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| Author(s) | Kaji, Shizuo |
| Citation | Osaka Journal of Mathematics. 2015, 52(1), p. <br> $31-41$ |
| Version Type | VoR |
| URL | https://doi.org/10.18910/57674 |
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# WEYL GROUP SYMMETRY ON THE GKM GRAPH OF A GKM MANIFOLD WITH AN EXTENDED LIE GROUP ACTION 

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(Received January 23, 2013, revised June 10, 2013)


#### Abstract

We consider the class of manifolds with compact Lie group actions which restrict to GKM-actions on the maximal torus. First, we see their GKM-graphs admit symmetry of the Weyl groups. And then, we study its combinatorial abstraction; starting with abstract GKM-graphs with symmetry, we derive certain properties which reflect topology in a purely combinatorial way.


## 1. Introduction

Let $T$ be a compact $n$-dimensional torus acting on a closed oriented $m$-manifold $M$ with finite fixed points. If $M$ satisfies certain conditions (see $\S 2$ ), it is called a GKM-manifold. Goresky, Kottwitz and MacPherson ([4]) developed a powerful method to compute the torus equivariant cohomology $H_{T}^{*}(M ; \mathbb{R})$ for a GKM-manifold $M$, by associating it a combinatorial data called the GKM-graph.

What we focus on in this note are the GKM-manifolds with extended Lie group actions. Let $G$ be a compact, simple, simply-connected Lie group with the maximal torus $T$. If $M$ admits a $G$-action whose restriction to $T$ equips $M$ a GKM-manifold structure, we call $M$ a GKM-manifold with an extended $G$-action (see [9]). This class of manifolds includes interesting examples of flag varieties, and more generally, maximal rank homogeneous spaces. Our goal is to see what additional structures are imposed on the GKM-graph, and to what extent the topology of $M$ is captured combinatorially.

The organization of this paper is as follows: After briefly recalling the GKMtheory in $\S 2$, we see in $\S 3$ that the GKM-graph of a GKM-manifold with an extended $G$-action has a symmetry of the Weyl group $W$ of $G$, and that the localization map is compatible with it. In $\S 4$, we start with an abstract GKM-graph $\Gamma$ with the symmetry of a finite Coxeter group $W$. We discuss the cohomology ring $H^{*}(\Gamma ; \mathbb{R})$ of the graph and derive its properties combinatorially. In particular, we see a necessary condition for $\Gamma$ to admit the symmetry of $W$. Finally in $\S 5$, we see that with our definition of the $W$-symmetry, a series of operators on $H^{*}(\Gamma ; \mathbb{R})$ indexed by $W$ are defined.

[^0]They correspond topologically to the left divided difference operators, and $H^{*}(\Gamma ; \mathbb{R})$ is equipped with an action of the nil-Hecke ring ([8]). As an application of this fact, we obtain a retraction $H^{*}(\Gamma ; \mathbb{R}) \rightarrow H^{*}(\Gamma ; \mathbb{R})^{W}$, which is reminiscent of the BeckerGottlieb transfer ([1]) $\tau: H_{T}^{*}(M ; \mathbb{R}) \rightarrow H_{G}^{*}(M ; \mathbb{R})$.

We assume that the coefficients for the cohomology ring is $\mathbb{R}$ unless otherwise stated.

## 2. GKM theory

For a closed oriented manifold $M$ with a $T$-action, we consider its $T$-equivariant cohomology $H_{T}^{*}(M):=H^{*}\left(E T \times_{T} M\right)$. Here $E T \times_{T} M$ is the Borel construction, which is the quotient space of $E T \times M$ by the equivalence relation $(u, m) \sim\left(u t^{-1}, t m\right)$, where $u \in E T, m \in M$, and $t \in T$. When $M$ is a point, $H_{T}^{*}(p t)=H^{*}(B T)$ is the symmetric algebra over the dual Lie algebra $\mathfrak{t}^{*}$ of $T$.
$M$ is said to be a GKM-manifold if it satisfies the following conditions:

- it admits a $T$-invariant almost complex structure,
- it is equivariant formal, i.e., $H_{T}^{*}(M)$ is a free $H_{T}^{*}(p t)$-module,
- the fixed points set $M^{T}$ is finite,
- the weights of the isotropic $T$-action on the tangent space $\mathrm{T}_{p}(M)$ are pairwise linearly independent for $\forall p \in M^{T}$.
It follows (see [5]) that the set of the one-dimensional orbits $\{x \in M \mid \operatorname{dim}(T x)=1\}$ is the disjoint union $\bigsqcup_{\hat{e} \in \hat{E}} X_{\hat{e}}^{o}$, where $\hat{E}$ is a finite index set and the closure $X_{\hat{e}}$ of each component $X_{\hat{e}}^{o}$ is diffeomorphic to $S^{2}$ containing exactly two fixed points $e_{p}$ and $e_{q}$ at the north and the south poles. Then, the GKM-graph $\Gamma(M)=(V, E, \alpha, \theta)$ for $M$ is constructed as follows:
- The vertex set $V$ is the fixed points set $M^{T}$.
- The edge set $E$; for each two sphere $X_{\hat{e}}(\hat{e} \in \hat{E})$, we draw an edge $e_{p} \rightarrow e_{q}$.
- The axial function $\alpha: E \rightarrow \mathfrak{t}^{*}$ is defined by assigning to $e_{p} \rightarrow e_{q} \in E$ the weight of the $T$-action on $T_{e_{p}} X_{\hat{e}}$. Note that $\alpha_{e}$ annihilates the Lie algebra of the stabilizer group of $X_{\hat{e}}$. We graphically denote an edge with the value of axial function by $p \xrightarrow{\alpha(e)} q$.
- Let out $(p)$ be the set of the outgoing edges from $p \in V$. The restriction of $T M$ around $p$ splits into the sum of the plane bundles $\bigoplus_{e^{\prime} \in \text { out }\left(e_{p}\right)} L_{e^{\prime}}$, where the restriction of $L_{e^{\prime}}$ at $e_{p}$ is isomorphic to $T_{e_{p}} X_{\hat{e}^{\prime}}$. For each edge $e \in E$, the connection $\theta_{e}$ : out $\left(e_{p}\right) \rightarrow$ $\operatorname{out}\left(e_{q}\right)$ is the bijection which assigns to $e^{\prime} \in \operatorname{out}\left(e_{p}\right)$ the edge corresponding to $L_{e^{\prime}} e_{e_{q}}$.

The relationship between the topology of a GKM-manifold and the combinatorics of its GKM-graph is bridged by the localization map. The inclusion map $i: M^{T} \hookrightarrow M$ is $T$-equivariant, and hence it induces a map on the equivariant cohomology. We call the induced map

$$
i^{*}=\bigoplus_{p \in M^{T}} i_{p}^{*}: H_{T}^{*}(M) \rightarrow \bigoplus_{p \in M^{T}} H_{T}^{*}(p) \cong \bigoplus_{p \in M^{T}} H^{*}(B T)
$$

the localization map. We often denote $i_{p}^{*}(h)$ by $h_{p}$ for $h \in H_{T}^{*}(M)$.

The main result in [4] states that the GKM-graph determines the image of the localization map, and thus encodes the $T$-equivariant cohomology:

Theorem 2.1 ([4]). The localization map gives the following isomorphism:

$$
H_{T}^{*}(M) \cong\left\{\bigoplus_{p \in M^{T}} h_{p} \mid h_{p} \in H^{*}(B T), h_{p}-h_{q} \in(\alpha(e)) \text { if } p \xrightarrow{\alpha(e)} q\right\},
$$

where $\mathfrak{t}^{*}$ is identified with $H^{2}(B T)$. Note that the right hand side is determined purely combinatorially, and is often called the GKM-description of the equivariant cohomology.

## 3. Weyl group action

Let $M$ be a GKM-manifold with an extended $G$ action, where $G$ is a compact, simple, simply-connected Lie group with the maximal torus $T$. The Weyl group $W$ of $G$ is by definition $N(T) / T$, where $N(T)$ is the normalizer of the maximal torus in $G$. We denote the positive roots by $\Pi^{+}$, and the reflection corresponding to $\alpha \in \Pi^{+}$by $s_{\alpha} \in \operatorname{Aut}\left(\mathrm{t}^{*}\right)$.

We define a right action of $W$ on $E T \times_{T} M$ by $w(u, x)=\left(u w, w^{-1} x\right)$, which in turn induces a left action on $H_{T}^{*}(M)$. In particular, $W$ acts on $H_{T}^{*}(p t)=H^{*}(B T)$. If we regard $H^{*}(B T)$ as the symmetric algebra over $\mathfrak{t}^{*}=\left\langle t_{1}, \ldots, t_{n}\right\rangle$, the action is nothing but the standard action on the variables:

$$
w(h)\left(t_{1}, \ldots, t_{n}\right)=h\left(w t_{1}, \ldots, w t_{n}\right), \quad w \in W, h \in H^{*}(B T)=\mathbb{R}\left[t_{1}, \ldots, t_{n}\right] .
$$

Note that the $W$-action is trivial on $E G \times_{G} M$, and so is on $H_{G}^{*}(M)$.
Remark 3.1. The $W$ action is not $T$-equivariant, and hence, it is well-defined only on $E T \times_{T} M$ but not on $M$.

We first investigate the $W$-action on $H_{T}^{*}(M)$ in terms of the GKM-description.
Lemma 3.2. For $w \in W, p \in M^{T}$, and $h \in H_{T}^{*}(M)$, we have

$$
w(h)_{p}=w\left(h_{w^{-1} p}\right) \in H^{*}(B T) .
$$

Proof. By the following commutative diagram

we obtain $i_{p}^{*} \circ w=w \circ i_{w^{-1} p}^{*}$.

The GKM-graph $\Gamma(M)$ admits the following symmetry of $W$.

Theorem 3.3. (1) If there is an edge $p \xrightarrow{\alpha(e)} q$, then so is $w p \xrightarrow{w(\alpha(e))} w q$.
(2) If $q=s_{\alpha} p$ for some $\alpha \in \Pi^{+}$, there is an edge $p \xrightarrow{\alpha} q$.
(3) $w\left(\theta_{e}\left(e^{\prime}\right)\right)=\theta_{w(e)}\left(w\left(e^{\prime}\right)\right)$ for all $e, e^{\prime} \in E$ and $w \in W$.

Proof. (1) Assume that $T_{p}(M)=\bigoplus_{e \in \text { out } p} \mathbb{C}_{\alpha(e)}$, where $\mathbb{C}_{\alpha}$ is the complex onedimensional $T$-module with the weight $\alpha(e)$. Then the tangent space at $w p$ decomposes to $T_{w p}(M)=\bigoplus_{e \in \text { out } p} \mathbb{C}_{w \alpha(e)}$.
(2) Let $P_{\alpha} \subset G$ be the subgroup corresponding to the root $\alpha$. The $P_{\alpha}$-orbit of $p$ is $P_{\alpha} / T \cong S^{2}$ with $p$ and $q$ as its north and south poles. Since, $T_{p}\left(P_{\alpha} / T\right) \cong \mathbb{C}_{\alpha}$ by definition, the assertion follows.
(3) The decomposition $\bigoplus_{e^{\prime} \in \operatorname{out}(p)} L_{e^{\prime}}$ in the definition of $\theta$ is mapped by $w$ to $\bigoplus_{e^{\prime} \in \operatorname{out}(p)} w L_{e^{\prime}}$.

The relationship between the $G$-equivariant cohomology and the $T$-equivariant cohomology is summarized as follows.

Proposition 3.4. (1) $H_{T}^{*}(M) \cong H^{*}(B T) \otimes H^{*}(M)$ as $H^{*}(B T)$-modules.
(2) $H_{G}^{*}(M) \cong H^{*}(B G) \otimes H^{*}(M)$ as $H^{*}(B G)$-modules.
(3) $H_{T}^{*}(M) \cong H^{*}(B T) \otimes_{H^{*}(B G)} H_{G}^{*}(M)$ as $H^{*}(B T)$-algebras.
(4) $H_{G}^{*}(M) \cong H_{T}^{*}(M)^{W}$ as $H^{*}(B G)$-algebras.

Proof. (1) Consider the fibration $M \hookrightarrow E T \times_{T} M \rightarrow B T$. Since $H_{T}^{*}(M)$ is a free $H^{*}(B T)$-module by assumption, we have $H_{T}^{*}(M) \cong H^{*}(B T) \otimes H^{*}(M)$ by the Serre spectral sequence.
(2) By Borel's localization theorem, $H^{*}(M)$ should be concentrated in even degrees. Then by the Serre spectral sequence for the fibration $M \hookrightarrow E G \times_{G} M \rightarrow B G$, we have $H_{G}^{*}(M) \cong H^{*}(B G) \otimes H^{*}(M)$.
(3) Consider the following pullback diagram:


The Eilenberg-Moore spectral sequence collapses at $E_{2}$-term and we have $H_{T}^{*}(M) \cong$ $H^{*}(B T) \otimes_{H^{*}(B G)} H_{G}^{*}(M)$ as $H^{*}(B T)$-algebras. Note that there are no extension problem in this case.
(4) Since $W$ acts on $H_{G}^{*}(M)$ trivially, the assertion follows from the previous isomorphism.

Example 3.5. We recover the well-known specialization formula for the double Schubert polynomials. Let $M$ be the flag variety $G / T$, with the $G$-action induced by the left multiplication. The fixed point set of the action restricted to the maximal torus $T$ is $\{w T / T \mid w \in W\}$. Since $H_{G}^{*}(G / T)=H^{*}(B T)$, we have $H_{T}^{*}(G / T)=H^{*}(B T) \otimes_{H^{*}(B G)}$ $H^{*}(B T)$ by Proposition 3.4. It is well-known that $H_{T}^{*}(G / T)$ admits a free $H^{*}(B T)$ module basis indexed by $W$ called the Schubert classes (see, for example, [6]). Let $\mathcal{S}_{w}(t ; x) \in \mathbb{R}\left[t_{1}, \ldots, t_{n}, x_{1}, \ldots, x_{n}\right]$ be a polynomial representing a Schubert class $\mathcal{S}_{w} \in$ $H_{T}^{*}(G / T)$, where $t_{i}$ 's and $x_{i}$ 's are the generators for the left and the right factors of $H^{*}(B T) \otimes H^{*}(B T)$. Then, the localization at a fixed point $v \in W$ is the specialization at $x_{i}=v^{-1} t_{i}$ :

$$
i_{v}^{*}\left(\mathcal{S}_{w}\right)=\mathcal{S}_{w}\left(t_{1}, \ldots, t_{n} ; v t_{1}, \ldots, v t_{n}\right)
$$

Proof. Since $E G \times_{T}\{*\} \rightarrow E G \times_{G} G / T$ induces the isomorphism $H^{*}(B T) \cong$ $H_{G}^{*}(G / T)$, the localization map at the identity element $i_{e}^{*}$ is the identity map on the right factor. Hence, by Lemma 3.2,

$$
\begin{aligned}
i_{v}^{*}\left(\mathcal{S}_{w}\right) & =v \circ i_{e}^{*} \circ v^{-1}\left(\mathcal{S}_{w}\right) \\
& =v \circ i_{e}^{*} \mathcal{S}_{w}\left(v^{-1} t_{1}, \ldots, v^{-1} t_{n} ; x_{1}, \ldots, x_{n}\right) \\
& =v \mathcal{S}_{w}\left(v^{-1} t_{1}, \ldots, v^{-1} t_{n} ; t_{1}, \ldots, t_{n}\right) \\
& =\mathcal{S}_{w}\left(t_{1}, \ldots, t_{n} ; v t_{1}, \ldots, v t_{n}\right) .
\end{aligned}
$$

## 4. Abstract GKM-graph with symmetry

Modeled after the topological setting we have just seen, we consider an abstract GKM-graph with the symmetry of a finite Coxeter group.

Definition 4.1. Let $\mathfrak{t}^{*}$ be the vector space generated on a basis $t_{1}, \ldots, t_{n}$. A connected, finite, $m$-valent graph $\Gamma=(V, E, \alpha, \theta)$ with the following additional structures is said to be a GKM-graph:
(1) The axial function $\alpha: E \rightarrow \mathfrak{t}^{*}$ satisfies that $\{\alpha(e) \mid e \in \operatorname{out}(p)\}$ are pairwise linearly independent for all $p \in V$.
(2) The connection $\theta$ assigns for each edge $e \in E$ a bijection between out $\left(e_{p}\right)$ and out $\left(e_{q}\right)$, satisfying that $\alpha\left(\theta_{e}\left(e^{\prime}\right)\right)-\alpha\left(e^{\prime}\right)$ is a scalar multiple of $\alpha(e)$ for any adjacent edges $e$ and $e^{\prime}$.
We say that $\Gamma$ has the symmetry of a rank $n$ finite Coxeter group $W$ if $W$ acts as an automorphism on the vertices and edges and further it satisfies
(1) $\alpha(w(e))=w(\alpha(e))$ for $\forall e \in E$.
(2) If $q=s_{\alpha} p$ and $q \neq p$ for some $\alpha \in \Pi^{+}$, there is an edge $p \xrightarrow{\alpha} q$.
(3) $w\left(\theta_{e}\left(e^{\prime}\right)\right)=\theta_{w(e)}\left(w\left(e^{\prime}\right)\right)$ for all $e, e^{\prime} \in E$ and $w \in W$. In the above, $W$ action on $\mathfrak{t}^{*}$ is given by the standard representation.

We see in $\S 3$, for a GKM manifold $M$ with an extended action of a Lie group $G$, the GKM-graph $\Gamma(M)$ admits the symmetry of the Weyl group of $G$.

The cohomology of $\Gamma$ is defined following Theorem 2.1:

DEFINITION 4.2. Let $\mathbb{R}\left[t_{1}, \ldots, t_{n}\right]$ be the symmetric algebra over $\mathfrak{t}^{*}$, where the generators $t_{i}$ 's are considered to be of degree two.

$$
H^{*}(\Gamma)=\left\{\bigoplus_{p \in V} h_{p} \mid h_{p} \in \mathbb{R}\left[t_{1}, \ldots, t_{n}\right], h_{p}-h_{q} \in(\alpha(e)) \text { if } p \xrightarrow{\alpha(e)} q\right\}
$$

We consider $H^{*}(\Gamma)$ as a $\mathbb{R}\left[t_{1}, \ldots, t_{n}\right]$-submodule of $\bigoplus_{p \in V} \mathbb{R}\left[t_{1}, \ldots, t_{n}\right]$, where the module structure is diagonal.

Note that $\mathbf{1}=\bigoplus_{p \in V} 1$ is the generator for $H^{0}(\Gamma) \cong \mathbb{R}$.
If $\Gamma$ has the symmetry of $W$, then $W$ acts on the cohomology.

Lemma 4.3. For $h \in H^{*}(\Gamma)$, the element $w(h) \in \bigoplus_{p \in V} \mathbb{R}\left[t_{1}, \ldots, t_{n}\right]$ defined by

$$
\begin{equation*}
w(h)_{p}=w\left(h_{w^{-1} p}\right) \tag{4.1}
\end{equation*}
$$

belongs to $H^{*}(\Gamma)$.
Proof. Suppose that there is an edge $p \xrightarrow{\alpha} q$. Then there should be an edge $w^{-1} p \xrightarrow{w^{-1} \alpha} w^{-1} q . w(h)_{p}-w(h)_{q}=w\left(h_{w^{-1} p}-h_{w^{-1} q}\right)$ is divisible by $w \circ w^{-1} \alpha=\alpha$.

The action reduces trivially on the "ordinary cohomology."
Lemma 4.4. For $h \in H^{*}(\Gamma)$ and $\alpha \in \Pi^{+}$, we have

$$
h-s_{\alpha}(h) \in(\alpha \cdot \mathbf{1})
$$

Proof. Since

$$
\left(h-s_{\alpha}(h)\right)_{p}=h_{p}-s_{\alpha}\left(h_{s_{\alpha} p}\right)=\left(h_{p}-h_{s_{\alpha} p}\right)+\left(h_{s_{\alpha} p}-s_{\alpha}\left(h_{s_{\alpha} p}\right)\right)
$$

is divisible by $\alpha$, the assertion follows.

Since $w_{1} w_{2}(h)-h=\left(w_{1} w_{2}(h)-w_{2}(h)\right)+\left(w_{2}(h)-h\right)$ for $w_{1}, w_{2} \in W$, it follows that $w(h)-h \in\left(\mathbb{R}^{+}\left[t_{1}, \ldots, t_{n}\right] \cdot \mathbf{1}\right)$ for any $w \in W$, that is, $W$ acts trivially on $H^{*}(\Gamma) /\left(t_{1}, \ldots, t_{n}\right)$.

REmARK 4.5. Note that if $\Gamma$ is a GKM-graph of some GKM $G$-manifold, it is an easy topological consequence; The fiber inclusions $\iota: M \rightarrow E T \times_{T} M$ and $w \circ \iota$ is homotopic since $G$ is connected and there exists a path from (a representative of) $w$ to the identity element. Therefore, they induces the same map $H_{T}^{*}(M) \rightarrow H^{*}(M) \cong$ $H_{T}^{*}(M) /\left(H^{+}(B T)\right)$.

From this lemma, we deduce a necessary condition for $\Gamma$ to admit the symmetry of $W$. For this purpose, we define a graph theoretical analogy for torus invariant submanifolds. A connected subgraph $\Lambda$ of $\Gamma$ is said to be closed under the connection $\theta$ when $\theta_{e}\left(e^{\prime}\right) \in \Lambda$ for $\forall e, e^{\prime} \in \Lambda$. Since $\theta_{e}$ is bijection for all $e \in E, \Lambda$ is always a regular graph. Let $\Gamma_{k}$ be the set of all the $k$-valent subgraphs which are closed under the connection. For $\Lambda \in \Gamma_{k}$, we obtain the class $\lambda \in H^{2 m-2 k}(\Gamma)$ which is supported on $\Lambda$ by

$$
\lambda_{p}= \begin{cases}\prod_{e \in \text { out } p \backslash \Lambda} \alpha(e) & (p \in \Lambda), \\ 0 & (p \notin \Lambda) .\end{cases}
$$

Note that $\lambda \in H^{*}(\Gamma)$ is clear from the definition of $\theta$. Suppose that another class $\lambda^{\prime} \in$ $H^{2 m-2 k}(\Gamma)$ has the same support as $\lambda$, that is, $\lambda_{p}^{\prime} \neq 0$ iff $p \in \Lambda$. Then it must be a scalar multiple of $\lambda$. To see this, observe that there exists $c_{p} \in \mathbb{R}$ for each $p \in \Lambda$ such that $\lambda_{p}^{\prime}=c_{p} \lambda_{p}$. Take $q \in \Lambda$ which is adjacent to $p$ by an edge $e$. Then $\left(\lambda^{\prime}-c_{p} \lambda\right)_{q}$ must be divisible by $\alpha(e) \prod_{e^{\prime} \in \text { out } q \backslash \Lambda} \alpha\left(e^{\prime}\right)$ and so must be 0 by degree reason. It follows that $c_{q}=c_{p}$, and by the connectivity of $\Lambda$ all the $c_{p}$ 's for $p \in \Lambda$ must be equal. Since $W$ permutes $\Gamma_{k}$, we have

Proposition 4.6. $W$ acts on the set $\left\{ \pm \lambda \in H^{2 m-2 k}(\Gamma) \mid \Lambda \in \Gamma_{k}\right\}$ as a signed permutation.

In some cases, this imposes a cohomological restriction on what kind of $W$ can act on $\Gamma$.

Example 4.7. Assume that the classes corresponding to $\Gamma_{m-1}$ gives a free basis for $H^{2}(\Gamma)$. Since $H^{2}(\Gamma)$ contains $t^{*} \cdot \mathbf{1}$, the action is faithful. Furthermore, each reflection $s_{\alpha} \in W$ fixes all but one or two elements. To see this, first observe that the trace of the action of $s_{\alpha}$ on $\mathfrak{t}^{*}$ is $n-2$ since it is a reflection. In addition, by the previous Lemma, $s_{\alpha}$ acts trivially on $H^{2}(\Gamma) / t^{*}$. Therefore, the trace of the action of $s_{\alpha}$ on $H^{2}(\Gamma)$ must be two less than the rank of $H^{2}(\Gamma)$. It follows that we can choose a subset of $\left\{ \pm \lambda \in H^{2}(\Gamma)\right\}$ of cardinality equal to or less than $2(n+1)$, on which $W$ acts faithfully as a signed permutation. In particular, such $W$ is restricted to of classical types. (Readers also refer to [12] and [10]).

## 5. Divided difference operators

In this section, we construct a series of operators on $H^{*}(\Gamma)$. Topologically, they coincide with the left divided difference operators.

First, we need the following lemma:
Lemma 5.1. $\left(h-s_{\alpha}(h)\right) / \alpha$ is in $H^{*}(\Gamma)$.
Proof. First see

$$
\left(h-s_{\alpha}(h)\right)_{p}-\left(h-s_{\alpha}(h)\right)_{q}=h_{p}-h_{q}+s_{\alpha}\left(h_{s_{\alpha} q}-h_{s_{\alpha} p}\right) .
$$

If $q \neq s_{\alpha} p$ and $p \xrightarrow{\beta} q$, it follows $s_{\alpha} p \xrightarrow{s_{\alpha}(\beta)} s_{\alpha} q$. Hence, both $h_{p}-h_{q}$ and $s_{\alpha}\left(h_{s_{\alpha} q}-h_{s_{\alpha} p}\right)$ is divisible by $\beta$ and since the labels are pairwise linearly independent, $\left(h-s_{\alpha}(h)\right)_{p} / \alpha-$ $\left(h-s_{\alpha}(h)\right)_{q} / \alpha$ is divisible by $\beta$. If $q=s_{\alpha} p$ and $p \xrightarrow{\alpha} q$, we put $h_{p}-h_{s_{\alpha} p}=\alpha \cdot f$ with $f \in \mathbb{R}\left[t_{1}, \ldots, t_{n}\right]$. Then we see

$$
h_{p}-h_{q}+s_{\alpha}\left(h_{s_{\alpha} q}-h_{s_{\alpha} p}\right)=\alpha \cdot f+s_{\alpha}(\alpha \cdot f)=\alpha\left(f-s_{\alpha}(f)\right)
$$

is divisible by $\alpha^{2}$.
Now, we can define the operators.

Definition 5.2. The left divided difference operator associated to $\alpha \in \Pi^{+}$is defined to be

$$
\begin{aligned}
\partial_{\alpha}: H^{*}(\Gamma) & \rightarrow H^{*-2}(\Gamma), \\
h & \mapsto \frac{h-s_{\alpha}(h)}{\alpha},
\end{aligned}
$$

or equivalently,

$$
\partial_{\alpha}(h)_{p}=\frac{1}{\alpha}\left(h_{p}-s_{\alpha} h_{s_{\alpha} p}\right) .
$$

It is well-known (see [8], for example) that they satisfy the braid relation for $W$, and hence, we can define $\partial_{w}$ for $w \in W$ as the composition $\partial_{\alpha_{1}} \partial_{\alpha_{2}} \cdots \partial_{\alpha_{l}}$, where $w=$ $s_{\alpha_{1}} s_{\alpha_{2}} \cdots s_{\alpha_{l}}$ is a reduced decomposition for $w$.

Let us recall the definition of the nil-Hecke ring from [8]: Let $S=\mathbb{R}\left[t_{1}, \ldots, t_{n}\right]$ be the symmetric algebra over $t^{*}$ and $Q$ be its quotient ring. We denote the smash product of the group ring $\mathbb{R}[W]$ and $Q$ by $Q_{W}$. Then the nil-Hecke ring $R$ is the sub ring of $Q_{W}$ generated freely as a left $S$-module by $\partial_{w}$ 's for $\forall w \in W$.

Theorem 5.3. Let $\Gamma$ be a GKM-graph with the symmetry of a finite Coxeter group $W$ (Definition 4.1). Then, $H^{*}(\Gamma)$ is a module over the nil-Hecke ring $R$.

Example 5.4. When $M=G / T$, the left divided difference operator $\partial_{w}$ exhibits the hierarchy of the Schubert classes: (We use the same notation as in Example 3.5.)

$$
\partial_{v}\left(\mathcal{S}_{w}\right)= \begin{cases}\mathcal{S}_{v w} & (v \leq w) \\ 0 & \text { (otherwise) } .\end{cases}
$$

This is easily seen by comparing the support, since the Schubert classes are known to be characterized by the upper-triangularity of the support (see, for example, [6]). Note that the operators here are so-called the left divided difference, and in [8] they consider the right divided difference.

Now we consider the $W$-invariant subalgebra of $H^{*}(\Gamma)$.
Definition 5.5. Let $H_{W}^{*}(\Gamma)=H^{*}(\Gamma)^{W}$. More explicitly, it is described as follows: Fix a set of representatives $p_{1}, \ldots, p_{k}$ for the right $W$-cosets $W \backslash V$ of $W \curvearrowright V$ and denote by $W^{p_{i}}=\left\{w \in W \mid w p_{i}=p_{i}\right\}$ the isotropy subgroup at $p_{i}$. Since, by (4.1), $h_{w p}=w\left(h_{p}\right)$ for $h \in H^{*}(\Gamma)^{W}$, we have

$$
\begin{gathered}
H_{W}^{*}(\Gamma)=\left\{\bigoplus_{i=1}^{k} h_{p_{i}} \mid h_{p_{i}} \in \mathbb{R}\left[t_{1}, \ldots, t_{n}\right]^{W^{p_{i}}}, h_{p_{i}}-w h_{p_{j}} \in(\alpha(e))\right. \\
\text { if } \left.\exists w \in W / W^{p_{j}} \text { s.t. } p_{i} \xrightarrow{\alpha(e)} w p_{j}\right\} .
\end{gathered}
$$

REmark 5.6. Since $H_{G}^{*}(M) \cong H_{T}^{*}(M)^{W}$ by Proposition 3.4, we have $H_{W}^{*}(\Gamma(M)) \cong$ $H_{G}^{*}(M)$. The above definition gives a combinatorial description for $H_{G}^{*}(M)$.

As an application of the left divided difference operators, we can define the BeckerGottlieb transfer. Let $w_{0} \in W$ be the longest element.

Definition 5.7. Define

$$
\tau: H^{*}(\Gamma) \rightarrow H_{W}^{*}(\Gamma)
$$

by

$$
\tau(h)=\partial_{w_{0}}(d \mathbf{1} \cdot h),
$$

where $d=\prod_{\alpha \in \Pi^{+}} \alpha$.
Then, we recover the famous theorem by Brumfiel and Madsen ([2]).
Proposition 5.8. $\tau$ is given by

$$
\tau(h)=\sum_{w \in W} w(h) .
$$

In particular, $\tau /|W|$ is a left inverse to the inclusion $\iota: H_{W}^{*}(\Gamma) \hookrightarrow H^{*}(\Gamma)$, that is, $\tau \circ \iota /|W|$ is the identity map.

Proof. Fix a reduced expression $w_{0}=s_{\alpha_{1}} s_{\alpha_{2}} \cdots s_{\alpha_{l\left(w_{0}\right)}}$. It is well-known (see [3], for example) that

$$
\partial_{w_{0}}=\alpha_{1}^{-1}\left(1-s_{\alpha_{1}}\right) \alpha_{2}^{-1}\left(1-s_{\alpha_{2}}\right) \cdots \alpha_{l\left(w_{0}\right)}^{-1}\left(1-s_{\alpha_{l\left(w_{0}\right)}}\right)=\frac{1}{d} \sum_{w \in W}(-1)^{l(w)} w
$$

Since $w(d)=(-1)^{l(w)} d$, we have the assertion.

Example 5.9. In [7, §7], it is shown that

$$
\frac{1}{|W|} \tau\left(\mathcal{S}_{w}\right)=(-1)^{l(w)} s_{w^{-1}}
$$

where $\mathcal{S}_{w}$ is the equivariant Schubert class associated to $w \in W$ and $s_{w^{-1}}$ is the ordinary Schubert class associated to $w^{-1} \in W$ as in Example 3.5.

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[^0]:    2010 Mathematics Subject Classification. Primary 55N91; Secondary 57S15.
    This work was supported by KAKENHI, Grant-in-Aid for Young Scientists (B) 22740051, and Bilateral joint research project with Russia (2010-2012), Toric topology with applications in combinatorics.

