Vernicos, C.
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LIPSCHITZ CHARACTERISATION OF POLYTOPAL HILBERT GEOMETRIES

CONSTANTIN VERNICOS

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Abstract

We study Hilbert geometries admitting similar singularities on their boundary to those of a simplex. We show that in an adapted neighborhood of those singularities, two such geometries are bi-Lipschitz. As a corollary we prove that the Hilbert geometry of a convex set is bi-Lipschitz equivalent to a normed vector space if and only if the convex is a polytope.

Introduction and statement of results

A Hilbert geometry is a particularly simple metric space on the interior of a compact convex set $\mathcal{C}$ modeled on the construction of the Klein model of hyperbolic geometry inside an euclidean ball. This metric happens to be a complete Finsler metric whose set of geodesics contains the straight lines. Since the definition of the Hilbert geometry only uses cross-ratios, the Hilbert metric is a projective invariant.

In addition to ellipsoids, a second family of convex sets play a distinct role among Hilbert geometries: the simplexes. If the ellipsoids' geometry is isometric to the hyperbolic geometry and are the only Riemannian Hilbert geometries (see D.C. Kay [14, Corollary 1]), at the opposite side simplexes happen to be the only ones whose geometry is isometric to a normed vector space (e.g. see De la Harpe [12] for the existence and Foertsch and Karlsson [11] for the uniqueness).

Many of the recent works done in the context of these geometries focus on finding out how close they are to the hyperbolic geometry, from different viewpoints (see, e.g., A. Karlsson and G. Noskov [13], Y. Benoist [1, 2] for δ-hyperbolicity, E. Socié-Méthou [16, 17] for automorphisms and B. Colbois and C. Vernicos [5, 6] for the spectrum). It is now quite well understood that this is closely related to regularity properties of the boundary of the convex set. For instance if the boundary is C^2 with positive Gaussian curvature, then B. Colbois and P. Verovic [9] have shown that the Hilbert geometry is bi-Lipschitz equivalent to the hyperbolic geometry.

The present work investigates those Hilbert geometries close to a norm vector space.

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Along that path it has been noticed than any polytopal Hilbert geometry can be isometrically embedded in a normed vector space of dimension twice the number of it faces (see B.C. Lins [15]). Then B. Colbois and P. Verovic [10] showed that in fact no other Hilbert geometry could be quasi-isometrically embedded into a normed vector space. Furthermore with B. Colbois and P. Verovic [8] we have shown that the Hilbert geometries of plane polygons are bi-Lipschitz to the euclidean plane. Even though we saw no reason for this result not to hold in higher dimension, our point of view made it difficult to obtain a generalisation due to the computations it involved. The present works aims at filling that gap by giving a slightly different proofs which holds in all dimension, with less computations, but at the cost of a longer study of simplexes.

The first main result on our paper is the following comparison theorem around some specific singularity on the boundary, which we call *conical flag*, and which can be stated in the following rather informal way:

Theorem (Comparison Theorem 5)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be two Hilbert geometries with a common extremal point x such that should one apply the dilations of ratio λ centred at x, as λ goes to infinity the images of both convex sets would converge to an orthant. Let $\mathcal{S}$ be any simplex contained in $\mathcal{A}$ as well as $\mathcal{B}$, such that x is a vertex and at most one $(n-1)$-dimensional face adjacent to x lies on the boundary of both boundaries of $\mathcal{A}$ and $\mathcal{B}$. Then inside $\mathcal{S}$, the Hilbert geometries of $\mathcal{A}$ and $\mathcal{B}$ are bi-Lipschitz equivalent.*

The precise definition of those singularities can be found in Section 1.2. As a corollary we then get our second main theorem.

Theorem 1. *Let $\mathcal{P} \subset \mathbb{R}^d$ be a convex polytope, its Hilbert geometry $(\mathcal{P}, d_{\mathcal{P}})$ is bi-Lipschitz to the d-dimensional euclidean geometry $(\mathbb{R}^d, \|\cdot\|)$. In other words there exist a map $F\colon \mathcal{P} \to \mathbb{R}^d$ and a constant $L \geqslant 1$ such that for any two points x and y in $\mathcal{P}$,*

$$\frac{1}{L} \cdot \|F(x) - F(y)\| \leqslant d_{\mathcal{P}}(x, y) \leqslant L \cdot \|F(x) - F(y)\|.$$

The main idea is that a polytopal convex set can be decomposed into pyramids with apex its barycentre and base its faces, and then to prove that each pyramid is bi-Lipschitz to the cone it defines. However due to the multitude of available faces in dimension higher than two, a reduction is needed and consists in using the barycentric subdivision to decompose each of these pyramids into similar simplexes, and to prove that each of these simplexes is bi-Lipschitz to the cone it defines.

The following corollary “à l” Bourbaki sums up the known characterisations of the polytopal Hilbert geometries

Corollary 2. *Let $\mathcal{C} \in \mathbb{R}^d$ be an open convex set which does not contain any straight line and $(\mathcal{C}, d_\mathcal{C})$ its Hilbert geometry. Then the following are equivalent*
(1) *$\mathcal{C}$ is a polytopal convex domain*;
(2) *$(\mathcal{C}, d_\mathcal{C})$ is bi-Lipschitz equivalent to an d-dimensional vector space*;
(3) *$(\mathcal{C}, d_\mathcal{C})$ is quasi-isometric to the euclidean d-dimensional vector space*;
(4) *$(\mathcal{C}, d_\mathcal{C})$ isometrically embeds into a normed vector space*;
(5) *$(\mathcal{C}, d_\mathcal{C})$ quasi-isometrically embeds into a normed vector space*;

The author believes that Theorem 5 should help in the bi-Lipschitz classification of Hilbert geometries.

NOTE. Theorem 1 was found and proved with a completely different approach by Andreas Bernig [4]. The two approaches are somewhat dual to one another: where Bernig uses faces, we use vertices.

1. Definition of a Hilbert geometry and notations

1.1. Hilbert geometries. Let us recall that a Hilbert geometry $(\mathcal{C}, d_\mathcal{C})$ is a non-empty bounded open convex set $\mathcal{C}$ on $\mathbb{R}^d$ (that we shall call *convex domain*) with the Hilbert distance $d_\mathcal{C}$ defined as follows: for any distinct points p and q in $\mathcal{C}$, the line passing through p and q meets the boundary $\partial\mathcal{C}$ of $\mathcal{C}$ at two points a and b, such that one walking on the line goes consecutively by a, p, q b (Fig. 1). Then we define

$$d_\mathcal{C}(p, q) = \frac{1}{2}\ln[a, p, q, b],$$

where $[a, p, q, b]$ is the cross ratio of (a, p, q, b), i.e.,

$$[a, p, q, b] = \frac{\|q-a\|}{\|p-a\|} \times \frac{\|p-b\|}{\|q-b\|} > 1,$$

with $\|\cdot\|$ the canonical euclidean norm in $\mathbb{R}^d$.

Note that the invariance of the cross-ratio by a projective map implies the invariance of $d_\mathcal{C}$ by such a map.

These geometries are naturally endowed with a C^0 Finsler metric $F_\mathcal{C}$ as follows: if $p \in \mathcal{C}$ and $v \in T_p\mathcal{C} = \mathbb{R}^d$ with $v \neq 0$, the straight line passing by p and directed by v meets $\partial\mathcal{C}$ at two points p^+ and p^-; we then define

$$F_\mathcal{C}(p, v) = \frac{1}{2}\|v\|\left(\frac{1}{\|p-p^-\|} + \frac{1}{\|p-p^+\|}\right)$$

and

$$F_\mathcal{C}(p, 0) = 0.$$

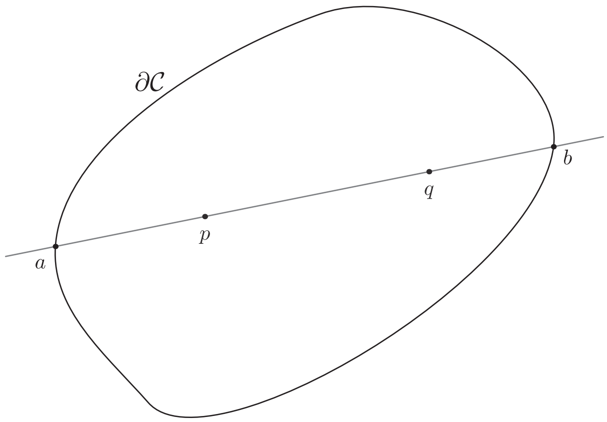

Fig. 1. The Hilbert distance.

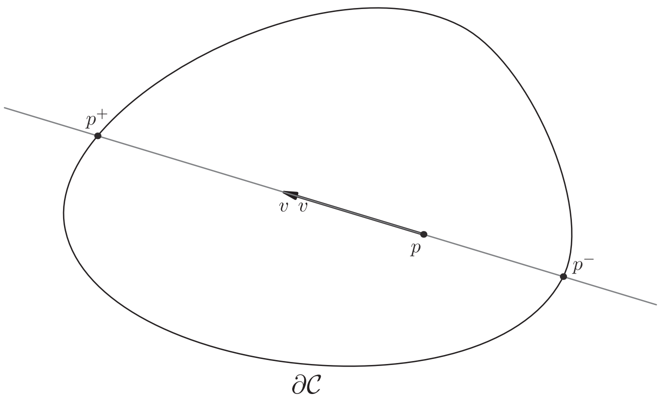

Fig. 2. The Finsler structure.

The Hilbert distance $d_{\mathcal{C}}$ is the length distance associated to $F_{\mathcal{C}}$.

Let us remark that by an abuse of notation if $\mathcal{C}$ is a closed convex set with non empty interior, then we still denote by $(\mathcal{C}, d_{\mathcal{C}})$ the Hilbert geometry associated to its interior $\mathring{\mathcal{C}}$.

1.2. Faces. Recall that to a closed convex set $K \subset \mathbb{R}^d$ we can associate an equivalence relation, stating that two points A and B are equivalent if they are equal or if there exists a segment $[C, D] \subset K$ containing the segment $[A, B]$ such that $C \neq A, B$ and $D \neq A, B$. The equivalent classes are called *faces*. A face is called a *k-face*, if the dimension of the smallest affine space containing it is k.

A 0-face is usually called an *extremal point* and for a convex polytopes it is called a *vertex*.

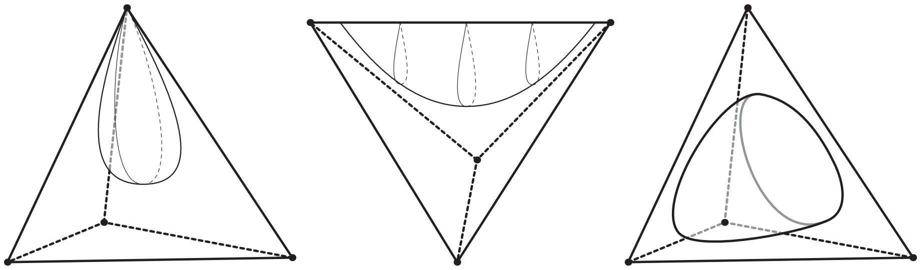

Fig. 3. Conical faces in dimension 3.

Thus defined all faces are open sets in their affine hull, that is in the smallest affine set containing them. For instance the segment $[a, b]$ in $\mathbb{R}$ admits three faces, which are $\{a\}$, $\{b\}$ and the open segment (a, b).

Notice that if K has non-empty interior (that is $K \setminus \partial K \neq \emptyset$), then its d-dimensional face is its interior.

In this paper a simplex in $\mathbb{R}^d$ is the convex closure of $d+1$ projectively independent points, i.e., a triangle in $\mathbb{R}^2$, a tetrahedron in $\mathbb{R}^3$, etc.

DEFINITION 1 (Conical faces). Let $\mathcal{C}$ be a closed convex set. Let $k < d$. Suppose that a simplex S contains $\mathcal{C}$ and that a non-empty k-face $f \subset \partial\mathcal{C}$, is included in a k-face of S. Then we say that f is a *conical face* of $\mathcal{C}$ and that $\mathcal{C}$ admits a conical face.

When a face f is in the boundary of another face F we write $f < F$.

DEFINITION 2 (Conical flag). Let $\mathcal{C}$ be a closed convex set in $\mathbb{R}^d$. If there exist a simplex $\mathcal{S}$ contained in $\mathcal{C}$, with a family of faces $f_0, f_1, \ldots, f_{d-1}$ such that for any $k = 0, \ldots, d-1$,

(1) $\emptyset < f_0 < f_1 < f_2 < \cdots < f_{d-1} < \mathcal{S}$;

(2) f_k is a subset of a k-conical face of $\mathcal{C}$;

(3) no other k-face of $\mathcal{S}$ is inside a k-conical face of $\mathcal{C}$;

then we call the sequence of faces $x = f_0 < f_1 < f_2 < \cdots < f_{d-1} < \mathcal{C}$ a *conical flag* and say that $\mathcal{C}$ admits a *conical flag at* x. Furthermore we will call the simplex $\mathcal{S}$ a *conical flag neighborhood of* the point x in the convex $\mathcal{C}$.

1.3. Prismatic neighborhoods and cones.

DEFINITION 3 (Prismatic neighborhoods). Let $\mathcal{S}$ be a simplex in $\mathbb{R}^d$ and let x_k be a point in a k-face of $\mathcal{S}$. Let A_k be the k-dimensional affine space containing x_k and its k-face. Let $(e_1, e_2, \ldots, e_k)$ be an orthonormal basis of the vector space $A_k - x_k$. We complete it into an orthonormal basis of $\mathbb{R}^d$ with $v_1, \ldots, v_{d-k}$ chosen as follows:

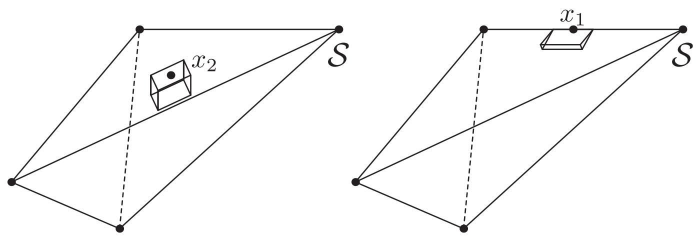

Fig. 4. Prismatic neighborhoods of a point in a 1-face and a 2-face in dimension 3.

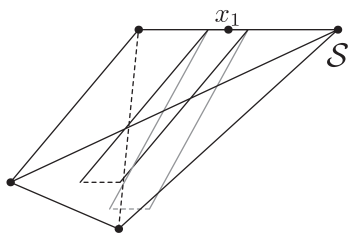

Fig. 5. Prismatic cone of a 1-face in dimension 3.

each of these vectors is parallel to one of the $(k+1)$-faces of $\mathcal{S}$ which contain x_k in their boundary.

- An (ε,α)-prismatic neighborhood with k-dimensional apex of x_k is the convex closure of a k-cube of diameter $2\sqrt{k}\varepsilon$ centred at x_k in A_k and its translates by αv_i, $i = 1, \dots, d-k$.
- An ε-prismatic cone with k-dimensional apex centred at x_k is the union of all (ε, α)-prismatic neighborhoods with k-dimensional apex of x_k for $\alpha \in \mathbb{R}^+$.

The following lemma, which compares the Hilbert geometries of a prismatic neighborhood of a point x and its corresponding cone around that point x, will play a critical role in the sequel.

Lemma 3. *Let $\mathcal{S}$ be a simplex in $\mathbb{R}^d$ and let x_k be a point in a k-face of $\mathcal{S}$. For any pair of positive numbers $\epsilon, \alpha > 0$ let $\mathcal{P}_k$ be an (ε, α)-prismatic neighborhood with k-dimensional apex of x_k, and $\mathcal{PC}_k$ the corresponding prismatic cone. Then for any sequence $(y_n, w_n)_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$, y_n is in the interior of $\mathcal{P}_k$; $w_n \in \mathbb{R}^d$; the sequence $(y_n)_{n\in\mathbb{N}}$ tends to x_k, one has*

$$\lim_{n\to\infty} \frac{F_{\mathcal{P}_k}(y_n, w_n)}{F_{\mathcal{PC}_k}(y_n, w_n)} = 1.$$

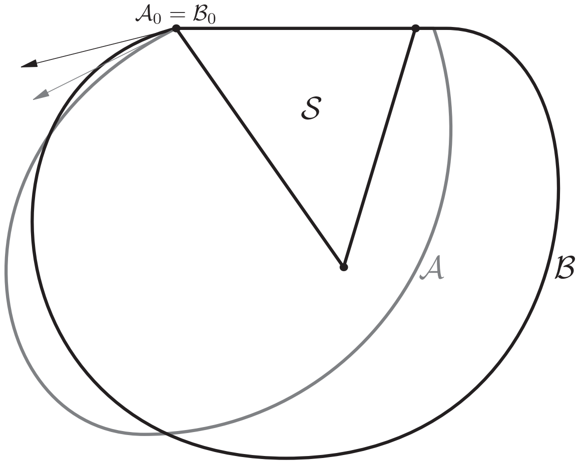

Fig. 6. Illustration of Example 1.

This lemma is a straightforward consequence of Proposition 2.6's proof in [3] which can be restated in the following way

Proposition 4. *Let K, K' be closed convex sets with non-empty interior and not containing any straight line. For any point x in the interior of $K \cap K'$, let $\| \cdot \|_x$, $\| \cdot \|'_x$ be their respective Finsler norm induced by the their respective Hilbert geometries. Let $p \in \partial K$, let E_0 be a supporting hyperplane of K at p and let E_1 be a hyperplane parallel to E_0 intersecting the interior of K. Let $\mathcal{E}$ be the strip obtained as the convex closure of E_0 and E_1. Suppose that K and K' have the same intersection with the strip $\mathcal{E}$, that is $\mathcal{E} \cap K = \mathcal{E} \cap K'$. Then as functions on $\mathbb{R}P^{n-1}$, the ratio $\| \cdot \|_x / \| \cdot \|'_x$ uniformly converge to 1 as x goes to p.*

2. Metric comparison around a conical flag

Theorem 5. *Let $\mathcal{A}$ and $\mathcal{B}$ be two convex sets with a common conical flag neighborhood $\mathcal{S}$ then there exists a constant C such that for any x in the interior of the simplex $\mathcal{S}$ and $v \in \mathbb{R}^d$ one has*

$$\frac{1}{C} \cdot F_{\mathcal{B}}(x, v) \leqslant F_{\mathcal{A}}(x, v) \leqslant C \cdot F_{\mathcal{B}}(x, v). \tag{1}$$

EXAMPLE 1. In the two-dimensional case the condition is that $\mathcal{A}$ and $\mathcal{B}$ contain a triangle $\mathcal{S}$, one of its edges on their boundaries, a vertex of which, and only one, is an extremal point of both of them which is a conical point (i.e. 0-conical face), that is to say that it admits two supporting lines.

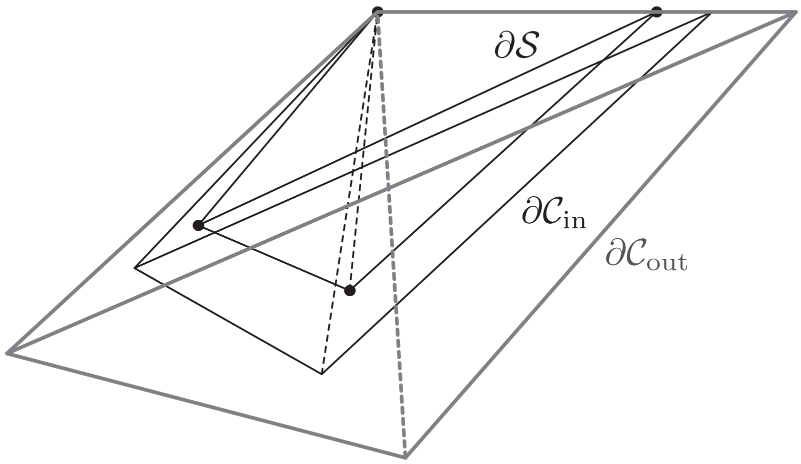

Fig. 7. The simplexes of Lemma 6.

To prove Theorem 5 we will reduce to the case where both $\mathcal{A}$ and $\mathcal{B}$ are simplexes and $\mathcal{A} \subset \mathcal{B}$ (see Fig. 7). This is the intermediate Lemma 6 whose statement and illustration follow.

Lemma 6. *Suppose that $\mathcal{S}$, $\mathcal{C}_{\mathrm{in}}$ and $\mathcal{C}_{\mathrm{out}}$ are three n-simplexes such that $\mathcal{S} \subset \mathcal{C}_{\mathrm{in}} \subset \mathcal{C}_{\mathrm{out}}$ and $\mathcal{S}$ is a conical flag neighborhood of both $\mathcal{C}_{\mathrm{in}}$ and $\mathcal{C}_{\mathrm{out}}$. Then there exists a constant M such that for any x in the interior of $\mathcal{S}$ and any vector $v \in \mathbb{R}^d$ one has*

$$F_{\mathcal{C}_{\mathrm{out}}}(x, v) \leqslant F_{\mathcal{C}_{\mathrm{in}}}(x, v) \leqslant M \cdot F_{\mathcal{C}_{\mathrm{out}}}(x, v).$$

We can now present Theorem 5's proof as a corollary.

Proof of Theorem 5. We are going to build a simplex $\mathcal{C}_{\mathrm{in}}$ in $\mathcal{A} \cap \mathcal{B}$ containing $\mathcal{S}$ and a simplex $\mathcal{C}_{\mathrm{out}}$ containing $\mathcal{A} \cup \mathcal{B}$ satisfying the assumptions required by Lemma 6.

Let us suppose these two simplexes exist. The inclusions $\mathcal{C}_{\mathrm{in}} \subset \mathcal{A} \cap \mathcal{B}$ and $\mathcal{A} \cup \mathcal{B} \subset \mathcal{C}_{\mathrm{out}}$ give the following sets of inequalities

$$F_{\mathcal{C}_{\mathrm{out}}}(x, v) \leqslant F_{\mathcal{A}}(x, v) \leqslant F_{\mathcal{C}_{\mathrm{in}}}(x, v),$$

and

$$F_{\mathcal{C}_{\mathrm{out}}}(x, v) \leqslant F_{\mathcal{B}}(x, v) \leqslant F_{\mathcal{C}_{\mathrm{in}}}(x, v).$$

We combine these inequalities to obtain

$$\frac{F_{\mathcal{C}_{\mathrm{out}}}(x, v)}{F_{\mathcal{C}_{\mathrm{in}}}(x, v)} \leqslant \frac{F_{\mathcal{A}}(x, v)}{F_{\mathcal{B}}(x, v)} \leqslant \frac{F_{\mathcal{C}_{\mathrm{in}}}(x, v)}{F_{\mathcal{C}_{\mathrm{out}}}(x, v)},$$

and the conclusion follows from Lemma 6.

Let us now make the construction of $\mathcal{C}_{\text{in}}$ and $\mathcal{C}_{\text{out}}$ precise. To do so, let us consider the conical flag $f_0 < f_1 < \cdots < f_{d-1} < S$. Then we will denote the k-face of $\mathcal{A}$ containing f_k by $\mathcal{A}_k$ and similarly by $\mathcal{B}_k$ the corresponding face of $\mathcal{B}$.

For $n \geqslant k \geqslant 0$, let us denote by v_k the vertex of $\mathcal{S}$ in $\mathcal{A}_k \cap \mathcal{B}_k$, but not in $\mathcal{A}_{k-1} \cap \mathcal{B}_{k-1}$ and by p_k the barycentre of the vertexes $v_k, \dots, v_0$. Then as p_k and v_k belong to the same face, there exists a point $v_{k,1} \in \mathcal{A}_k \cap \mathcal{B}_k$ and $v_{k,1} \neq v_k$ such that the segment $[p_k, v_{k,1}]$ contains v_k. We take for $\mathcal{C}_{\text{in}}$ the convex hull of $v_{d,1}, \dots, v_{0,1}$.

For $\mathcal{C}_{\text{out}}$, we will actually build its convex dual (i.e. the convex closure of the set of supporting hyperplanes in the dual vector space). Indeed, if we take convex sets that are dual to $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{S}$ with respect to some point in the interior of $\mathcal{S}$, we obtain respectively three convex sets $\mathcal{B}^*$, $\mathcal{A}^*$ and $\mathcal{S}^*$ such that both $\mathcal{B}^*$ and $\mathcal{A}^*$ are subsets of $\mathcal{S}^*$. In the sequel, for $k = 0, \dots, d-1$, let us denote by $\mathcal{S}_k^*$ the k-face of $\mathcal{S}^*$ corresponding to the hyperplanes tangent to f_{d-k-1}. Then as $\mathcal{S}$ is a conical flag neighborhood of both $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{S}_k^*$ contains the hyperplanes tangent to $\mathcal{A}_{d-k-1}$ and to $\mathcal{B}_{d-k-1}$ but not to $\mathcal{A}_{d-k}$ or $\mathcal{B}_{d-k}$.

Let us also remark that f_{d-k-1} is in the intersection of $\mathcal{A}_{d-k-1}$ and $\mathcal{B}_{d-k-1}$, which are both conical faces of $\mathcal{A}$ and $\mathcal{B}$ respectively. Therefore the intersection of the hyperplanes containing both these faces but no other faces of either $\mathcal{A}$ or $\mathcal{B}$, and simultaneously tangent to $\mathcal{A}$ and $\mathcal{B}$ is an open and nonempty subset of $\mathcal{S}_k^*$, which we shall denote by $\mathcal{O}_k^*$.

In particular the vertex $\mathcal{S}_0^* = \mathcal{O}_k^*$ corresponds to the common supporting hyperplane containing the three faces $\mathcal{A}_{d-1}$, $\mathcal{B}_{d-1}$ and f_{d-1}.

Now, let w_0 be the vertex $\mathcal{S}_0^*$, and for $k = 1, \dots, d-1$ take a point w_k in $\mathcal{O}_k$. Let also take a point w_d in the intersection of the convex sets $\mathcal{A}^*$ and $\mathcal{B}^*$. Then by construction, if we let $\mathcal{C}_{\text{out}}^*$ be the convex hull of $w_0, \dots, w_d$, it is a simplex, which is a common conical flag of $\mathcal{A}^*$, $\mathcal{B}^*$ and $\mathcal{S}^*$. Thus its dual will contain both $\mathcal{A}$ and $\mathcal{B}$, and admits $\mathcal{S}$ as a conical flag neighborhood. Therefore we can take it as our simplex $\mathcal{C}_{\text{out}}$. □

2.1. Proof of Lemma 6.

2.1.1. Notation needed along the proof. Recall that $\mathcal{S}$, $\mathcal{C}_{\text{in}}$ and $\mathcal{C}_{\text{out}}$ are three d-dimensional simplexes such that $\mathcal{S} \subset \mathcal{C}_{\text{in}} \subset \mathcal{C}_{\text{out}}$. By assumption the closure of one of the $(d-1)$-dimensional faces of $\mathcal{S}$ is the intersection of the closure of these three simplexes. In fact, for every $k \leqslant d-1$, there is a unique k dimensional face of $\mathcal{S}$, denoted by f_k, which is also a subset of a k dimensional face of $\mathcal{C}_{\text{in}}$ and $\mathcal{C}_{\text{out}}$, denoted respectively by $\varphi_{\text{in},k}$ and $\varphi_{\text{out},k}$. The assumptions of Lemma 6 imply

$$f_k \subset \varphi_{\text{in},k} \subset \varphi_{\text{out},k}.$$

We will denote by A_k the k-dimensional affine space containing the three faces f_k, $\varphi_{\mathrm{in},k}$ and $\varphi_{\mathrm{out},k}$ for $0 \leqslant k \leqslant d$. Hence A_0 is a common vertex to the three simplexes and A_d the whole space $\mathbb{R}^d$.

2.1.2. Step 0: Ignition. The left inequality of Lemma 6 is a straightforward consequence of the fact that $\mathcal{C}_{\mathrm{in}} \subset \mathcal{C}_{\mathrm{out}}$.

For the right inequality, by homogeneity we can restrict to vectors v in the unit euclidean sphere $\mathcal{B}_d$. Hence we will focus on the following ratio, where x is in the interior of $\mathcal{S}$ and v a unit vector

$$\mathcal{Q}(x, v) = \frac{F_{\mathcal{C}_{\mathrm{in}}}(x, v)}{F_{\mathcal{C}_{\mathrm{out}}}(x, v)}.$$

We want to show that $\mathcal{Q}$ remains bounded on $\mathring{\mathcal{S}} \times \mathcal{B}_d$

HYPOTHESIS. Let us suppose by contradiction that $\mathcal{Q}$ is not bounded.

Thanks to that hypothesis we can find a sequence $(x_n, w_n)_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$, x_n is in the interior of $\mathcal{S}$, $w_n \in \mathcal{B}_d$ and most importantly

$$\mathcal{Q}(x_n, w_n) \to +\infty. \tag{2}$$

Due to the compactness of $\mathcal{S}\times\mathcal{B}_d$, at the cost of taking a sub-sequence, we can assume that this sequence converges to (x_∞, w_∞)

REMARK 1. If the sequence $(x_n)_{n\in\mathbb{N}}$ remains in a compact set U contained in the interior of $\mathcal{C}_{\mathrm{in}}$, then $\mathcal{Q}$ remains bounded as a continuous function of two variables over the compact set $U \times \mathcal{B}_d$.

2.1.3. Step 1: Focusing on faces. Following the above Remark 1, if $(x_n)_{n\in\mathbb{N}}$ were to converge toward a point in $\mathcal{C}_{\mathrm{in}}$, we would get a contradiction. Hence x_∞ has to be on the boundary of $\mathcal{C}_{\mathrm{in}}$, which implies that x_∞ is on a common face of the three simplexes.

We will suppose that x_∞ belongs to the k-dimensional face f_k of $\mathcal{S}$ and obtain a contradiction.

To do so we will make two simplifications:

(1) We first replace the three simplexes by three prismatic neighborhoods of x_∞, in such a way that the sequence $(\mathcal{Q}(x_n, w_n))_{n\in\mathbb{N}}$ remains bounded if the quotient defined in the same way for these prismatic neighborhoods does (Steps 2 and 3).

(2) We then replace the three prismatic neighborhoods by their three corresponding prismatic cones centred at x_∞ and then we prove that the corresponding quotient remains bounded (Step 4, Lemmata 3 and 8).

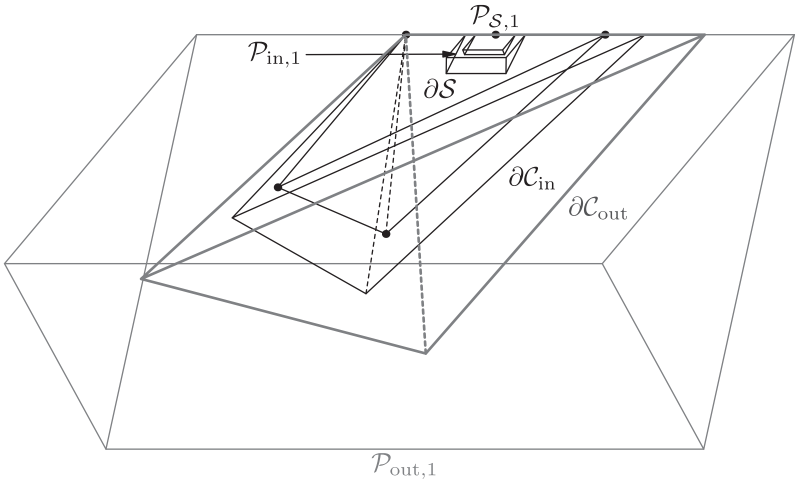

Fig. 8. Prismatic neighborhoods of a 1-face in dimension 3.

2.1.4. Step 2: The prismatic neighborhoods. For the following constructions we fix k and we suppose that the limit point x_∞ belongs to the k-dimensional face of $\mathcal{S}$, i.e., $x_\infty \in f_k$.

If $k \neq 0$, choose $0 < \alpha < \beta < \gamma$ such that

(i) the (α, α)-prismatic neighborhood of x_∞ with respect to $\mathcal{S}$ is a subset of $\mathcal{S}$;

(ii) the (β, β)-prismatic neighborhood of x_∞ with respect to $\mathcal{C}_{\mathrm{in}}$ is a subset of $\mathcal{C}_{\mathrm{in}}$;

(iii) the (γ, γ)-prismatic neighborhood of x_∞ with respect to $\mathcal{C}_{\mathrm{out}}$ *contains* $\mathcal{C}_{\mathrm{out}}$;

(iv) the (β, β)-prismatic neighborhood of x_∞ with respect to $\mathcal{C}_{\mathrm{in}}$ contains the (α, α)-prismatic neighborhood of x_∞ with respect to $\mathcal{S}$.

Then we will denote by

(i) $\mathcal{P}_{\mathcal{S},k}$ the $(\alpha/2, \alpha/2)$-prismatic neighborhood of x_∞ with respect to $\mathcal{S}$;

(ii) $\mathcal{P}_{\mathrm{in},k}$ the $(\beta/2, \beta/2)$-prismatic neighborhood of x_∞ with respect to $\mathcal{C}_{\mathrm{in}}$;

(iii) $\mathcal{P}_{\mathrm{out},k}$ the $(2\gamma, 2\gamma)$-prismatic neighborhood of x_∞ with respect to $\mathcal{C}_{\mathrm{out}}$;

For $k = 0$, we take $\mathcal{P}_{\mathcal{S},0} = \mathcal{S}$, $\mathcal{P}_{\mathrm{in},0} = \mathcal{C}_{\mathrm{in}}$ and $\mathcal{P}_{\mathrm{out},0} = \mathcal{C}_{\mathrm{out}}$.

Now, for any point x in the interior of $\mathcal{P}_{\mathcal{S},k}$ and any unit vector v in $\mathcal{B}_d$ we define

$$R_k(x, v) = \frac{F_{\mathcal{P}_{\mathrm{in},k}}(x, v)}{F_{\mathcal{P}_{\mathrm{out},k}}(x, v)}. \tag{3}$$

We introduce this ratio because for any point x in the interior of $\mathcal{P}_{\mathcal{S},k}$ and any vector v it bounds from above $\mathcal{Q}(x, v)$, i.e.,

$$\mathcal{Q}(x, v) \leqslant R_k(x, v). \tag{4}$$

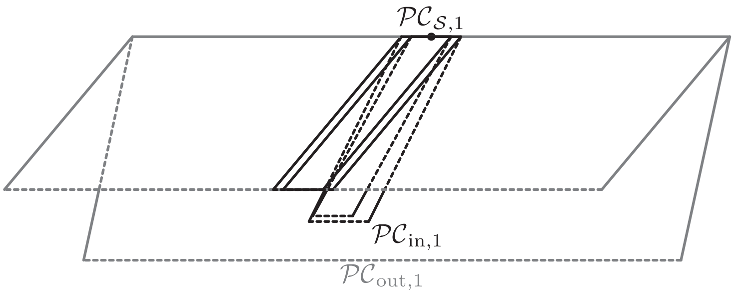

Fig. 9. Prismatic cones of a 1-face in dimension 3.

2.1.5. Step 3: The prismatic cones. Let us denote by

(i) $\mathcal{PC}_{\mathcal{S},k}$ the $\alpha/2$-prismatic cone centred at x_∞ with respect to $\mathcal{S}$;
(ii) $\mathcal{PC}_{\mathrm{in},k}$ the $\beta/2$-prismatic cone centred at x_∞ with respect to $\mathcal{C}_{\mathrm{in}}$;
(iii) $\mathcal{PC}_{\mathrm{out},k}$ the 2γ-prismatic cone centred at x_∞ with respect to $\mathcal{C}_{\mathrm{out}}$.

by construction we have $\mathcal{PC}_{\mathcal{S},k} \subset \mathcal{PC}_{\mathrm{in},k} \subset \mathcal{PC}_{\mathrm{out},k}$.

Finally we associate the following ratio with these prismatic cones.

$$\mathcal{R}_k(x, v) = \frac{F_{\mathcal{PC}_{\mathrm{in},k}}(x, v)}{F_{\mathcal{PC}_{\mathrm{out},k}}(x, v)}, \tag{5}$$

where x is in the interior of $\mathcal{PC}_{\mathcal{S},k}$ and $v \in \mathcal{B}_d$.

2.1.6. Step 4: Comparisons. First notice that there exist an integer N such that for all $n > N$, x_n is in the interior of $\mathcal{P}_{\mathcal{S},k}$. Hence, applying Lemma 3 we get the following equivalence.

Lemma 7. *Let us fix $0 \leqslant k \leqslant d$ and let $(y_n, u_n)_{n\in\mathbb{N}}$ be a sequence with y_n in the interior of the prismatic cone $\mathcal{PC}_{\mathcal{S},k}$ converging to (x_∞, u_∞) with $u_\infty \in \mathcal{B}_d$; then*

$$\lim_{n\to\infty} \frac{R_k(y_n, u_n)}{\mathcal{R}_k(y_n, u_n)} = 1.$$

The previous Lemma 7 allows us to focus on the prismatic cones, therefore the heart of our proof now lies in the following key lemma.

Lemma 8. *Let us fix $0 \leqslant k \leqslant d$ and let $(y_n, u_n)_{n\in\mathbb{N}}$ be a sequence with y_n in the interior of the prismatic cone $\mathcal{PC}_{\mathcal{S},k}$ converging to (x_∞, u_∞) with $u_\infty \in \mathcal{B}_d$; then there is a constant c such that for all $n \in \mathbb{N}$ one has*

$$\mathcal{R}_k(y_n, u_n) \leqslant c. \tag{6}$$

Proof of Lemma 8. We suppose that x_∞ is the origin thus the affine k-dimensional subspace A_k containing f_k is actually a sub-vector space. We then consider the decomposition

$$\mathbb{R}^d = A_k \oplus A_k^\perp,$$

and the *inhomogeneous scaling*[1] $VA_{k,\lambda}$ which is defined as the identity on A_k and as the dilation of ratio λ on $A_k^\perp$. When $k = 0$ this is just a dilation centred at the origin.

The three prismatic cones are invariant by these inhomogeneous scalings, hence $VA_{k,\lambda}$ is an isometry with respect to their Hilbert geometries.

Now consider a supporting hyperplane E_0 to these prismatic cones at the origin, and an affine hyperplane E_1 parallel to E_0 intersecting the prismatic cones and the face f_{k+1}. Then for any $n \in \mathbb{N}$, there is a λ such that y_n is is pushed away from the origin onto the hyperplanes E_1 while staying in the interior of the inside prismatic cone $\mathcal{PC}_{\mathcal{S},k}$, i.e.

$$\exists\lambda,\ VA_{k,\lambda}(y_n) \in E_1$$

and

$$VA_{k,\lambda}(y_n) \in \mathcal{PC}_{\mathcal{S},k}.$$

This gives a new sequence $(y'_n, u'_n)_{n\in\mathbb{N}}$, with $y'_n = VA_{k,\lambda}(y_n)$ and $u'_n = VA_{k,\lambda}(u_n)/\|VA_{k,\lambda}(u_n)\|$, which stays in the hyperplane E_1, and such that $\mathcal{R}_k(y_n, u_n) = \mathcal{R}_k(y'_n, u'_n)$.

By descending induction, suppose that for any triple of prismatic cones with k'-dimensional apex of type $\mathcal{PC}_{*,k'}$ which can occur in a construction in step 3, with $k' > k$, our conclusion holds.

CASE $k = d$: In that situation, the new sequence remains in the intersection of E_1 with the interior of the prismatic cone $\mathcal{PC}_{\mathcal{S},d}$, which is a common compact set of the prismatic cones $\mathcal{P}_{\text{in},d}$ and $\mathcal{P}_{\text{out},d}$, and thus we conclude following Remark 1, that the ratio $\mathcal{R}_d(y_n, u_n)$ remains bounded.

CASE $k < d$: Regarding the sequence $(y'_n, u'_n)_{n\in\mathbb{N}}$: either it stays away from the common hyperplane A_{n-1}, which means that the sequence remains in a common compact set in the interior of the prismatic cones $\mathcal{P}_{\text{in},k}$ and $\mathcal{P}_{\text{out},k}$, and thus again by Remark 1 we conclude that there is a constant c such that

$$\mathcal{R}_k(y_n, u_n) = \mathcal{R}_k(y'_n, u'_n) \leqslant c,$$

or the sequence admits a sub-sequence converging to the common hyperplane A_{n-1} while remaining in the hyperplane E_1, hence away from A_k. Without loss of generality we thus can suppose that the whole sequence $(y'_n, u'_n)_{n\in\mathbb{N}}$ converges to (y_∞, u_∞), with y_∞ in some common k'-dimensional, with $k' > k$ face of the three prismatic cones.

[1] Known as “Affinité vectorielle” in French.

We remark that from a projective point of view, the prismatic cones are actually prismatic polytopes, having another common face, the projective hyperplane at infinity. In other words, up to a change of affine charts, which is an isometry for the respective Hilbert geometries, we can suppose that the prismatic cones are prismatic polytopes.

Once we remarked this, we can now build three new prismatic polytopes of type $\mathcal{P}_{*,k'}$ and their corresponding prismatic cones of type $\mathcal{P}_{*,k'}$ containing y_∞, obtaining a new ratio of type $\mathcal{R}_{k'}$ which bounds from above our ratio $\mathcal{R}_k$. Now the induction assumption allows us to conclude that this new ratio is bounded from above, and therefore the sequence $\mathcal{R}_k(y_n, u_n)$ also stays bounded from above as n goes to infinity and our proof is complete. □

2.1.7. Step 5: Conclusion. Let us consider a converging sub-sequence $(x_n, w_n)_{n\in\mathbb{N}}$ satisfying the divergence property (2). Then for some $0 \leqslant k \leqslant d$, the limit x_∞ belongs to the face f_k.

Therefore Lemmata 7 and 8 imply that $R_k(x_n, w_n)$ remains bounded as $n \to \infty$, and by the inequality (4) that $\mathcal{Q}(x_n, w_n)$ as well, which is absurd.

Hence our initial hypothesis, that $\mathcal{Q}$ is not bounded is violated, which concludes our proof.

3. Polytopal Hilbert geometries are bi-Lipschitz to euclidean vector spaces

The barycentre of a polytope and its faces induce a decomposition of the polytope into pyramids with apex the barycentre and base the faces. These pyramids also give rise to cones with summit their apex which in turn decompose the ambient space. In this section we built a map which sends these pyramids to their corresponding cones and which is a bi-Lipschitz map between the Hilbert geometry of the polytope and the Euclidean geometry of the ambient space.

The proof of Theorem 1 consists in building a bi-Lipschitz map and take the following steps:

(1) Using the barycentric subdivision, in Section 3.1 we decompose a polytopal domain of $\mathbb{R}^d$ into a finite number of simplexes S_i, which we call *barycentric simplexes*.

(2) In Section 3.2 we prove that each barycentric simplex S_i of a polytope admits a bi-Lipschitz embedding onto a barycentric simplex $\mathcal{S}_d$ of the d-simplex.

(3) We show that we can send isometrically the barycentric simplex of a d-simplex onto a cone of a vector space W_d, using a known isometric map between the d-simplex and W_d (see Section 3.3). This cone is then sent isometrically to the cone associated to a barycentric simplex of a polytope.

(4) Finally this allows us in Section 3.4 to define a map from the polytopal domain to $\mathbb{R}^d$ by patching the bi-Lipschitz embeddings associated to each of its barycentric simplexes.

3.1. Cell decomposition of the polytope. Consider $\mathcal{P}$ a polytope in $\mathbb{R}^d$. We will denote by f_{ij} the i^{th} face of dimension j, $1 \leqslant j \leqslant d$.

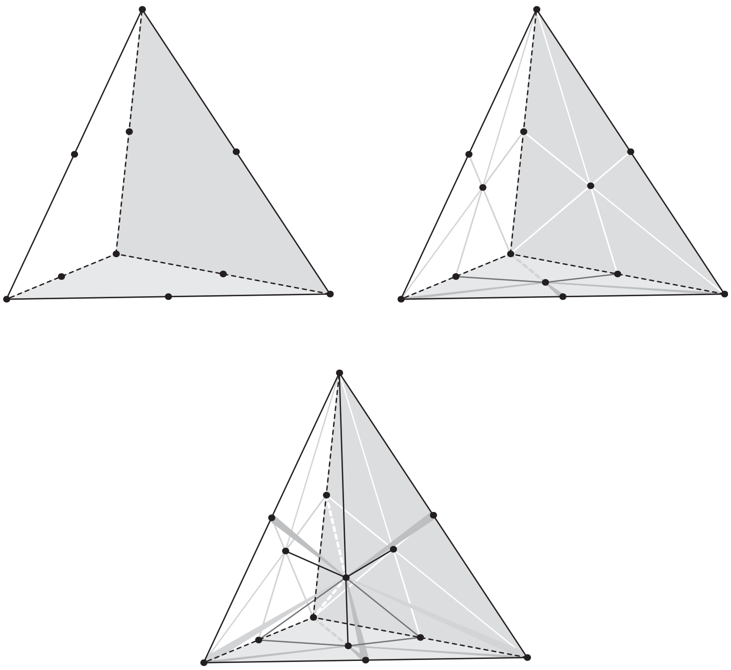

Fig. 10. The last three steps of the decomposition in dimension 3.

Let p_d be the barycentre of $\mathcal{P}$, and p_{ij} be the barycentre of the face f_{ij}. Let us denote by D_{ij} the half line from p_d to p_{ij}.

We recall the following well known property, emphasizing an aspect we need.

PROPERTY 9. A polytopal domain $\mathcal{P}$ in $\mathbb{R}^d$ can be uniquely decomposed as a union of d-dimensional simplexes, called *barycentric simplexes* or *cells*, such that the vertices are barycentres of the faces and each cell is a conical flag neighborhood of the polytope $\mathcal{P}$.

In the sequel let us adopt the following notations and conventions: If $\mathcal{P}$ is a polytope in $\mathbb{R}^d$, we will suppose that its barycentre is the origin and denote by S_i, for $i = 1, \dots, N$, its barycentric simplexes.

REMARK 2. The intersection of two barycentric simplexes is a lower-dimensional simplex: it is the closure of a common face containing the barycentre of the polytope.

S_i is the simplex whose vertexes are the point $v_{i,0}, \dots, v_{i,d}$, where $v_{i,d} = p_d$ is the barycentre of $\mathcal{P}$, and for $k = d-1, \dots, 0$, $v_{i,k}$ is the barycentre of a k-dimensional face, always on the boundary of the face to which $v_{i,k+1}$ belongs (see Fig. 11).

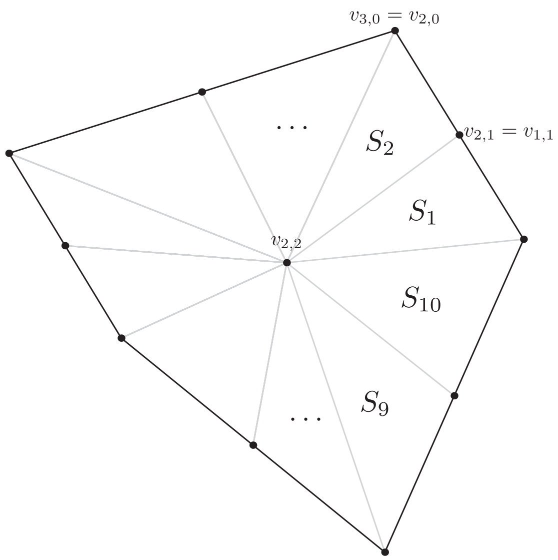

Fig. 11. Barycentric simplexes of a polygon.

To each $i = 1, \dots, N$ we will also associate the positive cone C_i based on p_d and defined by the vectors $\varpi_{i,k} = v_{i,k} - v_{i,d}$ for $k = d-1, \dots, 0$. We will call them the *barycentric cones* associated to the polytope (see Fig. 12).

The convex hull in $\mathbb{R}^{d+1}$ of the $d+1$ points $(1, 0, \dots, 0), (0, 1, \dots, 0), \dots, (0, 0, \dots, 1)$ will be denoted by $\mathfrak{S}_d$ and called *standard d-simplex*.

We will call *standard barycentric d-simplex* of the standard d-simplex, and denote it by $\mathcal{S}_d$, the convex hull of following the points (see Fig. 13):

$$\hat{v}_k := \Bigl(\underbrace{\frac{1}{k+1}, \dots, \frac{1}{k+1}}_{k+1 \text{ times}}, \underbrace{0, \dots, 0}_{d-k \text{ times}}\Bigr) \quad \text{for} \quad d \geqslant k \geqslant 0. \tag{7}$$

We will denote by W_d the d-dimensional hyperplane in $\mathbb{R}^{d+1}$ defined by the equation

$$x_1 + \dots + x_{d+1} = 0.$$

3.2. Embedding into the standard simplex. We keep the notations of the previous subsection. Let L_i be the linear map sending the barycentric simplex S_i onto the standard barycentric d-simplex $\mathcal{S}_d \subset \mathfrak{S}_n$ by mapping each point $v_{i,k}$ to $\hat{v}_k$.

Let $P_i = L_i(\mathcal{P})$ the image of the convex polytope by this linear map. L_i is an isometry between the Hilbert geometries of P_i and $\mathcal{P}$, in other words for any x in the interior of $\mathcal{P}$ we have (identifying L_i with its differential)

$$F_{P_i}(L_i(x), L_i(v)) = F_{\mathcal{P}}(x, v).$$

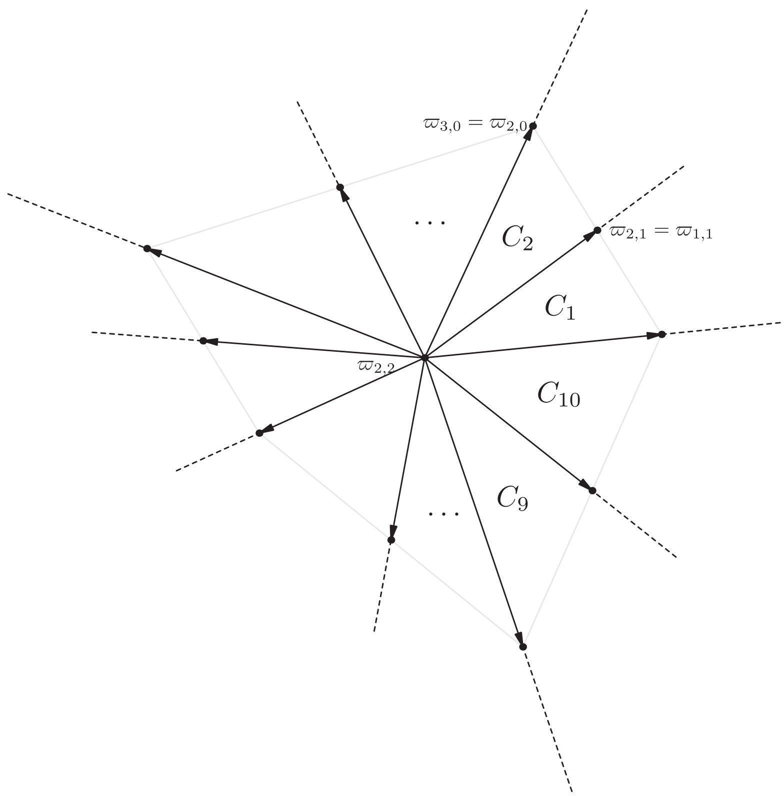

Fig. 12. Barycentric cones of a polygon.

This way, $\mathcal{S}_d$ is a common flag conical neighborhood of both P_i and $\mathfrak{S}_d$ and by Theorem 5 we obtain:

PROPERTY 10. There exists a constant k_i such that for any point x in the interior of the standard barycentric simplex $\mathcal{S}_d$ and any vector v one has

$$\frac{1}{k_i} \cdot F_{P_i}(x, v) \leqslant F_{\mathfrak{S}_d}(x, v) \leqslant k_i \cdot F_{P_i}(x, v).$$

3.3. From the standard simplex to W_d. Let $\Phi_d : \mathfrak{S}_d \to W_d \simeq \mathbb{R}^d \subset \mathbb{R}^{d+1}$ defined by

$$\begin{aligned}\Phi_d(x_1, \ldots, x_{d+1}) &= (X_1, \ldots, X_{d+1})\\ &= \left(\ln\left(\frac{x_1}{g}\right), \ldots, \ln\left(\frac{x_{d+1}}{g}\right)\right) \quad \text{with} \quad g = (x_1 \cdots x_{d+1})^{1/d+1}.\end{aligned}$$

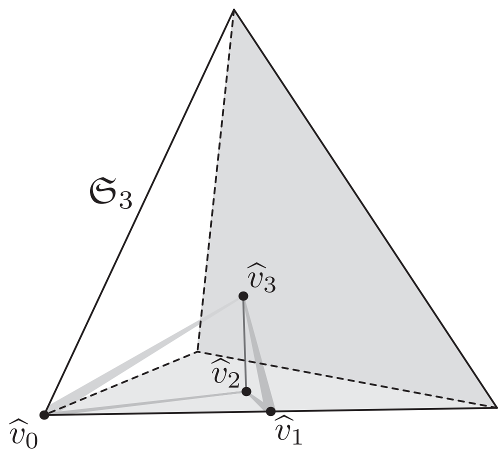

Fig. 13. The standard barycentric 3-simplex of the 3-simplex.

Thanks to P. de la Harpe [12] we know that Φ_d is an isometry from the simplex $\mathfrak{S}_d$ into W_d endowed with a norm whose unit ball is a centrally symmetric convex polytope.

For our purpose, let us remark that the image of the standard barycentric simplex $\mathcal{S}_d$ by Φ_d is the positive cone of W_d of summit at the origin and defined by the vectors

$$\tilde{v}_k := (\underbrace{d-k, \ldots, d-k}_{k+1 \text{ times}}, \underbrace{-(k+1), \ldots, -(k+1)}_{d-k \text{ times}}) \quad \text{for} \quad d > k \geqslant 0. \tag{8}$$

We denote by $\tilde{\mathcal{C}}_d = \Phi_d(\mathcal{S}_d)$ and call it *standard d-cone*.

Now for any polytopal convex set $\mathcal{P} \in \mathbb{R}^d$, consider the map M_i which maps the standard d-cone $\tilde{\mathcal{C}}_d$ into the barycentric cone C_i based on p_d, by sending the origin to p_d and the vector $\tilde{v}_k$ to the vector $\varpi_{i,k}$.

3.4. Conclusion. We can now define our bi-Lipschitz map

$$F\colon (\mathcal{P}, d_{\mathcal{P}}) \to (\mathbb{R}^d, \|\cdot\|)$$

in the following way.

$$\forall x \in S_i, \quad F(x) = M_i(\Phi_d(L_i(x))). \tag{9}$$

Following Remark 2, if $x \in \mathcal{P}$ is a common point of S_i and S_j, then necessarily $L_i(x) = L_j(x)$ thus,

$$\Phi_d(L_i(x)) = \Phi_d(L_j(x)) = y$$

and y is on boundary of the cone $\tilde{\mathcal{C}}_d$. Now $M_i(y) = M_j(y)$, because M_i and M_j send the corresponding boundary cone of $\tilde{\mathcal{C}}_d$ to the respective common boundary cone of the

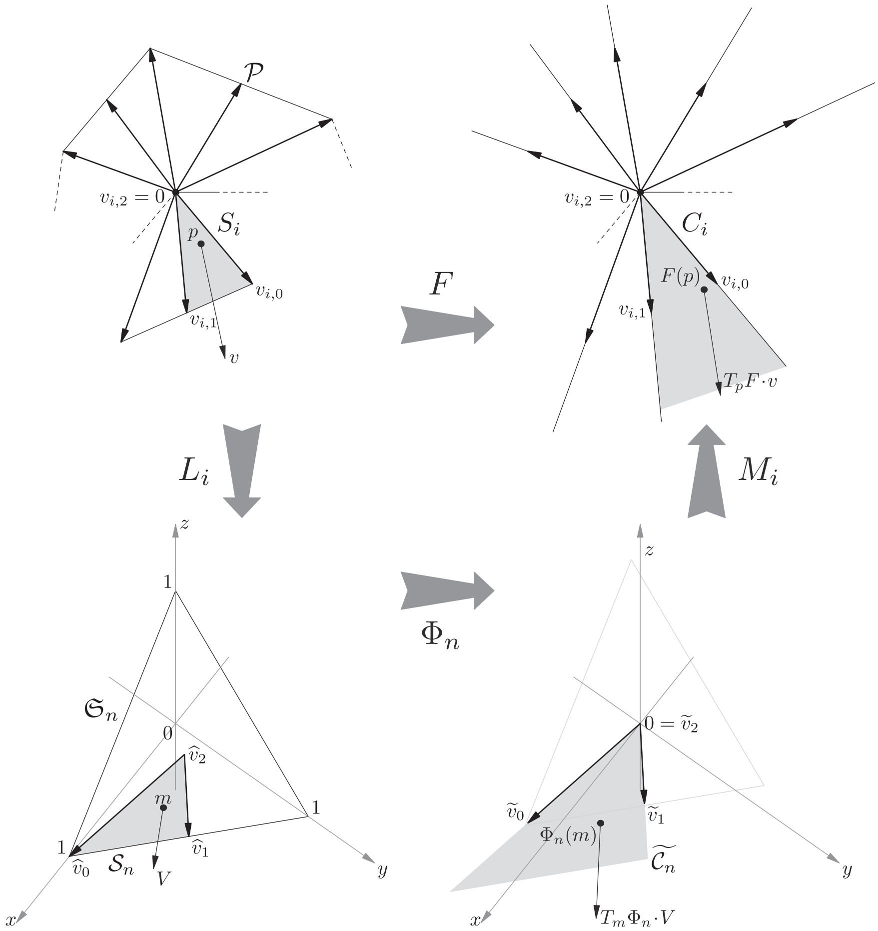

Fig. 14. The application F in dimension 2 illustrated.

cell-cones C_i and C_j in the same way. In other words,

$$\forall x \in S_i \cap S_j, \quad L_i(x) = L_j(x)$$

and

$$\forall z \in C_i \cap C_j, \quad M_i^{-1}(z) = M_j^{-1}(z)$$

thus F is well defined and it is a bijection.

To prove that it is bi-Lipschitz, we use the fact that line segments are geodesic and that both spaces are metric spaces.

Hence let p and q be two points in the polytope $\mathcal{P}$. Then there are M points $(p_j)_{j=1,\dots,M}$ on the segment $[p, q]$ such that $p = p_1$, $q = p_M$, and each segments

$[p_j, p_{j+1}]$, for $j = 1, \ldots, M-1$, belongs to a single simplex S_j of the simplex decomposition of $\mathcal{P}$.

Because of the key Property 10, and the fact that all norms in $\mathbb{R}^d$ are equivalent, we know that for each j, there is a constant k'_j such that, for $x, y \in S_j$, one has

$$\|F(x) - F(y)\| \leqslant k'_j \cdot d_{\mathcal{P}}(x, y).$$

Applying this to p_j, p_{j+1} for $j = 1, \ldots, M-1$, we obtain

$$\sum_{j=1}^{M-1} \|F(p_j) - F(p_{j+1})\| \leqslant (\sup_i k'_i) \cdot d_{\mathcal{P}}(p, q),$$

where the supremum is taken over all cells of the decomposition, then from the triangle inequality one concludes that

$$\|F(p) - F(q)\| \leqslant \left(\sup_i k'_i\right) \cdot d_{\mathcal{P}}(p, q).$$

Starting from a line from $F(p)$ to $F(q)$ and taking its inverse image after decomposing it in segments each of which is in a single barycentric-cone, we obtain in the same way the other inequality

$$d_{\mathcal{P}}(p, q) \leqslant \left(\sup_i k'_i\right) \cdot \|F(p) - F(q)\|.$$

4. Hilbert geometries bi-Lipschitz to a normed vector space

Let us make two remarks and give references on the reciprocal of Theorem 1.

Colbois and Verovic in [10] prove that a Hilbert geometry which quasi-isometrically embeds into a normed vector space is the Hilbert geometry of a polytope. Notice that in their paper they state a weaker result but actually prove this stronger statement.

In our paper [19] we prove that the asymptotic volume of a Hilbert geometry is finite if and only if it is the geometry of a polytope. Therefore, this allows us to conclude, without referring to the stronger result of Colbois and Verovic, that a Hilbert geometry bi-Lipschitz to a normed vector space comes from a polytope.

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Institut de mathématique et de modélisation de Montpellier
Université Montpellier 2
Case Courrier 051
Place Eugène Bataillon
F–34395 Montpellier Cedex
France
e-mail: Constantin.Vernicos@math.univ-montp2.fr