_{k+1}) \in \mathsf{P}(Q_G^{\mathrm{ud}})$, and set $\Gamma := \Delta\Delta_1 \in \mathsf{P}(Q_G^{\mathrm{ud}})$. Then we will examine $|\tilde{\mathfrak{G}}(\Gamma)|$. By induction, we have an equality $(*)$. Moreover, applying Lemma 7.2 to a path Γ and a sequence $(0 =: i_0 < k =: i_2 < k + 1 =: i_3)$ of indices, we have that

$$|\tilde{\mathfrak{G}}(\Gamma)| = |\tilde{\mathfrak{G}}(\Delta)| \times \left(\frac{1}{|G : L_k|} \times |\tilde{\mathfrak{G}}(\Delta_1)|\right).$$

CASE 1: $L_k < L_{k+1}$.

In this case $|\tilde{\mathfrak{G}}(\Delta_1)| = |G : L_k|$ and $|\tilde{\mathfrak{G}}(\Gamma)| = |\tilde{\mathfrak{G}}(\Delta)|$. If $k \in I(\Delta)^{\min}$ then $I(\Gamma)^{\max} = I(\Delta)^{\max} \cup \{k + 1\}$ and $I(\Gamma)^{\min} = I(\Delta)^{\min}$, so that $m_\Gamma = m_\Delta \times 1/|L_{k+1}|$. Thus

$$\begin{aligned}|\tilde{\mathfrak{G}}(\Gamma)| = |\tilde{\mathfrak{G}}(\Delta)| &= |G : L_0| \times \frac{\prod_{u \in I(\Delta)^{\max}} |L_u|}{\prod_{v \in I(\Delta)^{\min}} |L_v|} \times (|L_{k+1}| \times m_\Gamma)\\ &= |G : L_0| \times \frac{\prod_{u \in I(\Gamma)^{\max}} |L_u|}{\prod_{v \in I(\Gamma)^{\min}} |L_v|} \times m_\Gamma.\end{aligned}$$

If $k \in I(\Delta)^{\max}$ then $I(\Gamma)^{\max} = (I(\Delta)^{\max} \setminus \{k\}) \cup \{k + 1\}$ and $I(\Gamma)^{\min} = I(\Delta)^{\min}$, so that $m_\Gamma = m_\Delta \times |L_k|/|L_{k+1}|$. Thus

$$|\tilde{\mathfrak{G}}(\Gamma)| = |\tilde{\mathfrak{G}}(\Delta)| = |G : L_0| \times \frac{\prod_{u \in I(\Delta)^{\max}} |L_u|}{\prod_{v \in I(\Delta)^{\min}} |L_v|} \times \left(\frac{|L_{k+1}|}{|L_k|} \times m_\Gamma\right).$$

CASE 2: $L_k > L_{k+1}$.

In this case $|\tilde{\mathfrak{G}}(\Delta_1)| = |G : L_k| \times |L_k : L_{k+1}|$ and $|\tilde{\mathfrak{G}}(\Gamma)| = |\tilde{\mathfrak{G}}(\Delta)| \times |L_k|/|L_{k+1}|$. If $k \in I(\Delta)^{\min}$ then $I(\Gamma)^{\max} = I(\Delta)^{\max}$ and $I(\Gamma)^{\min} = (I(\Delta)^{\min} \setminus \{k\}) \cup \{k + 1\}$, so that $m_\Gamma = m_\Delta$. Thus

$$|\tilde{\mathfrak{G}}(\Gamma)| = |\tilde{\mathfrak{G}}(\Delta)| \times \frac{|L_k|}{|L_{k+1}|} = |G : L_0| \times \frac{\prod_{u \in I(\Delta)^{\max}} |L_u|}{\prod_{v \in I(\Delta)^{\min}} |L_v|} \times m_\Gamma \times \frac{|L_k|}{|L_{k+1}|}.$$

If $k \in I(\Delta)^{\max}$ then $I(\Gamma)^{\max} = I(\Delta)^{\max}$ and $I(\Gamma)^{\min} = I(\Delta)^{\min} \cup \{k + 1\}$, so that $m_\Gamma = m_\Delta \times |L_k|$. Thus

$$|\tilde{\mathfrak{G}}(\Gamma)| = |\tilde{\mathfrak{G}}(\Delta)| \times \frac{|L_k|}{|L_{k+1}|} = |G : L_0| \times \frac{\prod_{u \in I(\Delta)^{\max}} |L_u|}{\prod_{v \in I(\Delta)^{\min}} |L_v|} \times \left(\frac{1}{|L_k|} \times m_\Gamma\right) \times \frac{|L_k|}{|L_{k+1}|}.$$

This completes the proof. □

Proposition 7.5. *For a path* $\Delta = (L_0 - \cdots - L_k) \in \mathsf{P}(Q_G^{\mathrm{ud}})$ *in* Q_G^{ud} *and* $g_0 L_0 \in G/L_0$, *we have that*

$$r(\tilde{\mathfrak{G}}(\Delta)_{g_0L_0}) = g_0 L_{u_1} L_{u_2} \cdots L_{u_r}/L_k \subseteq G/L_k$$

where $I(\Delta)^{\max} = \{u_1, u_2, \ldots, u_r\}$ $(u_i < u_{i+1})$. *Note that* $L_{u_r} \geq L_k$ *always.*

Proof. Recall that

$$r(\tilde{\mathfrak{G}}(\Delta)_{g_0L_0}) = \left\{ A_k \in G/L_k \;\middle|\; \begin{array}{l} \exists A_j \in G/L_j \ (0 \leq j \leq k-1) \text{ such that} \\ (g_0L_0 =: A_0 - A_1 - \cdots - A_{k-1} - A_k) \in \tilde{\mathfrak{G}}(\Delta) \end{array} \right\}.$$

We proceed by induction on the length k of Δ. Suppose that $k = 1$, that is, $\Delta = (L_0 - L_1)$. If $L_0 < L_1$ then $I(\Delta)^{\max} = \{1\}$. There exists an unique coset $A \in G/L_1$ containing g_0L_0, that is, $A = g_0L_1$. Thus

$$r(\tilde{\mathfrak{G}}(\Delta)_{g_0L_0}) = \{g_0L_1\} = g_0L_1/L_1.$$

On the other hand, if $L_0 > L_1$ then $I(\Delta)^{\max} = \{0\}$. Let $L_0/L_1 = \{z_1L_1, \ldots, z_sL_1\}$. There are exactly $s = |L_0 : L_1|$ cosets $g_0z_iL_1 \in G/L_1$ $(1 \leq i \leq s)$ contained in g_0L_0. Thus

$$r(\tilde{\mathfrak{G}}(\Delta)_{g_0L_0}) = \{g_0z_iL_1 \mid 1 \leq i \leq s\} = g_0L_0/L_1.$$

Suppose next that $k \geq 1$. Take any $\Delta_1 := (L_k - L_{k+1}) \in \mathsf{P}(Q_G^{\mathrm{ud}})$, and set $\Gamma := \Delta\Delta_1 \in \mathsf{P}(Q_G^{\mathrm{ud}})$. Then we will examine $r(\tilde{\mathfrak{G}}(\Gamma)_{g_0L_0})$. Note that by induction we have that

$$r(\tilde{\mathfrak{G}}(\Delta)_{g_0L_0}) = g_0L_{u_1}L_{u_2}\cdots L_{u_r}/L_k =: \{y_1L_k, \ldots, y_tL_k\} \ (L_{u_r} \geq L_k).$$

CASE 1: $L_k > L_{k+1}$.

In this case $I(\Gamma)^{\max} = I(\Delta)^{\max}$. Let $L_k/L_{k+1} =: \{w_1L_{k+1}, \ldots, w_dL_{k+1}\}$. Thus we have that

$$\begin{aligned} &r(\tilde{\mathfrak{G}}(\Gamma)_{g_0L_0}) = \{y_iw_jL_{k+1} \mid 1 \leq i \leq t, 1 \leq j \leq d\}, \\ &y_iw_jL_{k+1} \in (g_0L_{u_1}L_{u_2}\cdots L_{u_r})L_k/L_{k+1} = g_0L_{u_1}L_{u_2}\cdots L_{u_r}/L_{k+1}. \end{aligned}$$

CASE 2: $L_k < L_{k+1}$.

In this case $r(\tilde{\mathfrak{G}}(\Gamma)_{g_0L_0}) = \{y_1L_{k+1}, \ldots, y_tL_{k+1}\}$. If $k \in I(\Delta)^{\min}$ then $I(\Gamma)^{\max} = I(\Delta)^{\max} \cup \{u_{r+1} := k+1\}$. Thus we have that

$$y_iL_{k+1} \in (g_0L_{u_1}L_{u_2}\cdots L_{u_r})L_{k+1}/L_{k+1} = g_0L_{u_1}L_{u_2}\cdots L_{u_r}L_{u_{r+1}}/L_{k+1}.$$

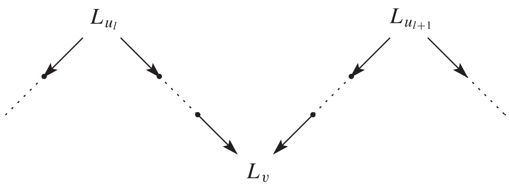
On the other hand, if $k \in I(\Delta)^{\max}$ (i.e. $u_r = k$) then $I(\Gamma)^{\max} = (I(\Delta)^{\max} \setminus \{k\}) \cup \{u_{r+1} := k+1\}$. Thus, using the fact that $L_kL_{k+1} = L_{k+1}$, we have that

$$y_iL_{k+1} \in (g_0L_{u_1}L_{u_2}\cdots L_{u_r})L_{k+1}/L_{k+1} = g_0L_{u_1}L_{u_2}\cdots L_{u_{r-1}}L_{u_{r+1}}/L_{k+1}.$$

This completes the proof. □

Next, we count the number of ranges in $r(\tilde{\mathfrak{G}}(\Delta)_{g_0 L_0})$ by using a way different from that in Proposition 7.5. The following integer M_v^Δ is kind of a multiplicity which plays an important role in Proposition 7.8.

DEFINITION 7.6. For a path $\Delta = (L_0 - \cdots - L_k) \in \mathsf{P}(Q_G^{\mathrm{ud}})$ in Q_G^{ud}, let $I(\Delta)^{\max} := \{u_1, u_2, \ldots, u_r\}$ ($u_i < u_{i+1}$). Then for any $v \in I'(\Delta)^{\min}$, there exists an integer l ($1 \leq l \leq r-1$) such that $(L_{u_l} \to \cdots \to L_v \leftarrow \cdots \leftarrow L_{u_{l+1}})$ is involved in Δ.



In this case, denote by

$$M_v^\Delta := |(L_{u_{l+1}} \cap (L_{u_l} L_{u_{l-1}} \cdots L_{u_1})(L_{u_1} L_{u_2} \cdots L_{u_l}))/L_v|.$$

REMARK 7.7. Let $A \subseteq G$ be a subset, and $B \leq G$ be a subgroup such that $A \supseteq B$ and $AB = A$. Then the number of cosets in $A/B = AB/B = \{aB \mid a \in A\}$ is $|A|/|B|$. In particular, $|A|$ is divisible by $|B|$. This situation can be applied to M_v^Δ in Definition 7.6.

Proposition 7.8. *For a path $\Delta = (L_0 - \cdots - L_k) \in \mathsf{P}(Q_G^{\mathrm{ud}})$ in Q_G^{ud} and $g_0 L_0 \in G/L_0$, we have that*

$$(**) \qquad |r(\tilde{\mathfrak{G}}(\Delta)_{g_0 L_0})| = \frac{|\tilde{\mathfrak{G}}(\Delta)|}{|G : L_0|} \times \prod_{v \in I'(\Delta)^{\min}} \frac{1}{M_v^\Delta}.$$

In particular, this value is independent of a choice of $g_0 L_0 \in G/L_0$.

Proof. The proof is similar to that of Proposition 7.5. We proceed by induction on the length k of Δ. Suppose that $k = 1$, that is, $\Delta = (L_0 - L_1)$. If $L_0 < L_1$ then as in the proof of Proposition 7.5, $r(\tilde{\mathfrak{G}}(\Delta)_{g_0 L_0}) = \{g_0 L_1\}$. Furthermore since $|\tilde{\mathfrak{G}}(\Delta)| = |G : L_0|$ and $I'(\Delta)^{\min} = \emptyset$, we have the result. On the other hand, if $L_0 > L_1$ then as in the proof of Proposition 7.5, $|r(\tilde{\mathfrak{G}}(\Delta)_{g_0 L_0})| = |L_0 : L_1|$. Furthermore since $|\tilde{\mathfrak{G}}(\Delta)| = |G : L_0| \times |L_0 : L_1|$ and $I'(\Delta)^{\min} = \emptyset$, we obtain the result.

Suppose next that $k \geq 1$. Take any $\Delta_1 := (L_k - L_{k+1}) \in \mathsf{P}(Q_G^{\mathrm{ud}})$, and set $\Gamma := \Delta\Delta_1 \in \mathsf{P}(Q_G^{\mathrm{ud}})$. Then we will examine $|r(\tilde{\mathfrak{G}}(\Gamma)_{g_0 L_0})|$. Note that by induction, we have an equality $(**)$.

CASE 1: $L_k > L_{k+1}$.

In this case, $|\tilde{\mathfrak{G}}(\Gamma)| = |\tilde{\mathfrak{G}}(\Delta)| \times |L_k : L_{k+1}|$ as in the proof of Proposition 7.4. Since $I'(\Gamma)^{\min} = I'(\Delta)^{\min}$ and $M_v^\Gamma = M_v^\Delta$ for any $v \in I'(\Gamma)^{\min}$, we have that

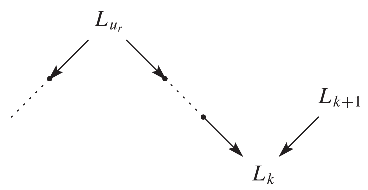
$$\begin{aligned} |r(\tilde{\mathfrak{G}}(\Gamma)_{g_0L_0})| &= |r(\tilde{\mathfrak{G}}(\Delta)_{g_0L_0})| \times |L_k : L_{k+1}| \\ &= \frac{1}{|G : L_0|} \times \left(\frac{|\tilde{\mathfrak{G}}(\Gamma)|}{|L_k : L_{k+1}|} \right) \times \prod_{v \in I'(\Delta)^{\min}} \frac{1}{M_v^\Delta} \times |L_k : L_{k+1}| \\ &= \frac{|\tilde{\mathfrak{G}}(\Gamma)|}{|G : L_0|} \times \prod_{v \in I'(\Gamma)^{\min}} \frac{1}{M_v^\Gamma}. \end{aligned}$$

CASE 2: $L_k < L_{k+1}$.

In this case, $|\tilde{\mathfrak{G}}(\Gamma)| = |\tilde{\mathfrak{G}}(\Delta)|$ as in the proof of Proposition 7.4. Furthermore by Proposition 7.5, we have that

$$r(\tilde{\mathfrak{G}}(\Delta)_{g_0L_0}) = g_0 L_{u_1} \cdots L_{u_r}/L_k =: \{y_1 L_k, \ldots, y_t L_k\}$$

where $I(\Delta)^{\max} = \{u_1, \ldots, u_r\}$ $(u_i < u_{i+1})$ and $y_i \in g_0 L_{u_1} \cdots L_{u_r}$ $(1 \leq i \leq t)$. It follows that $r(\tilde{\mathfrak{G}}(\Gamma)_{g_0L_0}) = \{y_1 L_{k+1}, \ldots, y_t L_{k+1}\}$ whose cardinality is less than or equal to t. Assume that $k \in I(\Delta)^{\min}$.



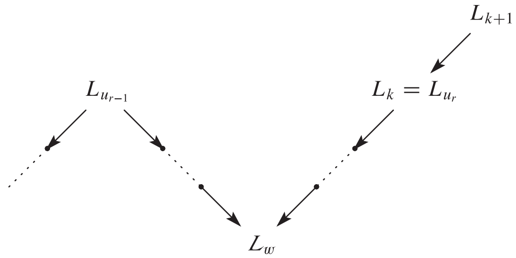
Then $I'(\Gamma)^{\min} = I'(\Delta)^{\min} \cup \{k\}$ and $I(\Gamma)^{\max} = I(\Delta)^{\max} \cup \{u_{r+1} := k+1\}$. Suppose that $y_i L_{k+1} = y_j L_{k+1}$ then

$$y_i^{-1} y_j \in L_{k+1} \cap (g_0 L_{u_1} \cdots L_{u_r})^{-1} (g_0 L_{u_1} \cdots L_{u_r}) = L_{k+1} \cap (L_{u_r} \cdots L_{u_1})(L_{u_1} \cdots L_{u_r}).$$

Thus $|r(\tilde{\mathfrak{G}}(\Gamma)_{g_0L_0})| = t/M_k^\Gamma$ where $M_k^\Gamma = |(L_{k+1} \cap (L_{u_r} \cdots L_{u_1})(L_{u_1} \cdots L_{u_r}))/L_k|$. Furthermore $M_v^\Delta = M_v^\Gamma$ for any $v \in I'(\Delta)^{\min}$, and hence

$$|r(\tilde{\mathfrak{G}}(\Gamma)_{g_0L_0})| = \frac{|\tilde{\mathfrak{G}}(\Delta)|}{|G : L_0|} \times \prod_{v \in I'(\Delta)^{\min}} \frac{1}{M_v^\Delta} \times \frac{1}{M_k^\Gamma} = \frac{|\tilde{\mathfrak{G}}(\Gamma)|}{|G : L_0|} \times \prod_{v \in I'(\Gamma)^{\min}} \frac{1}{M_v^\Gamma}.$$

On the other hand, assume that $k \in I(\Delta)^{\max}$.



Then $I'(\Gamma)^{\min} = I'(\Delta)^{\min}$ and $I(\Gamma)^{\max} = (I(\Delta)^{\max} \setminus \{k\}) \cup \{k+1\}$. Let $\Delta' = (L_0 - \cdots \leftarrow L_{u_{r-1}} \to \cdots \to L_w)$ involved in Δ. Then by Proposition 7.5, we have that

$$r(\tilde{\mathfrak{G}}(\Delta')_{g_0 L_0}) = g_0 L_{u_1} \cdots L_{u_{r-1}}/L_w =: \{z_1 L_w, \ldots, z_s L_w\}$$

where $z_i \in g_0 L_{u_1} \cdots L_{u_{r-1}}$ $(1 \le i \le s)$. By the same argument as above, the cardinalities of $r(\tilde{\mathfrak{G}}(\Delta)_{g_0 L_0})$ and $r(\tilde{\mathfrak{G}}(\Gamma)_{g_0 L_0})$ can be calculated as follows:

$$|r(\tilde{\mathfrak{G}}(\Delta)_{g_0 L_0})| = \frac{s}{|(L_k \cap (L_{u_{r-1}} \cdots L_{u_1})(L_{u_1} \cdots L_{u_{r-1}}))/L_w|} = \frac{s}{M_w^{\Delta}},$$
$$|r(\tilde{\mathfrak{G}}(\Gamma)_{g_0 L_0})| = \frac{s}{|(L_{k+1} \cap (L_{u_{r-1}} \cdots L_{u_1})(L_{u_1} \cdots L_{u_{r-1}}))/L_w|} = \frac{s}{M_w^{\Gamma}}.$$

Thus $|r(\tilde{\mathfrak{G}}(\Gamma)_{g_0 L_0})| = |r(\tilde{\mathfrak{G}}(\Delta)_{g_0 L_0})| \times (M_w^{\Delta}/M_w^{\Gamma})$. Note that $M_v^{\Delta} = M_v^{\Gamma}$ for any $v \in I'(\Delta)^{\min} \setminus \{w\}$. This yields that

$$|r(\tilde{\mathfrak{G}}(\Gamma)_{g_0 L_0})| = \frac{|\tilde{\mathfrak{G}}(\Delta)|}{|G : L_0|} \times \prod_{v \in I'(\Delta)^{\min}} \frac{1}{M_v^{\Delta}} \times \frac{M_w^{\Delta}}{M_w^{\Gamma}} = \frac{|\tilde{\mathfrak{G}}(\Gamma)|}{|G : L_0|} \times \prod_{v \in I'(\Gamma)^{\min}} \frac{1}{M_v^{\Gamma}}.$$

The proof is complete. □

7.2. The Euler characteristic of $\mathrm{T}_{Q_{CG}^{\mathrm{ud}}}(\tilde{\mathfrak{G}}(\Delta))$. In this section, for a special proper path Δ in Q_G^{ud}, the Euler characteristic of a coset complex of $\tilde{\mathfrak{G}}(\Delta)$ in Q_{CG}^{ud} is calculated. Since $\Delta = (L_0 - \cdots - L_k)$ is proper, we have that $G/L_i \cap G/L_j = \emptyset$ for all $0 \le i \ne j \le k$ (see Remark 4.8 (3)). This helps us to count simplices of $\mathrm{T}_{Q_{CG}^{\mathrm{ud}}}(\tilde{\mathfrak{G}}(\Delta))$. In the next, we recall the concept of types.

DEFINITION 7.9. For a proper path $\Delta = (L_0 - \cdots - L_k) \in \mathsf{P}(Q_G^{\mathrm{ud}})^{\mathrm{pr}}$ in Q_G^{ud}, let $\mathrm{T} := \mathrm{T}_{Q_{CG}^{\mathrm{ud}}}(\tilde{\mathfrak{G}}(\Delta))$ be a coset complex of $\tilde{\mathfrak{G}}(\Delta)$ in Q_{CG}^{ud}.
(1) We say that a q-simplex σ of T is of type $(i_0, \ldots, i_q)$ where $0 \le i_0 < \cdots < i_q \le k$, if $\sigma = \{A_{i_0}, \ldots, A_{i_q}\}$ for some $A_{i_s} \in G/L_{i_s}$ $(0 \le s \le q)$. For a subset $\emptyset \ne J =$

$\{j_0, \dots, j_q\} \subseteq I(\Delta)$, we identify J with the totally ordered sequence $(0 \leq j_0 < \cdots < j_q \leq k)$ of elements in J. Then we also use a term type J instead of type $(j_0, \dots, j_q)$.
(2) Denote by T^q $(0 \leq q \leq k)$ the set of all q-simplices of T. For a sequence $(0 \leq i_0 < \cdots < i_q \leq k)$, set $J := \{i_0, \dots, i_q\}$. Notations $\mathrm{T}_{(i_0,\dots,i_q)}$ and T_J stand for the set of all q-simplices of type J. In particular, we have that

$$|\mathrm{T}^q| = \sum_{J \subseteq I, |J| = q+1} |\mathrm{T}_J|.$$

Proposition 7.10. *For a proper path $\Delta = (L_0 - \cdots - L_k) \in \mathsf{P}(Q_G^{\mathrm{ud}})^{\mathrm{pr}}$ in Q_G^{ud}, let $\mathrm{T} := \mathrm{T}_{Q_{CG}^{\mathrm{ud}}}(\tilde{\mathfrak{G}}(\Delta))$ be a coset complex of $\tilde{\mathfrak{G}}(\Delta)$ in Q_{CG}^{ud}. For a sequence $(0 \leq i_0 < \cdots < i_q \leq k)$, we have that*

$$\left|\mathrm{T}_{(i_0,\dots,i_q)}\right| = |\tilde{\mathfrak{G}}(\Delta_{[i_0,i_q]})| \times \prod_{s=0}^{q-1} \prod_{v \in I'(\Delta_{[i_s,i_{s+1}]})^{\min}} \frac{1}{M_v^{\Delta_{[i_s,i_{s+1}]}}}$$

where M_v^{Γ} is an integer defined in Definition 7.6.

Proof. We count q-simplices $\sigma = \{A_{i_0}, \dots A_{i_q}\}$ of T of type $(i_0, \dots, i_q)$ where $A_{i_s} \in G/L_{i_s}$ $(0 \leq s \leq q)$. By the definition of simplices, there exists a path $\Gamma \in \tilde{\mathfrak{G}}(\Delta_{[i_0,i_q]})$ such that $\sigma \subseteq \mathsf{Ob}(\Gamma)$. Then by the same argument as in the proof of Lemma 7.2, Γ is identified with a path in the closure $\overline{Q_{CG}^{\mathrm{ud}}}$ (see Definition 3.5) as follows:

$$(A_{i_0} \xrightarrow{\Gamma_1} A_{i_1} \xrightarrow{\Gamma_2} A_{i_2} \to \cdots \to A_{i_{q-1}} \xrightarrow{\Gamma_q} A_{i_q})$$

where $\Gamma_{s+1} \in \tilde{\mathfrak{G}}(\Delta_{[i_s,i_{s+1}]})$ $(0 \leq s \leq q-1)$. Any coset $gL_{i_0} \in G/L_{i_0}$ can be taken as A_{i_0}. Furthermore

$$A_{i_{s+1}} = r(\Gamma_{s+1}) \in r(\tilde{\mathfrak{G}}(\Delta_{[i_s,i_{s+1}]})_{A_{i_s}}) \quad (0 \leq s \leq q-1).$$

By Proposition 7.8, the number of those ranges is independent of a choice of A_{i_s}. Therefore, applying Lemma 7.2 and Proposition 7.8, we can calculate as follows:

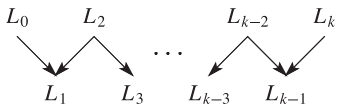
$$\begin{aligned}
|\mathrm{T}_{(i_0,\dots,i_q)}| &= |G : L_{i_0}| \times \prod_{s=0}^{q-1} |r(\tilde{\mathfrak{G}}(\Delta_{[i_s,i_{s+1}]})_{L_{i_s}})| \\
&= |G : L_{i_0}| \times \prod_{s=0}^{q-1} \left(\frac{|\tilde{\mathfrak{G}}(\Delta_{[i_s,i_{s+1}]})|}{|G : L_{i_s}|} \times \prod_{v \in I'(\Delta_{[i_s,i_{s+1}]})^{\min}} \frac{1}{M_v^{\Delta_{[i_s,i_{s+1}]}}} \right) \\
&= |\tilde{\mathfrak{G}}(\Delta_{[i_0,i_q]})| \times \prod_{s=0}^{q-1} \prod_{v \in I'(\Delta_{[i_s,i_{s+1}]})^{\min}} \frac{1}{M_v^{\Delta_{[i_s,i_{s+1}]}}}.
\end{aligned}$$

The proof is complete. □

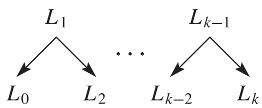
REMARK 7.11. For a proper path $\Delta = (L_0 - \cdots - L_k) \in \mathsf{P}(Q_G^{\mathrm{ud}})^{\mathrm{pr}}$ in Q_G^{ud}, let $\mathrm{T} := \mathrm{T}_{Q_{CG}^{\mathrm{ud}}}(\tilde{\mathfrak{G}}(\Delta))$ be a coset complex of $\tilde{\mathfrak{G}}(\Delta)$ in Q_{CG}^{ud}. It is clear that the numbers of 0-simplices (vertices) and k-simplices (those of maximal dimension) of T are $\sum_{i=0}^{k} |G/L_i|$ and $|\tilde{\mathfrak{G}}(\Delta)|$ respectively.

Proposition 7.12. *Let $\Delta \in \mathsf{P}(Q_G^{\mathrm{ud}})^{\mathrm{pr}}$ be one of the following four proper paths in Q_G^{ud}:*

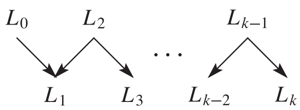
- Γ_1



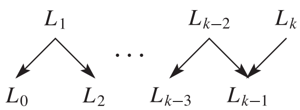
- Γ_2



- Γ_3



- Γ_4



Let $\mathrm{T} := \mathrm{T}_{Q_{CG}^{\mathrm{ud}}}(\tilde{\mathfrak{G}}(\Delta))$ *be a coset complex of* $\tilde{\mathfrak{G}}(\Delta)$ *in* Q_{CG}^{ud}. *Then we have the Euler characteristic of* T *as follows*:

$$\chi(\mathrm{T}) = \sum_{\emptyset \neq J \subseteq I(\Delta)^{\max}} (-1)^{|J|-1} |\mathrm{T}_J|.$$

Proof. Recall that the definition of the Euler characteristic is

$$\chi(\mathrm{T}) = \sum_{\emptyset \neq J \subseteq I(\Delta)} (-1)^{|J|-1} |\mathrm{T}_J|.$$

Suppose that $\Delta = \Gamma_2$ or $\Delta = \Gamma_3$, namely, $\Delta = (\cdots \to L_{k-2} \leftarrow L_{k-1} \to L_k)$. For any $xL_k \in G/L_k$, there exists a unique coset $A \in G/L_{k-1}$ containing xL_k (i.e. $A \to xL_k$), that is, $A = xL_{k-1}$. Let

$$\mathcal{J}_1 := \{J \subseteq I(\Delta) \mid k \in J,\ k-1 \notin J\},$$
$$\mathcal{J}_2 := \{J \subseteq I(\Delta) \mid k \in J,\ k-1 \in J\}.$$

Then there is a bijection $f : \mathcal{J}_1 \to \mathcal{J}_2$ defined by $J \mapsto f(J) := J \cup \{k-1\}$. For any $J \in \mathcal{J}_1$, we have that $|f(J)| = |J| + 1$. Furthermore, a map

$$\psi_J : \mathrm{T}_J \to \mathrm{T}_{f(J)}$$

defined by

$$\{\dots, xL_k\} \mapsto \{\dots, xL_{k-1}, xL_k\},$$

just inserting xL_{k-1}, is bijective. So $|\mathrm{T}_J| = |\mathrm{T}_{f(J)}|$ holds. It follows that the numbers of simplices of type J with $k \in J$ are cancelled out in the alternating sum. Thus we may suppose that $\Delta = \Gamma_1$ or $\Delta = \Gamma_4$, namely, $\Delta = (\cdots \leftarrow L_{k-2} \to L_{k-1} \leftarrow L_k)$.

For any $xL_{k-1} \in G/L_{k-1}$, there exists a unique coset $B \in G/L_k$ containing xL_{k-1} (i.e. $xL_{k-1} \leftarrow B$), that is, $B = xL_k$. Then by the same argument as above, the numbers of simplices of type J with $k-1 \in J$ are cancelled out in the alternating sum. Let $\mathcal{J} := \{\emptyset \neq J \subseteq I(\Delta) \mid k-1 \notin J\}$. Then we have that

$$\chi(\mathrm{T}) = \sum_{J \in \mathcal{J}} (-1)^{|J|-1} |T_J| = \mathfrak{X} + \mathfrak{Y}$$

where

$$\mathfrak{X} = \sum_{k \notin J \in \mathcal{J}} (-1)^{|J|-1} |\mathrm{T}_J|$$

and

$$\mathfrak{Y} = \sum_{k \in J \in \mathcal{J}} (-1)^{|J|-1} |\mathrm{T}_J|.$$

By induction, we get $\mathfrak{X} = \sum_{\emptyset \neq J \subseteq I(\Delta)^{\max} \setminus \{k\}} (-1)^{|J|-1} |\mathrm{T}_J|$. Thus it is enough to consider $\mathfrak{Y}$. But we can apply the same argument as above again. Indeed, for any $xL_{k-3} \in G/L_{k-3}$, there exists a unique coset $C \in G/L_{k-2}$ containing xL_{k-3} (i.e. $xL_{k-3} \leftarrow C$), that is, $C = xL_{k-2}$. Let

$$\mathcal{J}_{(1)} := \{J \in \mathcal{J} \mid k \in J,\ k-3 \in J,\ k-2 \notin J\},$$
$$\mathcal{J}_{(2)} := \{J \in \mathcal{J} \mid k \in J,\ k-3 \in J,\ k-2 \in J\}.$$

Then there is a bijection $g \colon \mathcal{J}_{(1)} \to \mathcal{J}_{(2)}$ defined by $J \mapsto g(J) := J \cup \{k-2\}$. For any

$J \in \mathcal{J}_{(1)}$, we have that $|g(J)| = |J| + 1$. Furthermore, a map

$$\theta_J : \mathrm{T}_J \to \mathrm{T}_{g(J)}$$

defined by

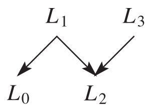
$$\{\ldots, xL_{k-3}, yL_k\} \mapsto \{\ldots, xL_{k-3}, xL_{k-2}, yL_k\},$$

just inserting xL_{k-2}, is bijective. So $|\mathrm{T}_J| = |\mathrm{T}_{g(J)}|$ holds. It follows that the numbers of simplices of type $k \in J \in \mathcal{J}$ with $k-3 \in J$ are cancelled out in the alternating sum. Repeating this process, we eventually conclude that

$$\mathfrak{Y} = \sum_{k \in J \subseteq I(\Delta)^{\max}} (-1)^{|J|-1} |\mathrm{T}_J|.$$

This completes the proof. □

EXAMPLE 7.13. Let $\Delta \in \mathsf{P}(Q_G^{\mathrm{ud}})^{\mathrm{pr}}$ be a proper path in Q_G^{ud} of the form



Then $I(\Delta)^{\max} = \{1, 3\}$. Set $\mathrm{T} := \mathrm{T}_{Q_{CG}^{\mathrm{ud}}}(\tilde{\mathfrak{G}}(\Delta))$. By Proposition 7.12, we have that

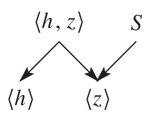
$$\chi(\mathrm{T}) = |\mathrm{T}_{\{1\}}| + |\mathrm{T}_{\{3\}}| - |\mathrm{T}_{\{1,3\}}|.$$

Furthermore, by Proposition 7.10, $|\mathrm{T}_{\{1\}}| = |G : L_1|$, $|\mathrm{T}_{\{3\}}| = |G : L_3|$, and

$$|\mathrm{T}_{\{1,3\}}| = |\tilde{\mathfrak{G}}(\Delta_{[1,3]})| \times \frac{1}{M_2^{\Delta_{[1,3]}}} = |G : L_2| \times \frac{1}{|L_1 \cap L_3 : L_2|} = |G : L_1 \cap L_3|.$$

It follows that $\chi(\mathrm{T}) = |G : L_1| + |G : L_3| - |G : L_1 \cap L_3|$.

Now, let $G := J_1$ be the first Janko group of order $2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$. There is a unique class of involutions, say a representative z, with the centralizer $C_G(z) \cong \langle z \rangle \times A$ where A is isomorphic to the alternating group of degree 5. Take Sylow subgroups $C_5 \cong \langle h \rangle \in \mathrm{Syl}_5(A)$ and $C_2 \times C_2 \times C_2 \cong S \in \mathrm{Syl}_2(G)$ with $z \in S$. Then the following proper path Δ in Q_G^{ud} is defined.



Note that $\langle h, z \rangle \cap S = \langle z \rangle$. From the above result, the Euler characteristic $\chi(\mathrm{T})$ of T is calculated as follows:

$$\chi(\mathrm{T}) = |G : \langle h, z\rangle| + |G : S| - |G : \langle z\rangle| = 83, \quad 391 = 3 \times 7 \times 11 \times 19^2,$$
$$\chi(\mathrm{T}) - 1 = 83, 390 = 2 \times 5 \times 31 \times 269.$$

7.3. The top homology of $\mathrm{T}_{Q^{\mathrm{ud}}_{CG}}(\tilde{\mathfrak{G}}(\Delta))$ is trivial. Let Δ be a proper path in Q^{ud}_G. In this section, it is shown that the top homology of a coset complex of $\tilde{\mathfrak{G}}(\Delta)$ in Q^{ud}_{CG} is trivial. This homology R-module can be realized as a homology of a quiver (see Section 5 and Example 5.13).

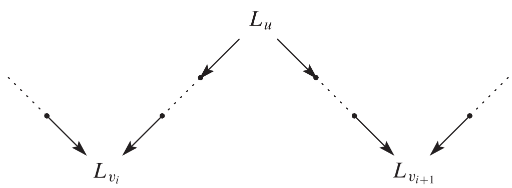
Proposition 7.14. *Let $\Delta = (L_0 - \cdots - L_k) \in \mathsf{P}(Q^{\mathrm{ud}}_G)^{\mathrm{pr}}$ be a proper path in Q^{ud}_G. Then the homology R-module of $\mathrm{T}_{Q^{\mathrm{ud}}_{CG}}(\tilde{\mathfrak{G}}(\Delta))$ of degree k is trivial.*

Proof. Put $\mathrm{T} := \mathrm{T}_{Q^{\mathrm{ud}}_{CG}}(\tilde{\mathfrak{G}}(\Delta))$, then $\dim \mathrm{T} = k$ (see Remark 6.22). Let $I(\Delta)^{\min} := \{v_1, \ldots, v_r\}$. Let $C_n(\mathrm{T})$ $(0 \le n \le k)$ be the R-free module with all n-dimensional oriented simplices $\langle\sigma\rangle$ of σ in T as basis, and let $\delta_n \colon C_n(\mathrm{T}) \to C_{n-1}(\mathrm{T})$ $(1 \le n \le k)$ be an R-homomorphism defined by

$$\delta_n(\langle x_0, x_1, \ldots, x_n\rangle) := \sum_{i=0}^{n} (-1)^i \langle x_0, \ldots, \hat{x}_i, \ldots, x_n\rangle$$

for $\langle x_0, x_1, \ldots, x_n\rangle \in C_n(\mathrm{T})$ (see Definition 5.7). It is enough to show that $\mathrm{Ker}\delta_k$ is trivial. Note that paths $\Gamma \in \tilde{\mathfrak{G}}(\Delta)$ can be identified with simplices of T of maximal dimension (see Remark 4.8 (3)).

First we recall that, for subgroups $H < K \le G$ and a coset $xH \in G/H$, there exists a unique coset $A \in G/K$ containing xH, that is, $A = xK$. This implies that a path $\Gamma \in \tilde{\mathfrak{G}}(\Delta)$ is uniquely determined by cosets $x_i L_{v_i} \in G/L_{v_i}$ $(0 \le i \le r)$ with the property that, in the following situation for Δ, $x_i L_u = x_{i+1} L_u$ namely $x_i^{-1} x_{i+1} \in L_u$ holds.



It follows that paths $(x_0 L_0 - \cdots - x_k L_k)$ and $(y_0 L_0 - \cdots - y_k L_k)$ in $\tilde{\mathfrak{G}}(\Delta)$ are the same if and only if $x_v L_v = y_v L_v$ for all $v \in I(\Delta)^{\min}$.

Take any $X = \sum_{\Gamma \in \tilde{\mathfrak{G}}(\Delta)} c_\Gamma \langle\Gamma\rangle \in \mathrm{Ker}\delta_k \subseteq C_k(\mathrm{T})$. Then

$$0 = \delta_k(X) = \sum_{\Gamma \in \tilde{\mathfrak{G}}(\Delta)} c_\Gamma \delta_k(\langle\Gamma\rangle) \in C_{k-1}(\mathrm{T}).$$

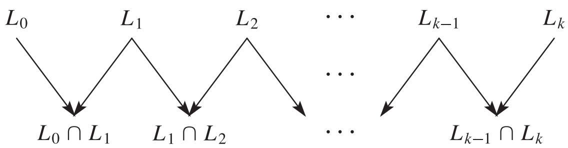
From the previous paragraph, we can see that, for a path $\Lambda = (y_0 L_0 - \cdots - y_k L_k) \in \tilde{\mathfrak{G}}(\Delta)$, $(k-1)$-simplices $\langle y_0 L_0, \ldots, \widehat{y_i L_i}, \ldots y_k L_k \rangle$ in $\delta_k(\langle \Lambda \rangle)$ which contain $y_v L_v$ for all $v \in I(\Delta)^{\min}$ must lie only in $c_\Lambda \delta_k(\langle \Lambda \rangle)$ in the sum $\sum_{\Gamma \in \tilde{\mathfrak{G}}(\Delta)} c_\Gamma \delta_k(\langle \Gamma \rangle)$. This forces that $c_\Gamma = 0$ for all $\Gamma \in \tilde{\mathfrak{G}}(\Delta)$. Thus $\mathrm{Ker}\delta_k$ is trivial. The proof is complete. □

7.4. A relation with coset geometries. Let $\mathcal{F} := \{L_0, \ldots, L_k\}$ be a family of subgroups of G, and let $I := \{0, \ldots, k\}$. A system $\mathcal{G}(G, \mathcal{F}) := (G/L_0, \ldots, G/L_k; *)$ is a coset geometry over I where $*$ is a binary reflective and symmetric relation on $V := G/L_0 \cup \cdots \cup G/L_k$ defined by the non-empty intersection, namely, cosets xL_i and yL_j are incident if $xL_i \cap yL_j \neq \emptyset$. Note that cosets $xL_i, yL_i \in G/L_i$ are incident if and only if $xL_i = yL_i$. In this section, we show that the automorphism group of $\mathcal{G}(G, \mathcal{F})$ is the intersection of those of our coset complexes $\mathrm{T}_{Q^{\mathrm{ud}}_{CG}}(\tilde{\mathfrak{G}}(\Gamma))$ for certain paths $\Gamma \in \mathsf{P}(Q^{\mathrm{ud}}_G)$.

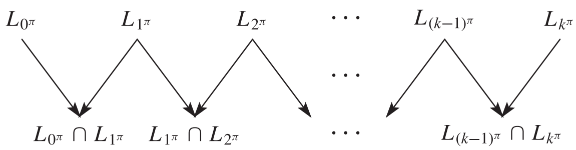
An automorphism of $\mathcal{G}(G,\mathcal{F})$ is an element $\psi \in \mathrm{Sym}(V)$ which preserves type and incidence, that is, $\psi(G/L_i) = G/L_i$ for all $0 \leq i \leq k$ and if $xL_i * yL_j$ then $\psi(xL_i) * \psi(yL_j)$. Denote by $\mathrm{Aut}(\mathcal{G}(G, \mathcal{F}))$ the group of all automorphisms of $\mathcal{G}(G, \mathcal{F})$. By the definition, we have that

$$\mathrm{Aut}(\mathcal{G}(G, \mathcal{F})) \leq \prod_{i=0}^{k} \mathrm{Sym}(G/L_i).$$

Let $\Delta \in \mathsf{P}(Q^{\mathrm{ud}}_G)$ be a path of the form



Furthermore, for a permutation $\pi \in \mathrm{Sym}(I)$, denote by $\Delta^{(\pi)}$ a path



where $i^\pi := \pi(i) \in I$ for all $i \in I$. For $\pi \in \mathrm{Sym}(I)$, set

$$\mathrm{T}_\pi := \mathrm{T}_{Q^{\mathrm{ud}}_{CG}}(\tilde{\mathfrak{G}}(\Delta^{(\pi)}))$$

and

$$W_\pi := \bigcup_{H \in \mathrm{Ob}(\Delta^{(\pi)})} G/H.$$

Then W_π is the vertex set of T_π. Denote by $\mathrm{Aut}(\mathrm{T}_\pi)$ the group of all elements of $\mathrm{Sym}(W_\pi)$ which preserves G/H for all $H \in \mathsf{Ob}(\Delta^{(\pi)})$, and preserves the set of simplices of T_π. In particular, we have that

$$\begin{aligned}\mathrm{Aut}(\mathrm{T}_\pi) &\leq \prod_{H\in \mathsf{Ob}(\Delta^{(\pi)})} \mathrm{Sym}(G/H)\\ &\hookrightarrow \left(\prod_{i=0}^{k} \mathrm{Sym}(G/L_i)\right) \times \left(\prod_{0\leq i<j\leq k} \mathrm{Sym}(G/L_i \cap L_j)\right).\end{aligned}$$

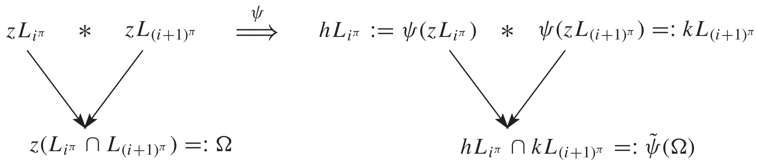
In this situation, we have the following.

Proposition 7.15. *The group* $\mathrm{Aut}(\mathcal{G}(G,\mathcal{F}))$ *is isomorphic to* $\bigcap_{\pi\in\mathrm{Sym}(I)} \mathrm{Aut}(\mathrm{T}_\pi)$.

Proof. Take any $\psi \in \mathrm{Aut}(\mathcal{G}(G,\mathcal{F}))$ and $\pi \in \mathrm{Sym}(I)$. For $z(L_{i^\pi} \cap L_{(i+1)^\pi}) \in G/(L_{i^\pi} \cap L_{(i+1)^\pi})$, elements zL_{i^π} and $zL_{(i+1)^\pi}$ are incident in $\mathcal{G}(G,\mathcal{F})$. So by the definition of ψ, $\psi(zL_{i^\pi}) =: hL_{i^\pi}$ and $\psi(zL_{(i+1)^\pi}) =: kL_{(i+1)^\pi}$ are incident. We set

$$\tilde{\psi}(z(L_{i^\pi} \cap L_{(i+1)^\pi})) := hL_{i^\pi} \cap kL_{(i+1)^\pi} = u(L_{i^\pi} \cap L_{(i+1)^\pi}) \in W_\pi$$

where $u \in hL_{i^\pi} \cap kL_{(i+1)^\pi} \neq \emptyset$.



Then a map $\tilde{\psi}\colon W_\pi \to W_\pi$, where $\tilde{\psi}|_V := \psi$, induces a bijection on W_π which preserves G/H for any $H \in \mathsf{Ob}(\Delta^{(\pi)})$. Furthermore, it is clear from the definition that $\tilde{\psi}$ acts on a set $\tilde{\mathfrak{G}}(\Delta^{(\pi)})$ of paths, and on the set of simplices of T_π. Thus we have that $\tilde{\psi} \in \mathrm{Aut}(\mathrm{T}_\pi)$.

Conversely, take any $\eta \in \bigcap_{\pi\in\mathrm{Sym}(I)} \mathrm{Aut}(\mathrm{T}_\pi)$. Then, by the definition, $\eta|_V \in \prod_{i=0}^{k} \mathrm{Sym}(G/L_i)$. For $xL_i, yL_j \in V$, suppose that $xL_i * yL_j$ in $\mathcal{G}(G,\mathcal{F})$, that is, $\emptyset \neq xL_i \cap yL_j = u(L_i \cap L_j)$ for $u \in xL_i \cap yL_j$. Then a path $(xL_i \to u(L_i \cap L_j) \to yL_j)$ is defined which is an interval of a path in $\tilde{\mathfrak{G}}(\Delta^{(\pi)})$ for some $\pi \in \mathrm{Sym}(I)$. Thus, by the definition of $\eta \in \mathrm{Aut}(\mathrm{T}_\pi)$, we have a path

$$(\eta(xL_i) \to \eta(u(L_i \cap L_j)) \leftarrow \eta(yL_j)),$$

namely, the intersection of $\eta(xL_i)$ and $\eta(yL_j)$ contains $\eta(u(L_i \cap L_j))$. So they are incident. It follows that $\eta|_V \in \mathrm{Aut}(\mathcal{G}(G,\mathcal{F}))$. Then the process given in the above provides the desired isomorphism as groups. □

7.5. Connectedness. In this section, we examine the connectedness of our subgroup and coset complexes. But before doing this, we recall its definition.

DEFINITION 7.16 (cf. pp. 164–165 in [4]). Let $K = (V(K), S(K))$ be a simplicial complex where $V(K)$ and $S(K)$ are respectively the sets of vertices and simplices of K. An edge in K is just an ordered pair $e = (a,b)$ of vertices $a,b \in V(K)$ such that a and b lie in a simplex $\sigma \in S(K)$. Denote by $\mathrm{ori}(e) := a$ and $\mathrm{end}(e) := b$. An edge path in K is a finite sequence $\alpha = (e_1, \ldots, e_n)$ of edges e_i in K such that $\mathrm{end}(e_i) = \mathrm{ori}(e_{i+1})$ for all $i = 1, \ldots, n-1$. Denote by $o(\alpha) := \mathrm{ori}(e_1)$ and $e(\alpha) := \mathrm{end}(e_n)$. K is connected if, for any $a, b \in V(K)$, there exists an edge path α in K such that $o(\alpha) = a$ and $e(\alpha) = b$.

Recall that a subgroup quiver Q_G^{ud} is a G-quiver, and that $\mathsf{P}(Q_G^{\mathrm{ud}})$ is preserved by G-conjugate action (see Remark 4.6). For a G-invariant subset $\mathcal{D} \subseteq \mathsf{P}(Q_G^{\mathrm{ud}})$ of paths in Q_G^{ud}, denote by $\mathcal{D}/_{\sim_G}$ a complete set of representatives of G-conjugate classes of $\mathcal{D}$.

Proposition 7.17. *Let $\mathcal{D} \subseteq \mathsf{P}(Q_G^{\mathrm{ud}})$ be a G-invariant subset of paths in Q_G^{ud}. Suppose that $\mathrm{T}_{Q_G^{\mathrm{ud}}}(\mathcal{D})$ is connected then so is $\mathrm{T}_{Q_G^{\mathrm{ud}}}(\mathcal{D}/_{\sim_G})$ for some $\mathcal{D}/_{\sim_G}$.*

Proof. Let $\mathcal{D} = \Delta_1^G \cup \Delta_2^G \cup \cdots \cup \Delta_m^G$ be a decomposition into G-orbits where $\Delta_i^G := \{x \cdot \Delta_i \mid x \in G\}$ and $x \cdot \Delta_i := (L_0^{x^{-1}} - \cdots - L_k^{x^{-1}})$ for $\Delta_i = (L_0 - \cdots - L_k)$. We proceed by induction on $m = |\mathcal{D}/_{\sim_G}|$. Suppose that $m = 1$. Then $\mathcal{D}/_{\sim_G} = \{\Delta_1\}$, so $\mathrm{T}_{Q_G^{\mathrm{ud}}}(\mathcal{D}/_{\sim_G}) = \mathrm{T}_{Q_G^{\mathrm{ud}}}(\Delta_1)$ is clearly connected. Thus we may assume that $m \geq 2$. Put $\mathcal{D}_1 := \Delta_2^G \cup \cdots \cup \Delta_m^G$, so $\mathcal{D} = \Delta_1^G \cup \mathcal{D}_1$. Then by induction, we may assume that, for a complete set $\mathcal{D}_1/_{\sim_G} = \{\Delta_2, \ldots, \Delta_m\}$ of representatives, $\mathrm{T}_{Q_G^{\mathrm{ud}}}(\mathcal{D}_1/_{\sim_G})$ is connected. Furthermore by our assumption that $\mathrm{T}_{Q_G^{\mathrm{ud}}}(\mathcal{D})$ is connected, we have that

$$\left(\bigcup_{\Gamma \in \Delta_1^G} \mathsf{Ob}(\Gamma)\right) \cap \left(\bigcup_{\Lambda \in \mathcal{D}_1} \mathsf{Ob}(\Lambda)\right) \neq \emptyset.$$

Thus we may assume that $\mathsf{Ob}(x \cdot \Delta_1) \cap \mathsf{Ob}(y \cdot \Delta_2) \neq \emptyset$ for some $x, y \in G$, namely $\mathsf{Ob}(y^{-1}x \cdot \Delta_1) \cap \mathsf{Ob}(\Delta_2) \neq \emptyset$. It follows that, for a complete set $\mathcal{D}/_{\sim_G} = \{y^{-1}x \cdot \Delta_1, \Delta_2, \ldots, \Delta_m\}$ of representatives, $\mathrm{T}_{Q_G^{\mathrm{ud}}}(\mathcal{D}/_{\sim_G})$ is connected. The proof is complete. $\square$

Proposition 7.18. *Let $\mathcal{D} \subseteq \mathsf{P}(Q_G^{\mathrm{ud}})$ be a subset of paths in Q_G^{ud}. The followings are equivalent.*

(1) $\mathrm{T}_{Q_{CG}^{\mathrm{ud}}}(\tilde{\mathfrak{G}}(\mathcal{D}))$ *is connected.*

(2) $\mathrm{T}_{Q_G^{\mathrm{ud}}}(\mathcal{D})$ *is connected, and* $\left\langle \bigcup_{\Delta \in \mathcal{D}} \mathsf{Ob}(\Delta) \right\rangle = G$ *holds.*

Proof. (1) $\Rightarrow$ (2): Take any vertices L and L' of $\mathrm{T}_{Q_G^{\mathrm{ud}}}(\mathcal{D})$. Since $\mathcal{D} \subseteq \tilde{\mathfrak{G}}(\mathcal{D})$, we have that

$$\bigcup_{\Delta \in \mathcal{D}} \mathsf{Ob}(\Delta) \subseteq \bigcup_{\Gamma \in \tilde{\mathfrak{G}}(\mathcal{D})} \mathsf{Ob}(\Gamma).$$

This shows that L and L' are vertices of $\mathrm{T}_{Q_{CG}^{\mathrm{ud}}}(\tilde{\mathfrak{G}}(\mathcal{D}))$ which is connected by our assumption. So there exists in $\mathrm{T}_{Q_{CG}^{\mathrm{ud}}}(\tilde{\mathfrak{G}}(\mathcal{D}))$ an edge path

$$((x_0L_0, x_1L_1), (x_1L_1, x_2L_2), \dots, (x_{t-1}L_{t-1}, x_tL_t))$$

where $x_0L_0 = L$, $x_tL_t = L'$, and $L_1, \dots L_{t-1} \in \bigcup_{\Delta \in \mathcal{D}} \mathsf{Ob}(\Delta)$. Since $(x_iL_i, x_{i+1}L_{i+1})$ is an edge, both x_iL_i and $x_{i+1}L_{i+1}$ lie in $\mathsf{Ob}(\Gamma)$ for some path $\Gamma = (y_0H_0 - \cdots - y_dH_d) \in \tilde{\mathfrak{G}}(\Delta) \subseteq \tilde{\mathfrak{G}}(\mathcal{D})$ where $\Delta = (H_0 - \cdots - H_d) \in \mathcal{D}$. Thus L_i and L_{i+1} lie in $\mathsf{Ob}(\Delta) = \{H_0, \dots, H_d\}$ (see Remark 4.8 (3)). It follows that (L_i, L_{i+1}) is an edge in $\mathrm{T}_{Q_G^{\mathrm{ud}}}(\mathcal{D})$, and that $((L, L_1), (L_1, L_2), \dots, (L_{t-1}, L'))$ forms an edge path in $\mathrm{T}_{Q_G^{\mathrm{ud}}}(\mathcal{D})$. Therefore $\mathrm{T}_{Q_G^{\mathrm{ud}}}(\mathcal{D})$ is connected.

Set $\mathcal{V} := \bigcup_{\Delta \in \mathcal{D}} \mathsf{Ob}(\Delta)$ and $N := \langle \mathcal{V} \rangle \leq G$. We will show that $N = G$. First we claim that, for a path $(x_0L_0 - \cdots - x_kL_k) \in \tilde{\mathfrak{G}}(\mathcal{D})$, if $x_iL_i \subseteq N$ for some i then $x_jL_j \subseteq N$ for all $0 \leq j \leq k$. Indeed suppose that $x_iL_i \leftarrow x_{i+1}L_{i+1}$. Then since $x_iL_i \subset x_{i+1}L_{i+1}$, we have that $x_{i+1}L_{i+1} = x_iL_{i+1} \subseteq N$. On the other hand, suppose that $x_iL_i \to x_{i+1}L_{i+1}$. Then $x_{i+1}L_{i+1} \subset x_iL_i \subseteq N$ as required. Next we claim that, for a connected component $\mathcal{C}$ of $\mathrm{T}_{Q_{CG}^{\mathrm{ud}}}(\tilde{\mathfrak{G}}(\mathcal{D}))$ with $\mathcal{C} \cap \mathcal{V} \neq \emptyset$, any element in $\mathcal{C}$ is contained in N. Indeed take $L \in \mathcal{C} \cap \mathcal{V}$. For any path $(L - x_1L_1 - \cdots - x_tL_t)$ in $\tilde{\mathfrak{G}}(\mathcal{D})$, since $L \leq \langle \mathcal{V} \rangle = N$, we have from the previous claim that $x_jL_j \subseteq N$ for all $1 \leq j \leq t$. Furthermore any element $A \in \mathcal{C}$ is connected with L along an edge path in $\mathrm{T}_{Q_{CG}^{\mathrm{ud}}}(\tilde{\mathfrak{G}}(\mathcal{D}))$. So $A \subseteq N$ as wanted.

Now assume that $N < G$, and take $y \in G$ such that $yN \neq N$. Let $\mathcal{C}$ be a connected component of $\mathrm{T}_{Q_{CG}^{\mathrm{ud}}}(\tilde{\mathfrak{G}}(\mathcal{D}))$ with $\mathcal{C} \cap \mathcal{V} \neq \emptyset$. Then from the previous claim, $A \subseteq N$ for any $A \in \mathcal{C}$. Thus $yA \subseteq yN$. Since $N \cap yN = \emptyset$, we have that $\mathcal{C} \cap y\mathcal{C} = \emptyset$ where $y\mathcal{C} := \{yA \mid A \in \mathcal{C}\}$ is a connected component. This yields that $\mathrm{T}_{Q_{CG}^{\mathrm{ud}}}(\tilde{\mathfrak{G}}(\mathcal{D}))$ is disconnected, a contradiction.

(2) $\Rightarrow$ (1): Note that $\mathrm{K} := \mathrm{T}_{Q_G^{\mathrm{ud}}}(\mathcal{D})$ is a subcomplex of $\mathrm{T} := \mathrm{T}_{Q_{CG}^{\mathrm{ud}}}(\tilde{\mathfrak{G}}(\mathcal{D}))$. Since K is connected by our assumption, the vertex set $\mathcal{V} := \bigcup_{\Delta \in \mathcal{D}} \mathsf{Ob}(\Delta)$ of K is contained in a connected component $\mathcal{C}$ of T. For any $L \in \mathcal{V}$ and $x \in L$, we have that $L = xL \in \mathcal{C} \cap x\mathcal{C}$ where $x\mathcal{C} := \{xA \mid A \in \mathcal{C}\}$ is a connected component of T, so that $x\mathcal{C} = \mathcal{C}$. This implies that $\langle \mathcal{V} \rangle \leq \mathrm{Stab}_G(\mathcal{C}) := \{g \in G \mid g\mathcal{C} = \mathcal{C}\}$. But since $G = \langle \mathcal{V} \rangle$ by our assumption, $\mathrm{Stab}_G(\mathcal{C}) = G$ holds. Thus $\mathcal{C}$ contains the set $\bigcup_{L \in \mathcal{V}} G/L$ of all vertices of T. So T is connected. The proof is complete. □

REMARK 7.19. Let $H < G$ be a proper subgroup of G. Set $\mathcal{D} := \{(L_0 - \cdots - L_k) \in \mathsf{P}(Q_G^{\mathrm{ud}}) \mid L_i \leq H\} \subseteq \mathsf{P}(Q_G^{\mathrm{ud}})$. Then it is easy to see that, for a path $\Lambda =$

$(g_0L_0 - \cdots - g_kL_k) \in \tilde{\mathfrak{G}}(\mathcal{D})$, if $g_iL_i \not\subseteq H$ for some i then $g_jL_j \not\subseteq H$ for all $0 \le j \le k$, so that $\mathsf{Ob}(\Lambda) \cap 2^H = \emptyset$ where 2^H is the power set of a set H. In other words, if $g_iL_i \subseteq H$ for some i then $g_jL_j \subseteq H$ for all $0 \le j \le k$, so that $\mathsf{Ob}(\Lambda) \subseteq 2^H$. This implies that a complex $\mathrm{T}_{Q^{\mathrm{ud}}_{CG}}(\tilde{\mathfrak{G}}(\mathcal{D}))$ is disconnected.

As in the following, the number of connected components of a coset complex of $\tilde{\mathfrak{G}}(\Delta)$ in Q^{ud}_{CG} is completely determined by a certain subgroup H of G. This result can be applied when we calculate the zero homology of the complex.

Theorem 7.20. *For a path $\Delta = (L_0 - \cdots - L_k) \in \mathsf{P}(Q^{\mathrm{ud}}_G)$ in Q^{ud}_G, the number of the connected components of $\mathrm{T}_{Q^{\mathrm{ud}}_{CG}}(\tilde{\mathfrak{G}}(\Delta))$ is given by the index $|G : H|$ where $H := \langle L_j \mid j \in I(\Delta)^{\max}\rangle = \langle \mathsf{Ob}(\Delta)\rangle \le G$.*

Proof. Any L_i and L_j $(0 \le i < j \le k)$ are connected along Δ. So there exists a connected component $\mathcal{C}$ of $\mathrm{T} := \mathrm{T}_{Q^{\mathrm{ud}}_{CG}}(\tilde{\mathfrak{G}}(\Delta))$ containing $\mathsf{Ob}(\Delta) = \{L_0, \ldots, L_k\}$. Since G acts transitively on the set of all connected components of T, it suffices to show that $H = \mathrm{Stab}_G(\mathcal{C}) := \{g \in G \mid g\mathcal{C} = \mathcal{C}\}$ where $g\mathcal{C} := \{gA \mid A \in \mathcal{C}\}$ is a connected component of T.

For any $L_i \in \mathsf{Ob}(\Delta) \subseteq \mathcal{C}$ and $u \in L_i$, we have that $L_i = uL_i \in \mathcal{C} \cap u\mathcal{C}$. Thus $u\mathcal{C} = \mathcal{C}$ and $L_i \le \mathrm{Stab}_G(\mathcal{C})$. Since each element of $\mathsf{Ob}(\Delta)$ is contained in L_j for some $j \in I(\Delta)^{\max}$, we have that

$$H = \langle L_j \mid j \in I(\Delta)^{\max}\rangle = \langle L_i \mid 0 \le i \le k\rangle \le \mathrm{Stab}_G(\mathcal{C}).$$

On the other hand, for any $z \in \mathrm{Stab}_G(\mathcal{C})$, we have that $\mathcal{C} = z\mathcal{C} \ni zL_0$. Then there exists an edge path

$$((z_0L_{i_0}, z_1L_{i_1}), (z_1L_{i_1}, z_2L_{i_2}), \ldots, (z_{t-1}L_{i_{t-1}}, z_tL_{i_t}))$$

where $z_0L_{i_0} = L_0$, $z_tL_{i_t} = zL_0$, and $L_{i_1}, \ldots L_{i_{t-1}} \in \mathsf{Ob}(\Delta)$. For an edge $(z_sL_{i_s}, z_{s+1}L_{i_{s+1}})$, we may assume that $i_s < i_{s+1}$. Since both $z_sL_{i_s}$ and $z_{s+1}L_{i_{s+1}}$ lie in a simplex of T, there exists a path $\Lambda \in \tilde{\mathfrak{G}}(\Delta_{[i_s,i_{s+1}]})$ such that $s(\Lambda) = z_sL_{i_s}$ and $r(\Lambda) = z_{s+1}L_{i_{s+1}}$. Then by Proposition 7.5,

$$z_{s+1}L_{i_{s+1}} = r(\Lambda) \in r(\tilde{\mathfrak{G}}(\Delta_{[i_s,i_{s+1}]})_{z_sL_{i_s}}) = z_sL_{u_1} \cdots L_{u_r}/L_{i_{s+1}}$$

where $\{u_1, \ldots, u_r\} = I(\Delta_{[i_s,i_{s+1}]})^{\max}$. Thus $z_{s+1} \in z_sL_{u_1} \cdots L_{u_r} \subseteq z_sH$. Note that if $i_s > i_{s+1}$ then by the same way, we obtain $z_s \in z_{s+1}H$, namely $z_{s+1}^{-1} \in Hz_s^{-1}$. Since $z_0 \in L_0 \le H$, we have that $z_s \in H$ $(0 \le s \le t)$. In particular, $z = z_t \in H$. It follows that $\mathrm{Stab}_G(\mathcal{C}) \le H$. This completes the proof. $\square$

7.6. Preimages under $\varphi_{G,\Delta}$. Let $\Delta = (L_0 - \cdots - L_k) \in \mathsf{P}(Q_G^{\mathrm{ud}})$ be a path in Q_G^{ud}, and let

$$\varphi_{G,\Delta} \colon \mathrm{T}_{Q_{CG}^{\mathrm{ud}}}(\tilde{\mathfrak{G}}(\mathcal{C}_\Delta)) \to \mathrm{T}_{Q_G^{\mathrm{ud}}}(\mathsf{P}(Q_G^{\mathrm{ud}}))$$

be a G-simplicial map defined in Section 6.3. In this section, we describe the preimage under $\varphi_{G,\Delta}$. We expect that this result will be applied to homotopy theory like Quillen's fiber theorem on homotopy equivalences (see [3, Proposition 1.6]), and so on.

Proposition 7.21. *Let $\Delta = (L_0 - \cdots - L_k) \in \mathsf{P}(Q_G^{\mathrm{ud}})$ be a path in Q_G^{ud}, and let $\varphi_{G,\Delta}$ be a G-simplicial map stated in the above. Let*

$$\tau = \{L_{j_0}^{a_{j_0}}, \ldots, L_{j_q}^{a_{j_q}}\} \subseteq \mathrm{Im}(\varphi_{G,\Delta}) = \bigcup_{j=0}^{k} \{L_j^a \mid a \in G\} \quad (0 \le j_0 < \cdots < j_q \le k)$$

be a q-simplex of $\mathrm{T}_{Q_G^{\mathrm{ud}}}(\mathsf{P}(Q_G^{\mathrm{ud}}))$ which is in the image of $\varphi_{G,\Delta}$.

(1) *Any simplex in $\varphi_{G,\Delta}^{-1}(\tau)$ is of dimension q.*

(2) *Any simplex σ in $\varphi_{G,\Delta}^{-1}(\tau)$ is of the form $\sigma = \{g_{j_0} L_{j_0}, \ldots g_{j_q} L_{j_q}\}$ for some $g_{j_d} \in a_{j_d}^{-1} N_G(L_{j_d})$ $(0 \le d \le q)$. Furthermore, there exists $\Delta_{[s,t]} \in \mathcal{C}_\Delta$ such that $\Delta_{[j_0,j_q]} \preceq \Delta_{[s,t]}$.*

(3) *The preimage $\varphi_{G,\Delta}^{-1}(\tau)$ is given as follows*:

$$\left\{ \{g_{j_0} L_{j_0}, \ldots g_{j_q} L_{j_q}\} \;\middle|\; \begin{array}{l} g_{j_d} \in a_{j_d}^{-1} N_G(L_{j_d}) \ (0 \le d \le q) \\ g_{j_{d+1}} \in g_{j_d} L_{u_{d,1}} L_{u_{d,2}} \cdots L_{u_{d,r_d}} \ (0 \le d \le q-1) \\ \text{where } I(\Delta_{[j_d, j_{d+1}]})^{\max} = \{u_{d,1}, \ldots, u_{d,r_d}\} \ (u_{d,i} < u_{d,i+1}) \end{array} \right\}.$$

Proof. (1) This is due to Proposition 6.25.

(2) By the definition of a simplex σ in $\varphi_{G,\Delta}^{-1}(\tau) \subseteq \mathrm{T}_{Q_{CG}^{\mathrm{ud}}}(\tilde{\mathfrak{G}}(\mathcal{C}_\Delta))$, there exists $\Delta_{[s,t]} \in \mathcal{C}_\Delta$ such that $\sigma \subseteq \mathsf{Ob}(\Gamma)$ for a path

$$\Gamma = (g_s L_s - g_{s+1} L_{s+1} - g_{s+2} L_{s+2} - \cdots - g_{t-1} L_{t-1} - g_t L_t)$$

in $\tilde{\mathfrak{G}}(\Delta_{[s,t]}) \subseteq \tilde{\mathfrak{G}}(\mathcal{C}_\Delta) \subseteq \mathsf{P}(Q_{CG}^{\mathrm{ud}})$. Set $\sigma = \{g_{i_0} L_{i_0}, \ldots, g_{i_q} L_{i_q}\}$ $(s \le i_0, \ldots, i_q \le t)$. Then

$$\{L_{j_0}^{a_{j_0}}, \ldots, L_{j_q}^{a_{j_q}}\} = \tau = \varphi_{G,\Delta}(\sigma) \subseteq \varphi_{G,\Delta}(\Gamma) = \{L_s^{g_s^{-1}}, \ldots, L_t^{g_t^{-1}}\}.$$

We may assume that

$$L_{j_d}^{a_{j_d}} = \varphi_{G,\Delta}(g_{i_d} L_{i_d}) = L_{i_d}^{g_{i_d}^{-1}} \quad (0 \le d \le q).$$

Then by G-conjugate condition in Definition 6.23 of $\Delta_{[s,t]} \in \mathcal{C}_\Delta$, we have that $i_d = j_d$ for all $0 \le d \le q$. It follows that $g_{j_d} \in a_{j_d}^{-1} N_G(L_{j_d})$ $(0 \le d \le q)$, and that $s \le j_0 < \cdots < j_q \le t$, namely $\Delta_{[j_0,j_q]} \preceq \Delta_{[s,t]}$.

(3) Note that since $\Delta_{[j_0,j_q]} \preceq \Delta_{[s,t]} \in \mathcal{C}_\Delta$ by the previous claim (2), we have that $s \leq j_0 < \cdots < j_q \leq t$. Now a set $\sigma = \{g_{j_0}L_{j_0}, \ldots g_{j_q}L_{j_q}\}$ of cosets is a simplex in $\mathrm{T}_{Q^{\mathrm{ud}}_{CG}}(\tilde{\mathfrak{G}}(\mathcal{C}_\Delta))$ if and only if there exists $\Gamma_{d+1} \in \tilde{\mathfrak{G}}(\Delta_{[j_d,j_{d+1}]}) \subseteq \mathsf{P}(Q^{\mathrm{ud}}_{CG})$ $(0 \leq d \leq q-1)$ such that $s(\Gamma_{d+1}) = g_{j_d}L_{j_d}$ and $r(\Gamma_{d+1}) = g_{j_{d+1}}L_{j_{d+1}}$. Then we obtain a path

$$\Lambda := (g_{j_0}L_{j_0} \xrightarrow{\Gamma_1} g_{j_1}L_{j_1} \xrightarrow{\Gamma_2} g_{j_2}L_{j_2} \to \cdots \to g_{j_{q-1}}L_{j_{q-1}} \xrightarrow{\Gamma_q} g_{j_q}L_{j_q})$$

in the closure $\overline{Q^{\mathrm{ud}}_{CG}}$ such that $\mathsf{Ob}(\Lambda) = \sigma$ (see also the proof of Lemma 7.2). By Proposition 7.5, we have that

$$\begin{aligned} g_{j_{d+1}}L_{j_{d+1}} = r(\Gamma_{d+1}) &\in r(\tilde{\mathfrak{G}}(\Delta_{[j_d,j_{d+1}]})_{g_{j_d}L_{j_d}}) \\ &= g_{j_d}L_{u_{d,1}}L_{u_{d,2}} \cdots L_{u_{d,r_d}}/L_{j_{d+1}} \subseteq G/L_{j_{d+1}} \end{aligned}$$

where $I(\Delta_{[j_d,j_{d+1}]})^{\max} = \{u_{d,1}, u_{d,2}, \ldots, u_{d,r_d}\}$ $(u_{d,i} < u_{d,i+1})$. Thus

$$g_{j_{d+1}} \in g_{j_d}L_{u_{d,1}}L_{u_{d,2}} \cdots L_{u_{d,r_d}}.$$

The proof is complete. □

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