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HYPERELLIPTIC SURFACES WITH $K^2 < 4\chi - 6$

CARLOS RITO and MARÍA MARTÍ SÁNCHEZ

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Abstract

Let $S$ be a smooth minimal surface of general type with a (rational) pencil of hyperelliptic curves of minimal genus $g$. We prove that if $K^2_S < 4\chi(O_S) - 6$, then $g$ is bounded. The surface $S$ is determined by the branch locus of the covering $S \to S/i$, where $i$ is the hyperelliptic involution of $S$. For $K^2_S < 3\chi(O_S) - 6$, we show how to determine the possibilities for this branch curve. As an application, given $g > 4$ and $K^2_S - 3\chi(O_S) < -6$, we compute the maximum value for $\chi(O_S)$. This list of possibilities is sharp.

1. Introduction

For a smooth minimal hyperelliptic surface $S$ of general type, Xiao [8, Theorem 1] has proved that if

$$K^2_S < \frac{4g}{g+1}(\chi(O_S) - \epsilon g - 2),$$

where either $\epsilon = 1$ if $\chi(O_S) > (2g - 1)(g + 1) + 2$, or $\epsilon = 9/8$, then $S$ has a pencil of hyperelliptic curves of genus $\leq g$. This result is not very useful for $g > 4$ and $\chi(O_S)$ small. For example, in [1] Ashikaga and Konno consider surfaces $S$ of general type with $K^2_S = 3\chi(O_S) - 10$. For these surfaces the canonical map is of degree 1 or 2. In the degree 2 case, the canonical image is a ruled surface, thus if $S$ is regular, it has a pencil of hyperelliptic curves. By the above inequality, if $\chi(O_S) \geq 47$, then $S$ has such a hyperelliptic pencil of curves of genus $\leq 4$. But for $\chi(O_S) \leq 46$ this result gives no information (for $\chi(O_S) = 46$ the slope formula [7, Theorem 2] implies $g \leq 5 \vee g \geq 9$; we show that in this case $S$ has a hyperelliptic pencil of minimal genus $g \leq 10$ and the cases $g = 9, g = 10$ do occur). Ashikaga and Konno study only the case $g \leq 4$ (there is an infinite number of possibilities). Nothing is said for the possibilities with $g \geq 5$ and $\chi(O_S) \leq 46$. A similar situation occurs in [5].

In this paper we study smooth minimal surfaces $S$ of general type which have a pencil of hyperelliptic curves (by pencil we mean a linear system of dimension 1). We say that $S$ has such a pencil of minimal genus $g$ if it has a hyperelliptic pencil of genus $g$ and all hyperelliptic pencils of $S$ are of genus $\geq g$. For $S$ such that $K^2_S < 4\chi(O_S) - 6$,
we give bounds for the minimal genus \( g \) (Theorem 1), improving Xiao’s inequality in the cases \( g > 4 \) and \( \chi(O_S) \) small.

The surface \( S \) is the smooth minimal model of a double cover of an Hirzebruch surface \( \mathbb{F}_e \) ramified over a curve \( \tilde{B} \) (which determines \( S \)). We prove that if \( K_S^2 < 3\chi(O_S) - 6 \), then \( \tilde{B} \) has at most points of multiplicity 8 and we show how to determine the possibilities for \( \tilde{B} \) (Proposition 2).

As an application, given \( g > 4 \) and \( K_S^2 - 3\chi(O_S) < -6 \), we compute the maximum value for \( \chi(O_S) \); this list of possibilities is sharp (Theorem 3).

The paper is organized as follows. In Section 2 we present the main results of the paper. The hyperelliptic involutions of the fibres of \( S \) induce an involution \( i \) of \( S \), so in Section 3 we review some general facts on involutions. Since the quotient \( S/i \) is a rational surface, a smooth minimal model of \( S/i \) is not unique. We make a choice for this minimal model in Section 4 (which is due to Xiao [9]) and we show some consequences of it. Section 5 contains the key result of the paper, which allow us to compute bounds for the minimal genus of the hyperelliptic fibration. We perform a careful analysis of the possibilities for the branch locus of the covering \( S \to S/i \) considering the restrictions imposed by the choice of minimal model. Finally this is used in Section 6 to prove the main results, stated in Section 2.

Several calculations are made using a computational algebra system. The respective code lines are available at http://home.utad.pt/~crito/magma_code.html.

**Notation.** We work over the complex numbers; all varieties are assumed to be projective algebraic. A \((-2)\)-curve or nodal curve \( A \) on a surface is a curve isomorphic to \( \mathbb{P}^1 \) such that \( A^2 = -2 \). An \((m_1,m_2,...)\)-point of a curve, or point of type \((m_1,m_2,...)\), is a singular point of multiplicity \( m_1 \), which resolves to a point of multiplicity \( m_2 \) after one blow-up, etc. By double cover we mean a finite morphism of degree 2. The rest of the notation is standard in algebraic geometry.

**2. Main results**

**Theorem 1.** Let \( S \) be a smooth minimal surface of general type with a pencil of hyperelliptic curves of minimal genus \( g \). If \( K_S^2 < 4\chi(O_S) - 6 \), then \( g \) is not greater than

\[
\max \left\{ -1 + \frac{8\chi(O_S)}{4\chi(O_S) - K_S^2 - 6}, 1 + \frac{8\chi(O_S) - 16}{4\chi(O_S) - K_S^2 - 6}, 1 + \frac{8\chi(O_S)}{4\chi(O_S) - K_S^2 - 3}, \frac{3 + \sqrt{1 + 8\chi(O_S)}}{2} \right\}.
\]

Let \( B \subset W \) be the branch locus of a double cover \( V \to W \), where \( V \) and \( W \) are smooth surfaces (thus \( B \) is also smooth). Let \( \rho: W \to P \) be the projection of \( W \) onto a minimal model and denote by \( \tilde{B} \) the projection \( \rho(B) \).
Suppose that $\tilde{B}$ has singular points $x_1, \ldots, x_n$ (possibly infinitely near). For each $x_i$ there is an exceptional divisor $E_i$ and a number $r_i \in 2\mathbb{N}$ such that

$$E_i^2 = -1,$$

$$K_W \equiv \rho^*(K_P) + \sum E_i,$$

$$B = \rho^*(\tilde{B}) - \sum r_i E_i.$$

Notice that $r_i$ is not the multiplicity of the singular point $x_i$, it is the multiplicity of the corresponding singularity in the canonical resolution (see [2, III. 7]). For example, in the case of a point of type $(2r - 1, 2r - 1)$ one has $r_1 = 2r - 2$ and $r_2 = 2r$.

Since, from Theorem 1, we have a bound for the genus $g$, we also have a bound for the multiplicities $r_i$. For the case $K_S^2 < 3\chi(\mathcal{O}_S) - 6$, we prove the result below.

Proposition 2. Denote by $C_0$ and $F$ the negative section and a ruling of the Hirzebruch surface $\mathbb{F}_e$. Let $S$ be a minimal smooth surface of general type with a hyperelliptic pencil of minimal genus $(k - 2)/2$. If $K_S^2 < 3\chi(\mathcal{O}_S) - 6$, then $S$ is the smooth minimal model of a double cover $S' \to \mathbb{F}_e$ with branch curve $\tilde{B} \equiv kC_0 + (ek/2 + 1)F$ such that:

a) $r_i \leq \min\{8, k/2 + 2, l - k/2 + 2\} \forall i$;

b) $N_4 + N_6 = 15 + K_S^2 - 3\chi(\mathcal{O}_S) - (1/4)(k - 10)(l - 10)$;

c) $\chi(\mathcal{O}_S) = 1 + (1/4)(k - 2)(l - 2) - N_4 - 3N_6 - 6N_8$,

where $S'' \to S'$ is the canonical resolution.

Proposition 2 can be used to restrict possibilities for $\tilde{B}$. We show the following:

Theorem 3. Let $S$ be a smooth minimal surface of general type with a hyperelliptic pencil of minimal genus $g > 4$. If $K_S^2 < 3\chi(\mathcal{O}_S) - 6$, then $\chi(\mathcal{O}_S)$ is bounded by the number given in the table below (emptiness means non-existence). All these cases do exist.

<table>
<thead>
<tr>
<th>$K_S^2 - 3\chi$</th>
<th>$-7$</th>
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REMARK 4. This result gives three examples where Theorem 1 is almost sharp: in the cases \((g, K^2 - 3\chi) = (10, -10), (9, -13), (8, -15)\) we have \(\chi \leq 46, 37, 29\), thus Theorem 1 implies \(g \leq 11, 10, 9\), respectively (cf. Remark 10).

There is at least one case where Theorem 1 is sharp: a double plane with branch locus a curve of degree 18 with 8 points of multiplicity 6. In this case \(\chi = 5, K^2 = 8\) and \(g = 5\).

3. Involutions

Let \(S\) be a smooth minimal surface of general type with a (rational) pencil of hyperelliptic curves. This hyperelliptic structure induces an involution (i.e. an automorphism of order 2) \(i\) of \(S\). The quotient \(S/i\) is a rational surface.

Since \(S\) is minimal of general type, this involution is biregular. The fixed locus of \(i\) is the union of a smooth curve \(R''\) (possibly empty) and of \(t \geq 0\) isolated points \(P_1, \ldots, P_t\). Let \(p: S \to S/i\) be the projection onto the quotient. The surface \(S/i\) has nodes at the points \(Q_i := p(P_i), i = 1, \ldots, t\), and is smooth elsewhere. If \(R'' \neq \emptyset\), the image via \(p\) of \(R''\) is a smooth curve \(B''\) not containing the singular points \(Q_i, i = 1, \ldots, t\). Let now \(h: V \to S\) be the blow-up of \(S\) at \(P_1, \ldots, P_t\) and set \(R' = h^*(R'')\). The involution \(i\) induces a biregular involution \(\tilde{i}\) on \(V\) whose fixed locus is \(R'i\). The quotient \(W := V/\tilde{i}\) is smooth and one has a commutative diagram

\[
\begin{array}{ccc}
V & \xrightarrow{h} & S \\
\downarrow{\pi} & & \downarrow{p} \\
W & \xrightarrow{g} & S/i
\end{array}
\]

where \(\pi: V \to W\) is the projection onto the quotient and \(g: W \to S/i\) is the minimal desingularization map. Notice that

\[A_i := g^{-1}(Q_i), \quad i = 1, \ldots, t,\]

are \((-2)\)-curves and \(\pi^*(A_i) = 2 \cdot h^{-1}(P_i)\).

Set \(B' := g^*(B'')\). Since \(\pi\) is a double cover, its branch locus \(B' + \sum_i A_i\) is even, i.e. there is a line bundle \(L\) on \(W\) such that

\[2L = B := B' + \sum_i A_i.\]

4. Choice of minimal model

Part of this section may be found in [9]. We use the notation introduced so far. As above, \(W\) is a rational surface, thus either it is isomorphic to \(\mathbb{P}^2\) or its minimal model is an Hirzebruch surface \(\mathbb{F}_e\).
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Blowing-up, if necessary, $\mathbb{P}^2$ at a point, we can suppose that $W \neq \mathbb{P}^2$.

Notice that in this case the map $h: V \rightarrow S$ is the contraction of two $(-1)$-curves. With this assumption we do not need to consider the case $W = \mathbb{P}^2$ separately.

Thus there is a birational morphism

$$\rho: W \rightarrow \mathbb{F}_e.$$ 

Let $\tilde{B} := \rho(B)$ and consider the double cover $S' \rightarrow \mathbb{F}_e$ with branch locus $\tilde{B}$. If $\tilde{B}$ is singular then $S'$ is also singular and $S$ is isomorphic to the minimal smooth resolution of $S'$.

We can define $k$ and $l$ such that

$$\tilde{B} \equiv: kC_0 + \left( \frac{ek}{2} + l \right)F,$$

where $C_0$ and $F$ are, respectively, the negative section and a ruling of $\mathbb{F}_e$ (thus $C_0^2 = -e$, $C_0F = 1$, $F^2 = 0$). Notice that $\tilde{B}^2 = 2kl$ and $K_B \tilde{B} = -2k - 2l$.

$(\ast)$. Among all the possibilities for the map $\rho$, we choose one satisfying, in this order:

1) the degree $k$ of $\tilde{B}$ over a section is minimal;
2) the greatest order of the singularities of $\tilde{B}$ is minimal;
3) the number of singularities with greatest order is also minimal.

Recall that a $(2r - 1, 2r - 1)$ singularity of $\tilde{B}$ is a pair $(x_j, x_k)$ such that $x_k$ is infinitely near to $x_j$ and $r_j = 2r - 2$, $r_k = 2r$.

Let

$$r_m := \max\{r_i\}$$

or $r_m := 0$ if $\tilde{B}$ is smooth.

By elementary transformation over $x_i \in \mathbb{F}_e$ we mean the blow-up of $x_i$ followed by the blow-down of the strict transform of the ruling of $\mathbb{F}_e$ that contains $x_i$.

The following is a consequence of the two assumptions $(\ast)$ on the map $\rho$.

**Proposition 5** ([9]). We have:

a) If $k \equiv 0 \pmod{4}$, then $r_m \leq k/2 + 2$ and the equality holds only if $x_m$ belongs to a singularity $(k/2 + 1, k/2 + 1)$. In this last case $l \geq k + 2$ and all the branches of the singularity are tangent to the ruling of $\mathbb{F}_e$ that contains it.

b) If $k \equiv 2 \pmod{4}$, then $r_m \leq k/2 + 1$ and the equality holds only if $x_m$ belongs to a singularity $(k/2, k/2)$. In this case $l \geq k$.

In a similar vein:
Proposition 6. We have that:

a) if \( l = k + 2 \) and \( k > 8 \), there are at most two \((k/2 + 1, k/2 + 1)\)-points;

b) \( l \geq k/2 \) and \( l \geq k/2 + r_m - 2 \);

c) if \( l = k/2 + r_m - 2 \), then either:
   - \( e = 2 \), \( l = k - 2 \), the branch locus \( \tilde{B} \) has a \((k/2 - 1, k/2 - 1)\)-point and all singularities are of multiplicity \(< k/2\), or
   - we can suppose \( e = 1 \), the negative section \( C_0 \) of \( \mathbb{P}_1 \) is contained in \( \tilde{B} \), \( \tilde{B} \) has a point of multiplicity \( r_m \) contained in \( C_0 \) and the remaining singularities are of multiplicity \(< r_m \).

Proof. a) This is due to Borrelli ([3]). Suppose that there are three singularities \((k/2 + 1, k/2 + 1)\). The rulings of \( \mathbb{P}_e \) through these points are contained in \( \tilde{B} \) and then \( \tilde{B}C_0 = l - ek/2 \geq 4 \) (\( \tilde{B}C_0 \) is even). This implies \( e \leq 1 \). Making, if necessary, an elementary transformation over one of these points, we can suppose that \( e = 1 \).

Let \( \rho \) be as above and \( E_i, E'_i, i = 1, 2, 3 \), be the exceptional divisors corresponding to three singularities \((k/2 + 1, k/2 + 1)\) of \( \tilde{B} \). The general element of the linear system \([\rho^*(4C_0 + 5F) - \sum_i^3(2E_i + 2E'_i)]\) is a smooth and irreducible rational curve \( C \) such that \( CB < k \). This contradicts the choice (\( \ast \)) of the map \( \rho \).

b) If \( r_m > k/2 \) then the result follows from Proposition 5. Suppose now \( r_m \leq k/2 \). We have \( \tilde{B}C_0 \geq -e \), i.e. \( l - ek/2 \geq -e \). Therefore if \( e \geq 2 \), then

\[
  l \geq k - 2 \geq \frac{k}{2} \quad \text{and} \quad l \geq k - 2 \geq \frac{k}{2} + r_m - 2.
\]

When \( e = 0 \) we obtain immediately \( l \geq k \), by the choice of the map \( \rho \), thus \( l \geq k/2 + r_m \).

If \( e = 1 \) then \( \tilde{B}C_0 = l - k/2 \geq 0 \). Blowing-down \( C_0 \) we obtain a singularity of order at most \( l - k/2 + 1 \), hence the choice of the minimal model implies \( r_m \leq l - k/2 + 2 \) (notice that the equality happens only if the order of the singularity is \((r_m - 1, r_m - 1)\)).

c) Assume that \( l = k/2 + r_m - 2 \). Proposition 5 implies \( r_m \leq k/2 \). From \( \tilde{B}C_0 \geq -e \) we obtain \( k/2 + r_m - 2 = l \geq ek/2 - e \), thus either \( e = 1 \) or \( e = 2 \) and \( r_m = k/2 \) (notice that \( e = 0 \) implies \( l \geq k \)).

In the case \( e = 1 \) we can, as in the proof of b), contract the section with self-intersection \((-1)\) to obtain a branch curve in \( \mathbb{P}^2 \) with at most singularities of type \((l - k/2 + 1, l - k/2 + 1)\).

Suppose now that \( e = 2 \) and there is a point \( x_i \) of multiplicity \( k/2 \). In this case \( \tilde{B}C_0 = -2 \), hence \( x_i \notin C_0 \). We make an elementary transformation over \( x_i \) to obtain the case \( e = 1 \) also with \( l = k - 2 \).
5. Bound of genus

In this section we prove the key result to establish bounds for the minimal genus of the hyperelliptic fibrations.

From [6] (cf. also [4]), we get the following:

**Proposition 7.** Let \( S'' \to S' \) be the canonical resolution of a double cover \( S' \to \mathbb{P}^{1} \) with branch locus \( \tilde{B} = kC_0 + (ek/2 + l)F \). Let \( S \) be the minimal model of \( S'' \) and \( t := K_S^2 - K_S^2 \). If \( S \) is of general type, then:

a) \( \sum (r_i - 2) (k - r_i - 2) = H \);

b) \( 2l = G + \sum (r_i - 2) \),

where

\[
H = 2k^2 - 4k \chi(O_S) + t - K_S^2 + 8 + 2t - 2K_S^2
\]

and

\[
G = -2k + 4 \chi(O_S) + t - K_S^2 + 8.
\]

Proof. From [6, Propositions 2 and 3, a)] one gets:

a) \( 2kl = -48 + 12l + 12k - 8 \chi(O_S) + 4K_S^2 - 4t + \sum (r_i - 2)(r_i - 4) \);

b) \( 2k + 2l = 8 + 4 \chi(O_S) + t - K_S^2 + \sum (r_i - 2) \).

The result is obtained replacing (a) by (a) + (6 - k)(b). \( \square \)

The motivation for Lemma 8 and Proposition 9 below is the following. Among all the solutions of the equations of Proposition 7, the ones with biggest \( l \) correspond to the solutions with singularities of maximal order. This gives an upper bound for \( l \). But we also have a lower bound for \( l \), implied by the assumptions (*) on the map \( \rho \) (Propositions 5 and 6). We note that the arguments used in the proofs are mostly formal.

**Lemma 8.** Suppose that \( k > 8 \). With the above notation, we have

a) \( 2l \leq G + H/(k - r_m - 2) \), and

b) if \( r_m \) is obtained only from singularities of type \((r_m - 1, r_m - 1)\), then

\[
2l \leq G + \frac{H}{(r_m - 4)(k - r_m) + (r_m - 2)(k - r_m - 2)(2r_m - 6)}.
\]

Proof. a) Proposition 5 implies \( r_m \leq k/2 + 2 \). If \( k - r_m - 2 \leq 0 \), we get from \( k - 2 \leq r_m \leq k/2 + 2 \) that \( k \leq 8 \). Hence \( k - r_m - 2 > 0 \) and the statement follows from Proposition 7.

b) By the assumptions, if \( x_i \) does not belong to a \((r_m - 1, r_m - 1)\) singularity, we have \( r_i < r_m \). Let \( n \geq 1 \) be the number of singularities of type \((r_m - 1, r_m - 1)\) and \( s \geq 0 \) be the number of singular points \( x_j \) of another type. As seen in Section 4, each
singularity \( (r_m - 1, r_m - 1) \) corresponds to two infinitely near singular points \( x_k, x_{k+1} \) with \( r_k = r_m - 2, r_{k+1} = r_m \). Therefore

\[
\sum_{i=1}^{2n+s} (r_i - 2) = n(2r_m - 6) + \sum_{j=1}^{s} (r_j - 2),
\]

with \( r_j < r_m \). Thus from Proposition 7, b) we get

\[
2l = G + n(2r_m - 6) + \sum_{j=1}^{s} (r_j - 2).
\]

By Proposition 7, a),

\[
H = n((r_m - 4)(k - r_m) + (r_m - 2)(k - r_m - 2)) + \sum_{j=1}^{s} (r_j - 2)(k - r_j - 2),
\]

hence

\[
n = \frac{H - \sum_{j=1}^{s} (r_j - 2)(k - r_j - 2)}{(r_m - 4)(k - r_m) + (r_m - 2)(k - r_m - 2)}
\]

and then

\[
(1) \quad 2l = G + \frac{H - \sum_{j=1}^{s} (r_j - 2)(k - r_j - 2)}{(r_m - 4)(k - r_m) + (r_m - 2)(k - r_m - 2)} (2r_m - 6) + \sum_{j=1}^{s} (r_j - 2).
\]

Since \( r_j < r_m, j = 1, \ldots, s \),

\[
(r_m - 4)(k - r_m) + (r_m - 2)(k - r_m - 2) \leq (2r_m - 6)(k - r_j - 2).
\]

This implies

\[
\sum_{j=1}^{s} (r_j - 2) \leq \sum_{j=1}^{s} \frac{(r_j - 2)(k - r_j - 2)(2r_m - 6)}{(r_m - 4)(k - r_m) + (r_m - 2)(k - r_m - 2)}
\]

and the result follows from (1). \( \square \)

The next result will allow us to give bounds for \( k \). Notice that, since \( \tilde{B} \) is even and \( \tilde{B}C_0 = l - ek/2 \),

\[
k \equiv 0 \pmod{4} \quad \implies \quad l \equiv 0 \pmod{2}.
\]

**Proposition 9.** In the conditions of Proposition 7, suppose that \( k > 8 \). If \( k \equiv 0 \pmod{4} \), one of the following holds:
a) $r_m = k/2 + 2$, $l = k + 2$ and

$$ (4 \chi(O_S) + t - K_S^2 - 8)k \leq 16 \chi(O_S) - 16, \quad \text{with} \quad t \geq 2; $$

b) $r_m = k/2 + 2$, $l \geq k + 4$ and

$$ (4 \chi(O_S) + t - K_S^2 - 8)k^2 - 16 \chi(O_S)k + 32 \chi(O_S) \leq 0, \quad \text{with} \quad t \geq 2; $$

c) $r_m = k/2$, $l = k - 2$ and

$$ (4 \chi(O_S) + t - K_S^2 - 4)k^2 + (-48 \chi(O_S) - 8t + 8K_S^2 + 32)k $$

$$ + 160 \chi(O_S) + 16t - 16K_S^2 - 96 \leq 0, \quad \text{with} \quad t \geq 1, $$

or

$$ (4 \chi(O_S) + t - K_S^2 + 2)k \leq 32 \chi(O_S) + 4t - 4K_S^2 - 8, \quad \text{with} \quad t \geq 1, $$

or

$$ (4 \chi(O_S) + t - K_S^2 - 5)k^2 + (-48 \chi(O_S) - 8t + 8K_S^2 + 44)k $$

$$ + 160 \chi(O_S) + 16t - 16K_S^2 - 128 \leq 0, \quad \text{with} \quad t \geq 2; $$

d) $r_m = k/2$, $l = k + j$, $j \geq 0$, and

$$ (4 \chi(O_S) + t - K_S^2 + 8 + 2j - 2n)k \leq 32 \chi(O_S) + 4t - 4K_S^2 - 8n, $$

with $n \leq j + 7$, where $n$ is the number of points $x_i$ (possibly infinitely near) such that $r_i = k/2$;

e) $r_m \leq k/2 - 2$ and

$$ k \leq 5 + \sqrt{1 + 8 \chi(O_S)}, $$

or

$$ (4 \chi(O_S) + t - K_S^2)k \leq 32 \chi(O_S) + 4t - 4K_S^2. $$

If $k \equiv 2 \pmod{4}$, one of the following holds:

f) $r_m = k/2 + 1$ and

$$ (4 \chi(O_S) + t - K_S^2 - 2)k \leq 24 \chi(O_S) + 2t - 2K_S^2 - 20, \quad \text{with} \quad t \geq 1, $$

or

$$ (4 \chi(O_S) + t - K_S^2 - 8)k^2 + (-32 \chi(O_S) - 4t + 4K_S^2 + 48)k $$

$$ + 80 \chi(O_S) + 4t - 4K_S^2 - 96 \leq 0, \quad \text{with} \quad t \geq 2; $$

g) $r_m \leq k/2 - 1$ and

$$ k \leq 5 + \sqrt{1 + 8 \chi(O_S)}, $$

or

$$ 2(4 \chi(O_S) + t - K_S^2 - 6)k \leq 24 \chi(O_S) + 2t - 2K_S^2 - 28. $$
Remark 10. As noted in Remark 4, there are examples where cases e) and g) fail to be sharp by 1. The reason for not having a sharp result is the following: in these examples we have \( r_m = 0 \), thus we are using \( l \geq k/2 - 2 \) in the proof of e) and g). But in fact we have \( l \geq k/2 \) in these cases, from Proposition 6, b).

The last example referred in Remark 4 shows that case d) with \( k/2 - 1, j = 0, n = 7 \) is sharp.

Proof of Proposition 9. Let \( H, G \) be as defined in Proposition 7 and let

\[
P_1(l, r_m, G, H, k) := (2l - G)(k - r_m - 2) - H,\]

\[
P_2(l, r_m, G, H, k) := (2l - G)((r_m - 4)(k - r_m) + (r_m - 2)(k - r_m - 2)) - H(2r_m - 6).
\]

From Lemma 8,

\[ P_1 \leq 0 \quad \text{and} \quad P_2 \leq 0. \]

a) Let \( n \) be the number of \( (k/2 + 1, k/2 + 1) \) points. From Propositions 5, a) and 6, a), \( n = 1 \) or 2. From Proposition 7, we have

\[
\sum (r_i - 2)(k - r_i - 2) = H' \quad \text{and} \quad 2l = G' + \sum (r_i - 2),
\]

where

\[ H' = H - n(k/2(k/2 - 4) + (k/2 - 2)^2), \quad G' = G + n(k - 2) \]

and \( r_i \leq k/2, \forall i \).

The result follows from

\[ P_1(k + 2, k/2, G', H', k) \leq 0. \]

Notice that \( t \geq 2n \).

b) From Proposition 5, there are at most \( (k/2 + 1, k/2 + 1) \) singularities. The inequality

\[ P_2(k + 4, k/2 + 2, G, H, k) \leq 0 \]

gives the result.

c) Let \( n \) be the number of points of multiplicity \( k/2 \) and \( m \) be the number of \( (k/2 - 1, k/2 - 1) \) singularities. From Proposition 6, c), \( n = 0 \) or 1.

If \( n = 0 \), then \( r_m = k/2 \) implies \( m \geq 1 \) (thus \( t \geq 1 \)). From

\[ P_2(k - 2, k/2, G, H, k) \leq 0 \]

one gets the first inequality.
Suppose $n = 1$. Notice that, as shown in the proof of Proposition 6, c), the point of multiplicity $k/2$ is obtained from the blow-up of $\mathbb{P}^2$ at a point of type $(k/2 - 1, k/2 - 1)$. Hence $t \geq 1$.

Let 

$$H' := H - (k/2 - 2)^2, \quad G' = G + k/2 - 2$$

(we remove the contribution of the point of multiplicity $k/2$).

If $m = 0$, then 

$$P_1(k - 2, k/2 - 2, G', H', k) \leq 0$$

implies the second inequality.

If $m > 0$, then 

$$P_2(k - 2, k/2, G', H', k) \leq 0$$

gives the third inequality. In this case $t \geq 2$.

d) Let $j := l - k$ and let $n$ be the number of points $x_i$ (possibly infinitely near) such that $r_i = k/2$. From Proposition 7, we have 

$$\sum (r_i - 2)(k - r_i - 2) = H' \quad \text{and} \quad 2l = G' + \sum (r_i - 2),$$

where 

$$H' = H - n(k/2 - 2)^2, \quad G' = G + n(k/2 - 2)$$

and $r_i \leq k/2 - 2, \forall i$.

The inequality 

$$P_1(k + j, k/2 - 2, G', H', k) \leq 0$$

gives 

$$(4\chi(\mathcal{O}_S) + t - K_S^2 + 8 + 2j - 2n)k \leq 32\chi(\mathcal{O}_S) + 4t - 4K_S^2 - 8n.$$

It only remains to show that $n \leq j + 7$.

One can verify, using the double cover formulas (see e.g. [2, V. 22]), that $n \geq j + 8$ implies $\chi(\mathcal{O}_S) < 1$, except for $n = 8$, $l = k$ and $n = 10, k = 12, l = 14$. We claim that in these cases $K_S^2 \leq 0$. This is impossible because $S$ is of general type.

Proof of the claim. From the double cover formulas one gets that $\chi(\mathcal{O}_S) \leq 2$ and there is at least a $(-2)$-curve $A$ contained in the branch curve $B$, otherwise $K_S^2 \leq 0$. One has 

$$B \equiv -\frac{k}{2}K_W + (l - k)\bar{F} + \sum \left(\frac{k}{2} - r_i\right)E_i,$$

where $\bar{F}$ is the total transform of $F$ and each $E_i$ is an exceptional divisor with self-intersection $-1$. Since $AB = -2$, $AK_W = 0$, $l \geq k$ and $r_i \leq k/2 \forall i$, we have $AE_i < 0$.
for some \( i \) such that \( r_i < k/2 \). The only possibility is the existence of a \((3, 3)\)-point in \( \tilde{B} \) and \( \chi(\mathcal{O}_S) = 1 \). But the imposition of such a singularity in the branch locus decreases the self-intersection of the canonical divisor by 1, thus \( K_5^2 \leq 0 \).

e) From Proposition 6, b), \( l \geq k/2 + r_m - 2 \). Let

\[
f(r_m) := P_1(k/2 + r_m - 2, r_m, G, H, k).
\]

We have

\[
f(r_m) = -2r_m^2 + br_m + c \leq 0,
\]

where

\[
b = 4\chi(\mathcal{O}_S) + t - K_5^2 - k + 8
\]

and

\[
c = k^2 - 10k - 8\chi(\mathcal{O}_S) + 24.
\]

Suppose that \( c = f(0) > 0 \) (i.e. \( k > 5 + \sqrt{1 + 8\chi(\mathcal{O}_S)} \)). Then \( f(r_m) \) has exactly one positive root \( x \). One has

\[
4x - b = \sqrt{b^2 + 8c}
\]

and \( k/2 - 2 \geq r_m \geq x \) implies that

\[
(4(k/2 - 2) - b)^2 \geq b^2 + 8c.
\]

This inequality gives the result.

f) Let \( n \) be the number of points of type \((k/2, k/2)\).

If \( n = 1 \), we proceed as in a), with \( l \geq k \).

If \( n > 1 \), the inequality is given by

\[
P_2(k, k/2 + 1, G, H, k) \leq 0.
\]

g) It is analogous to the proof of e): in this case the result follows from \( k/2 - 1 \geq r_m \geq x \).

6. Proof of main results

Proof of Theorem 1. Consider the parabola given by \( f(x) = ax^2 + bx + c \), with \( a > 0 \). If \( f(k) \leq 0 \), \( f(z) \geq 0 \) and \( z \geq -b/2a \) (the first coordinate of the vertex), then \( k \leq z \).

This fact and Proposition 9 imply that, if \( K_5^2 < 4\chi(\mathcal{O}_S) - 6 \), one of the following holds:
a) \( k \leq (16\chi(\mathcal{O}_S) - 16)/(4\chi(\mathcal{O}_S) - K_5^2 - 6); \)
b) \( k \leq (16\chi(\mathcal{O}_S))/(4\chi(\mathcal{O}_S) + t - K_S^2 - 8), \ t \geq 2; \)

c) \( k \leq 4 + (16\chi(\mathcal{O}_S))/(4\chi(\mathcal{O}_S) + t - K_S^2 - 4), \ t \geq 1; \)

c') \( k \leq 4 + (16\chi(\mathcal{O}_S) - 4)/(4\chi(\mathcal{O}_S) + t - K_S^2 - 5), \ t \geq 2; \)

d) \( k \leq 4 + (16\chi(\mathcal{O}_S) - 32)/(4\chi(\mathcal{O}_S) - K_S^2 - 6); \)

e) \( k \leq 5 + \sqrt{1 + 8\chi(\mathcal{O}_S)}; \)

e') \( k \leq 4 + (16\chi(\mathcal{O}_S) - 2K_S^2); \)

\( f) \ k \leq 2 + (16\chi(\mathcal{O}_S) - 16)/(4\chi(\mathcal{O}_S) - K_S^2 - 1); \)

\( f') \ k \leq 2 + (16\chi(\mathcal{O}_S) - 16)/(4\chi(\mathcal{O}_S) + t - K_S^2 - 8), \ t \geq 2; \)

g) \( k \leq 5 + \sqrt{1 + 8\chi(\mathcal{O}_S)}; \)

g') \( k \leq 2 + (16\chi(\mathcal{O}_S) - 16)/(4\chi(\mathcal{O}_S) - K_S^2 - 6). \)

We want to show that \( k \) is not greater than

\[
\max \left\{ \frac{16\chi(\mathcal{O}_S)}{4\chi(\mathcal{O}_S) - K_S^2 - 6}, \ 4 + \frac{16\chi(\mathcal{O}_S) - 32}{4\chi(\mathcal{O}_S) - K_S^2 - 6}, \ 4 + \frac{16\chi(\mathcal{O}_S)}{4\chi(\mathcal{O}_S) - K_S^2 - 3}, \ 5 + \sqrt{1 + 8\chi(\mathcal{O}_S)} \right\}.
\]

The result follows easily. Just notice that

\[
4\chi(\mathcal{O}_S) - K_S^2 - 6 \leq 8 \implies 2 + \frac{16\chi(\mathcal{O}_S) - 16}{4\chi(\mathcal{O}_S) - K_S^2 - 6} \leq \frac{16\chi(\mathcal{O}_S)}{4\chi(\mathcal{O}_S) - K_S^2 - 6}
\]

and

\[
4\chi(\mathcal{O}_S) - K_S^2 - 6 \geq 8 \implies 2 + \frac{16\chi(\mathcal{O}_S) - 16}{4\chi(\mathcal{O}_S) - K_S^2 - 6} \leq 4 + \frac{16\chi(\mathcal{O}_S) - 32}{4\chi(\mathcal{O}_S) - K_S^2 - 6}. \quad \square
\]

Proof of Proposition 2. Let (\( \alpha \), (\( \beta \)) be the equations of Proposition 7, a), b), respectively. One has that \([\alpha] + (k - 10)(\beta)]/8 \) is equivalent to

\[
(2) \quad \frac{1}{8} \sum (r_i - 2)(8 - r_i) = 15 + K_S^2 - t - 3\chi(\mathcal{O}_S) - \frac{1}{4}(k - 10)(l - 10)
\]

and (\( \beta \) + (2)) is equivalent to

\[
(3) \quad \chi(\mathcal{O}_S) = 1 + \frac{1}{4}(k - 2)(l - 2) - \frac{1}{8} \sum r_i(r_i - 2).
\]

Now it suffices to show that \( r_m \leq 8. \)

Suppose that \( K_S^2 < 3\chi(\mathcal{O}_S) - 6. \)

From [8, Theorem 1] one gets that if \( \chi(\mathcal{O}_S) \geq 54, \) then \( S \) has a pencil of hyperelliptic curves of genus \( \leq 6. \) In this case \( k \leq 14, \) thus \( r_m \leq k/2 + 2 \) implies \( r_m \leq 8. \)

From the proof of Theorem 1 we obtain that if \( \chi(\mathcal{O}_S) \leq 31, \) then one of the possibilities below occur. In all cases \( r_m \leq 8. \)
a) and b) $k < 16$, $r_m < 8$;
c) c') and d) $k \leq 18$, $r_m = k/2 \leq 8$;
e) $k \leq 20$, $r_m \leq k/2 - 2 \leq 8$;
e') $k \leq 16$, $r_m \leq k/2 - 2 \leq 6$;
f) $k \leq 14$, $r_m = k/2 + 1 \leq 8$;
f') $k \leq 16$, $r_m = k/2 + 1 \leq 8$;
g) $k \leq 18$, $r_m \leq k/2 - 1 \leq 8$;
g') $k \leq 14$, $r_m \leq k/2 - 1 \leq 6$.

Suppose now that $32 \leq \chi(O_S) \leq 53$. From Theorem 1 we get that $k \leq 18$ or $k \leq 5 + \sqrt{1 + 8\chi(O_S)}$. In this last case $k \leq 24$ and $r_m \leq k/2 - 1$ (see Proposition 9 e), g)). Thus we have $r_m \leq 18/2 + 2$ or $r_m \leq 24/2 - 1$. Since $r_m$ is even, $r_m \leq 10$.

Let $N_j$ be the number of points $x_i$ such that $r_i = j$. We have

\[ \sum (r_i - 2) \geq 8N_{10} + 6N_8 \]

and, from (2),

\[ 8N_{10} \geq (k - 10)(l - 10) - 32. \]

Using Proposition 7, b) and the assumption $\chi(O_S) \geq 32$, this implies

\[ 2l + 2k \geq 15 + (k - 10)(l - 10) + 6N_8, \]

or equivalently

\[ (k - 12)(l - 12) \leq 29 - 6N_8. \]

Suppose $r_m = 10$. Then Propositions 5 and 6 give two possibilities:

- $k = 16$, $l \geq k + 2 = 18$, there is a singularity of type $(9, 9)$ ($N_8 \geq 1$);
- $k \geq 18$, $l \geq k/2 + r_m - 2 \geq 17$.

Both cases contradict (4). We conclude that $r_m \leq 8$. \hfill \Box

Proof of Theorem 3. First we claim that if $A$ is a $(-2)$-curve contained in the branch curve $B$, the image $\tilde{A}$ of $A$ in $\mathbb{F}_e$ does not intersect a negligible singularity of $\tilde{B}$, unless $\tilde{A}$ is the negative section of $\mathbb{F}_1$ and the only singularity of $\tilde{B}$ is a double point in $C_0$ (this corresponds to a smooth branch curve in $\mathbb{P}^2$). In fact otherwise there is a $(-1)$-curve $E$ such that $AE = 1$ or 2. If $AE = 1$, then $A + E$ can be contracted to a smooth point of the branch curve $\tilde{B} \subset \mathbb{F}_e$. This is a contradiction because the canonical resolution blows-up only singular points of $\tilde{B}$. Suppose $AE = 2$. The inverse image of $A$ is a $(-1)$-curve which contracts to a smooth point of $S$. The inverse image of $E$ is then contracted to a curve $\tilde{E}$ with arithmetic genus 1 and $\tilde{E}^2 = 2$. We obtain from the adjunction formula that $K_S \tilde{E} = -2$, which is impossible because $S$ is of general type.

Recall that $t := K_S^2 - K_{\tilde{S}}^2$. The following holds:
(1) \( l \geq k/2 \) (Because \( l - ek/2 = \bar{B}C_0 \geq -e \) and \( \bar{B}C_0 \) is even);
(2) \( l = k/2 \iff (t = 2 \land N_4 = N_6 = N_8 = 0) \) (In this case \( e = 1 \) and \( \bar{B}C_0 = 0 \);
(3) \( l = k/2 + 2 \iff (N_6 = N_8 = 0 \land t \geq N_4 \land (t = N_4 \lor N_4 > 1)) \); (If \( N_4 \neq 0 \), this corresponds to a branch curve in \( \mathbb{P}^2 \) with \( N_4 \) points of type \( (3, 3) \) (see Proposition 6, b), c)).
(4) \( l = k - 2 \land t = 0 \Longrightarrow k/2 \) even; (As in (1), \( l \geq ek/2 - e \), thus \( e \leq 2 \). If \( e = 2 \), \( \bar{B}C_0 = -2 \) implies \( t \geq 1 \). Hence \( e = 1 \) and then \( l \) even implies \( k/2 \) even.);
(5) \( l < k - 2 \Longrightarrow l - k/2 \) even; (As in (1), \( l \geq ek/2 - e \), thus \( e = 1 \) and then \( l - k/2 = \bar{B}C_0 \) is even.)
(6) \( t = 1 \land N_4 = N_6 = N_8 = 0 \Longrightarrow l = k - 2 \). (If there are only negligible singularities, \( t = 1 \) is only possible if the negative section of \( F_2 \) is an isolated component of the branch locus.)

For given values of \( K_S^2 - 3\chi(O_S) \) and \( k \), we want to choose the solution of the equation given in Proposition 2, b) which maximizes the value of \( \chi(O_S) \), given by the equation in Proposition 2, c). We can assume \( N_6 = N_8 = 0 \).

It suffices to compute the numerical possibilities for Proposition 2, b), c) which satisfy conditions (1), \ldots, (6). We note the following: since \( k \geq 12 \), [8, Theorem 1] implies \( \chi(O_S) \leq 69 \), then Theorem 1 gives \( k \leq 28 \); \( l \geq k/2 \), \( k \geq 12 \) and (2) imply \(-7 \geq K_S^2 - 3\chi(O_S) \geq -18 + t + N_4 \), thus \( K_S^2 - 3\chi(O_S) \geq -18 \), \( t \leq 11 \) and \( N_4 \leq 11 \).


It remains to prove the existence. All cases can be constructed as double covers of \( \mathbb{P}^2 \), \( F_0 \), \( F_1 \) or \( F_2 \). The table below contains information about \( l \) or the degree of the branch curve in \( \mathbb{P}^2 \) and about the singularities of the branch curve, if any.

<table>
<thead>
<tr>
<th>( K^2 - 3\chi )</th>
<th>( g )</th>
<th>( -7 )</th>
<th>( -8 )</th>
<th>( -9 )</th>
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<td>( F_0 ), ( l = 24 )</td>
<td>( F_0 ), ( l = 22 )</td>
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<tr>
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<td>( F_1 ), ( l = 17 )</td>
<td>( F_0 ), ( l = 16 )</td>
<td>( F_1 ), ( l = 15 )</td>
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</tr>
<tr>
<td>7</td>
<td>( F_1 ), ( l = 14 ), (3, 3)</td>
<td>( F_2 ), ( l = 14 )</td>
<td>( F_1 ), ( l = 14 )</td>
<td>( F_1 ), ( l = 12 ), (3, 3)</td>
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<tr>
<td>8</td>
<td>( F_1 ), ( l = 13 ), (3, 3)</td>
<td>( F_1 ), ( l = 13 ), (4)</td>
<td>( F_1 ), ( l = 13 )</td>
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<td></td>
</tr>
<tr>
<td>9</td>
<td>( \mathbb{P}^2 ), 22, (3, 3)</td>
<td>( F_1 ), ( l = 12 )</td>
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<td>10</td>
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<td>( \mathbb{P}^2 ), 22</td>
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<table>
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<th>( g )</th>
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<th>( -12 )</th>
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<th>( -15 )</th>
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<td>( F_0 ), ( l = 16 )</td>
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<td>( F_1 ), ( l = 13 )</td>
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<td>( \mathbb{P}^2 ), 18, (3, 3)</td>
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<td>( \mathbb{P}^2 ), 20</td>
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Suppose first that $S$ is smooth and is the double cover of an Hirzebruch surface $\mathbb{F}_e$ with branch locus $B = 2L = kC_0 + (ek/2 + 1)F$. We get from the double cover formulas (see e.g. [2]) that

$$\chi(\mathcal{O}_S) = 2\chi(\mathcal{O}_{\mathbb{F}_e}) + \frac{1}{2}L(K_{\mathbb{F}_e} + L) = 2 + \frac{1}{4}kl - \frac{1}{2}(k + l)$$

and

$$K_S^2 = 2(K_{\mathbb{F}_e} + L)^2 = 16 - 4(k + l) + kl.$$ 

Now we compute $\chi$ and $K^2$ for the cases given in the table above taking in account that a 4-uple point in the branch locus decreases $K^2$ by 2 and $\chi$ by 1 and a $(3,3)$-point decreases both $K^2$ and $\chi$ by 1. Notice that $k = 2g + 2$.

Finally if $S$ is a double cover of $\mathbb{P}^2$ with branch locus a smooth curve of degree $d$, then

$$\chi(\mathcal{O}_S) = 2 + \frac{1}{8}d(d - 6) \quad \text{and} \quad K_S^2 = \frac{1}{2}(d - 6)^2.$$ 

The result follows by computing $\chi$ and $K^2$ for $d = 16, 18, 20$ and 22.

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References

HYPERELLIPTIC SURFACES with $K^2 < 4g - 6$


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