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MINIMAL SURFACES OF GENUS ONE WITH CATENOIDAL ENDS II

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Abstract

In the previous paper, we classified n-noids of genus one into two classes, and considered one of the classes. As a sequel, we give a necessary and sufficient condition for the existence of an n-noid of genus one with prescribed flux in the other class. By using the condition, we give obstructions for a certain type of flux and arrangement of the ends. We also give new examples by deforming a quadruple covering of Berglund–Rossman's n-end catenoid.

1. Introduction

Let $\overline{M}$ be a compact Riemann surface, and $X\colon M = \overline{M} \setminus \{q_1, \dots, q_n\} \to \mathbf{R}^3$ an *n-noid*, that is a complete conformal minimal immersion with finite total curvature, which has embedded ends at $q_1, \dots, q_n \in \overline{M}$. We call X an *n-end catenoid*, if all the ends are catenoidal. Let $G\colon \overline{M} \to \mathbf{S}^2 \subset \mathbf{R}^3$ be the Gauss map of X. The *flux vector* at the end q_j is defined by the integral $\varphi_j := \int_{\gamma_{q_j}} \vec{n}\, ds$, where γ_{q_j} is a loop surrounding q_j from the left, $\vec{n}$ is a unit conormal vector field along γ_{q_j} such that $(\dot{\gamma}_{q_j}, \vec{n})$ is positively oriented, and ds is the line element of $X(M)$. By the divergence formula, we get the *flux formula* $\sum_{j=1}^{n} \varphi_j = \mathbf{0}$. Now, since we assume that each end q_j is an embedded end, $G(q_j)$ is parallel to φ_j (see [7], [11], [14]), and hence there exists a real number $w(q_j)$ satisfying $\varphi_j = 4\pi w(q_j) G(q_j)$. We call $w(q_j)$ the *weight* of the end q_j. We note here that the end q_j is catenoidal (resp. planar) if and only if $w(q_j) \neq \mathbf{0}$ (resp. $w(q_j) = \mathbf{0}$). By using the weights, we can rewrite the flux formula as follows:

$$\sum_{j=1}^{n} w(q_j) G(q_j) = \mathbf{0}. \tag{1.1}$$

Conversely, we can consider a problem of finding n-noids that realize given data $G(q_j)$ and $w(q_j)$ ($j = 1, \dots, n$) satisfying (1.1). By the Weierstrass representation formula, an n-noid is given by

$$ {}^{t}X(z) = \mathrm{Re} \int_{z_0}^{z} (1 - g^2, \sqrt{-1}(1 + g^2), 2g)\eta, \tag{1.2}$$

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where g is a meromorphic function on $\overline{M}$, and η is a meromorphic 1-form on $\overline{M}$, which is holomorphic on M. We call (g, η) the *Weierstrass data* of X. Here g is the composition of the Gauss map G and the stereographic projection $\Pi\colon \mathbf{S}^2 \to \hat{\mathbf{C}} := \mathbf{C}\cup\{\infty\}$ from the north pole $\mathbf{e}_3 := {}^t(0,0,1)$. The inverse of Π is given by the following:

$$v(p) := \Pi^{-1}(p) = \frac{1}{|p|^2+1}\begin{pmatrix} 2\,\mathrm{Re}\,p \\ 2\,\mathrm{Im}\,p \\ |p|^2-1 \end{pmatrix}.$$

Now, the inverse problem of the flux formula is stated as follows:

PROBLEM 1.1. Let $p_1, \dots, p_n$ be complex numbers or ∞, and let $a_1, \dots, a_n$ be real numbers satisfying

$$\sum_{j=1}^{n} a_j v(p_j) = \mathbf{0}. \tag{1.3}$$

Does there exist an n-noid $X\colon M = \overline{M} \setminus \{q_1, \dots, q_n\} \to \mathbf{R}^3$ satisfying the following condition?

$$g(q_j) = p_j, \quad w(q_j) = a_j, \quad \varphi(q_j) = 4\pi a_j v(p_j) \quad (j = 1, \dots, n). \tag{1.4}$$

In the previous paper [9], we classified n-noids of genus one into two classes as follows:

Let ω_1 and ω_2 be complex numbers such that

$$\mathrm{Im}\,\frac{\omega_2}{\omega_1} > 0. \tag{1.5}$$

Set $T^2 := \mathbf{C}/(\mathbf{Z}\omega_1 + \mathbf{Z}\omega_2)$. We call (ω_1, ω_2) the *fundamental period* of T^2. We denote each point in T^2 by one of its representatives, $u \in \mathbf{C}$. Let $u_1, \dots, u_n$ be distinct points in T^2, and set $M := T^2 \setminus \{u_1, \dots, u_n\}$. Let $X\colon M \to \mathbf{R}^3$ be an n-noid of genus one, one of whose complete systems of representatives of the ends is given by $\{u_1, \dots, u_n\}$. Let (g, η) be its Weierstrass data. Assume $G(u_j) \neq v(\infty) = {}^t(0, 0, 1)$, that is,

$$p_j = g(u_j) \neq \infty \quad (j = 1, \dots, n). \tag{1.6}$$

Since X is an n-noid and we assume (1.6), all of the ends $u_1, \dots, u_n$ are embedded ends, and each u_j is a pole of η (resp. $g^2\eta$) whose order is 2 (resp. at most 2). On the other hand, all of zeroes of η must be poles of g, and the order of each zero of η

is exactly a double of its order as a pole of g. Hence (g, η) is of the form

$$
\begin{cases} g(u) = C_1 \dfrac{\sigma(u-t_1)\cdots\sigma(u-t_n)}{\sigma(u-s_1)\cdots\sigma(u-s_n)}, \\ \eta = -\left(C_2 \dfrac{\sigma(u-s_1)\cdots\sigma(u-s_n)}{\sigma(u-u_1)\cdots\sigma(u-u_n)}\right)^2 \dfrac{\sigma(u-u_n)}{\sigma(u-u_n-\omega)}\, du, \end{cases} \tag{1.7}
$$

where C_1 and C_2 are nonzero constants, σ is the Weierstrass σ-function (cf. [6]), $\{s_1, \ldots, s_n\}$ is a complete system of representatives of the poles of g, and $\{t_1, \ldots, t_n\}$ is that of the zeroes of g, which satisfy $s_1 + \cdots + s_n = t_1 + \cdots + t_n$ and

$$
2(u_1 + \cdots + u_n) + \omega = 2(s_1 + \cdots + s_n) \tag{1.8}
$$

for some $\omega \in \{0, \omega_1, \omega_2, \omega_1 + \omega_2\}$. In the case that $\omega \neq 0$, by replacing (ω_1, ω_2) if necessary, we may assume $\omega = \omega_2$ without loss of generality. Now, we can classify n-noids of genus one into two classes $\omega = 0$ and $\omega = \omega_2$.

Costa's examples [3], Berglund and Rossman's examples [2], and almost all examples are included in the class $\omega = \omega_2$. In a weak sense, catenoid fences (cf. [5], [8]) are also included in the class. We formulated our problem in the similar way as in [10], the case of genus zero, and gave an equation with respect to elliptic functions which describes a necessary and sufficient condition for the existence of n-noids of genus one in the class. By applying our equation, we also gave new examples.

In this paper, we study the remaining class in which $\omega = 0$ holds. The first family of examples in this class was constructed also by Costa [4] (see also Abi-Khuzam [1]), and their generalization to arbitrary conformal classes was given by Kusner–Schmitt [12]. However any ends of these examples are planar ends, and, as the authors' knowledge, there are no previously known examples with catenoidal ends in this class.

In §§2–3, we give a necessary and sufficient condition for the existence of n-noids in this class (cf. Theorem 3.6). In §4, we rewrite the condition given in §3 in the case that n-noids are symmetric with respect to a plane. In §5, we prepare some equalities which we use in the calculations in §§7–8. In §6, we observe the correspondence between n-noids of genus one and their double or quadruple coverings.

In §7, by applying the conditions in §4, we give new obstructions for the existence of n-noids symmetric with respect to a plane. For instance, in the class $\omega = \omega_2$, Berglund–Rossman [2] constructed a Jorge–Meeks type n-end catenoid of genus one for each $n \geq 3$, which is invariant under the action of the direct product $D_n \times \mathbf{Z}_2$ of the dihedral group and a reflection. On the other hand, by new obstructions, we can see that there are no Jorge–Meeks type n-end catenoids defined on a rectangular torus in the class $\omega = 0$ (cf. Theorem 7.1).

In §8, by deforming quadruple coverings of Berglund–Rossman's n-end catenoids, we give various new examples defined on non-rectangular tori with catenoidal ends in the class $\omega = 0$ (cf. Theorem 8.3).

2. Local period problem

In this paper, we discuss the class $\omega = 0$, in which the sum of a complete system of representatives of the ends and that of the poles of g coincide with each other. In this class, the Weierstrass data (g, η) of any n-noid of genus one, $X\colon M = T^2 \setminus \{u_1, \dots, u_n\} \to \mathbf{R}^3$, is given by the form

$$g(u) = \frac{P(u)}{Q(u)}, \quad \eta = -Q(u)^2\, du, \tag{2.1}$$

where $P(u)$ and $Q(u)$ are elliptic functions of fundamental period (ω_1, ω_2) of the following form

$$P(u) = \sum_{j=1}^{n} c_j \zeta(u - u_j) + c_0, \quad Q(u) = \sum_{j=1}^{n} b_j \zeta(u - u_j) + b_0, \tag{2.2}$$

where ζ is the Weierstrass ζ-function (cf. [6]), $b_1, \dots, b_n, b_0, c_1, \dots, c_n, c_0$ are complex numbers satisfying $b_j \neq 0$, $c_j = p_j b_j = g(u_j) b_j$ $(j = 1, \dots, n)$, and $\sum_{j=1}^{n} b_j = \sum_{j=1}^{n} c_j = 0$ (see [9, Proposition 4.1]).

Conversely, let (g, η) be a data of the form (2.1) with (2.2). Then the map X given by (1.2) is well-defined on M if and only if

$$\operatorname{Re} \int_{\gamma} (1 - g^2, \sqrt{-1}(1 + g^2), 2g)\eta = \mathbf{0} \tag{2.3}$$

holds for any loop γ in M. Set

$$R_i = R_i(\gamma) := \frac{1}{2\pi\sqrt{-1}} \int_{\gamma} g^i \eta \quad (i = 0, 1, 2). \tag{2.4}$$

Then the condition (2.3) is rewritten as

$$R_0 - R_2 \in \mathbf{R}, \quad R_0 + R_2 \in \sqrt{-1}\mathbf{R}, \quad R_1 \in \mathbf{R}, \tag{2.5}$$

and this is equivalent to

$$R_0 + \overline{R_2} = 0, \quad R_1 = \overline{R_1}. \tag{2.6}$$

Let p be a complex number satisfying

$$p^2 R_0 - 2pR_1 + R_2 = 0. \tag{2.7}$$

Then the condition (2.3) is also rewritten as follows (see [9, Theorem 2.1]).

$$\begin{cases} w = w(\gamma) := -pR_0 + R_1 \in \mathbf{R}, \\ w^* = w^*(\gamma) := -\dfrac{1}{2}(|p|^2 - 1)R_0 + \overline{p}R_1 = 0. \end{cases} \tag{2.8}$$

In particular, for the loop γ_q surrounding an end q once from the left, the value w coincides with the weight of the end q defined in the introduction.

Since

$$\zeta(u) = \frac{1}{u} + O(u^3), \tag{2.9}$$

the Laurent expansion of $P(u)Q(u)$ at u_j is given by

$$\begin{aligned} &P(u)Q(u) \\ &= \frac{c_j b_j}{(u-u_j)^2} + \frac{1}{u-u_j}\left\{c_j\left(\sum_{k=1;\ k\neq j}^{n} b_k\zeta(u_j-u_k) + b_0\right)\right. \\ &\qquad\qquad \left. + b_j\left(\sum_{k=1;\ k\neq j}^{n} c_k\zeta(u_j-u_k) + c_0\right)\right\} + O(1) \\ &= \frac{c_j b_j}{(u-u_j)^2} + \frac{1}{u-u_j}\left\{\sum_{k=1;\ k\neq j}^{n} (c_j b_k + b_j c_k)\zeta(u_j-u_k) + (c_j b_0 + b_j c_0)\right\} + O(1). \end{aligned}$$

In the same way, we also have

$$Q(u)^2 = \frac{{b_j}^2}{(u-u_j)^2} + \frac{1}{u-u_j}\left\{\sum_{k=1;\ k\neq j}^{n} 2b_j b_k\zeta(u_j-u_k) + 2b_j b_0\right\} + O(1),$$

$$P(u)^2 = \frac{{c_j}^2}{(u-u_j)^2} + \frac{1}{u-u_j}\left\{\sum_{k=1;\ k\neq j}^{n} 2c_j c_k\zeta(u_j-u_k) + 2c_j c_0\right\} + O(1).$$

Now, for each end u_j, denote the corresponding R_0, R_1, R_2, w, w^* by R_{0j}, R_{1j}, R_{2j}, w_j, w_j^* respectively, for instance, $w_j := w(\gamma_{q_j})$. Then we have the following lemma.

Lemma 2.1. *The integrals R_{0j}, R_{1j} and R_{2j} are given by the following equalities*:

$$
(2.10)\qquad
\begin{cases}
R_{0j} = -\operatorname{Res}_{u=u_j} Q(u)^2\,du = -\left\{\displaystyle\sum_{k=1;\,k\neq j}^{n} 2b_jb_k\zeta(u_j-u_k) + 2b_jb_0\right\},\\
R_{1j} = -\operatorname{Res}_{u=u_j} P(u)Q(u)\,du\\
\qquad = -\left\{\displaystyle\sum_{k=1;\,k\neq j}^{n} (c_jb_k+b_jc_k)\zeta(u_j-u_k) + (c_jb_0+b_jc_0)\right\}\\
\qquad = -\left\{\displaystyle\sum_{k=1;\,k\neq j}^{n} (p_j+p_k)b_jb_k\zeta(u_j-u_k) + (p_jb_jb_0+b_jc_0)\right\},\\
R_{2j} = -\operatorname{Res}_{u=u_j} P(u)^2\,du = -\left\{\displaystyle\sum_{k=1;\,k\neq j}^{n} 2c_jc_k\zeta(u_j-u_k) + 2c_jc_0\right\}\\
\qquad = -\left\{\displaystyle\sum_{k=1;\,k\neq j}^{n} 2p_jp_kb_jb_k\zeta(u_j-u_k) + 2p_jb_jc_0\right\}.
\end{cases}
$$

Hence we get the following lemma:

Lemma 2.2. *The values w_j and w_j^* are given by the following equalities*:

$$
(2.11)\qquad
\begin{cases}
w_j = -p_jR_{0j} + R_{1j} = \displaystyle\sum_{k=1;\,k\neq j}^{n} (p_j-p_k)b_jb_k\zeta(u_j-u_k) + (p_jb_jb_0 - b_jc_0),\\
w_j^* = -\dfrac{1}{2}(|p_j|^2-1)R_{0j} + \overline{p_j}R_{1j}\\
\qquad = -\left\{\displaystyle\sum_{k=1;\,k\neq j}^{n} (\overline{p_j}p_k+1)b_jb_k\zeta(u_j-u_k) + (\overline{p_j}b_jc_0 + b_jb_0)\right\}.
\end{cases}
$$

The end q_j is well-defined if and only if $w_j \in \mathbf{R}$ and $w_j^* = 0$.

3. Global period problem

In this section, we calculate the global period around the generators of the first homology group of T^2. Let $\wp$ be the Weierstrass $\wp$-function. First, by the addition theorems of elliptic functions, we have the following lemma (see [6, p. 188 (6)]).

Lemma 3.1. *For any $u, u', u'' \in \mathbf{C}$ such that $u + u' + u'' = 0$, the following equality holds*:

$$
\wp(u) + \wp(u') + \wp(u'') = (\zeta(u) + \zeta(u') + \zeta(u''))^2.
$$

Now, we represent $Q(u)^2$, $P(u)^2$ and $P(u)Q(u)$ by using $\wp(u)$ and $\zeta(u)$.

Lemma 3.2. *Let $P(u)$ and $Q(u)$ be as in* (2.2), *and let R_{0j}, R_{1j} and R_{2j} be as in* (2.10). *Then the following equalities hold.*

$$Q(u)^2 = \sum_{j=1}^{n} {b_j}^2 \wp(u-u_j) - \sum_{j=1}^{n} R_{0j}\zeta(u-u_j)$$
$$+ \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1;\ k\neq j}^{n} b_j b_k(\zeta(u_j-u_k)^2 - \wp(u_j-u_k)) + {b_0}^2,$$
$$P(u)Q(u) = \sum_{j=1}^{n} c_j b_j \wp(u-u_j) - \sum_{j=1}^{n} R_{1j}\zeta(u-u_j)$$
$$+ \frac{1}{4}\sum_{j=1}^{n}\sum_{k=1;\ k\neq j}^{n} (c_j b_k + b_j c_k)(\zeta(u_j-u_k)^2 - \wp(u_j-u_k)) + c_0 b_0,$$
$$P(u)^2 = \sum_{j=1}^{n} {c_j}^2 \wp(u-u_j) - \sum_{j=1}^{n} R_{2j}\zeta(u-u_j)$$
$$+ \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1;\ k\neq j}^{n} c_j c_k(\zeta(u_j-u_k)^2 - \wp(u_j-u_k)) + {c_0}^2.$$

Proof. We prove our assertion for $P(u)Q(u)$. By (2.9) and $\zeta'(u) = -\wp(u)$, the Laurent expansion of $Q(u)$ at u_j is

$$Q(u) = b_j\zeta(u-u_j) + \sum_{k=1;\ k\neq j}^{n} b_k\zeta(u-u_k) + b_0$$
$$= \frac{b_j}{u-u_j} + \sum_{k=1;\ k\neq j}^{n} b_k\{\zeta(u_j-u_k) - \wp(u_j-u_k)(u-u_j)\} + b_0 + O((u-u_j)^2)$$
$$= \frac{b_j}{u-u_j} - \frac{R_{0j}}{2b_j} - (u-u_j)\sum_{k=1;\ k\neq j}^{n} b_k\wp(u_j-u_k) + O((u-u_j)^2),$$

and that of $P(u)$ is

$$P(u) = \frac{c_j}{u-u_j} - \frac{R_{2j}}{2c_j} - (u-u_j)\sum_{k=1;\ k\neq j}^{n} c_k\wp(u_j-u_k) + O((u-u_j)^2).$$

Hence that of $P(u)Q(u)$ is

$$\begin{aligned}
&P(u)Q(u)\\
&= \frac{c_j b_j}{(u-u_j)^2} - \frac{1}{u-u_j}\left(\frac{c_j R_{0j}}{2b_j} + \frac{b_j R_{2j}}{2c_j}\right) + \frac{R_{2j}}{2c_j}\frac{R_{0j}}{2b_j}\\
&\quad - c_j \sum_{k=1;\ k\neq j}^{n} b_k \wp(u_j-u_k) - b_j \sum_{k=1;\ k\neq j}^{n} c_k \wp(u_j-u_k) + O(u-u_j)\\
&= \frac{c_j b_j}{(u-u_j)^2} - \frac{R_{1j}}{u-u_j} + \frac{R_{2j}R_{0j}}{4c_j b_j} - \sum_{k=1;\ k\neq j}^{n} (c_j b_k + b_j c_k)\wp(u_j-u_k) + O(u-u_j).
\end{aligned}$$

On the other hand, by using

$$\wp(u) = \frac{1}{u^2} + O(u^2) \tag{3.1}$$

and (2.9), we have

$$\begin{aligned}
&\sum_{j=1}^{n} c_j b_j \wp(u-u_j) - \sum_{j=1}^{n} R_{1j}\zeta(u-u_j)\\
&= \frac{c_j b_j}{(u-u_j)^2} - \frac{R_{1j}}{u-u_j} + \sum_{k=1;\ k\neq j}^{n} c_k b_k \wp(u_j-u_k)\\
&\quad - \sum_{k=1;\ k\neq j}^{n} R_{1k}\zeta(u_j-u_k) + O(u-u_j).
\end{aligned}$$

Since $P(u)Q(u) - \sum_{j=1}^{n} c_j b_j \wp(u-u_j) + \sum_{j=1}^{n} R_{1j}\zeta(u-u_j)$ is an elliptic function with no poles, it holds that

$$\begin{aligned}
&P(u)Q(u) - \sum_{j=1}^{n} c_j b_j \wp(u-u_j) + \sum_{j=1}^{n} R_{1j}\zeta(u-u_j)\\
&= \frac{R_{2j}R_{0j}}{4c_j b_j} - \sum_{k=1;\ k\neq j}^{n} (c_j b_k + b_j c_k)\wp(u_j-u_k) - \sum_{k=1;\ k\neq j}^{n} c_k b_k \wp(u_j-u_k)\\
&\quad + \sum_{k=1;\ k\neq j}^{n} R_{1k}\zeta(u_j-u_k).
\end{aligned}$$

Note here that this constant does not depend on j. Since $\wp(-u)=\wp(u)$ and $\zeta(-u)=-\zeta(u)$, it holds that

$$\begin{aligned}
&\sum_{j=1}^{n}\left\{-\sum_{k=1;\,k\neq j}^{n}(c_jb_k+b_jc_k)\wp(u_j-u_k)-\sum_{k=1;\,k\neq j}^{n}c_kb_k\wp(u_j-u_k)\right\}\\
&=-\sum_{j=1}^{n}\sum_{k=1;\,k\neq j}^{n}(c_jb_k+b_jc_k+c_kb_k)\wp(u_j-u_k)\\
&=-\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1;\,k\neq j}^{n}(c_jb_j+c_kb_k+2c_jb_k+2c_kb_j)\wp(u_j-u_k),
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{j=1}^{n}\left(\frac{R_{2j}R_{0j}}{4c_jb_j}+\sum_{k=1;\,k\neq j}^{n}R_{1k}\zeta(u_j-u_k)\right)\\
&=\sum_{j=1}^{n}\Bigg[\left(\sum_{k=1;\,k\neq j}^{n}c_k\zeta(u_j-u_k)+c_0\right)\left(\sum_{l=1;\,l\neq j}^{n}b_l\zeta(u_j-u_l)+b_0\right)\\
&\qquad-\sum_{k=1;\,k\neq j}^{n}\zeta(u_j-u_k)\left\{\sum_{l=1;\,l\neq k}^{n}(c_kb_l+b_kc_l)\zeta(u_k-u_l)+(c_kb_0+b_kc_0)\right\}\Bigg]\\
&=\sum_{j=1}^{n}\sum_{k=1;\,k\neq j}^{n}\sum_{l=1;\,l\neq j,k}^{n}\{c_k\zeta(u_j-u_k)b_l\zeta(u_j-u_l)-\zeta(u_j-u_k)(c_kb_l+b_kc_l)\zeta(u_k-u_l)\}\\
&\quad+\sum_{j=1}^{n}\sum_{k=1;\,k\neq j}^{n}\{c_k\zeta(u_j-u_k)b_k\zeta(u_j-u_k)-\zeta(u_j-u_k)(c_kb_j+b_kc_j)\zeta(u_k-u_j)\}\\
&\quad+nc_0b_0\\
&=\sum_{j=1}^{n}\sum_{k=1;\,k\neq j}^{n}\sum_{l=1;\,l\neq j,k}^{n}(c_kb_l+c_jb_l+b_jc_l)\zeta(u_j-u_k)\zeta(u_j-u_l)\\
&\quad+\sum_{j=1}^{n}\sum_{k=1;\,k\neq j}^{n}(c_kb_k+c_kb_j+b_kc_j)\zeta(u_j-u_k)^2+nc_0b_0\\
&=\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1;\,k\neq j}^{n}\sum_{l=1;\,l\neq j,k}^{n}(c_kb_l+c_jb_l+b_jc_l+c_lb_k+c_jb_k+b_jc_k)\zeta(u_j-u_k)\zeta(u_j-u_l)\\
&\quad+\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1;\,k\neq j}^{n}(c_kb_k+c_kb_j+b_kc_j+c_jb_j+c_jb_k+b_jc_k)\zeta(u_j-u_k)^2+nc_0b_0
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{6}\sum_{j=1}^{n}\sum_{k=1;\,k\neq j}^{n}\sum_{l=1;\,l\neq j,k}^{n}(c_jb_k+c_kb_j+c_kb_l+c_lb_k+c_lb_j+c_jb_l)\\
&\qquad\qquad \times(\zeta(u_j-u_k)\zeta(u_j-u_l)+\zeta(u_k-u_l)\zeta(u_k-u_j)\\
&\qquad\qquad\quad +\zeta(u_l-u_j)\zeta(u_l-u_k))\\
&\quad +\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1;\,k\neq j}^{n}(c_jb_j+c_kb_k+2b_kc_j+2c_kb_j)\zeta(u_j-u_k)^2+nc_0b_0.
\end{aligned}
$$

Now, by Lemma 3.1, it holds that

$$
\begin{aligned}
&\zeta(u_j-u_k)\zeta(u_j-u_l)+\zeta(u_k-u_l)\zeta(u_k-u_j)+\zeta(u_l-u_j)\zeta(u_l-u_k)\\
&=\frac{1}{2}(\zeta(u_j-u_k)^2+\zeta(u_k-u_l)^2+\zeta(u_l-u_j)^2\\
&\qquad -\wp(u_j-u_k)-\wp(u_k-u_l)-\wp(u_l-u_j)).
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
&\frac{1}{6}\sum_{j=1}^{n}\sum_{k=1;\,k\neq j}^{n}\sum_{l=1;\,l\neq j,k}^{n}(c_jb_k+c_kb_j+c_kb_l+c_lb_k+c_lb_j+c_jb_l)\\
&\qquad\qquad \times(\zeta(u_j-u_k)\zeta(u_j-u_l)+\zeta(u_k-u_l)\zeta(u_k-u_j)\\
&\qquad\qquad\quad +\zeta(u_l-u_j)\zeta(u_l-u_k))\\
&=\frac{1}{12}\sum_{j=1}^{n}\sum_{k=1;\,k\neq j}^{n}\sum_{l=1;\,l\neq j,k}^{n}(c_jb_k+c_kb_j+c_kb_l+c_lb_k+c_lb_j+c_jb_l)\\
&\qquad\qquad \times(\zeta(u_j-u_k)^2+\zeta(u_k-u_l)^2+\zeta(u_l-u_j)^2\\
&\qquad\qquad\quad -\wp(u_j-u_k)-\wp(u_k-u_l)-\wp(u_l-u_j))\\
&=\frac{1}{4}\sum_{j=1}^{n}\sum_{k=1;\,k\neq j}^{n}\sum_{l=1;\,l\neq j,k}^{n}(c_jb_k+c_kb_j+c_kb_l+c_lb_k+c_lb_j+c_jb_l)\\
&\qquad\qquad \times(\zeta(u_j-u_k)^2-\wp(u_j-u_k))\\
&=\frac{1}{4}\sum_{j=1}^{n}\sum_{k=1;\,k\neq j}^{n}\{(n-2)(c_jb_k+c_kb_j)-2(c_j+c_k)(b_j+b_k)\}\\
&\qquad \times(\zeta(u_j-u_k)^2-\wp(u_j-u_k))\\
&=\frac{1}{4}\sum_{j=1}^{n}\sum_{k=1;\,k\neq j}^{n}\{(n-4)(c_jb_k+c_kb_j)-2(c_jb_j+c_kb_k)\}(\zeta(u_j-u_k)^2-\wp(u_j-u_k)),
\end{aligned}
$$

and hence

$$n\left\{P(u)Q(u)-\sum_{j=1}^{n}c_jb_j\wp(u-u_j)+\sum_{j=1}^{n}R_{1j}\zeta(u-u_j)\right\}$$
$$=\frac{1}{4}\sum_{j=1}^{n}\sum_{k=1;\ k\neq j}^{n}n(c_jb_k+c_kb_j)(\zeta(u_j-u_k)^2-\wp(u_j-u_k))+nc_0b_0.$$

Now we get our assertion for $P(u)Q(u)$. If we consider the case that $P(u)=Q(u)$, then we get our assertions for $Q(u)^2$ and $P(u)^2$. □

Henceforth, we choose a complete system of representatives of the ends $\{u_1,\ldots,u_n\}$ and a complex number u_0 to satisfy

$$\{u_1,\ldots,u_n\}\subset\{u_0+t_1\omega_1+t_2\omega_2\mid t_1,t_2\in(0,1)\}. \tag{3.2}$$

Lemma 3.3. *Assume* (3.2) *holds. Let* $P(u)$ *and* $Q(u)$ *be as in* (2.1), *and* γ_i *a line segment joining* u_0 *and* $u_0+\omega_i$, *and set*

$$\eta_i:=\zeta(u+\omega_i)-\zeta(u)\quad(i=1,2). \tag{3.3}$$

Then the following equalities hold.

$$\begin{aligned}R_0(\gamma_i)&=\frac{1}{2\pi\sqrt{-1}}\int_{\gamma_i}(-Q(u)^2)\,du\\&=\frac{-1}{2\pi\sqrt{-1}}\Bigg[-\sum_{j=1}^{n}({b_j}^2-R_{0j}u_j)\eta_i\\&\qquad+\left\{\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1;\ k\neq j}^{n}b_jb_k(\zeta(u_j-u_k)^2-\wp(u_j-u_k))+{b_0}^2\right\}\omega_i\Bigg],\end{aligned}$$

$$\begin{aligned}&R_1(\gamma_i)\\&=\frac{1}{2\pi\sqrt{-1}}\int_{\gamma_i}(-P(u)Q(u))\,du\\&=\frac{-1}{2\pi\sqrt{-1}}\Bigg[-\sum_{j=1}^{n}(c_jb_j-R_{1j}u_j)\eta_i\\&\qquad+\left\{\frac{1}{4}\sum_{j=1}^{n}\sum_{k=1;\ k\neq j}^{n}(c_jb_k+b_jc_k)(\zeta(u_j-u_k)^2-\wp(u_j-u_k))+c_0b_0\right\}\omega_i\Bigg],\end{aligned}$$

$$R_2(\gamma_i) = \frac{1}{2\pi\sqrt{-1}} \int_{\gamma_i} (-P(u)^2)\, du$$
$$= \frac{-1}{2\pi\sqrt{-1}} \Bigg[-\sum_{j=1}^{n} ({c_j}^2 - R_{2j} u_j)\eta_i$$
$$+ \left\{ \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1;\ k\neq j}^{n} c_j c_k (\zeta(u_j - u_k)^2 - \wp(u_j - u_k)) + {c_0}^2 \right\} \omega_i \Bigg]$$

$(i = 1, 2)$.

Proof. We prove our assertion for $R_1(\gamma_i)$ $(i = 1, 2)$. First, it holds that

$$\int P(u)Q(u)\, du$$
$$= -\sum_{j=1}^{n} c_j b_j \zeta(u - u_j) - \sum_{j=1}^{n} R_{1j} \log\sigma(u - u_j)$$
$$+ \left\{ \frac{1}{4} \sum_{j=1}^{n} \sum_{k=1;\ k\neq j}^{n} (c_j b_k + b_j c_k)(\zeta(u_j - u_k)^2 - \wp(u_j - u_k)) + c_0 b_0 \right\} u.$$

For $i = 1$ (resp. $i = 2$), we consider $\log\sigma(u)$ to be a single-valued continuous function on the strip $\{t_1\omega_1 + t_2\omega_2 \mid t_1 \in \mathbf{R}, t_2 \in (-1, 0)\}$ (resp. $\{t_1\omega_1 + t_2\omega_2 \mid t_1 \in (-1, 0), t_2 \in \mathbf{R}\}$). Then, by the choice of u_0, it holds that

$$\log\sigma(u_0 + \omega_i - u_j) - \log\sigma(u_0 - u_j) = \eta_i(u_0 + \omega_i/2 - u_j) + (2k_i + 1)\pi\sqrt{-1},$$

where k_i is an integer which only depends on i (cf. [6, p. 181, Satz 3]).

On the other hand, by the residue theorem, we have $\sum_{j=1}^{n} R_{1j} = 0$. Combining these equalities, we get our assertion. □

Let η_i be as in (3.3). Set

$$\xi_i(u) := \left(\wp(u) - \frac{2\eta_i}{\omega_i}\right) - \left(\zeta(u) - \frac{\eta_i u}{\omega_i}\right)^2 \quad (i = 1, 2). \tag{3.4}$$

We enumerate several properties of $\xi_i(u)$ in §5. Here we note that $\xi_i(u)$ has a period ω_i:

$$\xi_i(u + \omega_i) = \xi_i(u) \quad (i = 1, 2). \tag{3.5}$$

By using $\xi_i(u)$, we can describe $R_0(\gamma_i)$, $R_1(\gamma_i)$ and $R_2(\gamma_i)$ as follows:

Lemma 3.4. *The integrals $R_0(\gamma_i)$, $R_1(\gamma_i)$ and $R_2(\gamma_i)$ $(i = 1, 2)$ are given by the following equalities*:

$$\begin{cases} R_0(\gamma_i) = \dfrac{-\omega_i}{2\pi\sqrt{-1}}\left\{-\dfrac{1}{2}\displaystyle\sum_{j=1}^{n}\sum_{k=1;\,k\neq j}^{n} b_j b_k \xi_i(u_j - u_k) + \left(b_0 - \eta_i \sum_{j=1}^{n} b_j \dfrac{u_j}{\omega_i}\right)^2\right\}, \\ R_1(\gamma_i) = \dfrac{-\omega_i}{2\pi\sqrt{-1}}\left\{-\dfrac{1}{4}\displaystyle\sum_{j=1}^{n}\sum_{k=1;\,k\neq j}^{n} (c_j b_k + b_j c_k)\xi_i(u_j - u_k)\right. \\ \qquad\qquad\qquad \left. + \left(c_0 - \eta_i \displaystyle\sum_{j=1}^{n} c_j \dfrac{u_j}{\omega_i}\right)\left(b_0 - \eta_i \sum_{j=1}^{n} b_j \dfrac{u_j}{\omega_i}\right)\right\}, \\ R_2(\gamma_i) = \dfrac{-\omega_i}{2\pi\sqrt{-1}}\left\{-\dfrac{1}{2}\displaystyle\sum_{j=1}^{n}\sum_{k=1;\,k\neq j}^{n} c_j c_k \xi_i(u_j - u_k) + \left(c_0 - \eta_i \sum_{j=1}^{n} c_j \dfrac{u_j}{\omega_i}\right)^2\right\}. \end{cases}$$

Proof. Since $\sum_{j=1}^{n} c_j = 0$ and $\sum_{j=1}^{n} b_j = 0$, it holds that

$$\sum_{j=1}^{n} c_j b_j + \sum_{j=1}^{n}\sum_{k=1;\,k\neq j}^{n} c_j b_k = 0.$$

Hence, by Lemma 3.3, we have

$$\begin{aligned} &\frac{-2\pi\sqrt{-1}}{\omega_i} R_1(\gamma_i) \\ &= -\sum_{j=1}^{n}(c_j b_j - R_{1j} u_j)\frac{\eta_i}{\omega_i} \\ &\quad + \left\{\frac{1}{4}\sum_{j=1}^{n}\sum_{k=1;\,k\neq j}^{n}(c_j b_k + b_j c_k)(\zeta(u_j - u_k)^2 - \wp(u_j - u_k)) + c_0 b_0\right\} \\ &= c_0 b_0 - \frac{\eta_i}{\omega_i}\left(b_0 \sum_{j=1}^{n} c_j u_j + c_0 \sum_{j=1}^{n} b_j u_j\right) \\ &\quad + \frac{1}{4}\sum_{j=1}^{n}\sum_{k=1;\,k\neq j}^{n}(c_j b_k + b_j c_k)\left\{\zeta(u_j - u_k)^2 - 2\eta_i \zeta(u_j - u_k)\frac{u_j - u_k}{\omega_i}\right\} \\ &\quad - \frac{1}{4}\sum_{j=1}^{n}\sum_{k=1;\,k\neq j}^{n}(c_j b_k + b_j c_k)\wp(u_j - u_k) - \sum_{j=1}^{n} c_j b_j \frac{\eta_i}{\omega_i} \end{aligned}$$

$$
\begin{aligned}
&= \left(c_0 - \eta_i \sum_{j=1}^{n} c_j \frac{u_j}{\omega_i}\right)\left(b_0 - \eta_i \sum_{j=1}^{n} b_j \frac{u_j}{\omega_i}\right) - \left(\frac{\eta_i}{\omega_i}\right)^2 \sum_{j=1}^{n} c_j u_j \sum_{k=1}^{n} b_k u_k \\
&\quad + \frac{1}{4} \sum_{j=1}^{n} \sum_{k=1;\ k\neq j}^{n} (c_j b_k + b_j c_k)\left(\zeta(u_j - u_k) - \eta_i \frac{u_j - u_k}{\omega_i}\right)^2 \\
&\quad - \frac{1}{4}\left(\frac{\eta_i}{\omega_i}\right)^2 \sum_{j=1}^{n} \sum_{k=1;\ k\neq j}^{n} (c_j b_k + b_j c_k)(u_j - u_k)^2 \\
&\quad - \frac{1}{4} \sum_{j=1}^{n} \sum_{k=1;\ k\neq j}^{n} (c_j b_k + b_j c_k)\left(\wp(u_j - u_k) - \frac{2\eta_i}{\omega_i}\right) \\
&= -\frac{1}{4} \sum_{j=1}^{n} \sum_{k=1;\ k\neq j}^{n} (c_j b_k + b_j c_k)\xi_i(u_j - u_k) + \left(c_0 - \eta_i \sum_{j=1}^{n} c_j \frac{u_j}{\omega_i}\right)\left(b_0 - \eta_i \sum_{j=1}^{n} b_j \frac{u_j}{\omega_i}\right) \\
&\quad - \left(\frac{\eta_i}{\omega_i}\right)^2 \left\{\sum_{j=1}^{n} c_j u_j \sum_{k=1}^{n} b_k u_k + \frac{1}{4} \sum_{j=1}^{n} \sum_{k=1;\ k\neq j}^{n} (c_j b_k + b_j c_k)(u_j - u_k)^2\right\}.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\sum_{j=1}^{n} c_j u_j \sum_{k=1}^{n} b_k u_k + \frac{1}{4} \sum_{j=1}^{n} \sum_{k=1;\ k\neq j}^{n} (c_j b_k + b_j c_k)(u_j - u_k)^2 \\
&= \sum_{j=1}^{n} c_j u_j \sum_{k=1}^{n} b_k u_k + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1;\ k\neq j}^{n} c_j b_k (u_j - u_k)^2 \\
&= \sum_{j=1}^{n} c_j b_j u_j^2 + \sum_{j=1}^{n} \sum_{k=1;\ k\neq j}^{n} c_j b_k u_j u_k + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1;\ k\neq j}^{n} c_j b_k (u_j^2 - 2u_j u_k + u_k^2) \\
&= \sum_{j=1}^{n} c_j b_j u_j^2 + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1;\ k\neq j}^{n} (c_j b_k + b_j c_k) u_j^2 \\
&= \frac{1}{2} \sum_{j=1}^{n} \left(c_j \sum_{k=1}^{n} b_k + b_j \sum_{k=1}^{n} c_k\right) u_j^2 = 0,
\end{aligned}
$$

we get our assertion for $R_1(\gamma_i)$. If we consider the case that $b_j = c_j$, we get our assertions also for $R_0(\gamma_i)$ and $R_2(\gamma_i)$. □

Lemma 3.5. *The integrals* $R_0(\gamma_i)$, $R_1(\gamma_i)$ *and* $R_2(\gamma_i)$ *satisfy the following equalities*:

$$\begin{cases} R_0(\gamma_2) = \dfrac{\omega_2}{\omega_1} R_0(\gamma_1) - \dfrac{1}{\omega_1} \displaystyle\sum_{j=1}^{n} (b_j^2 - R_{0j} u_j), \\ R_1(\gamma_2) = \dfrac{\omega_2}{\omega_1} R_1(\gamma_1) - \dfrac{1}{\omega_1} \displaystyle\sum_{j=1}^{n} (c_j b_j - R_{1j} u_j), \\ R_2(\gamma_2) = \dfrac{\omega_2}{\omega_1} R_2(\gamma_1) - \dfrac{1}{\omega_1} \displaystyle\sum_{j=1}^{n} (c_j^2 - R_{2j} u_j). \end{cases}$$

Proof. By using the Legendre relation

$$\eta_1 \omega_2 - \eta_2 \omega_1 = 2\pi \sqrt{-1} \tag{3.6}$$

and Lemma 3.3, we have

$$\begin{aligned} &R_1(\gamma_2) \\ &= \frac{-1}{2\pi\sqrt{-1}} \Bigg[-\sum_{j=1}^{n} (c_j b_j - R_{1j} u_j) \eta_2 \\ &\qquad + \left\{ \frac{1}{4} \sum_{j=1}^{n} \sum_{k=1; k \neq j}^{n} (c_j b_k + b_j c_k)(\zeta(u_j - u_k)^2 - \wp(u_j - u_k)) + c_0 b_0 \right\} \omega_2 \Bigg] \\ &= \frac{-1}{2\pi\sqrt{-1}} \Bigg[-\sum_{j=1}^{n} (c_j b_j - R_{1j} u_j) \frac{\eta_1 \omega_2 - 2\pi\sqrt{-1}}{\omega_1} \\ &\qquad + \frac{\omega_2}{\omega_1} \left\{ \frac{1}{4} \sum_{j=1}^{n} \sum_{k=1; k \neq j}^{n} (c_j b_k + b_j c_k)(\zeta(u_j - u_k)^2 - \wp(u_j - u_k)) + c_0 b_0 \right\} \omega_1 \Bigg] \\ &= \frac{\omega_2}{\omega_1} R_1(\gamma_1) - \frac{1}{\omega_1} \sum_{j=1}^{n} (c_j b_j - R_{1j} u_j). \end{aligned}$$

□

The Weierstrass data (g, η) of an n-noid of the form (2.1) with (2.2) must satisfy the period conditions. In the case that $\overline{M} = T^2$ and $\omega = 0$, Problem 1.1 is reduced to a problem of finding u_j, b_j, c_j $(j = 1, \ldots, n)$, ω_1 and ω_2 satisfying the following conditions.

Theorem 3.6. *There exists an n-noid* $X\colon M = T^2\setminus\{u_1,\dots u_n\} \to \mathbf{R}^3$ *of type* $\omega = 0$ *satisfying* (1.4) *if and only if there exist* $q_j, b_j, c_j = p_j b_j$ $(j = 1, \dots, n)$ *satisfying*

$$\begin{cases} \left.\begin{aligned} w_j &= a_j, \\ w_j^* &= 0, \end{aligned}\right\} & (j = 1, \dots, n), \\ \left.\begin{aligned} &R_0(\gamma_i) + \overline{R_2(\gamma_i)} = 0, \\ &R_1(\gamma_i) \in \mathbf{R}, \end{aligned}\right\} & (i = 1, 2), \end{cases}$$

and the degree of g *given by* (2.1) *is* n.

4. *n*-noids symmetric with respect to the coordinate plane

In this section, we rewrite the condition in Theorem 3.6 in the case that n-noids are symmetric with respect to a plane. Theorem 4.2 is applied to construct various examples in Example 8.2, and Theorem 4.4 is applied to derive a new obstruction in Theorem 7.1. Both theorems treat essentially the same symmetry. But we present these theorems separatedly, since the criterions in the theorems are useful in a different situation from each other.

First we consider the symmetry with respect to the x_1x_2-plane. We can show the following fact in the same way as the condition for a minimal surface to be a double covering of a nonorientable minimal surface (cf. [13]).

Proposition 4.1. *Let* X *be a conformal minimal immersion into* $\mathbf{R}^3$, *defined on a Riemann surface* M *with the Weierstrass data* (g,η). *Then* X *is symmetric with respect to the* x_1x_2*-plane* (*up to parallel translations*) *if and only if* (g,η) *satisfies the condition*

$$g \circ I = \frac{1}{\bar{g}}, \quad I^*\eta = -\overline{g^2\eta} \tag{4.1}$$

for some antiholomorphic involution $I\colon M \to M$, *which satisfies* $I^2 = \mathrm{id}_M$ *and* $\partial I = 0$.

In the case of n-noids symmetric with respect to the x_1x_2-plane, the equation for the global period problem can be rewritten to simpler form.

Without loss of generality, we may assume $I(u) = \bar{u}$. Then, by changing the fundamental period if necessary, $\overline{M} = T^2$ must satisfy either of the following:
(1) $\omega_1 \in \mathbf{R}_+$ and $\omega_2 \in \sqrt{-1}\mathbf{R}_+$.
(2) $\omega_1 \in \mathbf{R}_+$ and $\omega_2 - \omega_1/2 \in \sqrt{-1}\mathbf{R}_+$.

Here we assume the former condition (1). In this case, it holds that $\overline{\zeta(u)} = \zeta(\bar{u})$, $\eta_1 \in \mathbf{R}$ and $\eta_2 \in \sqrt{-1}\mathbf{R}$.

Let X be an n-noid of genus one whose Weierstrass data (g, η) is of the form (2.1) with (2.2). The data (g, η) satisfies the condition (4.1) with $I(u) = \bar{u}$ if and only if $Q \circ I(u) = \pm\sqrt{-1}\,\overline{P(u)}$, that is

$$\sum_{j=1}^{n} b_j \zeta(\overline{u} - u_j) + b_0 = \pm\sqrt{-1}\left(\sum_{j=1}^{n} \overline{c_j}\zeta(\overline{u} - \overline{u_j}) + \overline{c_0}\right).$$

Here we choose $u_1, \dots, u_n$ and u_0 to satisfy both (3.2) and

$$\{u_1, \dots, u_n\} \subset \{t_1'\omega_1 + t_2'\omega_2 \mid t_1', t_2' \in [0, 1)\}. \tag{4.2}$$

Now, to realize such an n-noid, we may assume that u_j, p_j, b_j, c_j $(j = 1, \dots, n)$, b_0, c_0 satisfy

$$\begin{cases} u_j \in \mathbf{R}, \quad |p_j| = 1 \quad (j = 1, \dots, N_1), \\ u_j \in \mathbf{R} + \dfrac{1}{2}\omega_2, \quad |p_j| = 1 \quad (j = N_1 + 1, \dots, N_1 + N_2), \\ u_j = \overline{u_{j'}} + \omega_2, \quad p_j\overline{p_{j'}} = 1 \quad (j = N_1 + N_2 + 1, \dots, n), \\ \pm b_j = \sqrt{-1}\overline{c_{j'}} \quad (j = 1, \dots, n), \\ \pm b_0 = \sqrt{-1}\left(\overline{c_0} + \displaystyle\sum_{j=N_1+1}^{n} \overline{c_j}\eta_2\right) \\ \text{where} \quad N_1 + N_2 + 2N_3 = n, \\ j' = \begin{cases} j & (j = 1, \dots, N_1 + N_2), \\ j + N_3 & (j = N_1 + N_2 + 1, \dots, N_1 + N_2 + N_3), \\ j - N_3 & (j = N_1 + N_2 + N_3 + 1, \dots, n). \end{cases} \end{cases} \tag{4.3}$$

In this case, our problem is reduced to the following:

Theorem 4.2. *Assume that $\omega_1 \in \mathbf{R}_+$ and $\omega_2 \in \sqrt{-1}\mathbf{R}_+$. There exists an n-noid $X\colon M = T^2 \setminus \{u_1, \dots, u_n\} \to \mathbf{R}^3$ of type $\omega = 0$ satisfying* (1.4), (4.1) *with $I(u) = \overline{u}$,* (3.2) *and* (4.2) *if and only if there exist u_j, b_j, $c_j = p_j b_j$ $(j = 1, \dots, n)$ satisfying* (4.3) *and*

$$\begin{cases} \left.\begin{array}{l} w_j = a_j, \\ w_j^* = 0, \end{array}\right\} \quad (j = 1, \dots, N_1 + N_2 + N_3), \\ P_1 := R_0(\gamma_1) + \overline{R_2(\gamma_1)} = 2R_0(\gamma_1) + \displaystyle\sum_{j=N_1+1}^{n} R_{0j} = 0, \\ P_2 := R_1(\gamma_2) - \overline{R_1(\gamma_2)} = 2R_1(\gamma_2) = 0, \end{cases} \tag{4.4}$$

and the degree of g given by (2.1) *is n.*

Proof. By the assumption (4.3) and Lemma 2.1, it holds that

$$\begin{cases} \pm c_0 = \sqrt{-1}\left(\overline{b_0} + \displaystyle\sum_{j=N_1+1}^{n} \overline{b_j}\eta_2\right), \\ \overline{c_j c_k} = -b_{j'} b_{k'}, \quad \overline{c_j b_k} = -b_{j'} c_{k'} \quad (j = 1, \dots, n;\ k = 1, \dots, n), \\ \overline{R_{2j}} = -R_{0j}, \quad \overline{R_{1j}} = -R_{1j} \quad (j = 1, \dots, n), \end{cases}$$

from which, and from Lemma 3.4 and (3.5), it also holds that

$$\overline{R_2(\gamma_2)} = -R_0(\gamma_2), \quad \overline{R_1(\gamma_2)} = -R_1(\gamma_2).$$

By using Lemma 3.5 we also have

$$\begin{cases} R_1(\gamma_1) - \overline{R_1(\gamma_1)} = -\displaystyle\sum_{j=N_1+1}^{n} R_{1j}, \\ R_0(\gamma_1) + \overline{R_2(\gamma_1)} = 2R_0(\gamma_1) + \displaystyle\sum_{j=N_1+1}^{n} R_{0j}. \end{cases}$$

Now, if $w_j = a_j \in \mathbf{R}$ and $w_j^* = 0$ hold for $j = 1, \dots, N_1 + N_2 + N_3$, then they also hold for $j = N_1 + N_2 + N_3 + 1, \dots, n$, and hence $R_{1j} = \overline{R_{1j}}$ holds for $j = 1, \dots, n$. Therefore we have

$$\begin{cases} R_{1j} = 0 & (j = 1, \dots, N_1 + N_2), \\ R_{1j} + R_{1j'} = 0 & (j = N_1 + N_2 + 1, \dots, n). \end{cases}$$

Hence we get

$$R_1(\gamma_1) - \overline{R_1(\gamma_1)} = -\sum_{j=N_1+1}^{n} R_{1j} = \sum_{j=1}^{N_1} R_{1j} = 0. \qquad \square$$

Now we consider the symmetry with respect to the x_1x_3-plane.

Proposition 4.3. *Let X be a conformal minimal immersion into $\mathbf{R}^3$, defined on a Riemann surface M with the Weierstrass data (g, η). Then X is symmetric with respect to the x_1x_3-plane* (*up to parallel translations*) *if and only if (g, η) satisfies the condition*

$$g \circ I = \overline{g}, \quad I^*\eta = \overline{\eta} \tag{4.5}$$

for some antiholomorphic involution $I\colon M \to M$, which satisfies $I^2 = \mathrm{id}_M$ and $\partial I = 0$.

Here, we also assume that $\overline{M} = T^2$ satisfies $\omega_1 \in \mathbf{R}_+$ and $\omega_2 \in \sqrt{-1}\mathbf{R}_+$.

Let X be an n-noid of genus one whose Weierstrass data (g, η) is of the form (2.1) with (2.2). The data (g, η) satisfies the condition (4.5) with $I(u) = \overline{u}$ if and only if $Q \circ I(u) = \pm\overline{Q(u)}$ and $P \circ I(u) = \pm\overline{P(u)}$, that is

$$\begin{cases} \displaystyle\sum_{j=1}^{n} b_j \zeta(\overline{u} - u_j) + b_0 = \pm\left(\sum_{j=1}^{n} \overline{b_j}\zeta(\overline{u} - \overline{u_j}) + \overline{b_0}\right), \\ \displaystyle\sum_{j=1}^{n} c_j \zeta(\overline{u} - u_j) + c_0 = \pm\left(\sum_{j=1}^{n} \overline{c_j}\zeta(\overline{u} - \overline{u_j}) + \overline{c_0}\right). \end{cases}$$

Here we assume (3.2) and (4.2) again. Now, to realize such an n-noid, we may assume that $u_j,\ p_j,\ b_j,\ c_j\ (j = 1, \dots, n)$, b_0, c_0 satisfy

$$\text{(4.6)}\qquad \begin{cases} u_j \in \mathbf{R}, \quad p_j \in \mathbf{R} \quad (j = 1, \dots, N_1), \\ u_j \in \mathbf{R} + \dfrac{1}{2}\omega_2, \quad p_j \in \mathbf{R} \quad (j = N_1 + 1, \dots, N_1 + N_2), \\ u_j = \overline{u_{j'}} + \omega_2, \quad p_j = \overline{p_{j'}} \quad (j = N_1 + N_2 + 1, \dots, n), \\ \left.\begin{array}{l} \pm b_j = \overline{b_{j'}}, \\ \pm c_j = \overline{c_{j'}}, \end{array}\right\} \quad (j = 1, \dots, n), \\ \pm b_0 = \overline{b_0} + \displaystyle\sum_{j=N_1+1}^{n} \overline{b_j}\eta_2, \\ \pm c_0 = \overline{c_0} + \displaystyle\sum_{j=N_1+1}^{n} \overline{c_j}\eta_2 \\ \text{where} \quad N_1 + N_2 + 2N_3 = n, \\ j' = \begin{cases} j & (j = 1, \dots, N_1 + N_2), \\ j + N_3 & (j = N_1 + N_2 + 1, \dots, N_1 + N_2 + N_3), \\ j - N_3 & (j = N_1 + N_2 + N_3 + 1, \dots, n). \end{cases} \end{cases}$$

In this case, our problem is reduced to the following:

Theorem 4.4. *Assume that $\omega_1 \in \mathbf{R}_+$ and $\omega_2 \in \sqrt{-1}\mathbf{R}_+$. There exists an n-noid X: $M = T^2 \setminus \{u_1, \dots, u_n\} \to \mathbf{R}^3$ of type $\omega = 0$ satisfying* (1.4), (4.5) *with $I(u) = \overline{u}$,* (3.2) *and* (4.2) *if and only if there exist* u_j, b_j, $c_j = p_j b_j$ $(j = 1,\dots,n)$ *satisfying* (4.6) *and*

$$
\text{(4.7)}\qquad
\begin{cases}
\left.\begin{aligned} w_j &= a_j, \\ w_j^* &= 0, \end{aligned}\right\} \quad (j = 1, \dots, N_1 + N_2 + N_3), \\
P_1' := R_0(\gamma_1) + \overline{R_2(\gamma_1)} = R_0(\gamma_1) - R_2(\gamma_1) - \displaystyle\sum_{j=N_1+1}^{n} R_{2j} = 0, \\
P_2' := R_1(\gamma_1) - \overline{R_1(\gamma_1)} = 2R_1(\gamma_1) + \displaystyle\sum_{j=N_1+1}^{n} R_{1j} = 0, \\
P_3' := R_0(\gamma_2) + \overline{R_2(\gamma_2)} = R_0(\gamma_2) + R_2(\gamma_2) = 0,
\end{cases}
$$

and the degree of g *given by* (2.1) *is* n.

Proof. By the assumption (4.6) and Lemma 2.1, it holds that

$$
\begin{cases}
\overline{b_j b_k} = b_{j'} b_{k'}, \quad \overline{c_j b_k} = c_{j'} b_{k'}, \quad \overline{c_j c_k} = c_{j'} c_{k'} \quad (j = 1, \dots, n;\ k = 1, \dots, n), \\
\overline{R_{0j}} = R_{0j'}, \quad \overline{R_{1j}} = R_{1j'}, \quad \overline{R_{2j}} = R_{2j'} \quad (j = 1, \dots, n),
\end{cases}
$$

from which, and from Lemma 3.4 and (3.5), it also holds that

$$
\overline{R_0(\gamma_2)} = R_0(\gamma_2), \quad \overline{R_1(\gamma_2)} = R_1(\gamma_2), \quad \overline{R_2(\gamma_2)} = R_2(\gamma_2).
$$

By using Lemma 3.5 we also have

$$
\begin{cases}
\overline{R_0(\gamma_1)} = -R_0(\gamma_1) - \displaystyle\sum_{j=N_1+1}^{n} R_{0j}, \\
\overline{R_1(\gamma_1)} = -R_1(\gamma_1) - \displaystyle\sum_{j=N_1+1}^{n} R_{1j}, \\
\overline{R_2(\gamma_1)} = -R_2(\gamma_1) - \displaystyle\sum_{j=N_1+1}^{n} R_{2j}.
\end{cases}
$$

Now, if $w_j = a_j \in \mathbf{R}$ and $w_j^* = 0$ hold for $j = 1, \dots, N_1 + N_2 + N_3$, then they also hold for $j = N_1 + N_2 + N_3 + 1, \dots, n$, and hence $R_{0j} + \overline{R_{2j}} = 0$, $R_{1j} = \overline{R_{1j}}$ holds for $j = 1, \dots, n$. Therefore we have

$$
\begin{cases}
R_{0j} + R_{2j'} = 0 \quad (j = 1, \dots, n), \\
R_{1j} = R_{1j'} \qquad\quad (j = N_1 + N_2 + 1, \dots, n).
\end{cases}
\qquad \square
$$

5. The functions $\xi_i(u)$ and $\chi_i(u)$

In this section, we describe some properties of elliptic functions for later use. Let $T^2 := \mathbf{C}/(\mathbf{Z}\omega_1 + \mathbf{Z}\omega_2)$ with (1.5). In §3, we introduce the function $\xi_i(u)$ $(i = 1, 2)$ to write down the global periods of the given Weierstrass data (g, η) (see (3.4)). Here, we also introduce the function $\chi_i(u)$ $(i = 1, 2)$ by which we write down the weights of the given data in §8. We enumerate several properties of $\xi_i(u)$ and $\chi_i(u)$ in Lemmas 5.1–5.6, which we use repeatedly in the calculations in §§7–8.

First, we cite some equalities of elliptic functions. Set $\tau := \omega_2/\omega_1$ and $h := \exp(\pi\sqrt{-1}\tau)$ (see [6, p. 190]). The Jacobi ϑ-function ϑ_1 is given as follows:

$$\begin{aligned}
&\vartheta_1(v, h) \\
&= 2Ch^{1/4} \sin \pi v \prod_{m=1}^{\infty}(1 - 2h^{2m} \cos 2\pi v + h^{4m}) \\
&= -\sqrt{-1}Ch^{1/4}(e^{\pi\sqrt{-1}v} - e^{-\pi\sqrt{-1}v}) \prod_{m=1}^{\infty}(1 - e^{2\pi\sqrt{-1}v}h^{2m})(1 - e^{-2\pi\sqrt{-1}v}h^{2m}),
\end{aligned} \tag{5.1}$$

where

$$C = \prod_{m=1}^{\infty}(1 - h^{2m})$$

(cf. [6, p. 204]). Set $\vartheta_1'(v, h) := \partial\vartheta_1(v, h)/\partial v$. Then it holds that

$$\zeta(u) = \frac{\eta_1 u}{\omega_1} + \frac{1}{\omega_1}\frac{\vartheta_1'(u/\omega_1, h)}{\vartheta_1(u/\omega_1, h)} \tag{5.2}$$

and

$$\eta_1 = \frac{\pi^2}{\omega_1}\left(\frac{1}{3} - 8\sum_{m=1}^{\infty}\frac{mh^{2m}}{1 - h^{2m}}\right).$$

(cf. [6, p. 207, p. 210]). We also note that ϑ_1 satisfies the heat equation:

$$\frac{\partial^2}{\partial v^2}\vartheta_1(v, h) = 4\pi\sqrt{-1}\frac{\partial}{\partial \tau}\vartheta_1(v, h) \quad \left(= -4\pi^2 h\frac{\partial}{\partial h}\vartheta_1(v, h)\right)$$

(cf. [6, pp. 197–198]). Combining these equalities with $\wp(u) = -\zeta'(u)$, we have

$$\begin{aligned}
\xi_1(u) &= \left(\wp(u) - \frac{2\eta_1}{\omega_1}\right) - \left(\zeta(u) - \frac{\eta_1 u}{\omega_1}\right)^2 = -\frac{3\eta_1}{\omega_1} - \frac{1}{\omega_1^2}\frac{\vartheta_1''(u/\omega_1, h)}{\vartheta_1(u/\omega_1, h)} \\
&= -\frac{3\eta_1}{\omega_1} + \frac{4\pi^2}{\omega_1^2}\frac{h}{\vartheta_1(u/\omega_1, h)}\frac{\partial}{\partial h}\vartheta_1(u/\omega_1, h) \\
&= -8\left(\frac{\pi}{\omega_1}\right)^2\sum_{m=1}^{\infty}\left(-2\frac{mh^{2m}}{1 - h^{2m}} + \frac{me^{2\pi\sqrt{-1}u/\omega_1}h^{2m}}{1 - e^{2\pi\sqrt{-1}u/\omega_1}h^{2m}} + \frac{me^{-2\pi\sqrt{-1}u/\omega_1}h^{2m}}{1 - e^{-2\pi\sqrt{-1}u/\omega_1}h^{2m}}\right).
\end{aligned} \tag{5.3}$$

We can express $\xi_1(u)$ in terms of another expansion. Henceforth, we use our notation $r := h^{-2}$ as in the previous paper ([9, §3]).

Lemma 5.1. *Let* $z(u) := \exp(2\pi\sqrt{-1}u/\omega_1)$ *and* $r := \exp(-2\pi\sqrt{-1}\omega_2/\omega_1)$. *For any* u *such that* $|\mathrm{Im}(u/\omega_1)| < \mathrm{Im}(\omega_2/\omega_1)$, *the function* $\xi_1(u)$ *satisfies the following*:

$$
\begin{aligned}
\xi_1(u) &= \left(\wp(u) - \frac{2\eta_1}{\omega_1}\right) - \left(\zeta(u) - \frac{\eta_1 u}{\omega_1}\right)^2 \\
&= -8\left(\frac{\pi}{\omega_1}\right)^2 \sum_{l=1}^{\infty} \frac{r^l}{(r^l-1)^2}(-2 + z(u)^l + z(u)^{-l}) \\
&= 32\left(\frac{\pi}{\omega_1}\right)^2 \sum_{l=1}^{\infty} \frac{r^l}{(r^l-1)^2} \sin^2 \frac{l\pi u}{\omega_1}.
\end{aligned}
$$

Proof. First, we note that for any x such that $|x| > 1$, we have

$$
\frac{1}{x-1} = \sum_{k=1}^{\infty} \frac{1}{x^k}, \quad \frac{x}{(x-1)^2} = \sum_{k=1}^{\infty} \frac{k}{x^k}. \tag{5.4}
$$

By (1.5), it holds that $|r| > 1$ and $|r|^{-1} < |z(u)| < |r|$. Combining (5.4) with (5.3), we have

$$
\begin{aligned}
\xi_1(u) &= -8\left(\frac{\pi}{\omega_1}\right)^2 \sum_{m=1}^{\infty} \left(-2\frac{m}{r^m-1} + \frac{m}{z(u)^{-1}r^m - 1} + \frac{m}{z(u)r^m - 1}\right) \\
&= -8\left(\frac{\pi}{\omega_1}\right)^2 \sum_{m=1}^{\infty}\sum_{l=1}^{\infty} \left(-2\frac{m}{r^{lm}} + \frac{m}{z(u)^{-l}r^{lm}} + \frac{m}{z(u)^l r^{lm}}\right) \\
&= -8\left(\frac{\pi}{\omega_1}\right)^2 \sum_{l=1}^{\infty}\sum_{m=1}^{\infty} \frac{m}{r^{lm}}(-2 + z(u)^l + z(u)^{-l}) \\
&= -8\left(\frac{\pi}{\omega_1}\right)^2 \sum_{l=1}^{\infty} \frac{r^l}{(r^l-1)^2}(-2 + z(u)^l + z(u)^{-l}) \\
&= 32\left(\frac{\pi}{\omega_1}\right)^2 \sum_{l=1}^{\infty} \frac{r^l}{(r^l-1)^2} \sin^2 \frac{l\pi u}{\omega_1}. \qquad \square
\end{aligned}
$$

Here, we define the function χ_i $(i = 1, 2)$ as follows:

$$
\chi_i(u) := \zeta(u) - \frac{\eta_i u}{\omega_i}.
$$

By using (5.1) and (5.2), we have

$$
\begin{aligned}
\chi_1(u) &= \frac{1}{\omega_1}\frac{\vartheta_1'(u/\omega_1, h)}{\vartheta_1(u/\omega_1, h)}\\
&= \frac{\pi\sqrt{-1}}{\omega_1}\left\{\frac{e^{\pi\sqrt{-1}u/\omega_1}+e^{-\pi\sqrt{-1}u/\omega_1}}{e^{\pi\sqrt{-1}u/\omega_1}-e^{-\pi\sqrt{-1}u/\omega_1}}\right.\\
&\qquad\left. -2\sum_{m=1}^{\infty}\left(\frac{e^{2\pi\sqrt{-1}u/\omega_1}h^{2m}}{1-e^{2\pi\sqrt{-1}u/\omega_1}h^{2m}}-\frac{e^{-2\pi\sqrt{-1}u/\omega_1}h^{2m}}{1-e^{-2\pi\sqrt{-1}u/\omega_1}h^{2m}}\right)\right\}
\end{aligned}
\tag{5.5}
$$

(cf. [6, p. 208]).

Lemma 5.2. *Let $z(u)$ and r as in* Lemma 5.1. *For any u such that* $|\mathrm{Im}(u/\omega_1)| < \mathrm{Im}(\omega_2/\omega_1)$, *the function $\chi_1(u)$ satisfies the following*:

$$
\begin{aligned}
\chi_1(u) &= \zeta(u) - \frac{\eta_1 u}{\omega_1}\\
&= \frac{\pi\sqrt{-1}}{\omega_1}\left\{\frac{z(u)+1}{z(u)-1} - 2\sum_{l=1}^{\infty}\frac{1}{r^l-1}(z(u)^l - z(u)^{-l})\right\}.
\end{aligned}
$$

Proof. By (1.5), it holds that $|r| > 1$ and $|r|^{-1} < |z(u)| < |r|$. Combining (5.4) with (5.5), we have

$$
\begin{aligned}
\chi_1(u) &= \frac{\pi\sqrt{-1}}{\omega_1}\left\{\frac{z(u)+1}{z(u)-1} - 2\sum_{m=1}^{\infty}\left(\frac{1}{z(u)^{-1}r^m-1}-\frac{1}{z(u)r^m-1}\right)\right\}\\
&= \frac{\pi\sqrt{-1}}{\omega_1}\left\{\frac{z(u)+1}{z(u)-1} - 2\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}\left(\frac{1}{z(u)^{-l}r^{lm}}-\frac{1}{z(u)^l r^{lm}}\right)\right\}\\
&= \frac{\pi\sqrt{-1}}{\omega_1}\left\{\frac{z(u)+1}{z(u)-1} - 2\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{r^{lm}}(z(u)^l - z(u)^{-l})\right\}\\
&= \frac{\pi\sqrt{-1}}{\omega_1}\left\{\frac{z(u)+1}{z(u)-1} - 2\sum_{l=1}^{\infty}\frac{1}{r^l-1}(z(u)^l - z(u)^{-l})\right\}. \qquad \square
\end{aligned}
$$

Replacing the fundamental period (ω_1, ω_2) with $(\omega_2, -\omega_1)$, we also have the following lemmas:

Lemma 5.3. *Let* $\tilde{z}(u) := \exp(2\pi\sqrt{-1}u/\omega_2)$ *and* $\tilde{r} = \exp(2\pi\sqrt{-1}\omega_1/\omega_2)$. *For any* u *such that* $|\mathrm{Im}(u/\omega_2)| < -\mathrm{Im}(\omega_1/\omega_2)$, *the function* $\xi_2(u)$ *satisfies the following*:

$$\begin{aligned}\xi_2(u) &= \left(\wp(u) - \frac{2\eta_2}{\omega_2}\right) - \left(\zeta(u) - \frac{\eta_2 u}{\omega_2}\right)^2 \\ &= -8\left(\frac{\pi}{\omega_2}\right)^2 \sum_{l=1}^{\infty} \frac{\tilde{r}^l}{(\tilde{r}^l-1)^2}(-2 + \tilde{z}(u)^l + \tilde{z}(u)^{-l}) \\ &= 32\left(\frac{\pi}{\omega_2}\right)^2 \sum_{l=1}^{\infty} \frac{\tilde{r}^l}{(\tilde{r}^l-1)^2} \sin^2 \frac{l\pi u}{\omega_2}.\end{aligned}$$

Lemma 5.4. *Let* $\tilde{z}(u)$ *and* $\tilde{r}$ *as in* Lemma 5.3. *For any* u *such that* $|\mathrm{Im}(u/\omega_1)| < \mathrm{Im}(\omega_2/\omega_1)$, *the function* $\chi_2(u)$ *satisfies the following*:

$$\chi_2(u) = \zeta(u) - \frac{\eta_2 u}{\omega_2} = \frac{\pi\sqrt{-1}}{\omega_2}\left\{\frac{\tilde{z}(u)+1}{\tilde{z}(u)-1} - 2\sum_{l=1}^{\infty} \frac{1}{\tilde{r}^l - 1}(\tilde{z}(u)^l - \tilde{z}(u)^{-l})\right\}.$$

By straightforward calculations, we see that $\xi_i(u)$ and $\chi_i(u)$ $(i = 1, 2)$ have the following properties.

Lemma 5.5. *The function* $\xi_i(u)$ $(i = 1, 2)$ *satisfies the following*:
(i) $\xi_i(0) = 0$.
(ii) $\xi_i(-u) = \xi_i(u)$.

Lemma 5.6. *The function* $\chi_i(u)$ $(i = 1, 2)$ *satisfies the following*:
(i) $\chi_i(-u) = -\chi_i(u)$.
(ii) $\chi_i(u + \omega_i) = \chi_i(u)$.
(iii) $\chi_1(u + \omega_2) = \chi_1(u) - 2\pi\sqrt{-1}/\omega_1$.
(iv) $\chi_1(u) = -\chi_1(\omega_1 - u)$.
(v) $\chi_1(u \pm \omega_2/2) = -\chi_1((\omega_1 - u) \pm \omega_2/2) \mp 2\pi\sqrt{-1}/\omega_1$.
(vi) $\chi_1(u) = \chi_2(u) - \{2\pi\sqrt{-1}/(\omega_1\omega_2)\}u$.

6. Covering spaces of *n*-noids

In this section, we observe the correspondence between the type of ω of an n-noid of genus one and that of a double or quadruple covering space of the n-noid. Since the covering space depends on the choice of the fundamental period (ω_1, ω_2) of the n-noid, we fix the fundamental period and distinguish the four types of ω.

First, we consider the double covering of an n-noid of each type.

Lemma 6.1. *Assume that* X *is an* n*-noid of type* $\omega = m_1\omega_1 + m_2\omega_2$ $(m_1, m_2 \in \{0,1\})$. *Let* $\tilde{\omega}_1 := 2\omega_1$, $\tilde{T}^2 := \mathbf{C}/(\mathbf{Z}\tilde{\omega}_1 + \mathbf{Z}\omega_2)$ *and* $u_{n+j} := u_j + \omega_1$ $(j = 1, \dots, n)$. *Then the double covering of* X, $\tilde{X}\colon \tilde{T}^2 \setminus \{u_1, \dots, u_{2n}\} \to \mathbf{R}^3$, *is a* $2n$*-noid of type* $\tilde{\omega} = m_1\tilde{\omega}_1$.

Proof. Since X is an n-noid of type $\omega = m_1\omega_1 + m_2\omega_2$, it follows that

$$2(u_1 + \cdots + u_n) + m_1\omega_1 + m_2\omega_2 = 2(s_1 + \cdots + s_n)$$

(see (1.8)). Let $s_{n+j} := s_j + \omega_1$, $t_{n+j} := t_j + \omega_1$ ($j = 1, \dots, n-1$), $s_{2n} := s_n + \omega_1 - m_2\omega_2$ and $t_{2n} := t_n + \omega_1 - m_2\omega_2$. Then, $\{s_1, \dots, s_{2n}\}$ (resp. $\{t_1, \dots, t_{2n}\}$) is a complete system of representatives of the poles (resp. zeroes) of $\tilde{g}$, the Gauss map of $\tilde{X}$, and it holds that $s_1 + \cdots + s_{2n} = t_1 + \cdots + t_{2n}$ and

$$2(u_{n+1} + \cdots + u_{2n}) + m_1\omega_1 - m_2\omega_2 = 2(s_{n+1} + \cdots + s_{2n}).$$

Combining these equations we have

$$2(u_1 + \cdots + u_{2n}) + m_1\tilde{\omega}_1 = 2(s_1 + \cdots + s_{2n}).$$

Hence $\tilde{X}$ is an $2n$-noid of type $\tilde{\omega} = m_1\tilde{\omega}_1$. □

In the same way, we have the following lemma:

Lemma 6.2. *Assume that X is an n-noid of type $\omega = m_1\omega_1 + m_2\omega_2$ ($m_1, m_2 \in \{0, 1\}$). Let $\tilde{\omega}_2 := 2\omega_2$, $\tilde{T}^2 := \mathbf{C}/(\mathbf{Z}\omega_1 + \mathbf{Z}\tilde{\omega}_2)$ and $u_{n+j} := u_j + \omega_2$ ($j = 1, \dots, n$). Then the double covering of X, $\tilde{X}\colon \tilde{T}^2 \setminus \{u_1, \dots, u_{2n}\} \to \mathbf{R}^3$, is a $2n$-noid of type $\tilde{\omega} = m_2\tilde{\omega}_2$.*

By Lemmas 6.1 and 6.2, we also have the following corollary:

Corollary 6.3. *Assume that X is an n-noid of type $\omega = \omega_1 + \omega_2$. Let $\tilde{\omega}_1 := 2\omega_1$, $\tilde{\omega}_2 := 2\omega_2$, $\tilde{T}^2 := \mathbf{C}/(\mathbf{Z}\tilde{\omega}_1 + \mathbf{Z}\tilde{\omega}_2)$ and $u_{n+j} := u_j + \omega_1$, $u_{2n+j} := u_j + \omega_2$, $u_{3n+j} := u_j + \omega_1 + \omega_2$ ($j = 1, \dots, n$). Then the quadruple covering of X, $\tilde{X}\colon \tilde{T}^2 \setminus \{u_1, \dots, u_{4n}\} \to \mathbf{R}^3$, is a $4n$-noid of type $\tilde{\omega} = 0$.*

Conversely, for a given $2N$-noid (resp. $4N$-noid) X of type ω, we can denote the necessary and sufficient condition for the existence of an N-noid $\check{X}$ of type $\check{\omega}$, whose double covering (resp. quadruple covering) is X.

Here, by choosing a branch, we consider elliptic functions

$$\text{(6.1)} \qquad \begin{cases} \tilde{P}(u) := C_1C_2\dfrac{\sigma(u-t_1)\cdots\sigma(u-t_n)}{\sigma(u-u_1)\cdots\sigma(u-u_n)}\sqrt{\dfrac{\sigma(u-u_n)}{\sigma(u-u_n-\omega)}}, \\ \tilde{Q}(u) := C_2\dfrac{\sigma(u-s_1)\cdots\sigma(u-s_n)}{\sigma(u-u_1)\cdots\sigma(u-u_n)}\sqrt{\dfrac{\sigma(u-u_n)}{\sigma(u-u_n-\omega)}}, \end{cases}$$

both of which have a fundamental period $(2\omega_1, 2\omega_2)$. By (1.7), the Weierstrass data of X is given by

$$g(u) = \frac{\tilde{P}(u)}{\tilde{Q}(u)}, \quad \eta = -\tilde{Q}(u)^2\, du. \tag{6.2}$$

Now, we describe the properties of $\tilde{P}$ and $\tilde{Q}$ for n-noids of each type.

In the case that $\omega = 0$, we have $\tilde{P}(u) = P(u)$, $\tilde{Q}(u) = Q(u)$, where $P(u)$ and $Q(u)$ are of the form (2.2) (cf. [9, Proposition 4.1]). Hence it holds that

$$\tilde{P}(u+\omega_i) = \tilde{P}(u), \quad \tilde{Q}(u+\omega_i) = \tilde{Q}(u) \quad (i = 1, 2).$$

In the case that $\omega = m_1\omega_1 + m_2\omega_2$ ($(m_1, m_2) = (1, 0)$ or $(0, 1)$ or $(1, 1)$), we have

$$\sigma(u + m_1\omega_1 + m_2\omega_2) = -\exp\left\{(m_1\eta_1 + m_2\eta_2)\left(u + \frac{m_1\omega_1 + m_2\omega_2}{2}\right)\right\}\sigma(u)$$

(cf. [6, p. 181, Satz 3]). Therefore, it holds that

$$\begin{aligned}&\sqrt{\frac{\sigma(u-u_n)}{\sigma(u-u_n-(m_1\omega_1+m_2\omega_2))}}\\ &= \sqrt{-1}\exp\left\{\frac{m_1\eta_1+m_2\eta_2}{2}\left(u-u_n-\frac{m_1\omega_1+m_2\omega_2}{2}\right)\right\}.\end{aligned}$$

Combining this equality with the Legendre relation (3.6), we have

$$\begin{aligned}&\sqrt{\frac{\sigma((u+\omega_1)-u_n)}{\sigma((u+\omega_1)-u_n-(m_1\omega_1+m_2\omega_2))}}\\ &= \sqrt{-1}\exp\left\{\frac{m_1\eta_1+m_2\eta_2}{2}\left(u-u_n-\frac{m_1\omega_1+m_2\omega_2}{2}\right)\right\}\exp\left\{\frac{(m_1\eta_1+m_2\eta_2)\omega_1}{2}\right\}\\ &= (-1)^{m_2}\sqrt{-1}\exp\left\{\frac{m_1\eta_1+m_2\eta_2}{2}\left(u-u_n-\frac{m_1\omega_1+m_2\omega_2}{2}\right)\right\}\\ &\quad\times\exp\left\{\frac{\eta_1(m_1\omega_1+m_2\omega_2)}{2}\right\}\end{aligned}$$

and

$$\begin{aligned}&\sqrt{\frac{\sigma((u+\omega_2)-u_n)}{\sigma((u+\omega_2)-u_n-(m_1\omega_1+m_2\omega_2))}}\\ &= \sqrt{-1}\exp\left\{\frac{m_1\eta_1+m_2\eta_2}{2}\left(u-u_n-\frac{m_1\omega_1+m_2\omega_2}{2}\right)\right\}\exp\left\{\frac{(m_1\eta_1+m_2\eta_2)\omega_1}{2}\right\}\end{aligned}$$

$$= (-1)^{m_1}\sqrt{-1}\exp\left\{\frac{m_1\eta_1+m_2\eta_2}{2}\left(u-u_n-\frac{m_1\omega_1+m_2\omega_2}{2}\right)\right\}$$
$$\times\exp\left\{\frac{\eta_2(m_1\omega_1+m_2\omega_2)}{2}\right\}.$$

Hence it holds that

$$\begin{cases}\tilde P(u+\omega_1)=(-1)^{m_2}\tilde P(u), & \tilde Q(u+\omega_1)=(-1)^{m_2}\tilde Q(u),\\ \tilde P(u+\omega_2)=(-1)^{m_1}\tilde P(u), & \tilde Q(u+\omega_2)=(-1)^{m_1}\tilde Q(u).\end{cases}$$

Combining these results, we get the following lemma.

Lemma 6.4. *Let X: $T^2\setminus\{u_1,\dots,u_n\}\to\mathbf{R}^3$ be an n-noid of type $\omega=m_1\omega_1+m_2\omega_2$ (m_1, $m_2\in\{0,1\}$), whose Weierstrass data is of the form* (6.2) *with* (6.1)*. Then the following equalities hold.*

$$\begin{cases}\tilde P(u+\omega_1)=(-1)^{m_2}\tilde P(u), & \tilde Q(u+\omega_1)=(-1)^{m_2}\tilde Q(u),\\ \tilde P(u+\omega_2)=(-1)^{m_1}\tilde P(u), & \tilde Q(u+\omega_2)=(-1)^{m_1}\tilde Q(u).\end{cases}$$

Applying this lemma, we can show the following.

Lemma 6.5. *Let X be a $2N$-noid of type $\omega=m_2\omega_2$ ($m_2\in\{0,1\}$), (ω_1,ω_2) a fundamental period, (g,η) the Weierstrass data given by* (6.1) *and* (6.2)*. Then, X is a double covering of an N-noid $\check X$ of type $\check\omega=m_2\check\omega_2$, whose fundamental period is $(\check\omega_1,\check\omega_2):=(\omega_1,\omega_2/2)$, if and only if*

$$\tilde P\left(u+\frac{\omega_2}{2}\right)=\tilde P(u),\quad \tilde Q\left(u+\frac{\omega_2}{2}\right)=\tilde Q(u).$$

In particular, in the case $\omega=0$, this condition is rewritten as

$$\begin{cases}u_{N+j}=u_j+\dfrac{\omega_2}{2},\\ c_{N+j}=c_j,\\ b_{N+j}=b_j\end{cases}\qquad (j=1,\dots,N)$$

by changing indices.

Proof. We only have to prove the sufficiency. Assume $\tilde P(u+\omega_2/2)=\tilde P(u)$ and $\tilde Q(u+\omega_2/2)=\tilde Q(u)$. Then, the Weierstrass data (g,η) given by (6.2) has a period $(\omega_1,\omega_2/2)$, and it holds that $R_k(\gamma_2)=2R_k(\check\gamma_2)$ $(k=0,1,2)$, where $\check\gamma_2$ is a line segment joining u_0 and $u_0+\omega_2/2$. Combining with $R_0(\gamma_2)+\overline{R_2(\gamma_2)}=0$ and $R_1(\gamma_2)=\overline{R_1(\gamma_2)}$, we have

$$R_0(\check\gamma_2)+\overline{R_2(\check\gamma_2)}=0,\quad R_1(\check\gamma_2)=\overline{R_1(\check\gamma_2)}.$$

Therefore, (g, η) realizes an N-noid $\check{X}$ of type $\check{\omega} = m_2\check{\omega}_2$, whose fundamental period is $(\check{\omega}_1, \check{\omega}_2) = (\omega_1, \omega_2/2)$. □

In the same way, we also have following lemmas.

Lemma 6.6. *Let X be a $2N$-noid of type $\omega = m_1\omega_1$ ($m_1 \in \{0, 1\}$), (ω_1, ω_2) a fundamental period, (g, η) the Weierstrass data given by* (6.1) *and* (6.2)*. Then, X is a double covering of an N-noid $\check{X}$ of type $\check{\omega} = m_1\check{\omega}_1$, whose fundamental period is $(\check{\omega}_1, \check{\omega}_2) := (\omega_1/2, \omega_2)$, if and only if*

$$\tilde{P}\left(u + \frac{\omega_1}{2}\right) = \tilde{P}(u), \quad \tilde{Q}\left(u + \frac{\omega_1}{2}\right) = \tilde{Q}(u).$$

In particular, in the case $\omega = 0$, this condition is rewritten as

$$\begin{cases} u_{N+j} = u_j + \dfrac{\omega_1}{2}, \\ c_{N+j} = c_j, \\ b_{N+j} = b_j \end{cases} \quad (j = 1, \dots, N)$$

by changing indices.

Lemma 6.7. *Let X be a $2N$-noid of type $\omega = m_2\omega_2$ ($m_2 \in \{0, 1\}$), (ω_1, ω_2) a fundamental period, (g, η) the Weierstrass data given by* (6.1) *and* (6.2)*. Then, X is a double covering of an N-noid $\check{X}$ of type $\check{\omega} = \check{\omega}_1 + m_2\check{\omega}_2$, whose fundamental period is $(\check{\omega}_1, \check{\omega}_2) := (\omega_1, \omega_2/2)$, if and only if*

$$\tilde{P}\left(u + \frac{\omega_2}{2}\right) = -\tilde{P}(u), \quad \tilde{Q}\left(u + \frac{\omega_2}{2}\right) = -\tilde{Q}(u).$$

In particular, in the case $\omega = 0$, this condition is rewritten as

$$\begin{cases} \left.\begin{aligned} u_{N+j} &= u_j + \frac{\omega_2}{2}, \\ c_{N+j} &= -c_j, \\ b_{N+j} &= -b_j, \end{aligned}\right\} \quad (j = 1, \dots, N), \\ c_0 = -\dfrac{1}{2}\displaystyle\sum_{j=1}^{N} c_j\eta_2, \\ b_0 = -\dfrac{1}{2}\displaystyle\sum_{j=1}^{N} b_j\eta_2 \end{cases}$$

by changing indices.

Lemma 6.8. *Let X be a $2N$-noid of type $\omega = m_1\omega_1$ ($m_1 \in \{0, 1\}$), (ω_1, ω_2) a fundamental period, (g, η) the Weierstrass data given by* (6.1) *and* (6.2). *Then, X is a double covering of an N-noid $\check{X}$ of type $\check{\omega} = m_1\check{\omega}_1 + \check{\omega}_2$, whose fundamental period is $(\check{\omega}_1, \check{\omega}_2) := (\omega_1/2, \omega_2)$, if and only if*

$$\tilde{P}\left(u + \frac{\omega_1}{2}\right) = -\tilde{P}(u), \quad \tilde{Q}\left(u + \frac{\omega_1}{2}\right) = -\tilde{Q}(u).$$

In particular, in the case $\omega = 0$, this condition is rewritten as

$$\begin{cases} \left.\begin{aligned} u_{N+j} &= u_j + \frac{\omega_1}{2}, \\ c_{N+j} &= -c_j, \\ b_{N+j} &= -b_j, \end{aligned}\right\} \quad (j = 1, \dots, N), \\ c_0 = -\dfrac{1}{2}\displaystyle\sum_{j=1}^{N} c_j\eta_1, \\ b_0 = -\dfrac{1}{2}\displaystyle\sum_{j=1}^{N} b_j\eta_1 \end{cases}$$

by changing indices.

Lemma 6.9. *Let X be a $4N$-noid of type $\omega = 0$, (ω_1, ω_2) a fundamental period, (g, η) the Weierstrass data given by* (2.1) *and* (2.2). *Then, X is a quadruple covering of an N-noid $\check{X}$ of type $\check{\omega} = \check{\omega}_1 + \check{\omega}_2$, whose fundamental period is $(\check{\omega}_1, \check{\omega}_2) := (\omega_1/2, \omega_2/2)$, if and only if*

$$\begin{cases} \left.\begin{aligned} u_{N+j} &= u_j + \frac{\omega_1}{2}, \\ u_{2N+j} &= u_j + \frac{\omega_2}{2}, \\ u_{3N+j} &= u_j + \frac{\omega_1 + \omega_2}{2}, \\ c_j &= -c_{N+j} = -c_{2N+j} = c_{3N+j}, \\ b_j &= -b_{N+j} = -b_{2N+j} = b_{3N+j}, \end{aligned}\right\} \quad (j = 1, \dots, N), \\ c_0 = 0, \\ b_0 = 0 \end{cases}$$

by changing indices.

7. Nonexistence

By Lemmas 6.1, 6.2 and Corollary 6.3, a double or quadruple covering of an n-noid of each type is of type $\omega = 0$. However, without such examples, almost all previously known examples are not included in the class $\omega = 0$. In this section, we give obstructions for the existence of n-noids symmetric with respect to a plane. As in the previous section, we distinguish the four types of $\omega = m_1\omega_1 + m_2\omega_2$ $(m_1, m_2 \in \{0, 1\})$.

Theorem 7.1. *Assume that $\omega_1 \in \mathbf{R}_+$, $\omega_2 \in \sqrt{-1}\mathbf{R}_+$ and $u_j \in \mathbf{R}$ $(j = 1, \dots, n)$. Then, there does not exist an n-noid $X\colon T^2 \setminus \{u_1, \dots, u_n\} \to \mathbf{R}^3$ of type $\omega = 0$ which has plane symmetry with respect to the antiholomorphic involution $I(u) = \bar{u}$.*

Proof. By suitable motion and point reflection in $\mathbf{R}^3$, we may assume that X is symmetric with respect to the x_1x_3-plane and that X satisfies the condition (4.6) with all double signs plus. We assume $w_j \in \mathbf{R}$ and $w_j^* = 0$ $(j = 1, \dots, n)$. Then, by the proof of Theorem 4.4, we have $R_{0j} + R_{2j} = 0$ $(j = 1, \dots, n)$. Now, we estimate the signature of P_3'. By using Lemma 3.5, we have

$$\begin{aligned} P_3' &= R_0(\gamma_2) + R_2(\gamma_2) \\ &= \left\{ \frac{\omega_2}{\omega_1} R_0(\gamma_1) - \frac{1}{\omega_1} \sum_{j=1}^{n} (b_j^2 - R_{0j}u_j) \right\} + \left\{ \frac{\omega_2}{\omega_1} R_2(\gamma_1) - \frac{1}{\omega_1} \sum_{j=1}^{n} (c_j^2 - R_{2j}u_j) \right\} \\ &= \frac{\omega_2}{\omega_1} (R_0(\gamma_1) + R_2(\gamma_1)) - \frac{1}{\omega_1} \sum_{j=1}^{n} (b_j^2 + c_j^2). \end{aligned}$$

First, we estimate the signature of $(\omega_2/\omega_1)R_0(\gamma_1)$. By using Lemma 5.5 and $\sum_{j=1}^n b_j = 0$, we have

$$\begin{aligned} &\sum_{j=1}^{n} \sum_{k=1;\, k\neq j}^{n} b_j b_k \xi_1(u_j - u_k) \\ &= \sum_{j=1}^{n-1} \sum_{k=1;\, k\neq j}^{n-1} b_j b_k \xi_1(u_j - u_k) + \sum_{j=1}^{n-1} b_j b_n \xi_1(u_n - u_j) + \sum_{k=1}^{n-1} b_n b_k \xi_1(u_n - u_k) \\ &= \sum_{j=1}^{n-1} \sum_{k=1;\, k\neq j}^{n-1} b_j b_k \xi_1(u_j - u_k) - \sum_{j=1}^{n-1} b_j \sum_{k=1}^{n-1} b_k \xi_1(u_n - u_j) - \sum_{j=1}^{n-1} b_j \sum_{k=1}^{n-1} b_k \xi_1(u_n - u_k) \\ &= -\sum_{j=1}^{n-1} \sum_{k=1}^{n-1} b_j b_k (\xi_1(u_n - u_k) + \xi_1(u_n - u_j) - \xi_1((u_n - u_k) - (u_n - u_j))). \end{aligned}$$

Combining Lemma 5.1 with

$$\sin^2\alpha + \sin^2\beta - \sin^2(\alpha - \beta) = 2(\sin^2\alpha \sin^2\beta + \sin\alpha\cos\alpha\sin\beta\cos\beta),$$

we have

$$
\begin{aligned}
&-\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1;\,k\neq j}^{n} b_j b_k \xi_1(u_j-u_k)\\
&=\frac{1}{2}\sum_{j=1}^{n-1}\sum_{k=1}^{n-1} b_j b_k\big(\xi_1(u_n-u_k)+\xi_1(u_n-u_j)-\xi_1((u_n-u_k)-(u_n-u_j))\big)\\
&=16\left(\frac{\pi}{\omega_1}\right)^2\sum_{l=1}^{\infty}\frac{r^l}{(r^l-1)^2}\sum_{j=1}^{n-1}\sum_{k=1}^{n-1} b_j b_k\bigg(\sin^2\frac{l\pi(u_n-u_k)}{\omega_1}+\sin^2\frac{l\pi(u_n-u_j)}{\omega_1}\\
&\qquad\qquad -\sin^2\frac{l\pi\{(u_n-u_k)-(u_n-u_j)\}}{\omega_1}\bigg)\\
&=32\left(\frac{\pi}{\omega_1}\right)^2\sum_{l=1}^{\infty}\frac{r^l}{(r^l-1)^2}\sum_{j=1}^{n-1}\sum_{k=1}^{n-1} b_j b_k\bigg(\sin^2\frac{l\pi(u_n-u_j)}{\omega_1}\sin^2\frac{l\pi(u_n-u_k)}{\omega_1}\\
&\qquad\qquad +\sin\frac{l\pi(u_n-u_j)}{\omega_1}\cos\frac{l\pi(u_n-u_j)}{\omega_1}\\
&\qquad\qquad \times\sin\frac{l\pi(u_n-u_k)}{\omega_1}\cos\frac{l\pi(u_n-u_k)}{\omega_1}\bigg)\\
&=32\left(\frac{\pi}{\omega_1}\right)^2\sum_{l=1}^{\infty}\frac{r^l}{(r^l-1)^2}\left\{\left(\sum_{j=1}^{n-1} b_j\sin^2\frac{l\pi(u_n-u_j)}{\omega_1}\right)^2\right.\\
&\qquad\qquad \left.+\left(\sum_{j=1}^{n-1} b_j\sin\frac{l\pi(u_n-u_j)}{\omega_1}\cos\frac{l\pi(u_n-u_j)}{\omega_1}\right)^2\right\}\\
&\geq 0.
\end{aligned}
$$

Hence, by Lemma 3.4, we have

$$
\begin{aligned}
\frac{\omega_2}{\omega_1}R_0(\gamma_1)&=\frac{-\omega_2}{2\pi\sqrt{-1}}\left\{-\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1;\,k\neq j}^{n} b_j b_k\xi_1(u_j-u_k)+\left(b_0-\eta_1\sum_{j=1}^{n} b_j\frac{u_j}{\omega_1}\right)^2\right\}\\
&\leq 0.
\end{aligned}
$$

Similarly, we also have $(\omega_2/\omega_1)R_2(\gamma_1)\leq 0$.

On the other hand, since $b_j, c_j\in\mathbf{R}$ and $b_j\neq 0$ $(j=1,\ldots,n)$, we have

$$
-\frac{1}{\omega_1}\sum_{j=1}^{n}(b_j^2+c_j^2)<0,
$$

and hence $P_3'<0$. Therefore there do not exist u_j, b_j, c_j $(j=1,\ldots,n)$ satisfying (4.7). $\square$

REMARK. There exist n-end catenoids of type $\omega = 0$ if we assume $\omega_1 \in \mathbf{R}_+$, $\omega_2 - \omega_1/2 \in \sqrt{-1}\mathbf{R}_+$ and $u_j \in \mathbf{R}$ $(j = 1, \dots, n)$ (cf. Theorem 8.3).

By using Lemma 6.1, we also have the following:

Corollary 7.2. *Assume that $\omega_1 \in \mathbf{R}_+$, $\omega_2 \in \sqrt{-1}\mathbf{R}_+$ and $u_j \in \mathbf{R}$ $(j = 1, \dots, n)$. Then, there does not exist an n-noid $X\colon T^2 \setminus \{u_1, \dots, u_n\} \to \mathbf{R}^3$ of type $\omega = \omega_2$ which has plane symmetry with respect to the antiholomorphic involution $I(u) = \overline{u}$.*

REMARK. There exist n-end catenoids of type $\omega = \omega_2$ if we assume $\omega_1 \in \mathbf{R}_+$, $\omega_2 - \omega_1 \in \sqrt{-1}\mathbf{R}_+$ and $u_j \in \mathbf{R}$ $(j = 1, \dots, n)$ (cf. [2], [9]).

8. Examples

Throughout this section, we assume $\omega_1 \in \mathbf{R}_+$ and $\omega_2 \in \sqrt{-1}\mathbf{R}_+$, and set $y := \omega_2/(\sqrt{-1}\omega_1)$. As in §5, we denote $r = \exp(2\pi y)$ and $\tilde{r} = \exp(2\pi/y)$.

First, we describe the data of Costa's family of 4-noids (cf. [4]) by using our notation.

EXAMPLE 8.1. Consider the following flux data:

$$n = 4, \quad p_1 = p_3 := 0, \quad p_2 = \overline{p_4}, \quad a_1 = a_2 = a_3 = a_4 := 0.$$

Assume $\omega_1 = 1$ and set

$$\begin{aligned}
&u_1 := \frac{\sqrt{-1}y}{4}, \quad u_2 := \frac{1}{2} + \frac{\sqrt{-1}y}{4}, \quad u_3 := \frac{3\sqrt{-1}y}{4}, \quad u_4 := \frac{1}{2} + \frac{3\sqrt{-1}y}{4},\\
&b_1 := \beta(1 + \beta_1\sqrt{-1}), \quad b_2 := \beta(-1 - \beta_2\sqrt{-1}), \quad b_3 := \beta(1 - \beta_1\sqrt{-1}),\\
&b_4 := \beta(-1 + \beta_2\sqrt{-1}),\\
&c_1 := 0, \quad c_2 := \beta\beta_3\sqrt{-1}, \quad c_3 := 0, \quad c_4 := -\beta\beta_3\sqrt{-1},\\
&b_0 := -\beta\left(\eta_1 + \frac{\beta_1 - \beta_2}{2}\sqrt{-1}\eta_2\right), \quad c_0 := -\frac{\beta\beta_3}{2}\sqrt{-1}\eta_2,\\
&\beta, \beta_1, \beta_2, \beta_3 > 0.
\end{aligned}$$

Then the surface given by these data is symmetric with respect to the x_1x_3-plane, and we can apply Theorem 4.4. By the straightforward calculation, we see that $w_j = 0$ and $w_j^* = 0$ $(j = 1, 2, 3, 4)$ are satisfied, and the nontrivial condition is rewritten as follows:

$$\begin{aligned}
\frac{-2\pi\sqrt{-1}P_1'}{\beta^2} &\equiv (2\eta_1 - e_3)(\beta_1^2 + \beta_2^2) + 2(e_2 - e_1)\beta_1\beta_2 - (2\eta_1 - e_3)\beta_3^2 - 4(\eta_1 + e_3),\\
\frac{-\pi\sqrt{-1}P_2'}{\beta^2\beta_3} &\equiv (e_1 - e_2)\beta_1 - (2\eta_1 - e_3)\beta_2,
\end{aligned}$$

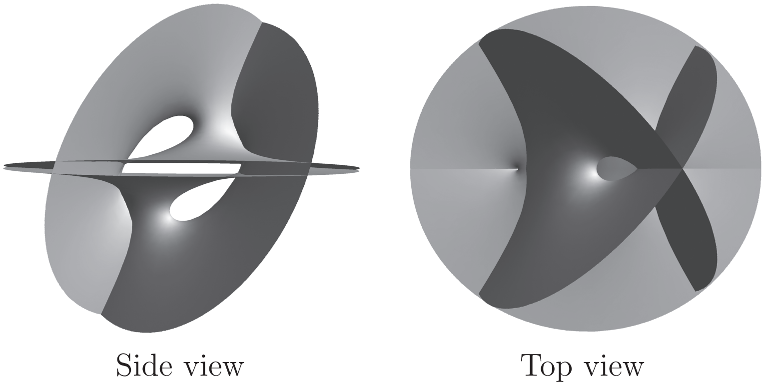


Fig. 8.1.

$$\frac{\pi(P_3' - y\sqrt{-1}P_1')}{\beta^2} \equiv 2\pi(\beta_1^2 + \beta_2^2) - \{(2\eta_1 - e_3)y - 2\pi\}\beta_3^2 - 4\pi,$$

where $e_1 := \wp(1/2)$, $e_2 := \wp(1/2 + y\sqrt{-1}/2)$, $e_3 := \wp(y\sqrt{-1}/2)$ and $e_1 + e_2 + e_3 = 0$. For any $y \geq 1$, there exists a $(\beta_1, \beta_2, \beta_3) \in (0, +\infty) \times (0, +\infty) \times (0, +\infty)$ satisfying the equations above (cf. [4]). Fig. 8.1 shows the case that $y = 1$.

For each integer $N \geq 3$, Berglund–Rossman [2] constructed a Jorge–Meeks type N-end catenoid of genus one, which is of type $\omega = \omega_1 + \omega_2$ with $\omega_1 \in \mathbf{R}_+$ and $\omega_2 \in \sqrt{-1}\mathbf{R}_+$, and invariant under the action of the direct product $D_N \times \mathbf{Z}_2$ of the dihedral group and a reflection. By Corollary 6.3, the quadruple covering of Berglund–Rossman's N-end catenoid is a $4N$-end catenoid of type $\omega = 0$. In Example 8.2 below, we consider a deformation family of such $4N$-end catenoid and construct a family of double coverings of $2N$-end catenoids of type $\check{\omega} = 0$ with $\check{\omega}_1 \in \mathbf{R}_+$ and $\check{\omega}_2 - \check{\omega}_1/2 \in \sqrt{-1}\mathbf{R}_+$, whose ends have alternating weights, for each odd integer $N \geq 3$. As a special case of this family, we get double coverings of N-end catenoids of type $\check{\omega} = 0$, which has the same flux as that of Jorge–Meeks' N-end catenoid (cf. [7]) and Berglund–Rossman's N-end catenoid, for each odd integer $N \geq 5$. We also construct a double covering of N-end catenoid of type $\check{\omega} = \check{\omega}_1$ with $\check{\omega}_1 \in \mathbf{R}_+$ and $\check{\omega}_2 \in \sqrt{-1}\mathbf{R}_+$ with the same flux above for each odd integer $N \geq 5$. These results are concluded in Theorem 8.3.

EXAMPLE 8.2. Let $N \geq 3$, and set $n := 4N$, $\zeta_N := e^{2\pi\sqrt{-1}/N}$ and $\zeta_{2N} := e^{2\pi\sqrt{-1}/(2N)}$. Here, we denote $p_{4(J-1)+L}$ (resp. $u_{4(J-1)+L}$, $b_{4(J-1)+L}$, $c_{4(J-1)+L}$, $w_{4(J-1)+L}$) by $p_{J,L}$ (resp. $u_{J,L}$, $b_{J,L}$, $c_{J,L}$, $w_{J,L}$) for $J = 1,\dots,N$; $L = 1,2,3,4$. For any $L = 1,2,3,4$, let L_1, L_2 be the unique numbers satisfying $L = 1 + L_1 + 2L_2$ and $L_1, L_2 \in \{0, 1\}$, that is

$$(L_1, L_2) := \begin{cases} (0,0) & \text{if} \quad L = 1, \\ (1,0) & \text{if} \quad L = 2, \\ (0,1) & \text{if} \quad L = 3, \\ (1,1) & \text{if} \quad L = 4. \end{cases}$$

We define permutations τ_2, τ_3 and $\tau_4 \in S_4$ by $\tau_2 := (1,2)(3,4)$, $\tau_3 := (1,3)(2,4)$ and $\tau_4 := (1,4)(2,3)$. For any $J = 1, \ldots, N$ and $L = 1, 2, 3, 4$, set

$$p_{J,L} := \zeta_N^{2(J-1)+L_1}, \tag{8.1}$$

$$u_{J,L} := \frac{2(J-1)+L_1}{2N}\omega_1 + \frac{L_2}{2}\omega_2, \tag{8.2}$$

$$\begin{cases} b_{J,L} := (-1)^{L_2}\zeta_{2N}^{-2(J-1)-L_1} e^{\pi\sqrt{-1}/4}\beta\beta_L, \\ c_{J,L} := p_{J,L} b_{J,L} = (-1)^{L_2}\zeta_{2N}^{2(J-1)+L_1} e^{\pi\sqrt{-1}/4}\beta\beta_L, \\ \beta > 0, \quad \beta_L \in \mathbf{R}. \end{cases} \tag{8.3}$$

By the assumption, we have

$$b_{J,L} = \zeta_N^{-(J-1)} b_{1,L}, \quad c_{J,L} = \zeta_N^{(J-1)} c_{1,L} \quad (J = 1, \ldots, N;\ L = 1, 2, 3, 4), \tag{8.4}$$

$$\sum_{J=1}^{N} b_{J,L} = 0, \quad \sum_{J=1}^{N} c_{J,L} = 0 \quad (L = 1, 2, 3, 4), \tag{8.5}$$

$$c_j = \sqrt{-1}\,\overline{b_j} \quad (j = 1, \ldots, n), \tag{8.6}$$

$$\sum_{j=1}^{n} c_j b_j = \sqrt{-1} \sum_{j=1}^{n} |b_j|^2, \tag{8.7}$$

$$\sum_{j=1}^{n} |b_j|^2 = N\beta^2 \sum_{L=1}^{4} \beta_L^2. \tag{8.8}$$

For the symmetry of the dihedral group D_N, it is natural to set

$$\begin{cases} b_0 := \eta_1 \dfrac{\zeta_N}{1-\zeta_N} \displaystyle\sum_{L=1}^{4} b_L, \\ c_0 := \eta_1 \dfrac{\zeta_N^{-1}}{1-\zeta_N^{-1}} \displaystyle\sum_{L=1}^{4} c_L. \end{cases} \tag{8.9}$$

By a straightforward calculation, we have

$$
\begin{aligned}
\frac{1}{\omega_1}\sum_{j=1}^{n} b_j u_j &= \sum_{J=1}^{N}\sum_{L=1}^{4} \zeta_N^{-(J-1)} b_{1,L}\left\{\frac{2(J-1)+L_1}{2N} + \frac{L_2}{2}\cdot\frac{\omega_2}{\omega_1}\right\} \\
&= \sum_{J=1}^{N}\sum_{L=1}^{4} \zeta_N^{-(J-1)} b_L \frac{J}{N} = \sum_{L=1}^{4} b_L \sum_{J=1}^{N} \frac{J}{N}\zeta_N^{-(J-1)} \\
&= \frac{\zeta_N}{1-\zeta_N}\sum_{L=1}^{4} b_L = \frac{\zeta_N}{1-\zeta_N} e^{\pi\sqrt{-1}/4}\beta\{(\beta_1-\beta_3)+\zeta_{2N}^{-1}(\beta_2-\beta_4)\}.
\end{aligned} \tag{8.10}
$$

In the same way, we also have

$$
\sum_{j=1}^{n} c_j u_j = \sqrt{-1}\sum_{j=1}^{n} \overline{b_j} u_j = \sqrt{-1}\overline{\sum_{j=1}^{n} b_j u_j}. \tag{8.11}
$$

Hence we get

$$
\begin{cases}
b_0 = \dfrac{\eta_1}{\omega_1}\displaystyle\sum_{k=1}^{n} b_k u_k = -\dfrac{\eta_1}{\omega_1}\sum_{k=1;\ k\neq j}^{n} b_k(u_j-u_k), \\
c_0 = \dfrac{\eta_1}{\omega_1}\displaystyle\sum_{k=1}^{n} c_k u_k = -\dfrac{\eta_1}{\omega_1}\sum_{k=1;\ k\neq j}^{n} c_k(u_j-u_k).
\end{cases} \tag{8.12}
$$

By (8.2), we have

$$
u_{J,L} - u_{K,M} = \frac{2(J-K)+(L_1-M_1)}{2N}\omega_1 + \frac{L_2-M_2}{2}\omega_2, \tag{8.13}
$$

and, by (8.4) and (8.3), we have

$$
b_{J,L} b_{K,M} = b_{1,L} b_{1,M} \zeta_N^{-(J+K-2)} = (-1)^{L_2+M_2}\sqrt{-1}\beta^2\beta_L\beta_M\zeta_{2N}^{-2(J+K-2)-(L_1+M_1)}, \tag{8.14}
$$

$$
c_{J,L} b_{K,M} = (-1)^{L_2+M_2}\sqrt{-1}\beta^2\beta_L\beta_M\zeta_{2N}^{2(J-K)+(L_1-M_1)}, \tag{8.15}
$$

$$
\begin{aligned}
b_L b_M &= b_{1,L} b_{1,M} \\
&= \begin{cases}
\sqrt{-1}\zeta_N^{-L_1}\beta^2\beta_L^2 & \text{if } L = M, \\
\sqrt{-1}\zeta_{2N}^{-(L_1+M_1)}\beta^2\beta_L\beta_M & \text{if } L_1 \neq M_1,\ L_2 = M_2, \text{ i.e. } M = \tau_2(L), \\
-\sqrt{-1}\zeta_{2N}^{-(L_1+M_1)}\beta^2\beta_L\beta_M & \text{if } L_2 \neq M_2, \text{ i.e. } M = \tau_3(L) \text{ or } \tau_4(L).
\end{cases}
\end{aligned} \tag{8.16}
$$

We note here that, for any $l, K, M_2 \in \mathbf{Z}$, it holds that

$$\exp\left\{\frac{2l\pi\sqrt{-1}}{\omega_1}\left(\frac{K}{2N}\omega_1 + \frac{M_2}{2}\omega_2\right)\right\} = \zeta_{2N}^{Kl} r^{-M_2 l/2}, \tag{8.17}$$

$$\exp\left\{\frac{2l\pi\sqrt{-1}}{\omega_2}\left(\frac{K}{2N}\omega_1 + \frac{M_2}{2}\omega_2\right)\right\} = \tilde{r}^{Kl/(2N)}(-1)^{M_2 l}. \tag{8.18}$$

For these data, we first calculate local and global period conditions for the existence of $4N$-end catenoids, and prove that $R_1(\gamma_2) = 0$ is the only nontrivial condition. Here we note that we calculate w_j and $R_1(\gamma_2)$ with respect to r and $\tilde{r}$, since both of them are necessary in the concrete estimates.

Calculation of R_{1j} and w_j^*. Combining Lemma 2.1 with (8.12), we have

$$\begin{aligned}
R_{1j} &= -\left\{\sum_{k=1;\ k\neq j}^{n} (p_j + p_k) b_j b_k \zeta(u_j - u_k) + (p_j b_j b_0 + b_j c_0)\right\} \\
&= -\sum_{k=1;\ k\neq j}^{n} (p_j + p_k) b_j b_k \left(\zeta(u_j - u_k) - (u_j - u_k)\frac{\eta_1}{\omega_1}\right) \\
&= -\sum_{k=1;\ k\neq j}^{n} (p_j + p_k) b_j b_k \chi_1(u_j - u_k).
\end{aligned}$$

Now, we consider R_{1j} with $j = 4(J-1) + L$.

$$\begin{aligned}
R_{1j} = -\Bigg\{ &\sum_{K=1;\ K\neq J}^{N} (p_{J,L} + p_{K,L}) b_{J,L} b_{K,L} \chi_1(u_{J,L} - u_{K,L}) \\
&+ \sum_{M=1;\ M\neq L}^{4} \sum_{K=1}^{N} (p_{J,L} + p_{K,M}) b_{J,L} b_{K,M} \chi_1(u_{J,L} - u_{K,M}) \Bigg\}.
\end{aligned}$$

By using (8.1), (8.13), (8.14) and Lemma 5.6, we have

$$\begin{aligned}
&\sum_{K=1;\ K\neq J}^{N} (p_{J,L} + p_{K,L}) b_{J,L} b_{K,L} \chi_1(u_{J,L} - u_{K,L}) \\
&= \sum_{K=1;\ K\neq J}^{N} (\zeta_N^{2(J-1)+L_1} + \zeta_N^{2(K-1)+L_1}) b_{1,L}^2 \zeta_N^{-(J+K-2)} \chi_1\left(\frac{J-K}{N}\omega_1\right) \\
&= b_L^2 \zeta_N^{L_1} \sum_{K=1;\ K\neq J}^{N} (\zeta_N^{J-K} + \zeta_N^{-(J-K)}) \chi_1\left(\frac{J-K}{N}\omega_1\right)
\end{aligned}$$

$$= b_L^2 \zeta_N^{L_1} \sum_{K'=1}^{N-1} (\zeta_N^{K'} + \zeta_N^{-K'}) \chi_1\left(\frac{K'}{N}\omega_1\right)$$

$$= b_L^2 \zeta_N^{L_1} \left\{ \sum_{K'=1}^{N-1} \zeta_N^{K'} \chi_1\left(\frac{K'}{N}\omega_1\right) - \sum_{K'=1}^{N-1} \zeta_N^{N-K'} \chi_1\left(\frac{N-K'}{N}\omega_1\right) \right\} = 0,$$

and, for $M \neq L$,

$$\sum_{K=1}^{N} (p_{J,L} + p_{K,M}) b_{J,L} b_{K,M} \chi_1(u_{J,L} - u_{K,M})$$

$$= \sum_{K=1}^{N} (\zeta_N^{2(J-1)+L_1} + \zeta_N^{2(K-1)+M_1}) b_{1,L} b_{1,M} \zeta_N^{-(J+K-2)}$$
$$\times \chi_1\left(\frac{2(J-K)+(L_1-M_1)}{2N}\omega_1 + \frac{L_2-M_2}{2}\omega_2\right)$$

$$= b_L b_M \zeta_{2N}^{L_1+M_1} \sum_{K=1}^{N} (\zeta_{2N}^{2(J-K)+(L_1-M_1)} + \zeta_{2N}^{-[2(J-K)+(L_1-M_1)]})$$
$$\times \chi_1\left(\frac{2(J-K)+(L_1-M_1)}{2N}\omega_1 + \frac{L_2-M_2}{2}\omega_2\right)$$

$$= b_L b_M \zeta_{2N}^{L_1+M_1} \sum_{K'=1}^{N} (\zeta_{2N}^{2K'+(L_1-M_1)} + \zeta_{2N}^{-[2K'+(L_1-M_1)]})$$
$$\times \chi_1\left(\frac{2K'+(L_1-M_1)}{2N}\omega_1 + \frac{L_2-M_2}{2}\omega_2\right)$$

$$= b_L b_M \zeta_{2N}^{L_1+M_1} \left\{ \sum_{K'=1}^{N} \zeta_{2N}^{2K'+(L_1-M_1)} \chi_1\left(\frac{2K'+(L_1-M_1)}{2N}\omega_1 + \frac{L_2-M_2}{2}\omega_2\right) \right.$$
$$- \sum_{K'=1}^{N} \zeta_{2N}^{2(N-K')-(L_1-M_1)} \chi_1\left(\frac{2(N-K')-(L_1-M_1)}{2N}\omega_1 + \frac{L_2-M_2}{2}\omega_2\right)$$
$$\left. - \sum_{K'=1}^{N} \zeta_{2N}^{-[2K'+(L_1-M_1)]} (L_2-M_2) \frac{2\pi\sqrt{-1}}{\omega_1} \right\}$$

$$= 0.$$

Hence we get $R_{1j} = 0$. Combining this equality with (2.11) and $|p_j| = 1$, we also get $w_j^* = \overline{p_j} R_{1j} = 0$ and $w_j = -p_j R_{0j}$ $(j = 1, \ldots, n)$.

Calculation of R_{0j} and w_j with respect to r. Combining Lemma 2.1 with (8.12), we have

$$
\begin{aligned}
R_{0j} &= -\left\{\sum_{k=1;\ k\neq j}^{n} 2b_j b_k \zeta(u_j-u_k) + 2b_j b_0\right\} \\
&= -\sum_{k=1;\ k\neq j}^{n} 2b_j b_k\left\{\zeta(u_j-u_k)-(u_j-u_k)\frac{\eta_1}{\omega_1}\right\} \qquad (8.19)\\
&= -\sum_{k=1;\ k\neq j}^{n} 2b_j b_k \chi_1(u_j-u_k).
\end{aligned}
$$

Now, we compute R_{0j} with $j=4(J-1)+L$.

$$
\begin{aligned}
R_{0j} = -\Bigg\{&\sum_{K=1;\ K\neq J}^{N} 2b_{J,L}b_{K,L}\chi_1(u_{J,L}-u_{K,L}) \\
&+\sum_{M=1;\ M\neq L}^{4}\sum_{K=1}^{N} 2b_{J,L}b_{K,M}\chi_1(u_{J,L}-u_{K,M})\Bigg\}.
\end{aligned}
$$

By using (8.13), (8.14) and Lemma 5.6, we have

$$
\begin{aligned}
&\sum_{K=1;\ K\neq J}^{N} 2b_{J,L}b_{K,L}\chi_1(u_{J,L}-u_{K,L}) \\
&= \sum_{K=1;\ K\neq J}^{N} 2b_{1,L}^2\zeta_N^{-(J+K-2)}\chi_1\left(\frac{J-K}{N}\omega_1\right) \\
&= 2b_L^2\zeta_N^{-2(J-1)}\sum_{K=1;\ K\neq J}^{N}\zeta_N^{J-K}\chi_1\left(\frac{J-K}{N}\omega_1\right) = 2b_L^2\zeta_N^{-2(J-1)}\sum_{K'=1}^{N-1}\zeta_N^{K'}\chi_1\left(\frac{K'}{N}\omega_1\right),
\end{aligned}
$$

and, for $M\neq L$,

$$
\begin{aligned}
&\sum_{K=1}^{N} 2b_{J,L}b_{K,M}\chi_1(u_{J,L}-u_{K,M}) \\
&=\sum_{K=1}^{N} 2b_{1,L}b_{1,M}\zeta_N^{-(J+K-2)}\chi_1\left(\frac{2(J-K)+(L_1-M_1)}{2N}\omega_1+\frac{L_2-M_2}{2}\omega_2\right) \\
&=2b_L b_M\zeta_{2N}^{-4(J-1)-(L_1-M_1)}\sum_{K=1}^{N}\zeta_{2N}^{2(J-K)+(L_1-M_1)}\chi_1\left(\frac{2(J-K)+(L_1-M_1)}{2N}\omega_1+\frac{L_2-M_2}{2}\omega_2\right) \\
&=2b_L b_M\zeta_{2N}^{-4(J-1)-(L_1-M_1)}\sum_{K'=1}^{N}\zeta_{2N}^{2K'+(L_1-M_1)}\chi_1\left(\frac{2K'+(L_1-M_1)}{2N}\omega_1+\frac{L_2-M_2}{2}\omega_2\right).
\end{aligned}
$$

We note here that, for any $l, L_1 \in \mathbf{Z}$, it holds that

$$\sum_{K=1}^{N} \zeta_{2N}^{(2K-L_1)l} = \begin{cases} 0 & \text{if} \quad l \not\equiv 0 \bmod N, \\ (-1)^{L_1} N & \text{if} \quad l \equiv N \bmod 2N, \\ N & \text{if} \quad l \equiv 0 \bmod 2N, \end{cases} \tag{8.20}$$

and

$$\begin{cases} \displaystyle\sum_{K=1}^{N-1} \frac{1}{1-\zeta_N^K} = \frac{N-1}{2}, \\ \displaystyle\sum_{K=1}^{N} \frac{1}{x-\zeta_{2N}^{2K-L_1}} = \frac{Nx^{N-1}}{x^N-(-1)^{L_1}} \quad (x^N \neq (-1)^{L_1}). \end{cases}$$

By Lemma 5.2, (8.17) and the equality above, we have

$$\begin{aligned} &\frac{\omega_1}{\pi\sqrt{-1}} \sum_{K=1}^{N-1} \zeta_N^K \chi_1\left(\frac{K}{N}\omega_1\right) \\ &= \sum_{K=1}^{N-1} \zeta_N^K \left\{ \frac{\zeta_N^K+1}{\zeta_N^K-1} - 2\sum_{l=1}^{\infty} \frac{1}{r^l-1}(\zeta_N^{Kl}-\zeta_N^{-Kl}) \right\} \\ &= \sum_{K=1}^{N-1} \left(\zeta_N^K + \frac{2}{1-\zeta_N^{-K}}\right) - 2\sum_{l=1}^{\infty} \frac{1}{r^l-1} \sum_{K=1}^{N-1} (\zeta_N^{K(l+1)} - \zeta_N^{-K(l-1)}) \\ &= -1 + 2\cdot\frac{N-1}{2} - 2\sum_{l'=1}^{\infty} \left\{ \frac{(-1)-(N-1)}{r^{l'N-(N-1)}-1} + \frac{(N-1)-(-1)}{r^{l'N-1}-1} \right\} \\ &= N-2+2N\sum_{l'=1}^{\infty} \left(\frac{1}{r^{l'N-(N-1)}-1} - \frac{1}{r^{l'N-1}-1} \right). \end{aligned}$$

In the same way, we also have, for $M \neq L$,

$$\begin{aligned} &\frac{\omega_1}{\pi\sqrt{-1}} \sum_{K=1}^{N} \zeta_{2N}^{2K+(L_1-M_1)} \chi_1\left(\frac{2K+(L_1-M_1)}{2N}\omega_1 + \frac{L_2-M_2}{2}\omega_2 \right) \\ &= \sum_{K=1}^{N} \zeta_{2N}^{2K+(L_1-M_1)} \left\{ \frac{\zeta_{2N}^{2K+(L_1-M_1)} r^{-(L_2-M_2)/2}+1}{\zeta_{2N}^{2K+(L_1-M_1)} r^{-(L_2-M_2)/2}-1} \right. \\ &\qquad - 2\sum_{l=1}^{\infty} \frac{1}{r^l-1} (\zeta_{2N}^{[2K+(L_1-M_1)]l} r^{-(L_2-M_2)l/2} \\ &\qquad\qquad \left. - \zeta_{2N}^{-[2K+(L_1-M_1)]l} r^{(L_2-M_2)l/2}) \right\} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{K=1}^{N}\left(\zeta_{2N}^{2K+(L_1-M_1)} + \frac{2}{r^{-(L_2-M_2)/2} - \zeta_{2N}^{-[2K+(L_1-M_1)]}}\right) \\
&\quad - 2\sum_{l=1}^{\infty}\frac{1}{r^l-1}\sum_{K=1}^{N}(\zeta_{2N}^{[2K+(L_1-M_1)](l+1)}r^{-(L_2-M_2)l/2} - \zeta_{2N}^{-[2K+(L_1-M_1)](l-1)}r^{(L_2-M_2)l/2}) \\
&= 0 + \frac{2Nr^{-(L_2-M_2)(N-1)/2}}{r^{-(L_2-M_2)N/2} - (-1)^{L_1-M_1}} \\
&\quad - 2\sum_{l'=1}^{\infty}\Bigg\{\frac{0 - (-1)^{(l'-1)(L_1-M_1)}Nr^{(L_2-M_2)[(l'-1)N+1]/2}}{r^{(l'-1)N+1}-1} \\
&\qquad\qquad + \frac{(-1)^{l'(L_1-M_1)}Nr^{-(L_2-M_2)(l'N-1)/2} - 0}{r^{l'N-1}-1}\Bigg\} \\
&= 2N\frac{-(-1)^{L_1-M_1}r^{(L_2-M_2)/2}}{r^{(L_2-M_2)N/2} - (-1)^{L_1-M_1}} \\
&\quad + 2N\sum_{l'=1}^{\infty}\left\{\frac{(-1)^{(l'-1)(L_1-M_1)}r^{(L_2-M_2)[l'N-(N-1)]/2}}{r^{l'N-(N-1)}-1} + \frac{-(-1)^{l'(L_1-M_1)}r^{-(L_2-M_2)(l'N-1)/2}}{r^{l'N-1}-1}\right\}.
\end{aligned}
$$

A direct computation with (5.4) yields

$$
\begin{aligned}
&2N\frac{-(-1)^{L_1-M_1}r^{(L_2-M_2)/2}}{r^{(L_2-M_2)N/2} - (-1)^{L_1-M_1}} \\
&= \begin{cases}
N & \text{if} \quad L_1 \neq M_1,\ L_2 = M_2, \\
2N\displaystyle\sum_{l'=1}^{\infty}\frac{(-1)^{(l'-1)(L_1-M_1)}(r^{[l'N-(N-1)]/2} - r^{-[l'N-(N-1)]/2})}{r^{l'N-(N-1)}-1} & \text{if} \quad L_2 = 0,\ M_2 = 1, \\
-2N\displaystyle\sum_{l'=1}^{\infty}\frac{(-1)^{l'(L_1-M_1)}(r^{(l'N-1)/2} - r^{-(l'N-1)/2})}{r^{l'N-1}-1} & \text{if} \quad L_2 = 1,\ M_2 = 0.
\end{cases}
\end{aligned}
$$

Combining these equalities with (8.16), we have

(8.21)

$$
\begin{aligned}
R_{0j} = \frac{2\pi}{\omega_1}\beta^2\beta_L\zeta_N^{-[2(J-1)+L_1]}\Bigg[&\beta_L\left\{(N-2) + 2N\sum_{l=1}^{\infty}\left(\frac{1}{r^{lN-(N-1)}-1} - \frac{1}{r^{lN-1}-1}\right)\right\} \\
&+ \beta_{\tau_2(L)}\left\{N + 2N\sum_{l=1}^{\infty}(-1)^{l-1}\left(\frac{1}{r^{lN-(N-1)}-1} + \frac{1}{r^{lN-1}-1}\right)\right\} \\
&-2N\beta_{\tau_3(L)}\sum_{l=1}^{\infty}\left(\frac{r^{[lN-(N-1)]/2}}{r^{lN-(N-1)}-1} - \frac{r^{(lN-1)/2}}{r^{lN-1}-1}\right) \\
&-2N\beta_{\tau_4(L)}\sum_{l=1}^{\infty}(-1)^{l-1}\left(\frac{r^{[lN-(N-1)]/2}}{r^{lN-(N-1)}-1} + \frac{r^{(lN-1)/2}}{r^{lN-1}-1}\right)\Bigg].
\end{aligned}
$$

Now, since $w_j = -p_j R_{0j}$ and $p_j = p_{J,L} = \zeta_N^{2(J-1)+L_1}$, we get $w_{J,L} = w_{1,L} = w_L$, and, in particular, $w_j \in \mathbf{R}$ ($j = 1, \dots, n$).

Now, we calculate w_L in a certain special case for later use. Assume that $\beta_L = \beta_{\tau_4(L)}$ and $\beta_{\tau_2(L)} = \beta_{\tau_3(L)}$. Then we have

(8.22)

$$
\begin{aligned}
w_L &= -p_L R_{0L} \\
&= -\frac{2\pi}{\omega_1}\beta^2\beta_L\Bigg[\beta_L\left\{(N-2)+2N\sum_{l=1}^{\infty}\left(\frac{1-(-1)^{l-1}r^{[lN-(N-1)]/2}}{r^{lN-(N-1)}-1}-\frac{1+(-1)^{l-1}r^{(lN-1)/2}}{r^{lN-1}-1}\right)\right\} \\
&\qquad + \beta_{\tau_2(L)}\left\{N+2N\sum_{l=1}^{\infty}\left(\frac{(-1)^{l-1}-r^{[lN-(N-1)]/2}}{r^{lN-(N-1)}-1}+\frac{(-1)^{l-1}+r^{(lN-1)/2}}{r^{lN-1}-1}\right)\right\}\Bigg] \\
&= -\frac{2\pi}{\omega_1}\beta^2\beta_L\Bigg[(N-2)\beta_L + N\beta_{\tau_2(L)} \\
&\qquad + 2N\sum_{l=1}^{\infty}\left\{\frac{-(-1)^{l-1}\beta_L-\beta_{\tau_2(L)}}{r^{[lN-(N-1)]/2}+(-1)^{l-1}}+\frac{-(-1)^{l-1}\beta_L+\beta_{\tau_2(L)}}{r^{(lN-1)/2}-(-1)^{l-1}}\right\}\Bigg].
\end{aligned}
$$

Calculation of R_{0j} and w_j with respect to $\tilde{r}$. Combining (8.19) with Lemma 5.6, we have

$$
\begin{aligned}
R_{0j} &= -\sum_{k=1;\ k\neq j}^{n} 2b_j b_k \chi_1(u_j - u_k) \\
&= -\sum_{k=1;\ k\neq j}^{n} 2b_j b_k \left\{\chi_2(u_j-u_k) - \frac{2\pi\sqrt{-1}}{\omega_1\omega_2}(u_j-u_k)\right\}.
\end{aligned}
\tag{8.23}
$$

Now, we compute R_{0j} with $j = 1 = 4(1-1)+1$ by using $\tilde{r}$.

$$
\begin{aligned}
R_{01} = &-\left\{\sum_{K=2}^{N} 2b_{1,1}b_{K,1}\chi_2(u_{1,1}-u_{K,1}) + \sum_{M=2}^{4}\sum_{K=1}^{N} 2b_{1,1}b_{K,M}\chi_2(u_{1,1}-u_{K,M})\right\} \\
&+ \frac{2\pi\sqrt{-1}}{\omega_2}\sum_{k=2}^{n} 2b_1 b_k \frac{u_1-u_k}{\omega_1}.
\end{aligned}
$$

By using (8.14), (8.13) and Lemma 5.6, we have

$$
\begin{aligned}
\sum_{K=2}^{N} 2b_{1,1}b_{K,1}\chi_2(u_{1,1}-u_{K,1}) &= \sum_{K=2}^{N} 2b_1^2\zeta_N^{-(1+K-2)}\chi_2\left(\frac{1-K}{N}\omega_1\right) \\
= 2b_1^2\sum_{K=2}^{N}\zeta_N^{1-K}\chi_2\left(\frac{1-K}{N}\omega_1\right) &= -2b_1^2\sum_{K'=1}^{N-1}\zeta_N^{-K'}\chi_2\left(\frac{K'}{N}\omega_1\right),
\end{aligned}
$$

and, for $M \neq 1$,

$$
\begin{aligned}
&\sum_{K=1}^{N} 2b_{1,1}b_{K,M}\chi_2(u_{1,1}-u_{K,M}) \\
&= \sum_{K=1}^{N} 2b_1 b_{1,M}\zeta_N^{-(1+K-2)}\chi_2\left(\frac{2(1-K)+(0-M_1)}{2N}\omega_1+\frac{0-M_2}{2}\omega_2\right) \\
&= 2b_1 b_M \sum_{K=1}^{N} \zeta_N^{1-K}\chi_2\left(\frac{2(1-K)-M_1}{2N}\omega_1-\frac{M_2}{2}\omega_2\right) \\
&= -2b_1 b_M \sum_{K'=0}^{N-1} \zeta_N^{-K'}\chi_2\left(\frac{2K'+M_1}{2N}\omega_1+\frac{M_2}{2}\omega_2\right).
\end{aligned}
$$

By Lemma 5.4, (5.4), (8.18) and (8.20), we have

$$
\begin{aligned}
&\frac{\omega_2}{\pi\sqrt{-1}}\sum_{K=1}^{N-1}\zeta_N^{-K}\chi_2\left(\frac{K}{N}\omega_1\right) \\
&= \sum_{K=1}^{N-1}\zeta_N^{-K}\left\{\frac{\tilde{r}^{K/N}+1}{\tilde{r}^{K/N}-1}-2\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{\tilde{r}^{lm}}(\tilde{r}^{Kl/N}-\tilde{r}^{-Kl/N})\right\} \\
&= \sum_{K=1}^{N-1}\zeta_N^{-K}\left\{\left(1+\frac{2}{\tilde{r}^{K/N}-1}\right)-2\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{1}{\tilde{r}^{(Nm-K)l/N}}-\frac{1}{\tilde{r}^{(Nm+K)l/N}}\right)\right\} \\
&= -1+2\sum_{K=1}^{N-1}\zeta_N^{-K}\sum_{l=1}^{\infty}\frac{1}{\tilde{r}^{Kl/N}}-2\sum_{K=1}^{N-1}\zeta_N^{-K}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{1}{\tilde{r}^{(Nm-K)l/N}}-\frac{1}{\tilde{r}^{(Nm+K)l/N}}\right) \\
&= -1+2\sum_{K'=1;\ K'\not\equiv 0 \bmod N}^{\infty}\zeta_N^{-K'}\sum_{l=1}^{\infty}\frac{1}{\tilde{r}^{K'l/N}}-2\sum_{m=1}^{\infty}\sum_{K=1}^{N-1}\zeta_N^{N-K}\sum_{l=1}^{\infty}\frac{1}{\tilde{r}^{[N(m-1)+(N-K)]l/N}} \\
&= -1+2\sum_{K''=1}^{\infty}(\zeta_N^{-K''}-\zeta_N^{K''})\sum_{l=1}^{\infty}\frac{1}{\tilde{r}^{K''l/N}} \\
&= -1-4\sqrt{-1}\sum_{K''=1}^{\infty}\sin\frac{2K''\pi}{N}\sum_{l=1}^{\infty}\frac{1}{\tilde{r}^{K''l/N}}.
\end{aligned}
$$

In the same way, we also have, for $M \neq 1$,

$$
\begin{aligned}
&\frac{\omega_2}{\pi\sqrt{-1}}\sum_{K=0}^{N-1}\zeta_N^{-K}\chi_2\left(\frac{2K+M_1}{2N}\omega_1+\frac{M_2}{2}\omega_2\right) \\
&= \sum_{K=0}^{N-1}\zeta_N^{-K}\left[\frac{(-1)^{M_2}\tilde{r}^{(2K+M_1)/(2N)}+1}{(-1)^{M_2}\tilde{r}^{(2K+M_1)/(2N)}-1}\right. \\
&\qquad\left.-2\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{\tilde{r}^{lm}}\{(-1)^{M_2 l}\tilde{r}^{(2K+M_1)l/(2N)}-(-1)^{-M_2 l}\tilde{r}^{-(2K+M_1)l/(2N)}\}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{K=0}^{N-1} \zeta_N^{-K} \left\{ \left(1 + \frac{2}{(-1)^{M_2} \tilde{r}^{(2K+M_1)/(2N)} - 1} \right) \right. \\
&\qquad \left. - 2 \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{(-1)^{M_2 l}}{\tilde{r}^{[2Nm-(2K+M_1)]l/(2N)}} - \frac{(-1)^{M_2 l}}{\tilde{r}^{[2Nm+(2K+M_1)]l/(2N)}} \right) \right\} \\
&= 0 - (1 - M_1) + 2 \sum_{K=1-M_1}^{N-1} \zeta_N^{-K} \sum_{l=1}^{\infty} \frac{(-1)^{M_2 l}}{\tilde{r}^{(2K+M_1)l/(2N)}} \\
&\quad - 2 \sum_{K=0}^{N-1} \zeta_N^{-K} \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{(-1)^{M_2 l}}{\tilde{r}^{[2(Nm-K)-M_1]l/(2N)}} - \frac{(-1)^{M_2 l}}{\tilde{r}^{[2(Nm+K+M_1)-M_1]l/(2N)}} \right) \\
&= -(1 - M_1) + 2 \sum_{K'=1-M_1}^{\infty} \zeta_N^{-K'} \sum_{l=1}^{\infty} \frac{(-1)^{M_2 l}}{\tilde{r}^{[2(K'+M_1)-M_1]l/(2N)}} \\
&\quad - 2 \sum_{m=1}^{\infty} \sum_{K=0}^{N-1} \zeta_N^{N-K} \sum_{l=1}^{\infty} \frac{(-1)^{M_2 l}}{\tilde{r}^{[2\{N(m-1)+(N-K)\}-M_1]l/(2N)}} \\
&= -(1 - M_1) + 2 \sum_{K''=1}^{\infty} (\zeta_N^{M_1-K''} - \zeta_N^{K''}) \sum_{l=1}^{\infty} \frac{(-1)^{M_2 l}}{\tilde{r}^{(2K''-M_1)l/(2N)}} \\
&= -(1 - M_1) - 4\sqrt{-1}\zeta_{2N}^{M_1} \sum_{K''=1}^{\infty} \sin \frac{(2K'' - M_1)\pi}{N} \sum_{l=1}^{\infty} \frac{(-1)^{M_2 l}}{\tilde{r}^{(2K''-M_1)l/(2N)}}.
\end{aligned}
$$

By using $u_1 = 0$, $b_1 = e^{\pi\sqrt{-1}/4}\beta\beta_1$ and (8.10), we also have

$$
\begin{aligned}
\sum_{k=2}^{n} 2b_1 b_k \frac{u_1 - u_k}{\omega_1} &= -\frac{2b_1}{\omega_1} \sum_{k=2}^{n} b_k u_k = -\frac{2b_1}{\omega_1} \sum_{k=1}^{n} b_k u_k \\
&= -2\sqrt{-1}\beta^2\beta_1 \frac{\zeta_N}{1 - \zeta_N} \{(\beta_1 - \beta_3) + \zeta_{2N}^{-1}(\beta_2 - \beta_4)\} \\
&= -2\sqrt{-1}\beta^2\beta_1 \left\{ \frac{\zeta_N}{1 - \zeta_N}(\beta_1 - \beta_3) + \frac{\zeta_{2N}}{1 - \zeta_N}(\beta_2 - \beta_4) \right\} \\
&= \beta^2\beta_1 \left\{ \left(\sqrt{-1} + \frac{\cos(\pi/N)}{\sin(\pi/N)} \right)(\beta_1 - \beta_3) + \frac{1}{\sin(\pi/N)}(\beta_2 - \beta_4) \right\}.
\end{aligned}
$$

Combining these equalities with (8.16), we have

$$
\begin{aligned}
-w_1 &= R_{01} \\
&= \frac{2\pi\sqrt{-1}}{\omega_2}\beta^2\beta_1 \left[\frac{1}{\sin(\pi/N)} \left\{ (\beta_1 - \beta_3)\cos\frac{\pi}{N} + (\beta_2 - \beta_4) \right\} \right. \\
&\qquad + 4 \sum_{l=1}^{\infty} \{\beta_1 + (-1)^{l-1}\beta_3\} \sum_{K=1}^{\infty} \frac{1}{\tilde{r}^{Kl/N}} \sin\frac{2K\pi}{N} \\
&\qquad \left. + 4 \sum_{l=1}^{\infty} \{\beta_2 + (-1)^{l-1}\beta_4\} \sum_{K=1}^{\infty} \frac{1}{\tilde{r}^{(2K-1)l/(2N)}} \sin\frac{(2K-1)\pi}{N} \right].
\end{aligned}
$$

Changing the coordinate by $\tilde{u} := u - u_{1,L}$, we have a similar equality for each $w_{1,L}$, and we get

(8.24)

$$
\begin{aligned}
w_{J,L} &= w_{1,L} = w_L \\
&= \frac{2\pi\sqrt{-1}}{\omega_2}\beta^2\beta_L\Bigg[-\frac{1}{\sin(\pi/N)}\left\{(\beta_L - \beta_{\tau_3(L)})\cos\frac{\pi}{N} + (\beta_{\tau_2(L)} - \beta_{\tau_4(L)})\right\} \\
&\qquad - 4\sum_{l=1}^{\infty}\{\beta_L + (-1)^{l-1}\beta_{\tau_3(L)}\}\sum_{K=1}^{\infty}\frac{1}{\tilde{r}^{Kl/N}}\sin\frac{2K\pi}{N} \\
&\qquad - 4\sum_{l=1}^{\infty}\{\beta_{\tau_2(L)} + (-1)^{l-1}\beta_{\tau_4(L)}\}\sum_{K=1}^{\infty}\frac{1}{\tilde{r}^{(2K-1)l/(2N)}}\sin\frac{(2K-1)\pi}{N}\Bigg].
\end{aligned}
$$

Now, we note some estimates of w_L which we use later. For any C such that $|C| > 1$, we have

$$
\sum_{K=1}^{\infty}\frac{1}{C^K}\sin Kx = \frac{C\sin x}{1 - 2C\cos x + C^2}.
$$

Assume that $\beta_L = \beta_{\tau_4(L)}$ and $\beta_{\tau_2(L)} = \beta_{\tau_3(L)}$. Then we have

(8.25)

$$
\begin{aligned}
w_L &= \frac{2\pi\sqrt{-1}}{\omega_2}\beta^2\beta_L\Bigg[-(\beta_L - \beta_{\tau_2(L)})\frac{\cos(\pi/N) - 1}{\sin(\pi/N)} \\
&\qquad - 4\sum_{l'=1}^{\infty}(\beta_L + \beta_{\tau_2(L)})\sum_{K'=1}^{\infty}\frac{1}{(\tilde{r}^{(2l'-1)/(2N)})^{K'}}\sin\left(K'\cdot\frac{\pi}{N}\right) \\
&\qquad - 4\sum_{l'=1}^{\infty}(\beta_L - \beta_{\tau_2(L)})\sum_{K'=1}^{\infty}\frac{1}{(-\tilde{r}^{2l'/(2N)})^{K'}}\sin\left(K'\cdot\frac{\pi}{N}\right)\Bigg] \\
&= \frac{2\pi\sqrt{-1}}{\omega_2}\beta^2\beta_L\Bigg[\frac{\sin(\pi/N)}{1 + \cos(\pi/N)}(\beta_L - \beta_{\tau_2(L)}) \\
&\qquad - 4\sum_{l=1}^{\infty}(\beta_L + \beta_{\tau_2(L)})\frac{\tilde{r}^{(2l-1)/(2N)}\sin(\pi/N)}{1 - 2\tilde{r}^{(2l-1)/(2N)}\cos(\pi/N) + \tilde{r}^{(2l-1)/N}} \\
&\qquad + 4\sum_{l=1}^{\infty}(\beta_L - \beta_{\tau_2(L)})\frac{\tilde{r}^{l/N}\sin(\pi/N)}{1 + 2\tilde{r}^{l/N}\cos(\pi/N) + \tilde{r}^{2l/N}}\Bigg].
\end{aligned}
$$

Hence we see that, if $0 < \beta_L \le \beta_{\tau_2(L)}$ or $\beta_{\tau_2(L)} \le \beta_L < 0$ (resp. $0 < \beta_L \le -\beta_{\tau_2(L)}$ or $-\beta_{\tau_2(L)} \le \beta_L < 0$), then $w_L < 0$ (resp. $w_L > 0$).

Calculation of $R_0(\gamma_1)$. First, we note here that $\sum_{J=1}^{N} R_{0\,4(J-1)+L} = 0$ $(L = 3, 4)$. For the loop γ_1, by using (4.4), Lemma 3.4, Lemma 5.5, (8.12), (8.14), (8.13), Lemma 5.1 and (8.17), we have

$$
\begin{aligned}
-\pi\sqrt{-1}P_1 &= -2\pi\sqrt{-1}R_0(\gamma_1)\\
&= -\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1;k\neq j}^{n} b_j b_k \xi_1(u_j-u_k) + \left(b_0-\eta_1\sum_{j=1}^{n} b_j\frac{u_j}{\omega_1}\right)^2\\
&= -\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} b_j b_k \xi_1(u_j-u_k)\\
&= -\frac{1}{2}\sum_{J=1}^{N}\sum_{L=1}^{4}\sum_{K=1}^{N}\sum_{M=1}^{4} b_{J,L}b_{K,M}\xi_1(u_{J,L}-u_{K,M})\\
&= \sum_{L=1}^{4}\sum_{M=1}^{4} b_{1,L}b_{1,M}\left(-\frac{1}{2}\right)\\
&\qquad \times\sum_{J=1}^{N}\sum_{K=1}^{N}\zeta_N^{-(J+K-2)}\xi_1\left(\frac{2(J-K)+(L_1-M_1)}{2N}\omega_1+\frac{L_2-M_2}{2}\omega_2\right),
\end{aligned}
$$

and

$$
\begin{aligned}
&-\frac{1}{2}\sum_{J=1}^{N}\sum_{K=1}^{N}\zeta_N^{-(J+K-2)}\xi_1\left(\frac{2(J-K)+(L_1-M_1)}{2N}\omega_1+\frac{L_2-M_2}{2}\omega_2\right)\\
&= 4\left(\frac{\pi}{\omega_1}\right)^2\sum_{J=1}^{N}\sum_{K=1}^{N}\zeta_N^{-(J+K-2)}\sum_{l=1}^{\infty}\frac{r^l}{(r^l-1)^2}\big(\zeta_{2N}^{[2(J-K)+(L_1-M_1)]l}r^{-(L_2-M_2)l/2}\\
&\qquad\qquad + \zeta_{2N}^{-[2(J-K)+(L_1-M_1)]l}r^{(L_2-M_2)l/2}\big)\\
&= 4\left(\frac{\pi}{\omega_1}\right)^2\sum_{J=1}^{N}\zeta_N^{-2J+2}\sum_{K'=J-N}^{J-1}\zeta_N^{K'}\sum_{l=1}^{\infty}\frac{r^l}{(r^l-1)^2}\big(\zeta_{2N}^{[2K'+(L_1-M_1)]l}r^{-(L_2-M_2)l/2}\\
&\qquad\qquad + \zeta_{2N}^{-[2K'+(L_1-M_1)]l}r^{(L_2-M_2)l/2}\big)\\
&= 4\left(\frac{\pi}{\omega_1}\right)^2\sum_{J=1}^{N}\zeta_N^{-2J+2}\sum_{K'=1}^{N}\zeta_N^{K'}\sum_{l=1}^{\infty}\frac{r^l}{(r^l-1)^2}\big(\zeta_{2N}^{[2K'+(L_1-M_1)]l}r^{-(L_2-M_2)l/2}\\
&\qquad\qquad + \zeta_{2N}^{-[2K'+(L_1-M_1)]l}r^{(L_2-M_2)l/2}\big)\\
&= 0.
\end{aligned}
$$

Hence we get $R_0(\gamma_1)=0$.

Calculation of $R_1(\gamma_2)$ with respect to r. First, recall here that $R_{1j} = 0$ ($j = 1, \dots, n$). For the loop γ_2, by using Lemmas 3.5, 3.4 and (8.12), we have

$$\begin{aligned} R_1(\gamma_2) &= \frac{\omega_2}{\omega_1} R_1(\gamma_1) - \frac{1}{\omega_1} \sum_{j=1}^{n} c_j b_j \\ &= \frac{-\omega_2}{2\pi\sqrt{-1}} \left\{ -\frac{1}{4} \sum_{j=1}^{n} \sum_{k=1;\, k \neq j}^{n} (c_j b_k + b_j c_k) \xi_1(u_j - u_k) \right\} - \frac{1}{\omega_1} \sum_{j=1}^{n} c_j b_j . \end{aligned}$$

By Lemma 5.5, (8.17), (8.15), (8.13), Lemma 5.1 and (8.20), we have

(8.26)

$$\begin{aligned} &-\frac{1}{4} \sum_{j=1}^{n} \sum_{k=1;k\neq j}^{n} (c_j b_k + b_j c_k) \xi_1(u_j - u_k) = -\frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} c_j b_k \xi_1(u_j - u_k) \\ &= -\frac{1}{2} \sum_{J=1}^{N} \sum_{L=1}^{4} \sum_{K=1}^{N} \sum_{M=1}^{4} c_{J,L} b_{K,M} \xi_1(u_{J,L} - u_{K,M}) \\ &= \sum_{L=1}^{4} \sum_{M=1}^{4} (-1)^{L_2+M_2} \sqrt{-1} \beta^2 \beta_L \beta_M \\ &\qquad \times \left(-\frac{1}{2}\right) \sum_{J=1}^{N} \sum_{K=1}^{N} \zeta_{2N}^{2(J-K)+(L_1-M_1)} \xi_1 \left(\frac{2(J-K)+(L_1-M_1)}{2N} \omega_1 + \frac{L_2 - M_2}{2} \omega_2 \right), \end{aligned}$$

and

$$\begin{aligned} &-\frac{1}{2} \sum_{J=1}^{N} \sum_{K=1}^{N} \zeta_{2N}^{2(J-K)+(L_1-M_1)} \xi_1 \left(\frac{2(J-K)+(L_1-M_1)}{2N} \omega_1 + \frac{L_2 - M_2}{2} \omega_2 \right) \\ &= 4 \left(\frac{\pi}{\omega_1} \right)^2 \sum_{J=1}^{N} \sum_{K=1}^{N} \zeta_{2N}^{2(J-K)+(L_1-M_1)} \sum_{l=1}^{\infty} \frac{r^l}{(r^l - 1)^2} \Big(\zeta_{2N}^{[2(J-K)+(L_1-M_1)]l} r^{-(L_2-M_2)l/2} \\ &\qquad\qquad + \zeta_{2N}^{-[2(J-K)+(L_1-M_1)]l} r^{(L_2-M_2)l/2} \Big) \\ &= 4 \left(\frac{\pi}{\omega_1} \right)^2 \sum_{J=1}^{N} \sum_{K'=J-N}^{J-1} \zeta_{2N}^{2K'+(L_1-M_1)} \sum_{l=1}^{\infty} \frac{r^l}{(r^l - 1)^2} \Big(\zeta_{2N}^{[2K'+(L_1-M_1)]l} r^{-(L_2-M_2)l/2} \\ &\qquad\qquad + \zeta_{2N}^{-[2K'+(L_1-M_1)]l} r^{(L_2-M_2)l/2} \Big) \\ &= 4N \left(\frac{\pi}{\omega_1} \right)^2 \sum_{l=1}^{\infty} \frac{r^l}{(r^l - 1)^2} \left(\sum_{K'=1}^{N} \zeta_{2N}^{[2K'+(L_1-M_1)](l+1)} r^{-(L_2-M_2)l/2} \right. \\ &\qquad\qquad \left. + \sum_{K'=1}^{N} \zeta_{2N}^{-[2K'+(L_1-M_1)](l-1)} r^{(L_2-M_2)l/2} \right) \end{aligned}$$

$$
=4N^2\left(\frac{\pi}{\omega_1}\right)^2\sum_{l'=1}^{\infty}\left\{\frac{(-1)^{(l'-1)(L_1-M_1)}r^{[l'N-(N-1)][1+(L_2-M_2)/2]}}{(r^{l'N-(N-1)}-1)^2}\right.
\left.+\frac{(-1)^{l'(L_1-M_1)}r^{(l'N-1)[1-(L_2-M_2)/2]}}{(r^{l'N-1}-1)^2}\right\}.
$$

We also note that $\log r = 2\pi\omega_2/(\sqrt{-1}\omega_1)$. Applying this equality, (8.7) and (8.8) to the above, we get

(8.27)

$$
\begin{aligned}
&R_1(\gamma_2)\\
&=-\frac{\beta^2N\sqrt{-1}}{\omega_1}\Big[(\beta_1^2+\beta_2^2+\beta_3^2+\beta_4^2)\\
&\qquad+N\log r\Big\{\frac{1}{2}(\beta_1^2+\beta_2^2+\beta_3^2+\beta_4^2)k_1\\
&\qquad\qquad+(\beta_1\beta_2+\beta_3\beta_4)k_2-(\beta_1\beta_3+\beta_2\beta_4)k_3-(\beta_1\beta_4+\beta_2\beta_3)k_4\Big\}\Big],
\end{aligned}
$$

where we set

$$
(8.28)\quad
\begin{cases}
k_1 := 2\displaystyle\sum_{l=1}^{\infty}\left\{\frac{r^{lN-(N-1)}}{(r^{lN-(N-1)}-1)^2}+\frac{r^{lN-1}}{(r^{lN-1}-1)^2}\right\},\\
k_2 := 2\displaystyle\sum_{l=1}^{\infty}\left\{\frac{(-1)^{l-1}r^{lN-(N-1)}}{(r^{lN-(N-1)}-1)^2}+\frac{(-1)^{l}r^{lN-1}}{(r^{lN-1}-1)^2}\right\},\\
k_3 := \displaystyle\sum_{l=1}^{\infty}\left\{\frac{r^{[lN-(N-1)]/2}(r^{lN-(N-1)}+1)}{(r^{lN-(N-1)}-1)^2}+\frac{r^{(lN-1)/2}(r^{lN-1}+1)}{(r^{lN-1}-1)^2}\right\},\\
k_4 := \displaystyle\sum_{l=1}^{\infty}\left\{\frac{(-1)^{l-1}r^{[lN-(N-1)]/2}(r^{lN-(N-1)}+1)}{(r^{lN-(N-1)}-1)^2}+\frac{(-1)^{l}r^{(lN-1)/2}(r^{lN-1}+1)}{(r^{lN-1}-1)^2}\right\}.
\end{cases}
$$

Now, we note some estimates of k_i which we use later. Since

$$
(8.29)\quad
\begin{aligned}
k_2-k_3 &= -\sum_{l=1}^{\infty}\left[\frac{r^{[lN-(N-1)]/2}\{r^{[lN-(N-1)]/2}+(-1)^l\}^2}{(r^{lN-(N-1)}-1)^2}\right.\\
&\qquad\left.+\frac{r^{(lN-1)/2}\{r^{(lN-1)/2}+(-1)^{l-1}\}^2}{(r^{lN-1}-1)^2}\right]\\
&= -\sum_{l=1}^{\infty}\left[\frac{r^{[lN-(N-1)]/2}}{\{r^{[lN-(N-1)]/2}+(-1)^{l-1}\}^2}+\frac{r^{(lN-1)/2}}{\{r^{(lN-1)/2}+(-1)^{l}\}^2}\right]<0,
\end{aligned}
$$

and

$$
(8.30)\quad
\begin{aligned}
k_1 - k_4 = & -\sum_{l=1}^{\infty}\Bigg[\frac{(-1)^{l-1} r^{[lN-(N-1)]/2}\{r^{[lN-(N-1)]/2}+(-1)^l\}^2}{(r^{lN-(N-1)}-1)^2} \\
& \qquad + \frac{(-1)^l r^{(lN-1)/2}\{r^{(lN-1)/2}+(-1)^{l-1}\}^2}{(r^{lN-1}-1)^2}\Bigg] \\
= & \sum_{l=1}^{\infty}\Bigg[(-1)^l \frac{r^{[lN-(N-1)]/2}}{\{r^{[lN-(N-1)]/2}+(-1)^{l-1}\}^2} + (-1)^{l-1}\frac{r^{(lN-1)/2}}{\{r^{(lN-1)/2}+(-1)^l\}^2}\Bigg],
\end{aligned}
$$

we have

$$
(8.31)\quad k_1 + k_2 - k_3 - k_4 = -2\sum_{l=1}^{\infty}\left\{\frac{r^{[2lN-(2N-1)]/2}}{(r^{[2lN-(2N-1)]/2}+1)^2} + \frac{r^{(2lN-1)/2}}{(r^{(2lN-1)/2}+1)^2}\right\} < 0.
$$

We also use

$$
(8.32)\quad
\begin{aligned}
k_1 - k_3 &= -\sum_{l=1}^{\infty}\left\{\frac{r^{[lN-(N-1)]/2}(r^{[lN-(N-1)]/2}-1)^2}{(r^{lN-(N-1)}-1)^2} + \frac{r^{(lN-1)/2}(r^{(lN-1)/2}-1)^2}{(r^{lN-1}-1)^2}\right\} \\
&= -\sum_{l=1}^{\infty}\left\{\frac{r^{[lN-(N-1)]/2}}{(r^{[lN-(N-1)]/2}+1)^2} + \frac{r^{(lN-1)/2}}{(r^{(lN-1)/2}+1)^2}\right\} < 0.
\end{aligned}
$$

Calculation of $R_1(\gamma_2)$ with respect to $\tilde{r}$. By Lemma 3.4, we have

$$
\begin{aligned}
R_1(\gamma_2) = \frac{-\omega_2}{2\pi\sqrt{-1}}\Bigg\{ & -\frac{1}{4}\sum_{j=1}^{n}\sum_{k=1;\ k\neq j}^{n}(c_j b_k + b_j c_k)\xi_2(u_j - u_k) \\
& + \left(c_0 - \eta_2\sum_{j=1}^{n} c_j\frac{u_j}{\omega_2}\right)\left(b_0 - \eta_2\sum_{j=1}^{n} b_j\frac{u_j}{\omega_2}\right)\Bigg\}.
\end{aligned}
$$

As (8.26), by Lemma 5.5, (8.18), (8.15), (8.13), Lemma 5.3 and (5.4), we have

$$
\begin{aligned}
-\frac{1}{4}\sum_{j=1}^{n}\sum_{k=1;k\neq j}^{n}(c_j b_k + b_j c_k)\xi_2(u_j - u_k) &= -\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} c_j b_k \xi_2(u_j - u_k) \\
= -\frac{1}{2}\sum_{J=1}^{N}\sum_{L=1}^{4}\sum_{K=1}^{N}\sum_{M=1}^{4} c_{J,L} b_{K,M}\xi_2(u_{J,L} - u_{K,M}) & \\
= \sum_{L=1}^{4}\sum_{M=1}^{4}(-1)^{L_2+M_2}\sqrt{-1}\beta^2\beta_L\beta_M & \\
\times\left(-\frac{1}{2}\right)\sum_{J=1}^{N}\sum_{K=1}^{N}\zeta_{2N}^{2(J-K)+(L_1-M_1)}\xi_2\left(\frac{2(J-K)+(L_1-M_1)}{2N}\omega_1 + \frac{L_2 - M_2}{2}\omega_2\right), &
\end{aligned}
$$

and

$$
\begin{aligned}
&-\frac{1}{2}\sum_{J=1}^{N}\sum_{K=1}^{N}\zeta_{2N}^{2(J-K)+(L_1-M_1)}\xi_2\left(\frac{2(J-K)+(L_1-M_1)}{2N}\omega_1+\frac{L_2-M_2}{2}\omega_2\right)\\
&=4\left(\frac{\pi}{\omega_2}\right)^2\sum_{J=1}^{N}\sum_{K=1}^{N}\zeta_{2N}^{2(J-K)+(L_1-M_1)}\\
&\qquad\qquad\times\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\frac{m}{\tilde{r}^{lm}}\left\{\tilde{r}^{[2(J-K)+(L_1-M_1)]l/(2N)}(-1)^{(L_2-M_2)l}\right.\\
&\qquad\qquad\qquad\left.+\tilde{r}^{-[2(J-K)+(L_1-M_1)]l/(2N)}(-1)^{-(L_2-M_2)l}\right\}\\
&=4\left(\frac{\pi}{\omega_2}\right)^2\sum_{l=1}^{\infty}(-1)^{(L_2-M_2)l}\sum_{m=1}^{\infty}\frac{m}{\tilde{r}^{lm}}\\
&\qquad\qquad\times\sum_{J=1}^{N}\sum_{K=1}^{N}\zeta_{2N}^{2(J-K)+(L_1-M_1)}\left(\tilde{r}^{[2(J-K)+(L_1-M_1)]l/(2N)}\right.\\
&\qquad\qquad\qquad\left.+\tilde{r}^{-[2(J-K)+(L_1-M_1)]l/(2N)}\right)\\
&=4\left(\frac{\pi}{\omega_2}\right)^2\sum_{l=1}^{\infty}(-1)^{(L_2-M_2)l}\sum_{m=1}^{\infty}\sum_{K'=-(N-1)}^{N-1}m(N-|K'|)\zeta_{2N}^{2K'+(L_1-M_1)}\\
&\qquad\qquad\times\left(\tilde{r}^{-[2mN-(2K'+L_1-M_1)]l/(2N)}\right.\\
&\qquad\qquad\qquad\left.+\tilde{r}^{-[2mN+(2K'+L_1-M_1)]l/(2N)}\right).
\end{aligned}
$$

Set $K'':=-K'+M_1$ ($K'=-(N-1),\dots,M_1-1$), and set $K''':=N-K'+M_1$ ($K'=M_1,\dots,N-1$). Then we get

$$
\begin{aligned}
&\sum_{m=1}^{\infty}\sum_{K'=-(N-1)}^{N-1}m(N-|K'|)\zeta_{2N}^{2K'+L_1-M_1}\tilde{r}^{-[2mN-(2K'+L_1-M_1)]l/(2N)}\\
&=\sum_{m=1}^{\infty}\sum_{K''=1}^{N-1+M_1}m(N-K''+M_1)\zeta_{2N}^{-(2K''-L_1-M_1)}\tilde{r}^{-[2mN+(2K''-L_1-M_1)]l/(2N)}\\
&\quad+\sum_{m=1}^{\infty}\sum_{K'''=1+M_1}^{N}m(K'''-M_1)\zeta_{2N}^{2N-(2K'''-L_1-M_1)}\tilde{r}^{-[2(m-1)N+(2K'''-L_1-M_1)]l/(2N)}\\
&=\sum_{m=0}^{\infty}\sum_{K''=1}^{N+M_1}m(N-K''+M_1)\zeta_{2N}^{-(2K''-L_1-M_1)}\tilde{r}^{-[2mN+(2K''-L_1-M_1)]l/(2N)}\\
&\quad+\sum_{m'=0}^{\infty}\sum_{K'''=M_1}^{N}(m'+1)(K'''-M_1)\zeta_{2N}^{-(2K'''-L_1-M_1)}\tilde{r}^{-[2m'N+(2K'''-L_1-M_1)]l/(2N)}\\
&=\sum_{m=0}^{\infty}\sum_{K=1}^{N}(mN+K-M_1)\zeta_{2N}^{-(2K-L_1-M_1)}\tilde{r}^{-[2mN+(2K-L_1-M_1)]l/(2N)}.
\end{aligned}
$$

We note here that, if $L_1 + M_1 = 2$, then $L_1 = M_1 = 1$, and hence $mN + K - M_1 = 0$ for $m = 0$ and $K = 1$. Set $K'' := N + K' + L_1$ ($K' = -(N-1), \dots, M_1 - 1$), and set $K''' := K' + L_1$ ($K' = M_1, \dots, N-1$). Then we get

$$\begin{aligned}
&\sum_{m=1}^{\infty}\sum_{K'=-(N-1)}^{N-1} m(N-|K'|)\zeta_{2N}^{2K'+L_1-M_1}\tilde{r}^{-[2mN+(2K'+L_1-M_1)]l/(2N)}\\
&=\sum_{m=1}^{\infty}\sum_{K''=1+L_1}^{N-1+L_1+M_1} m(K''-L_1)\zeta_{2N}^{-2N+(2K''-L_1-M_1)}\tilde{r}^{-[2(m-1)N+(2K''-L_1-M_1)]l/(2N)}\\
&\quad+\sum_{m=1}^{\infty}\sum_{K'''=L_1+M_1}^{N-1+L_1} m(N-K'''+L_1)\zeta_{2N}^{2K'''-L_1-M_1}\tilde{r}^{-[2mN+(2K'''-L_1-M_1)]l/(2N)}\\
&=\sum_{m'=0}^{\infty}\sum_{K''=L_1}^{N-1+L_1+M_1} (m'+1)(K''-L_1)\zeta_{2N}^{2K''-L_1-M_1}\tilde{r}^{-[2m'N+(2K''-L_1-M_1)]l/(2N)}\\
&\quad+\sum_{m=0}^{\infty}\sum_{K'''=L_1+M_1}^{N+L_1} m(N-K'''+L_1)\zeta_{2N}^{2K'''-L_1-M_1}\tilde{r}^{-[2mN+(2K'''-L_1-M_1)]l/(2N)}\\
&=\sum_{m=0}^{\infty}\sum_{K=L_1+M_1}^{N-1+L_1+M_1} (mN+K-L_1)\zeta_{2N}^{2K-L_1-M_1}\tilde{r}^{-[2mN+(2K-L_1-M_1)]l/(2N)}.
\end{aligned}$$

We note here that, if $L_1 + M_1 = 0$, then $L_1 = M_1 = 0$, and hence $mN + K - L_1 = 0$ for $m = 0$ and $K = 0$. Combining these equalities, we have

$$\begin{aligned}
&-\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} c_j b_k \xi_2(u_j-u_k)\\
&=\sum_{L=1}^{4}\sum_{M=1}^{4}(-1)^{L_2+M_2}\sqrt{-1}\beta^2\beta_L\beta_M\\
&\qquad\times 4\left(\frac{\pi}{\omega_2}\right)^2\sum_{l=1}^{\infty}(-1)^{(L_2-M_2)l}\sum_{m=0}^{\infty}\sum_{K=1+\epsilon_1}^{N+\epsilon_1}\Big\{(mN+K-L_1)\zeta_{2N}^{2K-L_1-M_1}\\
&\qquad\qquad+(mN+K-M_1)\zeta_{2N}^{-(2K-L_1-M_1)}\Big\}\\
&\qquad\qquad\times\tilde{r}^{-[2mN+(2K-L_1-M_1)]l/(2N)}\\
&=4\sqrt{-1}\left(\frac{\pi}{\omega_2}\right)^2\beta^2\\
&\quad\times\sum_{L=1}^{4}\sum_{M=1}^{4}\beta_L\beta_M\sum_{l=1}^{\infty}(-1)^{(L_2-M_2)(l-1)}\\
&\qquad\times\sum_{m=0}^{\infty}\sum_{K=1+\epsilon_1}^{N+\epsilon_1}\frac{1}{2}(2mN+2K-L_1-M_1)\\
&\qquad\qquad\times(\zeta_{2N}^{2K-L_1-M_1}+\zeta_{2N}^{-(2K-L_1-M_1)})\\
&\qquad\qquad\times\tilde{r}^{-[2mN+(2K-L_1-M_1)]l/(2N)}
\end{aligned}$$

$$= \frac{2\pi\sqrt{-1}}{-\omega_2} \cdot \frac{-2\pi\beta^2}{\omega_2}$$
$$\times \sum_{L=1}^{4} \sum_{M=1}^{4} \beta_L \beta_M \sum_{l=1}^{\infty} (-1)^{(L_2-M_2)(l-1)}$$
$$\times \sum_{K=1+\epsilon_1}^{\infty} (2K - L_1 - M_1) \cos \frac{(2K - L_1 - M_1)\pi}{2N} \tilde{r}^{-[2K-L_1-M_1]l/(2N)},$$

where we set $\epsilon_1 := \max\{L_1 + M_1 - 1, 0\}$. By using (8.12) and the Legendre relation (3.6), we have

$$b_0 - \eta_2 \sum_{j=1}^{n} b_j \frac{u_j}{\omega_2} = \frac{\eta_1\omega_2 - \eta_2\omega_1}{\omega_1\omega_2} \sum_{j=1}^{n} b_j u_j = \frac{2\pi\sqrt{-1}}{\omega_1\omega_2} \sum_{j=1}^{n} b_j u_j,$$
$$c_0 - \eta_2 \sum_{j=1}^{n} c_j \frac{u_j}{\omega_2} = \frac{\eta_1\omega_2 - \eta_2\omega_1}{\omega_1\omega_2} \sum_{j=1}^{n} c_j u_j = \frac{2\pi\sqrt{-1}}{\omega_1\omega_2} \sum_{j=1}^{n} c_j u_j.$$

Therefore, by using (8.10) and (8.11), we have

$$\left(\frac{\omega_2}{2\pi\sqrt{-1}}\right)^2 \frac{1}{\sqrt{-1}} \left(c_0 - \eta_2 \sum_{j=1}^{n} c_j \frac{u_j}{\omega_2}\right)\left(b_0 - \eta_2 \sum_{j=1}^{n} b_j \frac{u_j}{\omega_2}\right)$$
$$= \left|\frac{1}{\omega_1} \sum_{j=1}^{n} b_j u_j\right|^2 = \left|\frac{\zeta_N}{1-\zeta_N}\left(\sum_{L=1}^{4} b_L\right)\right|^2$$
$$= \frac{1}{|\zeta_{2N}^{-1} - \zeta_{2N}|^2} \beta^2 \{(\beta_1 - \beta_3) + \zeta_{2N}^{-1}(\beta_2 - \beta_4)\}\{(\beta_1 - \beta_3) + \zeta_{2N}(\beta_2 - \beta_4)\}$$
$$= \frac{1}{4\sin^2(\pi/N)} \beta^2 \left\{(\beta_1 - \beta_3)^2 + 2(\beta_1 - \beta_3)(\beta_2 - \beta_4) \cos \frac{\pi}{N} + (\beta_2 - \beta_4)^2\right\}.$$

Combining these equalities, we have

(8.33)
$$R_1(\gamma_2) = \frac{2\pi\beta^2}{\omega_2}\left[\frac{1}{4\sin^2(\pi/N)}\left\{(\beta_1 - \beta_3)^2 + 2(\beta_1 - \beta_3)(\beta_2 - \beta_4)\cos\frac{\pi}{N} + (\beta_2 - \beta_4)^2\right\}\right.$$
$$- (\beta_1^2 + \beta_2^2 + \beta_3^2 + \beta_4^2)\tilde{k}_1 - 2(\beta_1\beta_2 + \beta_3\beta_4)\tilde{k}_2$$
$$\left. - 2(\beta_1\beta_3 + \beta_2\beta_4)\tilde{k}_3 - 2(\beta_1\beta_4 + \beta_2\beta_3)\tilde{k}_4\right],$$

where we set

$$\begin{cases} \tilde{k}_1 := \displaystyle\sum_{l=1}^{\infty}\sum_{K=1}^{\infty} \frac{2K}{\tilde{r}^{Kl/N}} \cos\frac{2K\pi}{N}, \\ \tilde{k}_2 := \displaystyle\sum_{l=1}^{\infty}\sum_{K=1}^{\infty} \frac{2K-1}{\tilde{r}^{(2K-1)l/(2N)}} \cos\frac{(2K-1)\pi}{N}, \\ \tilde{k}_3 := \displaystyle\sum_{l=1}^{\infty}(-1)^{l-1}\sum_{K=1}^{\infty} \frac{2K}{\tilde{r}^{Kl/N}} \cos\frac{2K\pi}{N}, \\ \tilde{k}_4 := \displaystyle\sum_{l=1}^{\infty}(-1)^{l-1}\sum_{K=1}^{\infty} \frac{2K-1}{\tilde{r}^{(2K-1)l/(2N)}} \cos\frac{(2K-1)\pi}{N}. \end{cases} \tag{8.34}$$

Now, we note some estimates of $\tilde{k}_i$ which we use later. For any C such that $|C| > 1$, we have

$$\sum_{K=1}^{\infty} \frac{K}{C^K} \cos Kx = \frac{C(\cos x - 2C + C^2 \cos x)}{(1 - 2C \cos x + C^2)^2}.$$

Hence we have

$$\begin{aligned} & -\tilde{k}_1 + \tilde{k}_2 + \tilde{k}_3 - \tilde{k}_4 \\ &= \sum_{l=1}^{\infty}\{1-(-1)^{l-1}\}\sum_{K=1}^{\infty} \frac{2K-1}{\tilde{r}^{(2K-1)l/(2N)}} \cos\frac{(2K-1)\pi}{N} \\ &\quad - \sum_{l=1}^{\infty}\{1-(-1)^{l-1}\}\sum_{K=1}^{\infty} \frac{2K}{\tilde{r}^{Kl/N}} \cos\frac{2K\pi}{N} \\ &= 2\sum_{l=1}^{\infty}\sum_{K=1}^{\infty} \frac{2K-1}{\tilde{r}^{(2K-1)l/N}} \cos\frac{(2K-1)\pi}{N} - 2\sum_{l=1}^{\infty}\sum_{K=1}^{\infty} \frac{2K}{\tilde{r}^{2Kl/N}} \cos\frac{2K\pi}{N} \\ &= -2\sum_{l=1}^{\infty}\sum_{K=1}^{\infty} \frac{K}{(-\tilde{r}^{l/N})^K} \cos\frac{K\pi}{N} \\ &= -2\sum_{l=1}^{\infty} \frac{(-\tilde{r}^{l/N})\{\cos(\pi/N) - 2(-\tilde{r}^{l/N}) + \tilde{r}^{2l/N}\cos(\pi/N)\}}{\{1 - 2(-\tilde{r}^{l/N})\cos(\pi/N) + \tilde{r}^{2l/N}\}^2} \\ &= 2\sum_{l=1}^{\infty} \frac{\tilde{r}^{l/N}\{\cos(\pi/N) + 2\tilde{r}^{l/N} + \tilde{r}^{2l/N}\cos(\pi/N)\}}{\{1 + 2\tilde{r}^{l/N}\cos(\pi/N) + \tilde{r}^{2l/N}\}^2} > 0, \end{aligned} \tag{8.35}$$

$$
\begin{aligned}
&\tilde{k}_1 + \tilde{k}_2 + \tilde{k}_3 + \tilde{k}_4 \\
&= \sum_{l=1}^{\infty}\{1+(-1)^{l-1}\}\sum_{K=1}^{\infty}\frac{2K}{\tilde{r}^{Kl/N}}\cos\frac{2K\pi}{N} \\
&\quad + \sum_{l=1}^{\infty}\{1+(-1)^{l-1}\}\sum_{K=1}^{\infty}\frac{2K-1}{\tilde{r}^{(2K-1)l/(2N)}}\cos\frac{(2K-1)\pi}{N} \\
&= 2\sum_{l=1}^{\infty}\sum_{K=1}^{\infty}\frac{K}{\tilde{r}^{K(2l-1)/(2N)}}\cos\frac{K\pi}{N} \qquad (8.36)\\
&= 2\sum_{l=1}^{\infty}\frac{\tilde{r}^{(2l-1)/(2N)}\{\cos(\pi/N)-2\tilde{r}^{(2l-1)/(2N)}+\tilde{r}^{(2l-1)/N}\cos(\pi/N)\}}{\{1-2\tilde{r}^{(2l-1)/(2N)}\cos(\pi/N)+\tilde{r}^{(2l-1)/N}\}^2} \\
&= 2\sum_{l=1}^{\infty}\frac{(\tilde{r}^{(2l-1)/(2N)}+\tilde{r}^{-(2l-1)/(2N)})\cos(\pi/N)-2}{\{(\tilde{r}^{(2l-1)/(2N)}+\tilde{r}^{-(2l-1)/(2N)})-2\cos(\pi/N)\}^2},
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{k}_1 + \tilde{k}_4 &= \frac{1}{2}(\tilde{k}_1+\tilde{k}_2+\tilde{k}_3+\tilde{k}_4) - \frac{1}{2}(-\tilde{k}_1+\tilde{k}_2+\tilde{k}_3-\tilde{k}_4) \\
&= \sum_{l=1}^{\infty}\frac{\tilde{r}^{(2l-1)/(2N)}\{\cos(\pi/N)-2\tilde{r}^{(2l-1)/(2N)}+\tilde{r}^{(2l-1)/N}\cos(\pi/N)\}}{\{1-2\tilde{r}^{(2l-1)/(2N)}\cos(\pi/N)+\tilde{r}^{(2l-1)/N}\}^2} \qquad (8.37)\\
&\quad - \sum_{l=1}^{\infty}\frac{\tilde{r}^{l/N}\{\cos(\pi/N)+2\tilde{r}^{l/N}+\tilde{r}^{2l/N}\cos(\pi/N)\}}{\{1+2\tilde{r}^{l/N}\cos(\pi/N)+\tilde{r}^{2l/N}\}^2}.
\end{aligned}
$$

Now we have shown that $R_1(\gamma_2) = 0$ is the only nontrivial condition for our data to realize a well-defined n-noid, since the data satisfies the condition (4.3), and, by Theorem 4.2, the other global period conditions are automatically satisfied.

Moreover, if we assume that $R_1(\gamma_2) = 0$ holds, then we see the following facts by applying Lemma 6.8 with $m_1 = 0$, Lemma 6.7 with $m_2 = 0$, Lemma 6.9, Lemma 6.7 with $m_2 = 0$ and replacing ω_2 by $\omega_1 + \omega_2$, Lemma 6.8 with $m_1 = 0$, and Lemma 6.9 respectively:

(1) If $\beta_1 = \beta_2 \neq 0$ and $\beta_3 = \beta_4 \neq 0$ (resp. $= 0$), then the data realizes a double covering of a $2N$-noid (resp. an N-noid) of type $\check{\omega} = \check{\omega}_2$ whose fundamental period is $(\check{\omega}_1, \check{\omega}_2) := (\omega_1/2, \omega_2)$.

(2a) Let N be an odd integer. If $\beta_1 = \beta_3 \neq 0$ and $\beta_2 = \beta_4 \neq 0$ (resp. $= 0$), then the data realizes a double covering of a $2N$-noid (resp. an N-noid) of type $\check{\omega} = \check{\omega}_1$ whose fundamental period is $(\check{\omega}_1, \check{\omega}_2) := (\omega_1, \omega_2/2)$.

(2b) Let N be an even integer. If $\beta_1 = \beta_3 \neq 0$ and $\beta_2 = \beta_4 \neq 0$ (resp. $= 0$), then the data realizes a quadruple covering of an N-noid (resp. an $(N/2)$-noid) of type $\check{\omega} = \check{\omega}_1 + \check{\omega}_2$ whose fundamental period is $(\check{\omega}_1, \check{\omega}_2) := (\omega_1/2, \omega_2/2)$.

(3a) Let N be an odd integer. If $\beta_1 = \beta_4 \neq 0$ and $\beta_2 = \beta_3 \neq 0$ (resp. $= 0$), then the data realizes a double covering of a $2N$-noid (resp. an N-noid) of type $\check{\omega} = 0$ whose fundamental period is $(\check{\omega}_1, \check{\omega}_2) := (\omega_1, (\omega_1 + \omega_2)/2)$.
(3b) Let N be an even integer. If $\beta_1 = \beta_4 \neq 0$ and $\beta_2 = \beta_3 \neq 0$ (resp. $= 0$), then the data realizes a double covering of a $2N$-noid (resp. an N-noid) of type $\check{\omega} = \check{\omega}_2$ whose fundamental period is $(\check{\omega}_1, \check{\omega}_2) := (\omega_1/2, \omega_2)$.
(4) If $\beta_1 = \beta_2 = \beta_3 = \beta_4 \neq 0$, then the data realizes a quadruple covering of an N-noid of type $\check{\omega} = \check{\omega}_1 + \check{\omega}_2$ whose fundamental period is $(\check{\omega}_1, \check{\omega}_2) := (\omega_1/2, \omega_2/2)$.

We may assume that $\omega_1 = 1$ without loss of generality. From this assumption and the assumption in the top of this section, it follows that $\omega_2 = y\sqrt{-1}$ for some $y \in \mathbf{R}_+$. Recall here that $r = e^{2\pi y}$, $\tilde{r} = e^{2\pi/y}$. It holds that $\log r \cdot \log \tilde{r} = 4\pi^2$. Set

$$P_N(\beta_1, \beta_2, \beta_3, \beta_4; r) := -\frac{1}{\beta^2 N\sqrt{-1}} R_1(\gamma_2),$$
$$\tilde{P}_N(\beta_1, \beta_2, \beta_3, \beta_4; \tilde{r}) := \frac{\omega_2}{4\pi\beta^2} R_1(\gamma_2)$$

(cf. (8.27) and (8.33)).

Since we treat mainly the type of $\check{\omega} = 0$ in this paper, we assume here that N is an odd integer and that $\beta_1 = \beta_4 = 1$ and $\beta_2 = \beta_3$, so that our data gives a $4N$-noid which is a double covering of some $2N$-noid or N-noid of type $\check{\omega} = 0$.

Set

$$\begin{aligned}
P_N(\beta_2, r) &:= P_N(1, \beta_2, \beta_2, 1; r) \\
&= 2(\beta_2^2 + 1) + N \log r\{(k_1 - k_4)(\beta_2^2 + 1) + (k_2 - k_3) \cdot 2\beta_2\}, \\
\tilde{P}_N(\beta_2, \tilde{r}) &:= \tilde{P}_N(1, \beta_2, \beta_2, 1; \tilde{r}) \\
&= -\left[\tilde{k}_1 + \tilde{k}_4 - \frac{1}{4\{1 + \cos(\pi/N)\}}\right](\beta_2^2 + 1) \\
&\quad - \left[\tilde{k}_2 + \tilde{k}_3 + \frac{1}{4\{1 + \cos(\pi/N)\}}\right] \cdot 2\beta_2.
\end{aligned}$$

Then, by the definitions, it holds that

$$R_1(\gamma_2)|_{\beta_1=\beta_4=1,\beta_2=\beta_3} = -\beta^2 N\sqrt{-1} P_N(\beta_2, r) = \frac{4\pi\beta^2}{\omega_2} \tilde{P}_N(\beta_2, \tilde{r}),$$

and hence the condition $R_1(\gamma_2) = 0$ is also rewritten as $P_N(\beta_2, r) = 0$ or $\tilde{P}_N(\beta_2, \tilde{r}) = 0$. Note here that

$$\tilde{P}_N(\beta_2, \tilde{r}) = \frac{Ny}{4\pi} P_N(\beta_2, r),$$

and hence $P_N(\beta_2, r)$ and $\tilde{P}_N(\beta_2, \tilde{r})$ have the same signature.

Claim 1 (Uniqueness of the Berglund–Rossman's N-end catenoid). *For each $N \geq 3$, there exists a unique $r_{\mathrm{BR},N} \in (1, +\infty)$ satisfying $P_N(1, r_{\mathrm{BR},N}) = 0$.*

Proof. Set $T_1(x) := (x \log x)/(x+1)^2$. Then, by (8.31), we have

(8.38)
$$\begin{aligned} P_N(1,r) &= 4 + 2N(k_1+k_2-k_3-k_4)\log r \\ &= 4 - 4N \sum_{l=1}^{\infty} \left\{ \frac{2}{2lN-(2N-1)} T_1(r^{[2lN-(2N-1)]/2}) + \frac{2}{2lN-1} T_1(r^{(2lN-1)/2}) \right\}, \end{aligned}$$

from which we get $\lim_{r\to+\infty} P_N(1,r) = 4 > 0$. Now, since

$$\begin{aligned} \frac{d}{dx}T_1(x) &= \frac{(1-x)\log x + (x+1)}{(x+1)^3} \\ &\leq \frac{2(1-x)+(x+1)}{(x+1)^3} = \frac{3-x}{(x+1)^3} < 0 \quad (x \in [e^2, +\infty)), \end{aligned}$$

$T_1(x)$ is monotone decreasing on $x \in [e^2, +\infty)$. We note that $r^{[2lN-(2N-1)]/2} \geq r^{1/2}$ and $r^{(2lN-1)/2} \geq r^{1/2}$ holds for any $l \in \mathbf{N}$. Hence $P_N(1,r)$ is monotone increasing on $r \in [e^4, +\infty)$.

On the other hand, by (8.36), we have

$$\begin{aligned} \tilde{P}_N(1,\tilde{r}) &= -2(\tilde{k}_1+\tilde{k}_2+\tilde{k}_3+\tilde{k}_4) \\ &= -4 \sum_{l=1}^{\infty} \frac{(\tilde{r}^{(2l-1)/(2N)} + \tilde{r}^{-(2l-1)/(2N)})\cos(\pi/N) - 2}{\{(\tilde{r}^{(2l-1)/(2N)} + \tilde{r}^{-(2l-1)/(2N)}) - 2\cos(\pi/N)\}^2}. \end{aligned} \tag{8.39}$$

Set

$$T_2(x) := \frac{3}{\pi}\log(2+\sqrt{3})\cdot x - \log\left(\frac{1+\sin x}{\cos x}\right).$$

Since $T_2(0) = T_2(\pi/3) = 0$, and

$$\frac{d^2}{dx^2}T_2(x) = -\frac{\sin x}{\cos^2 x} < 0 \quad (x \in (0, \pi/3)),$$

it holds that $T_2(x) > 0$ for $x \in (0, \pi/3)$. In particular, for any $N \geq 3$, $T_2(\pi/N) \geq 0$ which yields that

$$\left(\frac{1+\sin(\pi/N)}{\cos(\pi/N)}\right)^{2N} \leq (2+\sqrt{3})^6.$$

Hence, for any $\tilde{r} \in ((2+\sqrt{3})^6, +\infty)$, we have $\tilde{r}^{1/(2N)} > \{1+\sin(\pi/N)\}/\cos(\pi/N)$, and each numerator of (8.39) satisfies

$$
\begin{aligned}
&(\tilde r^{(2l-1)/(2N)} + \tilde r^{-(2l-1)/(2N)}) \cos\frac{\pi}{N} - 2\\
&\geq (\tilde r^{1/(2N)} + \tilde r^{-1/(2N)}) \cos\frac{\pi}{N} - 2\\
&> \left(\frac{1+\sin(\pi/N)}{\cos(\pi/N)} + \frac{\cos(\pi/N)}{1+\sin(\pi/N)}\right) \cos\frac{\pi}{N} - 2 = 0.
\end{aligned}
$$

Therefore we get $\tilde P_N(1,\tilde r) < 0$ on $\tilde r \in ((2+\sqrt{3})^6, +\infty)$. Since $\exp(\pi^2) = 19333.6\cdots > 2701.9\cdots = (2+\sqrt{3})^6$, $\tilde P_N(1,\tilde r) < 0$ on $\tilde r \in [\exp(\pi^2), +\infty)$, from which it also follows that $P_N(1,r) < 0$ on $r \in (1, e^4]$.

Now, by the intermediate value theorem, there exist a unique $r_{\mathrm{BR},N} \in (e^4, +\infty)$ satisfying $P_N(1, r_{\mathrm{BR},N}) = 0$. This solution gives us the data realizing a quadruple covering of Jorge–Meeks type N-end catenoid constructed by Berglund–Rossman. □

We note here that Claim 1 holds also in the case that N is even.

Claim 2 (One parameter family of $2N$-end catenoid of type $\check\omega = 0$)**.** *Let $N \geq 3$ be an odd integer. For each $r \in (1, r_{\mathrm{BR},N}]$, there exist a $\beta_2 = \beta_2(r) \in (-1, 1]$ satisfying $P_N(\beta_2(r), r) = 0$. In particular, $\beta_2(r) \neq 1$ for each $r \in (1, r_{\mathrm{BR},N})$.*

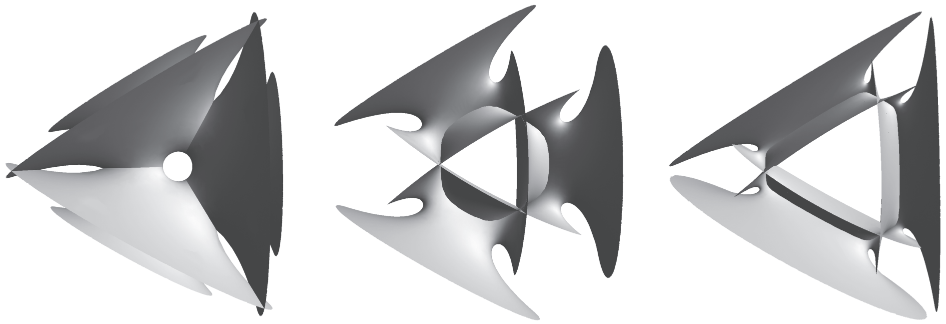
Proof. For any $r \in (1, r_{\mathrm{BR},N}]$ such that $\tilde k_1 + \tilde k_4 - 1/[4\{1+\cos(\pi/N)\}] \neq 0$, $P_N(\beta_2, r) = 0$ is a reciprocal quadratic equation of β_2. Let $\tilde D$ be the discriminant of this equation. Then

$$
\begin{aligned}
\frac{\tilde D}{4} &= \left[\tilde k_2 + \tilde k_3 + \frac{1}{4\{1+\cos(\pi/N)\}}\right]^2 - \left[\tilde k_1 + \tilde k_4 - \frac{1}{4\{1+\cos(\pi/N)\}}\right]^2\\
&= (\tilde k_1 + \tilde k_2 + \tilde k_3 + \tilde k_4)\left[(-\tilde k_1 + \tilde k_2 + \tilde k_3 - \tilde k_4) + \frac{1}{2\{1+\cos(\pi/N)\}}\right].
\end{aligned}
$$

By using (8.35) and Claim 1, we have $-\tilde k_1 + \tilde k_2 + \tilde k_3 - \tilde k_4 > 0$ for any $r \in (1, +\infty)$, and $\tilde k_1 + \tilde k_2 + \tilde k_3 + \tilde k_4 > 0$ (resp. $= 0$) for any $r \in (1, r_{\mathrm{BR},N})$ (resp. $r = r_{\mathrm{BR},N}$). Hence we get $\tilde D > 0$ for $r \in (1, r_{\mathrm{BR},N})$, and $\tilde D = 0$ for $r = r_{\mathrm{BR},N}$. Therefore we have a solution

$$
\beta_2 = \beta_2(r) = \frac{-[\tilde k_2 + \tilde k_3 + 1/[4\{1+\cos(\pi/N)\}]] + \sqrt{\tilde D/4}}{\tilde k_1 + \tilde k_4 - 1/[4\{1+\cos(\pi/N)\}]} \in (-1, 1] \setminus \{0\}. \tag{8.40}
$$

On the other hand, for $r \in (1, r_{\mathrm{BR},N})$ such that $\tilde k_1 + \tilde k_4 - 1/[4\{1+\cos(\pi/N)\}] = 0$, by using (8.35), we have $\tilde k_2 + \tilde k_3 + 1/[4\{1+\cos(\pi/N)\}] > \tilde k_1 + \tilde k_4 + 1/[4\{1+\cos(\pi/N)\}] = 1/[2\{1+\cos(\pi/N)\}] > 0$, and hence we have a solution $\beta_2 = \beta_2(r) = 0$. Since the



$w_2/w_1 \approx 0.85$ $w_2/w_1 \approx 7.3$ $w_2/w_1 \approx -2$

Fig. 8.2.

solution $\beta_2(r)$ in (8.40) is rewritten as

$$\beta_2(r) = \frac{\tilde{k}_1 + \tilde{k}_4 - 1/[4\{1+\cos(\pi/N)\}]}{-[\tilde{k}_2 + \tilde{k}_3 - 1/[4\{1+\cos(\pi/N)\}]] - \sqrt{\tilde{D}/4}},$$

$\beta_2 = \beta_2(r)$ is continuous with respect to the parameter $r \in (1, r_{\mathrm{BR},N}]$, and it gives us a 1-parameter family of double coverings of $D_N \times \mathbf{Z}_2$-invariant $2N$-noids of type $\check{\omega} = 0$. $\square$

The surfaces given by the other family of solutions $\beta_2(r)^{-1}$ of (8.40) are realized also by changing the indices by the permutation τ_2. Therefore we omit calculations for such solutions.

Fig. 8.2 shows some samples of the examples given by the above data in the case that $N = 3$.

Claim 3 (N-end catenoid of type $\check{\omega} = 0$). *For each odd integer* $N \geq 5$, *there exist* $r_{1,N} \in (e^{1/N}, 2^4)$ *and* $r_{2,N} \in (2^4, r_{\mathrm{BR},N})$ *satisfying* $P_N(0, r_{1,N}) = P_N(0, r_{2,N}) = 0$.

Proof. First, note here that $P_N(0, r) = 2 + N(k_1 - k_4)\log r$. Set

$$T_3(l, r) := \left[\frac{r^{(lN+1)/2}}{\{r^{(lN+1)/2} + (-1)^l\}^2} - \frac{r^{[(l+1)N+1]/2}}{\{r^{[(l+1)N+1]/2} + (-1)^{l+1}\}^2}\right] + \left[\frac{r^{(lN-1)/2}}{\{r^{(lN-1)/2} + (-1)^l\}^2} - \frac{r^{[(l+1)N-1]/2}}{\{r^{[(l+1)N-1]/2} + (-1)^{l+1}\}^2}\right].$$

Then, by (8.30), we have

$$k_1 - k_4 = -\frac{r^{1/2}}{(r^{1/2}+1)^2} + \sum_{l=1}^{\infty} T_3(2l-1, r)$$
$$= -\frac{r^{1/2}}{(r^{1/2}+1)^2} + \frac{r^{(N-1)/2}}{(r^{(N-1)/2}-1)^2} + \frac{r^{(N+1)/2}}{(r^{(N+1)/2}-1)^2} - \sum_{l=1}^{\infty} T_3(2l, r).$$

Since

$$\frac{x}{(x-1)^2} - \frac{Cx}{(Cx+1)^2} = \frac{x\{4Cx + (C-1)(Cx^2-1)\}}{(x-1)^2(Cx+1)^2} > 0 \quad (x > 1,\ C > 1),$$

$T_3(2l-1, r) > 0$ holds for any $r \in (1, +\infty)$ and $l \in \mathbf{N}$. By using this inequality, we see that

$$k_1 - k_4 > -\frac{r^{1/2}}{(r^{1/2}+1)^2} + 0 > -1 \quad (r \in (1, +\infty)).$$

Hence, for any $r \in (1, e^{1/N}]$, we get

$$P_N(0, r) > 2 + N \cdot (-1) \cdot \log r \geq 2 + N \cdot (-1) \cdot \frac{1}{N} = 1 > 0.$$

On the other hand, since

$$\frac{x}{(x+1)^2} - \frac{Cx}{(Cx-1)^2}$$
$$= \frac{x\{(C^2-6C+1)x + (C-1)(x-1)(Cx+1)\}}{(x+1)^2(Cx-1)^2} > 0 \quad (x > 1,\ C > 3 + 2\sqrt{2}),$$

$T_3(2l, r) > 0$ holds for any $r \in ((3+2\sqrt{2})^{2/N}, +\infty)$ and $l \in \mathbf{N}$. As $(2^4)^{N/2} = 2^{2N} > (3+2\sqrt{2})$ holds for $N \geq 2$, we can apply this inequality for $r = 2^4$. Since $N \geq 5$, we have

$$\left[-\frac{r^{1/2}}{(r^{1/2}+1)^2} + \frac{r^{(N-1)/2}}{(r^{(N-1)/2}-1)^2} + \frac{r^{(N+1)/2}}{(r^{(N+1)/2}-1)^2}\right]\Bigg|_{r=2^4}$$
$$\leq \left[-\frac{r^{1/2}}{(r^{1/2}+1)^2} + \frac{r^2}{(r^2-1)^2} + \frac{r^3}{(r^3-1)^2}\right]\Bigg|_{r=2^4}$$
$$= -\frac{4}{25} + \frac{2^8}{(2^8-1)^2} + \frac{2^{12}}{(2^{12}-1)^2} < -\frac{3}{20}.$$

We also have

$$\log 2^4 = \frac{8}{3} \log 2^{3/2} > \frac{8}{3}.$$

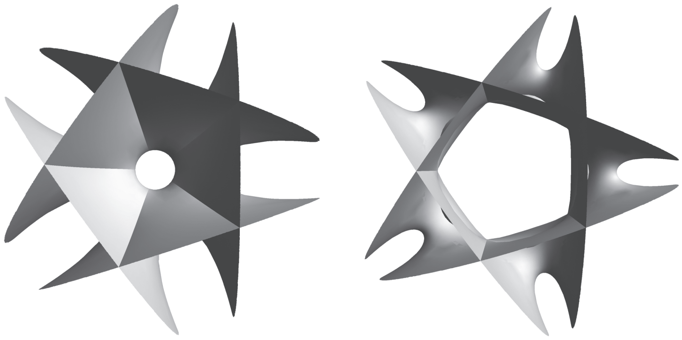


Fig. 8.3.

Combining these estimates we have

$$P_N(0, 2^4) < 2 - \frac{3}{20} N \log 2^4 < 0.$$

On the other hand, by (8.29), $k_2 - k_3 < 0$, and hence we have

$$\begin{aligned} P_N(0, r) &= 2 + N(k_1 + k_2 - k_3 - k_4) \log r - N(k_2 - k_3) \log r \\ &> 2 + N(k_1 + k_2 - k_3 - k_4) \log r = \frac{1}{2} P_N(1, r) \quad (r \in (1, +\infty)). \end{aligned}$$

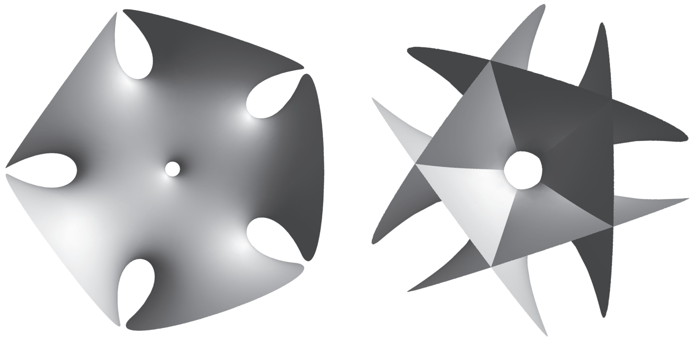
As we have already shown, it holds that $P_N(1, r) \geq 0$ for any $r \in [r_{\mathrm{BR},N}, +\infty)$. Therefore we get $P_N(0,r) > 0$ for any $r \in [r_{\mathrm{BR},N}, +\infty)$. Combining this fact with $P_N(0, 2^4) < 0$, we have $2^4 < r_{\mathrm{BR},N}$.

Now, by the intermediate value theorem, there exist $r_{1,N} \in (e^{1/N}, 2^4)$ and $r_{2,N} \in (2^4, r_{\mathrm{BR},N})$ satisfying $P_N(0, r_{1,N}) = P_N(0, r_{2,N}) = 0$. For the later use, we choose here them so that $r_{1,N} = \min\{r \in (1, 2^4) \mid P_N(0, r) = 0\}$ and $r_{2,N} = \max\{r \in (2^4, +\infty) \mid P_N(0, r) = 0\}$. $r_{1,N}$ and $r_{2,N}$ give us two $D_N \times \mathbf{Z}_2$-invariant N-end catenoids of type $\check{\omega} = 0$. $\square$

Fig. 8.3 shows two examples in the case that $N = 5$.

From the viewpoint of an inverse problem of the flux formula, we also have Berglund–Rossman's N-end catenoid as a solution of type $\check{\omega} = \check{\omega}_1 + \check{\omega}_2$, which enjoys the same flux data as the N-end catenoids above.

We can also construct an N-end catenoid of type $\check{\omega} = \check{\omega}_1$ with the same flux data, by setting $\beta_1 = \beta_3 = 1$ and $\beta_2 = \beta_4 = 0$. Indeed, in this case, $P_N(1, 0, 1, 0; r) =$



type $\check{\omega} = \check{\omega}_1 + \check{\omega}_2$
(Berglund–Rossman)

type $\check{\omega} = \check{\omega}_1$

Fig. 8.4.

$2 + N(k_1 - k_3)\log r$, and, by (8.32), we have

$$-\frac{r^{(N-1)/2} + r^{1/2}}{r^{N/2} - 1} = -\sum_{l=1}^{\infty}\left(\frac{1}{r^{[lN-(N-1)]/2}} + \frac{1}{r^{(lN-1)/2}}\right) < k_1 - k_3 < -\frac{r^{1/2}}{(r^{1/2}+1)^2},$$

from which we get $P_N(1,0,1,0;\,2^4) < 2-(32/75)N < 0$ and $\lim_{r\to+\infty} P_N(1,0,1,0;\,r) = 2 > 0$. Now, by the intermediate value theorem, there exists $r_{0,N} \in (2^4, +\infty)$ satisfying $P_N(1,0,1,0;\,r_{0,N}) = 0$. $r_{0,N}$ gives us a $D_N \times \mathbf{Z}_2$-invariant N-end catenoid of type $\check{\omega} = \check{\omega}_1$.

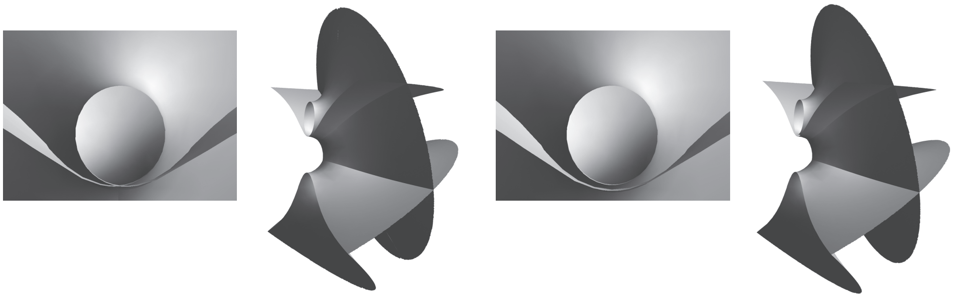
Fig. 8.4 shows these examples in the case that $N = 5$.

This N-end catenoid of type $\check{\omega} = \check{\omega}_1$ quite resembles one of the above N-end catenoids of type $\check{\omega} = 0$, and they seem to be the same surface at first glance. The difference between them appears in the holes. Indeed their handles on the plane of symmetry are attached to different positions. Fig. 8.5 shows the sectional views of them in the case that $N = 5$.

Claim 4 (Nonexistence of branch points)**.** *$2N$-end catenoids given by* Claim 2 *and N-end catenoids given by* Claim 3 *have no branch points.*

Proof. Here, any orbit of the action of $D_N \times \mathbf{Z}_2$ to each $2N$-end (resp. N-end) catenoid consists of $2N$ or $4N$ (resp. N, $2N$ or $4N$) points. On the other hand, the surface has no branch points if and only if the degree of its Gauss map is exactly $2N$ (resp. N). Since the Gauss map of each surface is not a constant map, each $2N$-end (resp. N-end) catenoid cannot be branched. □

Claim 5 (Estimates of the weights)**.** *Let $N \geq 5$ be an odd integer and set $\alpha(r) := w_2/w_1$. Then, for any $\alpha_0 \in (0, 1)$, there exist $r_{3,N}, r_{4,N}, r_{5,N}, r_{6,N} \in (1, r_{1,N})$ satisfying*



type $\check{\omega} = 0$ type $\check{\omega} = \check{\omega}_1$

Fig. 8.5.

$r_{3,N} < r_{4,N} < r_{5,N} < r_{6,N}$, $-\alpha(r_{3,N})^{-1} = \alpha(r_{5,N})^{-1} = \alpha(r_{6,N}) = \alpha_0$, $\alpha(r_{4,N}) = \infty$, *and there exists* $r_{7,N} \in (r_{2,N}, r_{\mathrm{BR},N})$ *satisfying* $\alpha(r_{7,N}) = \alpha_0$.

Proof. First, we observe the signature of w_2. By (8.34), we have

$$\begin{aligned}
&\tilde{k}_1, \quad \tilde{k}_3 = \cos\frac{2\pi}{N}\cdot\frac{2}{\tilde{r}^{1/N}} + O\left(\frac{1}{\tilde{r}^{2/N}}\right),\\
&\tilde{k}_2, \quad \tilde{k}_4 = \cos\frac{\pi}{N}\cdot\frac{1}{\tilde{r}^{1/(2N)}} + O\left(\frac{1}{\tilde{r}^{3/(2N)}}\right),\\
&\tilde{k}_1 + \tilde{k}_4 - \frac{1}{4\{1+\cos(\pi/N)\}} = -\frac{1}{4\{1+\cos(\pi/N)\}} + O\left(\frac{1}{\tilde{r}^{1/(2N)}}\right),\\
&\tilde{k}_2 + \tilde{k}_3 + \frac{1}{4\{1+\cos(\pi/N)\}} = \frac{1}{4\{1+\cos(\pi/N)\}} + O\left(\frac{1}{\tilde{r}^{1/(2N)}}\right),\\
&\tilde{k}_1 + \tilde{k}_2 + \tilde{k}_3 + \tilde{k}_4 = 2\cos\frac{\pi}{N}\cdot\frac{1}{\tilde{r}^{1/(2N)}} + O\left(\frac{1}{\tilde{r}^{1/N}}\right),\\
&-\tilde{k}_1 + \tilde{k}_2 + \tilde{k}_3 - \tilde{k}_4 = O\left(\frac{1}{\tilde{r}^{1/N}}\right),\\
&\frac{\tilde{D}}{4} = \frac{\cos(\pi/N)}{1+\cos(\pi/N)}\frac{1}{\tilde{r}^{1/(2N)}} + O\left(\frac{1}{\tilde{r}^{1/N}}\right).
\end{aligned}$$

Hence we have

$$\beta_2(r(\tilde{r})) = 1 - 4\sqrt{\cos\frac{\pi}{N}\left(1+\cos\frac{\pi}{N}\right)}\cdot\frac{1}{\tilde{r}^{1/(4N)}} + O\left(\frac{1}{\tilde{r}^{1/(2N)}}\right),$$

and, in particular, $\lim_{r\to 1+0}\beta_2(r) = 1 > 0$. By our choice of $r_{1,N}$, there are no $r \in (1, r_{1,N})$ satisfying $P_N(0, r) = 0$. Hence $\beta_2(r) > 0$ holds for $r \in (1, r_{1,N})$.

On the other hand, by the assumption, $\beta_2(r_{\mathrm{BR},N}) = \beta_1 = 1$. By our choice of $r_{2,N}$, there are no $r \in (r_{2,N}, +\infty)$ satisfying $P_N(0, r) = 0$. Hence $\beta_2(r) > 0$ also holds for $r \in (r_{2,N}, r_{\mathrm{BR},N}]$.

Now, by (8.25) and $\beta_2(r) \le 1 = \beta_1$ ($r \in (1, r_{\mathrm{BR},N}]$), we have $w_2 < 0$ for $r \in (1, r_{1,N}) \cup (r_{2,N}, r_{\mathrm{BR},N}]$.

Secondly, we observe the asymptotic behavior of w_1 as $r \to 1 + 0$. Also by using (8.25), we have

$$\frac{\omega_2 w_1}{2\pi\sqrt{-1}\beta^2}, \quad \frac{-\omega_2 w_2}{2\pi\sqrt{-1}\beta^2} = 4 \sin\frac{\pi}{N}\sqrt{\frac{\cos(\pi/N)}{1+\cos(\pi/N)}} + O\left(\frac{1}{\tilde{r}^{1/(2N)}}\right),$$

and hence $w_1 > 0$ holds if r is sufficiently close to 1, and $\alpha(r)^{-1} = w_1/w_2$ tends to -1 as $\tilde{r} \to +\infty$, that is $r \to 1 + 0$.

Thirdly, we observe the signature of w_1 in a neighborhood of $r = r_{1,N}$. Set

$$T_4(\tilde{r}) := \tilde{P}_N(\beta_2(r(\tilde{r})), \tilde{r}) - \frac{\omega_2}{8\pi\beta^2\sqrt{-1}\sin(\pi/N)} w_1.$$

Suppose that $\tilde{r}$ satisfies $\beta_2(r(\tilde{r})) = 0$. Then, combining with (8.33) and (8.37), we have

$$\begin{aligned}
T_4(\tilde{r}) = {} & \sum_{l=1}^{\infty}\Biggl[\frac{\tilde{r}^{(2l-1)/(2N)}\{1 - 2\tilde{r}^{(2l-1)/(2N)}\cos(\pi/N) + \tilde{r}^{(2l-1)/N}\}}{\{1 - 2\tilde{r}^{(2l-1)/(2N)}\cos(\pi/N) + \tilde{r}^{(2l-1)/N}\}^2} \\
& \qquad - \frac{\tilde{r}^{(2l-1)/(2N)}\{\cos(\pi/N) - 2\tilde{r}^{(2l-1)/(2N)} + \tilde{r}^{(2l-1)/N}\cos(\pi/N)\}}{\{1 - 2\tilde{r}^{(2l-1)/(2N)}\cos(\pi/N) + \tilde{r}^{(2l-1)/N}\}^2}\Biggr] \\
& - \sum_{l=1}^{\infty}\Biggl[\frac{\tilde{r}^{l/N}\{1 + 2\tilde{r}^{l/N}\cos(\pi/N) + \tilde{r}^{2l/N}\}}{\{1 + 2\tilde{r}^{l/N}\cos(\pi/N) + \tilde{r}^{2l/N}\}^2} \\
& \qquad - \frac{\tilde{r}^{l/N}\{\cos(\pi/N) + 2\tilde{r}^{l/N} + \tilde{r}^{2l/N}\cos(\pi/N)\}}{\{1 + 2\tilde{r}^{l/N}\cos(\pi/N) + \tilde{r}^{2l/N}\}^2}\Biggr] \\
= {} & \left(1 - \cos\frac{\pi}{N}\right)\sum_{l=1}^{\infty}\Biggl[\frac{\tilde{r}^{(2l-1)/(2N)}(1 + 2\tilde{r}^{(2l-1)/(2N)} + \tilde{r}^{(2l-1)/N})}{\{1 - 2\tilde{r}^{(2l-1)/(2N)}\cos(\pi/N) + \tilde{r}^{(2l-1)/N}\}^2} \\
& \qquad - \frac{\tilde{r}^{l/N}(1 - 2\tilde{r}^{l/N} + \tilde{r}^{2l/N})}{\{1 + 2\tilde{r}^{l/N}\cos(\pi/N) + \tilde{r}^{2l/N}\}^2}\Biggr] \\
= {} & \left(1 - \cos\frac{\pi}{N}\right)\sum_{l=1}^{\infty}\Biggl[\frac{(\tilde{r}^{(2l-1)/(4N)} + \tilde{r}^{-(2l-1)/(4N)})^2}{\{(\tilde{r}^{(2l-1)/(2N)} + \tilde{r}^{-(2l-1)/(2N)}) - 2\cos(\pi/N)\}^2} \\
& \qquad - \frac{(\tilde{r}^{l/(2N)} - \tilde{r}^{-l/(2N)})^2}{\{(\tilde{r}^{l/N} + \tilde{r}^{-l/N}) + 2\cos(\pi/N)\}^2}\Biggr].
\end{aligned}$$

Since

$$\frac{x+x^{-1}}{x^2+x^{-2}-2\cos(\pi/N)}=\frac{1}{(x+x^{-1})-2\{\cos(\pi/N)+1\}(x+x^{-1})^{-1}}$$

is monotone decreasing on $x \in (1,+\infty)$, it holds that

$$\begin{aligned}&\frac{\tilde{r}^{(2l-1)/(4N)}+\tilde{r}^{-(2l-1)/(4N)}}{\tilde{r}^{(2l-1)/(2N)}+\tilde{r}^{-(2l-1)/(2N)}-2\cos(\pi/N)}\\ &>\frac{\tilde{r}^{l/(2N)}+\tilde{r}^{-l/(2N)}}{\tilde{r}^{l/N}+\tilde{r}^{-l/N}-2\cos(\pi/N)}>\frac{\tilde{r}^{l/(2N)}+\tilde{r}^{-l/(2N)}}{\tilde{r}^{l/N}+\tilde{r}^{-l/N}+2\cos(\pi/N)}\\ &>\frac{\tilde{r}^{l/(2N)}-\tilde{r}^{-l/(2N)}}{\tilde{r}^{l/N}+\tilde{r}^{-l/N}+2\cos(\pi/N)},\end{aligned}$$

from which it follows that $T_4(\tilde{r})>0$. Therefore, if $\tilde{P}_N(\beta_2(r(\tilde{r})),\tilde{r})=0$, then it holds that $w_1<0$. Since $\beta_2(r_{1,N})=0$, we can apply this inequality to $r=r_{1,N}$, and we get $w_1|_{r=r_{1,N}}<0$. By combining $w_2|_{r=r_{1,N}}=0$, we see that $\alpha(r)^{-1}=w_1/w_2\to+\infty$ as $r\to r_{1,N}-0$.

Since the ratio of the weights $\alpha(r)^{-1}=w_1/w_2$ is a continuous function on $(1,r_{1,N})$, by applying the intermediate value theorem to $\alpha(r)^{-1}$, we see that, for any $\alpha_0\in(0,1)$, there exist $r_{3,N}$, $r_{4,N}$, $r_{5,N}$ and $r_{6,N}\in(1,r_{1,N})$ satisfying $r_{3,N}<r_{4,N}<r_{5,N}<r_{6,N}$, $\alpha(r_{3,N})^{-1}=-\alpha_0$, $\alpha(r_{4,N})^{-1}=0$, $\alpha(r_{5,N})^{-1}=\alpha_0$ and $\alpha(r_{6,N})^{-1}=\alpha_0^{-1}$.

Fourthly, we observe the signature of w_1 for $r\in[2^4,r_{\mathrm{BR},N}]$. Set

$$\begin{aligned}T_5(l,r)&:=\frac{1}{r^{(2lN-1)/2}+1}-\frac{1}{r^{(2lN+1)/2}+1},\\ T_6(l,r)&:=\frac{1}{r^{[2lN-(N-1)]/2}-1}-\frac{1}{r^{[2lN+(N-1)]/2}-1}.\end{aligned}$$

Then, by (8.22), we have

$$\begin{aligned}\frac{1}{2\pi\beta^2}w_1=&-(N-2)-N\beta_2+2N(1+\beta_2)\left(\frac{1}{r^{1/2}+1}-\sum_{l=1}^{\infty}T_5(l,r)\right)\\ &+2N(1-\beta_2)\left(\frac{1}{r^{(N-1)/2}-1}-\sum_{l=1}^{\infty}T_6(l,r)\right).\end{aligned}$$

Since both $1/(x+1)$ and $1/(x-1)$ are monotone decreasing on $(1,+\infty)$, $T_5(l,r)$, $T_6(l,r)>0$ holds for any $r\in(1,+\infty)$ and $l\in\mathbf{N}$. By using these inequalities, we see that

$$\begin{aligned}\frac{w_1}{2\pi\beta^2} &< -(N-2) - N\beta_2 + 2N(1+\beta_2)\frac{1}{r^{1/2}+1} + 2N(1-\beta_2)\frac{1}{r^{(N-1)/2}-1} \\ &\leq -(N-2) - N\beta_2 + 2N(1+\beta_2)\frac{1}{2^2+1} + 2N(1-\beta_2)\frac{1}{2^8-1} \\ &= 2 - \frac{151}{255}N - \frac{31}{51}N\beta_2 < 2 - \frac{151}{255}N < 0\end{aligned}$$

for $N \geq 5$ and $r \in [2^4, r_{\mathrm{BR},N}]$. By combining $2^4 < r_{2,N} < r_{\mathrm{BR},N}$, $w_2|_{r=r_{2,N}} = 0$ and $w_2|_{r=r_{\mathrm{BR},N}} = w_1|_{r=r_{\mathrm{BR},N}} < 0$, we see that $\alpha(r_{2,N}) = w_2/w_1|_{r=r_{2,N}} = 0$ and $\alpha(r_{\mathrm{BR},N}) = w_2/w_1|_{r=r_{\mathrm{BR},N}} = 1$.

Since the ratio of the weights $\alpha(r) = w_2/w_1$ is a continuous function on $[r_{2,N}, r_{\mathrm{BR},N}]$, by applying the intermediate value theorem to $\alpha(r)$, we see that, for any $\alpha_0 \in (0,1)$, there exists an $r_{7,N} \in (r_{2,N}, r_{\mathrm{BR},N})$ satisfying $\alpha(r_{7,N}) = \alpha_0$. $\square$

We note here that, if we change the indices by τ_2, or if we replace $\beta_2(r)$ by $\beta_2(r)^{-1}$, then we find that the data for $r = r_{5,N}$ (resp. $r_{3,N}$, $r_{4,N}$) realizes a surface such that $w_2/w_1 = \alpha_0$ (resp. $-\alpha_0$, 0). Hence, for any $\alpha_0 \in (0, 1)$, there exist at least three $2N$-end catenoids whose ratio of the weights are given by $w_2/w_1 = \alpha_0$. The fundamental periods of these three surfaces are given by $(\check{\omega}_1, \check{\omega}_2) = (\omega_1, (\omega_1 + \omega_2)/2) = (1, \{1 + (\sqrt{-1}\log r)/(2\pi)\}/2)$ with $r = r_{5,N}$, $r_{6,N}$ and $r_{7,N}$.

In summary of Example 8.2, we get the following theorem:

Theorem 8.3. *For each odd integer $N \geq 3$, there exists a deformation family of $2N$-end catenoids of type $\check{\omega} = 0$, which have $D_N \times \mathbf{Z}_2$ symmetry, and whose fundamental periods $(\check{\omega}_1, \check{\omega}_2)$ satisfy $\check{\omega}_1 \in \mathbf{R}_+$ and $\check{\omega}_2 - \check{\omega}_1/2 \in \sqrt{-1}\mathbf{R}_+$.*

In particular, for $N \geq 5$, we have the following examples in this family.

(i) *There exist two N-end catenoids.*

(ii) *For each $\alpha \in (0, 1)$, there exist three $2N$-end catenoids whose ratio of alternating weights of the ends is α.*

(iii) *There exists a $2N$-noid which has N catenoidal ends and N planar ends.*

(iv) *For each $\alpha \in (-1, 0)$, there exists a $2N$-end catenoid whose ratio of alternating weights of the ends is α.*

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