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<th>VARIETIES OF PICARD RANK ONE AS COMPONENTS OF AMPLE DIVISORS</th>
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Osaka University
Let $V$ be an integral normal complex projective variety of dimension $n \geq 3$ and denote by $L$ an ample line bundle on $V$. By imposing that the linear system $|L|$ contains an element $A = A_1 + \cdots + A_r$, $r \geq 1$, where all the $A_i$'s are distinct effective Cartier divisors with Pic$(A_i) = \mathbb{Z}$, we show that such a $V$ is as special as the components $A_i$ of $A \in |L|$. After making a list of some consequences about the positivity of the components $A_i$, we characterize pairs $(V, L)$ as above when either $A_1 \cong \mathbb{P}^{n-1}$ and Pic$(A_j) = \mathbb{Z}$ for $j = 2, \ldots, r$, or $V$ is smooth and each $A_i$ is a variety of small degree with respect to $[H_i]_{A_i}$, where $[H_i]_{A_i}$ is the restriction to $A_i$ of a suitable line bundle $H_i$ on $V$.

1. Introduction

Projective manifolds with an irreducible hyperplane section being a special variety have been studied since longtime (see, e.g., [2] and [6]), but the corresponding study for a reducible hyperplane section consisting of a simple normal crossing divisor whose components are special varieties started only recently by Chandler, Howard and Sommese [3]. Therefore, we continue here the study of varieties in terms of a hyperplane section $A$ which is not irreducible, assuming that $A$ is a union of distinct irreducible components $A_1, \ldots, A_r$, with $r \geq 1$. More precisely, let $V$ be an integral normal complex projective variety of dimension $n \geq 3$ endowed with an ample line bundle $L$. Assume that $|L|$ contains an element $A = A_1 + \cdots + A_r$, $r \geq 1$, where all the components $A_i$ are distinct and effective Cartier divisors with Pic$(A_i)$ of rank one.

Let us observe here that this assumption is a natural generalization of the classical hypothesis Pic$(A') \cong \mathbb{Z}$ on a hyperplane section $A'$ of $V$. Furthermore, if $(\star)$ holds then every component $A_i$ of $A \in |L|$ does not admit a non-trivial morphism onto a variety $W_i$ with $0 < \text{dim } W_i < n - 1$ and in general, also for the smooth case, the main known results on reducible hyperplane sections (see, e.g., [3]) can not be applied on any of the $A_i$'s. However, we can show that if a reducible subvariety $A$ as in $(\star)$ is contained in $V$ as an ample divisor, then it imposes severe restrictions to $V$, that is, the topological and geometric structures of $V$ are very closely related to those of each satisfaction of $(\star)$.
component of $A$. So, in §3 we prove first the following

**Theorem 1.1.** Let $V$ be an integral normal complex projective variety of dimension $n \geq 3$ and let $L$ be an ample line bundle on $V$. Assume that $(\diamondsuit)$ holds. Then all the $A_i$’s are nef and big Cartier divisors on $V$ and for any $i = 1, \ldots, r$ there exist proper birational morphisms $f_i: V \to V_i$ from $V$ to a projective normal variety $V_i$ given by the map associated to $|\mathcal{O}_V(m_iA_i)|$ for some $m_i \gg 0$. Furthermore, each map $f_i$ contracts at most a finite number of curves on $V$ and it is an isomorphism in a neighborhood of $A_i$ such that $f_i(A_i)$ is an effective ample divisor on $V_i$.

The above result shows that assumption $(\diamondsuit)$ implies that all the components $A_i$ of $A$ are either ample, or at worst big and 1-ample in the sense of [15, (1.3)]. By imposing some restrictions on the singularities of $V$, we are able to deduce that all the $A_i$’s are in fact ample Cartier divisors on $V$.

**Theorem 1.2.** Let $V$ be an integral normal complex projective variety of dimension $n \geq 3$ with at worst Cohen–Macaulay singularities. Let $L$ be an ample line bundle on $V$ and assume that $(\diamondsuit)$ holds. Furthermore, suppose that $V - F$ is a locally complete intersection for some finite, possibly empty, set $F \subset V - \text{Irr}(V)$ with $\dim \text{Irr}(V) \leq 0$, where $\text{Irr}(V)$ is the set of irrational singularities of $V$. Then all the $A_i$’s are ample Cartier divisors on $V$ and the maps $f_i: V \to V_i$ of Theorem 1.1 are all isomorphisms.

Furthermore, under some additional hypotheses on $V$ and on some $A_i$, we can finally obtain that $\text{Pic}(V) = \mathbb{Z}\langle \Lambda \rangle$ for an ample line bundle $\Lambda$ on $V$ (Corollary 3.1).

All of these results allow us to list in §4 some consequences about the positivity of the $A_i$’s and to obtain similar results as in [1, Theorem 1] (see also [14, Proposition VI]) for the case of reducible ample divisors on $V$ (Propositions 5.2 and 5.3).

We would like to note that the above results make use of weak hypotheses on $V$ and on each $A_i$, and that $(\diamondsuit)$ seems optimal a priori for Theorems 1.1 and 1.2, since easy examples show that these results do not hold assuming that $\text{Pic}(A_i) \neq \mathbb{Z}\langle H_i \rangle$ for some $i = 1, \ldots, r$, also when $r = 2$ and $V$ is smooth (Remark 3.2). Moreover, the techniques we employ in §3 leave out of account any special polarization on each $A_i$ by a (very) ample line bundle on $V$ and they allow us to assume that $L$ is simply ample and not necessarily ample and spanned or very ample on $V$.

Finally, as a by-product of §4 and some results obtained by many other authors about smooth complex projective variety $X$ containing ample divisors of special type (e.g., [1], [2], [6], [7], [9], [10], [11], [14]), in §5.1 we obtain similar results as in [1, Theorem 1] and [14, Proposition VI] (Propositions 5.2 and 5.3), and in §5.2 we classify smooth polarized pairs $(X, L)$ which admit an ample divisor $A \in [L]$ such that $A = A_1 + \cdots + A_r$, $r \geq 1$, and all the components $A_i$ have small degree with respect to suitable line bundles $H_i$ on $X$ for every $i = 1, \ldots, r$. 
Proposition 1.3. Let $L$ be an ample line bundle on a smooth complex projective variety $X$ of dimension $n$ with $n \geq 5$. Assume that there is a divisor $A = A_1 + \cdots + A_r \in |L|$, $r \geq 1$, where each $A_i$ is an irreducible and reduced normal Gorenstein projective variety with $\dim \operatorname{Irr}(A_i) \leq 0$, where $\operatorname{Irr}(A_i)$ is the set of irrational singularities of $A_i$. Suppose that for any $k = 1, \ldots, r$ there exist ample and spanned line bundles $H_k$ on $X$ such that $[H_k]_{A_k}$ is very ample and $[H_k]^{n-1}_{A_k} \leq 4$. Then one of the following possibilities holds:

1. $r \geq 1$, $H_1 = \cdots = H_r = H$ and $(X, H)$ is one of the following pairs:
   a. $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ and $A_i \in |\mathcal{O}_{\mathbb{P}^n}(a_i)|$ with $1 \leq a_i \leq 4$ for every $i = 1, \ldots, r$;
   b. $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$ and $A_i \in |\mathcal{O}_{\mathbb{Q}^n}(a_i)|$ with $a_i = 1, 2$ for every $i = 1, \ldots, r$;
   c. $X \subset \mathbb{P}^{n+1}$ is a hypersurface of degree 3 or 4, and $A_i = H \in |\mathcal{O}_{\mathbb{P}^{n+1}}(1)|$ for every $i = 1, \ldots, r$;
   d. $X = \mathbb{Q}_1 \cap \mathbb{Q}_2 \subset \mathbb{P}^{n+2}$ is a complete intersection of two quadric hypersurfaces $\mathbb{Q}_i \subset \mathbb{P}^{n+2}$ for $i = 1, 2$, and $A_i = H \in |\mathcal{O}_{\mathbb{P}^{n+2}}(1)|$ for every $i = 1, \ldots, r$;
   e. $\pi : X \to \mathbb{P}^n$ is a double cover of $\mathbb{P}^n$ with branch locus $\Delta \in |\mathcal{O}_{\mathbb{P}^n}(2b)|$, $b = 1, 2$, $H \in |\pi^*\mathcal{O}_{\mathbb{P}^n}(1)|$ and $A_i \in |\pi^*\mathcal{O}_{\mathbb{P}^n}(a_i)|$, $a_i = 1, 2$, for every $i = 1, \ldots, r$;
   f. $\pi : X \to \mathbb{Q}^n$ is a double cover of a quadric hypersurface $\mathbb{Q}^n \subset \mathbb{P}^{n+1}$ with branch locus $\Delta \in |\mathcal{O}_{\mathbb{Q}^n}(2b')|$, $b' = 1, 2$, and $A_i = H \in |\pi^*\mathcal{O}_{\mathbb{Q}^n}(1)|$ for every $i = 1, \ldots, r$;
   g. $\pi : X \to \mathbb{P}^n$ is a d-cover of $\mathbb{P}^n$ with $d = 3, 4$, and $A_i = H \in |\pi^*\mathcal{O}_{\mathbb{P}^n}(1)|$ for every $i = 1, \ldots, r$;

2. $r \geq 2$, $X \cong \mathbb{P}^1 \times \mathbb{P}^3$ and after renaming $(A_1, [H_1]_{A_1}) \cong (\mathbb{P}^1 \times \mathbb{P}^3, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(1, 1))$, with $A_1 \in |\mathcal{O}_{\mathbb{X}(0, 1)}|$ and $H_1 \in |\mathcal{O}_{\mathbb{X}(1, 1)}|$, $(A_2, [H_2]_{A_2}) \cong (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1))$ with $A_2 \in |\mathcal{O}_{\mathbb{X}(1, 0)}|$ and $H_2 \in |\mathcal{O}_{\mathbb{X}(t, 1)}|$ for some integer $t \geq 1$, and the remaining polarized pairs $(A_k, [H_k]_{A_k})$ are of these two types.

2. Notation

Let $V$ be an integral normal complex projective variety of dimension $n \geq 3$ endowed with an ample line bundle $L$. All notation and terminology used here are standard in algebraic geometry. We adopt the additive notation for line bundles and the numerical equivalence is denoted by $\equiv$, while the linear equivalence by $\simeq$. The pull-back $\pi^*F$ of a line bundle $F$ on $V$ by an embedding $\iota : Y \hookrightarrow V$ is sometimes denoted by $\mathcal{F}_Y$. We denote by $K_V$ the canonical bundle of $V$. For such a polarized variety $(V, \mathcal{L})$, we will use the adjunction theoretic terminology of [2] and we say that $V$ is an $n$-fold (denoted by $X$) when $V$ is smooth.

3. Proof of Theorems 1.1 and 1.2

First of all, let us show that $(\diamondsuit)$ implies that all the components of $A$ are actually nef and big Cartier divisors on $V$, obtaining the following
Proof of Theorem 1.1. Suppose that $A_1 \cap A_2 := W$ is nonempty. Define integers $a_{ij}$ putting $\mathcal{O}_V(A_i) := \mathcal{O}_{X_i}(a_{ij} \mathcal{H}_j)$, where $\mathcal{H}_j$ is the ample generator of $\text{Pic}(A_j)$ (mod torsion). Then

$$\mathcal{O}_W(a_{11} \mathcal{H}_1) = \mathcal{O}_W(A_1) = \mathcal{O}_W(a_{12} \mathcal{H}_2)$$

is an ample line bundle on $W$. Hence $a_{11}$ is positive. Since $A$ is a connected divisor on $V$ (see [8, III 7.9]), for every $i = 1, \ldots, r$ there exists a component $A_j$ of $A \in |\mathcal{L}|$ with $j \neq i$ such that $A_i \cap A_j \neq \emptyset$. This shows that $\mathcal{O}_V(A_i)_{|A_j}$ is ample for any $i = 1, \ldots, r$, i.e. $A_i$ is nef and big. From [7, III 4.2] (see also [2, (2.6.5)] or [12, (1.2.30)]), it follows that there exists a proper birational morphism $f_i : V \to V_i$ from $V$ to a projective normal variety $V_i$ given by the map associated to $|\mathcal{O}_V(m_i A_i)|$ for some $m_i > 0$, which is an isomorphism in a neighborhood of $A_i$ and such that $f_i(A_i)$ is an effective ample divisor on $V_i$.

Claim. The morphism $f_i$ contracts at most a finite number of curves on $V$.

By [2, (2.5.5)] note that $A_i \simeq H_i + D_i$ for any $i = 1, \ldots, r$, where $H_i$ is $\mathbb{Q}$-ample and $D_i$ is $\mathbb{Q}$-effective. This shows that $A_i \cdot A_j$ is nonzero and so $A_i \cap A_j$ can not be empty. Let $W_{ij} := A_i \cap A_j$ and note that $W_{ij}$ is an ample divisor both in $A_i$ and in $A_j$. Let $Z$ be an irreducible variety of dimension greater than or equal to two. Since $\mathcal{L}$ is ample, we have that $Z \cap A_k \neq \emptyset$ for some $k = 1, \ldots, r$, and then $W_{ik} \cap Z \neq \emptyset$ since dim $Z \cap A_k \geq 1$. Put $R := Z \cap A_i$. Thus $R$ is nonempty, dim $R \geq 1$ and $\mathcal{O}(R)_{|R}$ is ample. By [7, III 4.2] (or [12, (1.2.30)]), the linear system $|\mathcal{O}_V(m_i A_i)|$ restricted to $Z$ can not contract $Z$ to a lower dimensional variety. \hfill \square

Proof of Theorem 1.2. Put $\mathcal{O}_V(A_i)_{|A_j} = \mathcal{O}_{X_i}(a_{ij} \mathcal{H}_j)$ and $l_i := \mathcal{O}_{X_i}(\mathcal{H}_i)^{n-1} > 0$ for any $i, j, t \in \{1, \ldots, r\}$. From Theorem 1.1 we know that $a_{ii} > 0$ for every $i = 1, \ldots, r$. Moreover, since $A$ is a connected divisor on $X$ (see [8, III 7.9]), for every $j = 1, \ldots, r$ there exists a component $A_k$ of $A \in |\mathcal{L}|$ with $k \neq j$ such that $A_j \cap A_k \neq \emptyset$. So we get the following expressions

$$A_k^s A_j^{n-s} = \mathcal{O}_{A_i}(a_{kk} \mathcal{H}_k)^{s-1} \mathcal{O}_{A_i}(a_{jk} \mathcal{H}_j)^{n-s} = a_{kk}^s a_{jk}^{n-s-l_j},$$

(1)

with $1 \leq s \leq n - 1$. Note that $a_{jj} \neq 0$, $a_{kk} \neq 0$, $a_{kj} > 0$ and $a_{jk} > 0$. Moreover, from the equations (1) with $s = 1, 2$ we deduce that

$$a_{jk}(a_{jj}^2 a_{jj}^{n-2} l_j) = a_{jk}(a_{kk} a_{jk}^{n-2} l_k) = a_{kk}(a_{jk}^{n-1} l_k) = a_{kk}(a_{kj} a_{jj}^{n-2} l_j),$$

that is

(2)

$$a_{jk} a_{kj} = a_{jj} a_{kk}.$$
Since by (2) we get

\[ A_s^2 L^{n-2} = \sum_{h_1+\cdots+h_r=n-2} \frac{(n-2)!}{h_1! \cdots h_r!} A_{h_1}^{h_1} \cdots A_{h_r}^{h_r} \]

for every \( s = 1, \ldots, r \). Furthermore, for \( j \neq k \) we have also the following equations

3. \( A_j^2 L^{n-2} = [A_j]_A A_j L^{n-2} = \sum_{k_1+\cdots+k_r=n-2} \frac{(n-2)!}{k_1! \cdots k_r!} a_{k_1}^{h_1} \cdots a_{k_j}^{h_j} \cdots a_{k_r}^{h_r} \)

4. \( A_k^2 L^{n-2} = [A_k]_A A_k L^{n-2} = \sum_{h_1+\cdots+h_r=n-2} \frac{(n-2)!}{h_1! \cdots h_r!} a_{h_1}^{h_1} \cdots a_{h_k}^{h_k} \cdots a_{h_r}^{h_r} \)

5. \( A_j A_k L^{n-2} = [A_j]_A A_k L^{n-2} = \sum_{k_1+\cdots+k_r=n-2} \frac{(n-2)!}{k_1! \cdots k_r!} a_{k_1}^{h_1} \cdots a_{k_j}^{h_j} \cdots a_{k_k}^{h_k} \cdots a_{k_r}^{h_r} \)

6. \( A_j A_k L^{n-2} = [A_j]_A A_k L^{n-2} = \sum_{h_1+\cdots+h_r=n-2} \frac{(n-2)!}{h_1! \cdots h_r!} a_{h_1}^{h_1} \cdots a_{h_j}^{h_j} \cdots a_{h_k}^{h_k} \cdots a_{h_r}^{h_r} \)

Since by (2) we get

\[ (a_{i_1}^{k_1} \cdots a_{i_j}^{k_j} \cdots a_{i_r}^{k_r})(a_{k_1}^{h_1} \cdots a_{k_j}^{h_j} \cdots a_{k_r}^{h_r}) = a_{i_1} a_{i_2} (a_{i_3}^{k_1} \cdots a_{i_j}^{k_j} \cdots a_{i_r}^{k_r})(a_{k_1}^{h_1} \cdots a_{k_j}^{h_j} \cdots a_{k_r}^{h_r})
\]

\[ = a_{i_1} a_{i_2} a_{i_3}^{h_1} \cdots a_{i_j}^{h_j} \cdots a_{i_r}^{h_r} a_{i_1}^{k_1} \cdots a_{i_j}^{k_j} \cdots a_{i_r}^{k_r}
\]

\[ = (a_{i_1}^{k_1} \cdots a_{i_j}^{k_j} \cdots a_{i_r}^{k_r})(a_{j_1}^{h_1} \cdots a_{j_j}^{h_j} \cdots a_{j_r}^{h_r}) = a_{i_1} a_{i_2} a_{i_3}^{h_1} \cdots a_{i_j}^{h_j} \cdots a_{i_r}^{h_r} a_{i_1}^{k_1} \cdots a_{i_j}^{k_j} \cdots a_{i_r}^{k_r}
\]

from (3), (4), (5) and (6), we obtain that

\[ (A_k A_j L^{n-2})^2 = (A_k^2 L^{n-2})(A_j^2 L^{n-2}) > 0 \]

for every \( k \) and \( j \) such that \( A_j \cap A_k \neq \emptyset \).

Thus, by [2, (2.5.4)] we see that there exists a rational number \( \lambda_{jk} \) such that \( A_j \) is numerically equivalent to \( \lambda_{jk} A_k \). Since from (2) it follows that \( A_k \) meets all the other components of \( A \), by an inductive argument we get

\[ \mathcal{L} \simeq A_1 + \cdots + A_r \equiv (\lambda_{1k} + \cdots + \lambda_{kk} + \cdots + \lambda_{rk} + 1)A_k = \mu_k A_k, \]

where \( \mu_k := \lambda_{1k} + \cdots + \lambda_{kk} + \cdots + \lambda_{rk} + 1 \) and the symbol \( \lambda \) denotes suppression.

Moreover, since

\[ 0 < L_{\mathcal{A}_j}^{n-1} = A_j L^{n-1} = \lambda_{jk} A_k L^{n-1} = \lambda_{jk} L_{\mathcal{A}_k}^{n-1}, \]

\[ \text{Varieties of Picard Rank One} \]
we have that \(\lambda_{jk} > 0\) for every \(j \neq k\) and from (7) it follows that \(\mu_k \geq 1\), i.e. \(A_k\) is an ample Cartier divisor on \(V\) for any \(k = 1, \ldots, r\). By combining this with Theorem 1.1, we obtain that every \(f_i\) is an isomorphism.

**Corollary 3.1.** Let \(V\) be an integral normal complex projective variety of dimension \(n \geq 3\) with at worst Cohen–Macaulay singularities. Let \(L\) be an ample line bundle on \(V\) and assume that \((\diamond)\) holds. Furthermore, suppose that one of the following conditions holds:

(a) \(V\) is a locally complete intersection;

(b) \(n = 3\), \(V - A_k\) and \(V - F\) are locally complete intersections for some \(k = 1, \ldots, r\) and some finite set \(F \subset V - \text{Irr}(V)\).

Then \(\text{Pic}(V) = \mathbb{Z}(\Lambda)\), where \(\Lambda\) is an ample line bundle on \(V\). In particular, all the \(A_i\)'s are ample Cartier divisors on \(V\) and the maps \(f_i : V \to V_i\) of Theorem 1.1 are all isomorphisms.

**Proof.** In both cases (a) and (b), we simply use Theorem 1.2 and [2, (2.3.4)].

**Remark 3.2.** Let \(V \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(-d))\) with \(0 < d < n\) and denote by \(\pi : V \to \mathbb{P}^{n-1}\) the projection map. Note that there exists a smooth divisor \(E\) on \(V\) which is a section of \(\pi\) such that \(E \cong \mathbb{P}^{n-1}\) and \(E \in |\mathcal{O}_{\mathbb{P}^{n-1}}(-d)|\). Put \(L := E + \pi^* \mathcal{O}_{\mathbb{P}^{n-1}}(a)\). Then, for a suitable integer \(a \gg 0\), we have that \(L\) is ample on \(V\) and that there exists a divisor \(A \in |L|\) such that \(A = A_1 + A_2\), where \(A_1 = E\) and \(A_2 \in |\pi^* \mathcal{O}_{\mathbb{P}^{n-1}}(a)|\) is a smooth divisor on \(V\). This example shows that if \(r \geq 2\) then the above results can not be improved by assuming in \((\diamond)\) that \(\text{Pic}(A_i) \neq \mathbb{Z}\) for some \(i = 1, \ldots, r\), also when \(r = 2\) and \(V\) is a Fano \(n\)-fold with \(n \geq 3\).

### 4. Some immediate consequences

Let us collect here some results due to the different positiv ity of the components \(A_i\) of \(A \in |L|\) under the hypothesis \((\diamond)\).

**4.0.1. Nefness and bigness of the \(A_i\)'s.** From Theorem 1.1, we obtain the following

**Corollary 4.1.** Let \(V\) be an integral normal complex projective variety of dimension \(n \geq 3\) and let \(L\) be an ample line bundle on \(V\). Assume that \((\diamond)\) holds. Set \(D = \sum A_{i_k}\), where all the \(i_k \in \{1, \ldots, r\}\) are not necessarily distinct indexes. Moreover, let \(\text{Irr}(V)\) be the set of irrational singularities of \(V\). Then we have the following properties:

- **(Vanishing type theorems).**
  - (1) Let \(\varphi : V \to Y\) be a morphism from \(V\) to a projective variety \(Y\). Then 
    \[
    \varphi_{\ast}(K_V + D) = 0 \quad \text{for} \quad i \geq \max_{y \in \varphi(\text{Irr}(V))} \dim(\varphi^{-1}(y) \cap \text{Irr}(V)) + 1,
    \]
where $\varphi_i(T)$ is the $i$-th higher derived functor of the direct image $\varphi_*(T)$ of a sheaf $T$ on $V$;

(2) $H^1(\mathcal{V}, -D) = 0$;

(3) assuming that $\text{Irr}(\mathcal{V})$ is finite and nonempty, we have that

$$\dim H^0(\mathcal{V}, K_V + D) \geq \#(\text{Irr}(\mathcal{V})) > 0,$$

where $\#(\text{Irr}(\mathcal{V}))$ is the number of points in $\text{Irr}(\mathcal{V})$; in particular, if $\mathcal{A} \in |D|$ lies in the set of Cohen–Macaulay points of $\mathcal{V}$, then

$$\dim H^0(\mathcal{V}, K_V) + \dim H^0(\mathcal{A}, K_\mathcal{A}) \geq \#(\text{Irr}(\mathcal{V})) > 0.$$

• (Lefschetz type theorems).

(4) Let $\mathcal{A} \in |D|$ be a divisor such that $\mathcal{V} - \mathcal{A}$ is a local complete intersection. Then under the restriction map it follows that

$$H^j(\mathcal{V}, \mathbb{Z}) \cong H^j(\mathcal{A}, \mathbb{Z}) \quad \text{for} \quad j \leq n - 3,$$

and

$$H^j(\mathcal{V}, \mathbb{Z}) \to H^j(\mathcal{A}, \mathbb{Z}) \quad \text{for} \quad j = n - 2,$$

is injective with torsion free cokernel; moreover, we have that $\text{Pic}(\mathcal{V}) \cong \text{Pic}(\mathcal{A})$ for $n \geq 5$, and the restriction mapping $\text{Pic}(\mathcal{V}) \to \text{Pic}(\mathcal{A})$ is injective with torsion free cokernel for $n = 4$; in particular, if $\mathcal{V} - A_i$ is a local complete intersection for some $i = 1, \ldots, r$, then $\text{Pic}(\mathcal{V}) = \mathbb{Z}\langle \Lambda \rangle$ for $n \geq 4$ and some ample line bundle $\Lambda$ on $\mathcal{V}$;

• (The Albanese mapping).

(5) Assume that $\mathcal{A} \in |D|$ is normal. Moreover, suppose that $\mathcal{A}$ and $\mathcal{V}$ have at worst rational singularities. Then the map $\text{Alb}(\mathcal{A}) \to \text{Alb}(\mathcal{V})$ induced by inclusion is an isomorphism.

• (Hodge index type theorems).

(6) For $n_{i_1} + \cdots + n_{i_k} = n - 1$ and $n_{i_i} \geq 1$, we have

$$(A_{i_1}^{n_{i_1}} \cdots A_{i_k}^{n_{i_k}})^2 \geq (A_{i_1}^{2} \cdots A_{i_k}^{n_{i_i}-1} \cdots A_{i_k}^{n_{i_k}})(A_{i_1}^{n_{i_1}+1} \cdots A_{i_k}^{n_{i_k}}),$$

and for $n_{i_1} + \cdots + n_{i_k} = n$, we get also

$$(A_{i_1}^{n_{i_1}} \cdots A_{i_k}^{n_{i_k}})^n \geq (A_{i_1}^{n_{i_1}} \cdots A_{i_k}^{n_{i_k}})^{n_{i_i}} > 0,$$

where $i_k \in \{1, \ldots, r\}$;

(7) we have the following inequality: $A_{i-1}^{n-1} \cdot A_{j} \cdot A_{j-1}^{n-1} \geq A_{i}^{n} A_{j} > 0$;

(8) for $t \geq 1$ and any nef and big line bundle $H_i$ on $\mathcal{V}$ with $1 \leq i \leq t$, it follows that $\mathcal{O}_\mathcal{V}(A_j)^{n_{i-1}} \cdot \prod_{i=1}^{t} H_i > 0$, where $j \in \{1, \ldots, r\}$;
(9) let $H$ be a line bundle such that $\dim H^0(V, NH) \geq 2$ for some $N \geq 1$. Then $H \cdot \mathcal{O}_V(A_i) \cdots \mathcal{O}_V(A_{i_r}) > 0$, where $i_h \in \{1, \ldots, r\}$.

Proof. First of all, from Theorem 1.1 we deduce that the effective divisor $D = \sum A_i$ is nef and big on $V$. Thus (1), (2) and (3) of the statement follow from [2, (2.2.5), (2.2.7), (2.2.8)]. Finally, cases (4) to (9) follow from [2, (2.3.3), (2.3.4), (2.4.4), (2.5.1), (2.5.3), (2.5.2), (2.5.8) and (2.5.9)].

By applying Theorem 1.1 also to the zero locus of special sections of $k$-ample vector bundles on an $n$-fold $X$ for $k \geq 0$ (see [15, §1] and [2, §2.1]), we obtain the following

**Corollary 4.2** (Lefschetz–Sommese type theorem). Let $E$ be a $k$-ample vector bundle of rank $r \geq 1$ on an $n$-fold $X$ with $n \geq 4$. Assume that there exists a section $s \in \Gamma(E)$ whose zero locus $Z = (s)_0$ is an integral normal complex variety such that $\dim Z \geq 3$. Suppose that there exists a divisor $A = A_1 + \cdots + A_s$, $s \geq 1$, on $Z$ which satisfies $(\diamond)$. If $Z - A_i$ is a local complete intersection for some $i = 1, \ldots, s$, then

$$H^i(X, Z) \cong H^i(A, Z) \quad \text{for} \quad i \leq \min\{\dim Z - 3, n - r - k - 1\},$$

and the restriction maps $H^i(X, Z) \to H^i(A_i, Z)$ are injective with torsion free cokernel for $i = \min\{\dim Z - 2, n - r - k\}$.

Proof. From Theorem 1.1, we know that each $A_k$ is at worst 1-ample on $Z$. Thus by Corollary 4.1 (4), we have that

$$H^j(Z, Z) \cong H^j(A_i, Z) \quad \text{for} \quad j \leq \dim Z - 3,$$

and the restriction maps $H^j(Z, Z) \to H^j(A_i, Z)$ are injective with torsion free cokernel for $j = \dim Z - 2$. Moreover, from the Lefschetz–Sommese’s theorem for $k$-ample vector bundles on an $n$-fold $X$ (see [15, (1.16)] and [13, (7.1.1), (7.1.9)]), we deduce that

$$H^j(Z, Z) \cong H^j(X, Z) \quad \text{for} \quad j \leq n - r - k - 1,$$

and that the restriction maps $H^j(X, Z) \to H^j(Z, Z)$ are injective with torsion free cokernel for $j = n - r - k$.

**4.0.2. Ampleness of the $A_i$’s.** Under the same assumption of Theorem 1.2, we first deduce the following

**Corollary 4.3.** Let $V$ be an integral normal complex projective variety of dimension $n \geq 3$ with at worst Cohen–Macaulay singularities. Let $L$ be an ample line bundle on $V$ and assume that $(\diamond)$ holds. Moreover, suppose that $V - F$ is a local complete intersection for some finite, possibly empty, set $F \subset V - \text{Irr}(V)$ with $\dim \text{Irr}(V) \leq 0$,
where \( \text{Irr}(\mathcal{V}) \) is the set of irrational singularities of \( \mathcal{V} \). Set \( D = \sum A_{i_h} \), where all the \( i_h \in \{1, \ldots, r\} \) are not necessarily distinct indexes. Then we have the following properties:

(I) (Fujita’s vanishing type theorem). Given any coherent sheaf \( \mathcal{F} \) on \( \mathcal{V} \), there exists an integer \( m(\mathcal{F}, D) \) such that

\[
H^i(\mathcal{V}, \mathcal{F} \otimes \mathcal{O}_\mathcal{V}(mD + N)) = 0 \quad \text{for } i > 0, \ m \geq m(\mathcal{F}, D),
\]

where \( N \) is any nef divisor on \( \mathcal{V} \);

(II) \( \dim H^0(\mathcal{V}, D) \leq D^n + n, \) with equality if and only if \( \mathcal{V} \) is one of the following:

(a) \( \mathbb{P}^n \);
(b) a quadric hypersurface \( \mathbb{Q}^n \subset \mathbb{P}^{n+1} \);
(c) a \( \mathbb{P}^{n-1} \)-bundle over \( \mathbb{P}^1 \);
(d) a generalized cone over a smooth submanifold \( \mathcal{V} \subset \mathcal{V} \) as in (a), (b), (c);

(III) (Lefschetz type theorems). Let \( \mathcal{A} \in |D| \) be a divisor such that \( \mathcal{V} - \mathcal{A} \) is a local complete intersection. Then under the restriction map it follows that

\[
H^j(\mathcal{V}, \mathcal{Z}) \cong H^j(\mathcal{A}, \mathcal{Z}) \quad \text{for } j \leq n - 2,
\]

and

\[
H^j(\mathcal{V}, \mathcal{Z}) \rightarrow H^j(\mathcal{A}, \mathcal{Z}) \quad \text{for } j = n - 1,
\]

is injective with torsion free cokernel; moreover, we have that \( \text{Pic}(\mathcal{V}) \cong \text{Pic}(\mathcal{A}) \) for \( n \geq 4 \), and the restriction mapping \( \text{Pic}(\mathcal{V}) \rightarrow \text{Pic}(\mathcal{A}) \) is injective with torsion free cokernel for \( n = 3 \); in particular, if \( \mathcal{V} - A_i \) is a local complete intersection for some \( i = 1, \ldots, r \), then \( \text{Pic}(\mathcal{V}) = \mathbb{Z}<\Lambda> \) for \( n \geq 3 \) and some ample line bundle \( \Lambda \) on \( \mathcal{V} \).

Proof. By Nakai–Moishezon–Kleiman criterion and Theorem 1.2, we see that \( D = \sum A_{i_h} \) is ample on \( \mathcal{V} \). Thus case (I) of the statement follows from [5, §1, Theorem 1] (see also [12, (1.4.35)]). Finally, we obtain case (II) by [6, 1 (4.2), (5.10), (5.15)], while (III) follows from [2, (2.3.3)] and [2, (2.3.4)] respectively. \( \square \)

Finally, let us deduce also the following result for ample vector bundles on a smooth variety.

**Corollary 4.4.** Let \( \mathcal{E} \) be an ample vector bundle of rank \( r \geq 1 \) on an \( n \)-fold \( X \) with \( n \geq 4 \). Assume that there exists a section \( s \in \Gamma(\mathcal{E}) \) whose zero locus \( Z = (s)_0 \) is a smooth submanifold of \( X \). Suppose that there exists a divisor \( A = A_1 + \cdots + A_s, s \geq 1 \), on \( Z \) which satisfies (\( \diamond \)). If \( r < n - 2 \) then both \( \text{Pic}(X) \) and \( \text{Pic}(Z) \) have rank one.

Proof. It follows easily from Corollary 4.3 (III) and [15, (1.16)], or [13, (7.1.5) (ii)]. \( \square \)
5. Some applications

Here are two applications.

5.1. All the $A_i$'s are Fano varieties of Picard rank one. First of all, let us prove the following

**Lemma 5.1.** Let $\mathcal{L}, \mathcal{V}, A_i, \mathcal{V}_i, f_i$ be as in Theorem 1.1. If $\mathcal{V}_k$ is $\mathbb{Q}$-factorial for some $k = 1, \ldots, r$, then $A_k$ is ample on $\mathcal{V}$ and $f_k: \mathcal{V} \to \mathcal{V}_k$ is in fact an isomorphism.

**Proof.** Take any line bundle $\mathcal{D}$ on $\mathcal{V}$ and consider the following commutative diagram $(*)$:

\[
\begin{array}{ccc}
U & \xrightarrow{f_k} & U' \\
\downarrow j & & \downarrow j_k \\
\mathcal{V} & \xrightarrow{f_k} & \mathcal{V}_k
\end{array}
\]

where $j: U \to \mathcal{V}$ and $j_k: U' \to \mathcal{V}_k$ are the inclusion maps and $f_k|_U: U \to U'$ is the isomorphism induced by $f_k: \mathcal{V} \to \mathcal{V}_k$. Since $\mathcal{V}_k$ is $\mathbb{Q}$-factorial, we see that $f_k_* (N \mathcal{D}) = \mathcal{L}'$ is a line bundle on $\mathcal{V}_k$ for some positive integer $N$. Write $\mathcal{L}' = \sum_h a_h \mathcal{L}_h$, where $\mathcal{L}_h$ are the generators of Pic($\mathcal{V}_k$). Then by $(*)$ we get

\[
N \mathcal{D}|_U = j^*(N \mathcal{D}) = f_k|_U_* j^*(N \mathcal{D}) = j_k^* f_k_* (N \mathcal{D}) = \sum_h a_h j_k^* \mathcal{L}_h = \sum_h a_h j_k^* f_k^* \mathcal{L}_h.
\]

By Hartogs’ lemma (see, e.g., [4, (11.4)]), this gives $N \mathcal{D} = \sum_h a_h f_k^* (\mathcal{L}_h)$, i.e. $\mathcal{D} = \sum_h (a_h/N) f_k^* (\mathcal{L}_h)$. Therefore, if $A_k$ is not ample, then we deduce that there exists an irreducible curve $\Gamma \subset \mathcal{V}$ such that $m_k A_k \cdot \Gamma = 0$ for any positive integer $m_k$, i.e. the map $f_k$ contracts the curve $\Gamma$. Hence $f_k^* (\mathcal{L}_h)' \cdot \Gamma = 0$ for any $h$, i.e. $\mathcal{D} \cdot \Gamma = 0$, but this leads to a contradiction by taking $\mathcal{D} = \mathcal{L}$. \hfill $\square$

Similar results as in [1, Theorem 1] (see also [14, Proposition VI]) for the case of reducible ample divisors on $\mathcal{V}$ can be now proved.

**Proposition 5.2.** Let $\mathcal{V}$ be an integral normal complex projective variety of dimension $n \geq 3$ and let $\mathcal{L}$ be an ample line bundle on $\mathcal{V}$. Assume that $(\odot)$ holds. If, up to renaming, $A_1 \cong \mathbb{P}^{n-1}$, then $\mathcal{V}$ is the cone $C(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(s))$ on $\mathcal{L}$ with $A_1 = v_s(\mathbb{P}^{n-1})$ and $\mathcal{N}_{A_1/\mathcal{V}} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(s)$ for a suitable integer $s > 0$, where $v_s$ is the $s^{th}$ Veronese embedding of $\mathbb{P}^{n-1}$.
Proof. Since $(\diamondsuit)$ holds and $A_{1} \cong \mathbb{P}^{n-1}$, by Theorem 1.1 we have that $f_{1}(A_{1}) \cong \mathbb{P}^{n-1}$ is ample on $\mathcal{V}_{1}$. By [1, Theorem 1] we see that $\mathcal{V}_{1}$ is the cone $C(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(s))$ over $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(s))$, where $s$ is a positive integer such that $\mathcal{N}_{f_{1}(A_{1})/\mathcal{V}_{1}} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(s)$.

Since $\mathcal{V}_{1}$ is $\mathbb{Q}$-factorial and $\text{Pic}(\mathcal{V}_{1}) = \mathbb{Z}$, by Lemma 5.1 we deduce that $f_{1}$ is an isomorphism and $\text{Pic}(\mathcal{V}) = \mathbb{Z}$. Therefore, $A_{1}$ is an ample divisor on $\mathcal{V}$ and by applying now [1, Theorem 1] to the pair $(\mathcal{V}, A_{1})$, we get that $\mathcal{V} \cong C(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(s))$, $A_{1} \cong v_{s}(\mathbb{P}^{n-1})$ and $\mathcal{N}_{A_{1}/\mathcal{V}} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(s)$ for a suitable integer $s > 0$. \hfill$\square$

**Proposition 5.3.** Let $\mathcal{V}$ be an integral normal Gorenstein projective variety of dimension $n \geq 3$ and let $\mathcal{L}$ be an ample line bundle on $\mathcal{V}$. Suppose that $(\diamondsuit)$ holds and that $\dim \text{Irr}(\mathcal{V}) \leq 0$, where $\text{Irr}(\mathcal{V})$ is the set of irrational singularities of $\mathcal{V}$. Assume that each $A_{i}$ is a normal Gorenstein variety such that $K_{A_{i}} + \tau_{i} \mathcal{H}_{i} \simeq \mathcal{O}_{A_{i}}$ for some integer $\tau_{i}$. If $\mathcal{V} - A_{k}$ is a local complete intersection for some $k = 1, \ldots, r$ and either $n \geq 4$, or $n = 3$ and $\mathcal{V} - F$ is a local complete intersection for some finite, possibly empty set $F \subset \mathcal{V} - \text{Irr}(\mathcal{V})$, then $\text{Pic}(\mathcal{V}) = \mathbb{Z} \langle \Lambda \rangle$, where $\Lambda$ is an ample line bundle on $\mathcal{V}$, $K_{\mathcal{V}} = \rho \Lambda$, $A_{i} = a_{i} \Lambda$ and $\Lambda_{A_{i}} = h_{i} \mathcal{H}_{i}$ with $\tau_{i} = -h_{i}(\rho + a_{i})$, where $\rho, a_{i} > 0$ and $h_{i} > 0$ are integers. In particular, for $n \geq 5$ we have $h_{i} = 1$ for every $i = 1, \ldots, r$.

Proof. Assume that $n \geq 4$. From case (4) of Corollary 4.1 it follows that either (a) $n \geq 5$ and $\text{Pic}(\mathcal{V}) \cong \text{Pic}(A_{i}) = \mathbb{Z}$ for any $i = 1, \ldots, r$, or (b) $n = 4$ and $\text{Pic}(\mathcal{V})$ restricts injectively into $\text{Pic}(A_{i}) = \mathbb{Z}$. In both situations, we see that $\text{Pic}(\mathcal{V}) = \mathbb{Z} \langle \Lambda \rangle$ for some ample line bundle $\Lambda$ on $\mathcal{V}$. Moreover, in (a) we have that $\Lambda_{A_{i}} \cong \mathcal{H}_{i}$, while in (b) we have that $\Lambda_{A_{i}} \cong h_{i} \mathcal{H}_{i}$ for some positive integer $h_{i}$. By adjunction formula, we obtain that

$$\mathcal{O}_{A_{i}} \cong K_{A_{i}} + \tau_{i} \mathcal{H}_{i} \cong \begin{cases} (K_{\mathcal{V}} + A_{i} + \tau_{i} \Lambda)_{A_{i}} & \text{in case (a)}, \\ (K_{\mathcal{V}} + A_{i} + \frac{\tau_{i}}{h_{i}} \Lambda)_{A_{i}} & \text{in case (b)}. \end{cases}$$

i.e. $K_{\mathcal{V}} + A_{i} + (\tau_{i}/h_{i}) \Lambda \cong \mathcal{O}_{\mathcal{V}}$ for some $h_{i} \geq 1$. If we put $K_{\mathcal{V}} = \rho \Lambda$ and $A_{i} = a_{i} \Lambda$ for some integers $\rho$ and $a_{i} \geq 1$, then we see that $\tau_{i} = -h_{i}(\rho + a_{i})$, with $h_{i} = 1$ when $n \geq 5$.

As to the case $n = 3$, from (III) of Corollary 4.3 it follows that $\text{Pic}(\mathcal{V})$ restricts injectively into $\text{Pic}(A_{i}) = \mathbb{Z}$. Then by arguing as above, we conclude that also in this situation $\text{Pic}(\mathcal{V}) = \mathbb{Z} \langle \Lambda \rangle$ for some ample line bundle $\Lambda$ on $\mathcal{V}$ and, with the same notation as above, $\tau_{i} = -h_{i}(\rho + a_{i})$, where $h_{i}$ and $a_{i}$ are positive integers. \hfill$\square$

**Remark 5.4.** When $n \geq 6$, Proposition 5.3 generalizes [16, Theorem 1].

**5.2. All the $A_{i}$’s have small degrees.** Denote now by $X$ an $n$-fold with $n \geq 3$ and by $L$ an ample line bundle on $X$. In this subsection we prove the last result stated in the Introduction.
Proof of Proposition 1.3. Since \([H_i]_{A_i}^{n-1} \leq 4\), by [2, (8.10.1)] we see that 
\((A_i, [H_i]_{A_i})\) satisfies one of the following two conditions:
(a) \(\text{Pic}(A_i) = \mathbb{Z}([H_i]_{A_i})\);
(b) \((A_i, [H_i]_{A_i}) \cong (\mathbb{P}^1 \times \mathbb{P}^3, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(1, 1))\).

Step (1). First of all, assume that \(r = 1\). In Case (a), by Lefschetz's theorem we see that \(\text{Pic}(X) = \mathbb{Z}([H_i])\). Write \(A_1 = a_1 H_1\) for a positive integer \(a_1\). Since \(a_1 H_i^2 = A_1 H_i^{n-1} = [H_i]_{A_1}^{n-1} \leq 4\), we deduce that \(a_1 H_i^2 \leq 4\). Consider the map \(\varphi : X \to \mathbb{P}^N\) associated to \([H_i]\). Since \([H_i]\) is ample and spanned, the morphism \(\varphi\) is finite and such that
\([H_i]^n = \deg \varphi \cdot \deg \varphi(X) \leq 4\).

This gives the following three possibilities:
(1) \(\deg \varphi = 1\), \(\deg \varphi(X) \leq 4\);
(2) \(\deg \varphi = 2\), \(\deg \varphi(X) \leq 2\);
(3) \(\deg \varphi = 3, 4\) and \(\deg \varphi(X) = 1\).

In (1), since \(n \geq 5\) and \(\text{Pic}(X) = \mathbb{Z}([H_i])\), by [2, (8.10.1)] (see also [6], [9], [11]) we deduce that \((X, H_1)\) is one of the following pairs:
- \((\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))\), where \(A_1 \in [\mathcal{O}_{\mathbb{P}^n}(a_1)]\) with \(0 < a_1 \leq 4\);
- \((\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))\), where \(A_1 \in [\mathcal{O}_{\mathbb{Q}^n}(a_1)]\) with \(0 < a_1 \leq 2\);
- \((V_d, \mathcal{O}_{\mathbb{P}^{n+1}}(1)_{V_d})\) with \(A_1 \in [H_i]\), where \(V_d \subset \mathbb{P}^{n+1}\) is a smooth hypersurface of degree \(d = 3, 4\);
- \((W, \mathcal{O}_{\mathbb{P}^{n+2}}(1)_{W})\) with \(A_1 \in [H_i]\), where \(W = Q_1 \cap Q_2 \subset \mathbb{P}^{n+2}\) is a complete intersection of two quadric hypersurfaces \(Q_i \subset \mathbb{P}^{n+2}\) for \(i = 1, 2\).

In (2), the map \(\varphi\) is a double cover of either (i) \(\mathbb{P}^n\), or (ii) \(\mathbb{Q}^n \subset \mathbb{P}^{n+1}\). In case (i), we see that \(H_1 = \varphi^* \mathcal{O}_{\mathbb{P}^n}(1)\) and \(A_1 \in [a_1 H_i]\) with \(a_1 = 1, 2\). In case (ii), we get \(H_1 = \varphi^* \mathcal{O}_{\mathbb{Q}^n}(1)\) and \(A_1 \in [H_i]\). Finally, in (3) the morphism \(\varphi\) is a \(d\)-cover of \(\mathbb{P}^n\) with \(d = 3, 4\), and \(A_1 = H_1 = \varphi^* \mathcal{O}_{\mathbb{P}^n}(1)\).

Finally, consider Case (b). Note that \(\text{Pic}(X) \neq \mathbb{Z}\). Moreover, since \(A_1\) is ample, for any ample line bundle \(H\) on \(\mathbb{P}^1 \times \mathbb{P}^3\), we have
\[
0 = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(1, 0, 0) \cdot \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(2, 0, 0) \cdot H^2 = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(1, 0) \cdot (K_{A_1} + 4 H_{1, A_1}) \cdot H^2
\]
\[
= \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(1, 0) \cdot (K_X + 4 H_1 + A_1)_{A_1} \cdot H^2 \geq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(1, 0) \cdot (K_X + 5 H_1)_{A_1} \cdot H^2,
\]
i.e. \(K_X + 5 H_1\) can not be ample on \(X\). So by [10], [6] and [2, §7.2], we see that \((X, H_1)\) is a scroll over \(\mathbb{P}^1\) of dimension five, i.e. \(X \cong \mathbb{P}^{\mathcal{E}}\), where \(\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_5)\) with \(1 \leq a_1 \leq \cdots \leq a_5\), and \(H_1\) is the tautological line bundle \(\xi\) of \(\mathbb{P}(\mathcal{E})\). Put \(\xi' = \xi - a_1 F\). Since \(A_1\) is ample on \(X\), write \(A_1 = a \xi' + b F\) for some positive integers \(a\) and \(b\) (see [2, (3.2.4)]). Then
\[
[H_i]_{A_1} = (a \xi' + b F)(\xi')^4 = a(\xi - a_1 F)(\xi')^4 + b = a(a_2 + \cdots + a_5) + b \geq 5,
\]
but this is absurd. Thus Case (b) can not occur for \(r = 1\).
STEP (II). Suppose now that \( r \geq 2 \). First of all, let us prove the following

**Claim.** If \( (A_1, H_{1A_1}) \cong (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)) \), then \( K_X + 4H_i \) is not nef for some \( i = 1, \ldots, r \).

Proof. Suppose that \( K_X + 4H_k \) is nef for every \( k = 1, \ldots, r \). Let \( \Phi_1 : A_1 \to \mathbb{P}^1 \) be the nefvalue morphism of \( (A_1, H_{1A_1}) \). Up to renaming, assume that \( A_2, \ldots, A_r \) are the only components of \( A \) such that \( A_1 \cap A_h \neq \emptyset \) for \( h = 2, \ldots, s \) with \( s \leq r \). Put \( h_k := [A_k]_{A_1} \) for \( k = 2, \ldots, r \) and note that \( h_t \) is trivial for \( t = s + 1, \ldots, r \).

First of all, if \( \Phi_1(h_k) \) is a union of points of \( \mathbb{P}^1 \) for every \( k = 2, \ldots, s \), then for a general fiber \( F \cong \mathbb{P}^3 \) of \( \Phi_1 \) we have that

\[
(L_{A_1})_F = ([A_1]_{A_1})_F + [h_2]_F + \cdots + [h_s]_F = ([A_1]_{A_1})_F = \mathcal{O}_F(a),
\]

for a suitable integer \( a \geq 1 \). Moreover, note that \( K_{A_1} + 4H_{1A_1} \cong 2F \).

Thus by adjunction we obtain that

\[
[K_X + 4H_1]_F H_F^2 = ([K_X + 4H_1]_{A_1})_F H_{1A_1}^F = (2F - [A_1]_{A_1})_F H_{1A_1}^F
\]

\[
= -(A_{1A_1})_F H_{1A_1}^F = -\mathcal{O}_F(a) \mathcal{O}_F(1)^2 < 0,
\]

but this is absurd, since \( K_X + 4H_1 \) is nef and \( H_1 \) is ample on \( X \).

Assume now that \( \Phi_1(h_t) \) is a point \( t = 2, \ldots, s \). If \( \text{Pic}(A_t) = \mathbb{Z} \), then by [3, (4.2)] we get a contradiction by taking \( B = A_t \) and \( A = A_1 \). Thus we can assume that \( (A_t, H_{1A_t}) \cong (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)) \). Note that there exists a general fiber \( F \) of \( \Phi_1 \) such that \( F \not\subseteq h_t \) and \( [h_t]_F \) is effective and not trivial. Moreover, we have that \( \text{Pic}(F) \cong \mathbb{Z} \) and \( F \not\subseteq A_t \), since otherwise \( F \subseteq A_1 \cap A_t = h_t \). Therefore \( F \cap A_t \neq \emptyset \) and \( F \cap A_t \neq F \). Since

\[
\dim F + \dim A_t - \dim \Phi_1(A_1) = 3 + 4 - 1 = 6 > 5 = \dim X,
\]

by the same argument as in the proof of [3, (4.2)] with \( B = F \) and \( A = A_t \), we can conclude that \( F \subseteq A_t \), but this gives a contradiction. \( \square \)

By the above Claim, if one of the \( A_k \), say \( A_1 \), is such that

\[
(A_1, H_{1A_1}) \cong (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)),
\]

then \( n = 5 \) and \( K_X + 4H_i \) is not nef for some \( i = 1, \ldots, r \). Since \( \text{Pic}(A_1) \neq \mathbb{Z} \), by [10], [6] and [2, §7] we deduce that \( (X, H_i) \) is a \( \mathbb{P}^3 \)-bundle over a smooth curve \( C \). Moreover, \( A_1 \) dominates \( C \). So \( C \cong \mathbb{P}^1 \) and \( X \cong \mathbb{P}^1(E) \) for some vector bundle \( E \) of rank-5 on \( \mathbb{P}^1 \) such that \( E \cong \mathcal{O}_{\mathbb{P}^1}(\tilde{a}_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(\tilde{a}_4) \) with \( 0 \leq \tilde{a}_1 \leq \cdots \leq \tilde{a}_4 \). Let \( \xi \) be the tautological line bundle of \( \mathbb{P}(E) \). Write \( A_1 = a_1 \xi + b_1 F \) and \( H_i = a_i \xi + b_i F \) with \( b_1 \geq 0 \) and \( a_1, a_i, b_i \) positive integers (see [2, (3.2.4)]). Furthermore, put \( F_{A_1} \cong \mathbb{P}^1 \).
\[ \mathcal{O}_{A_1}(a, 0) \text{ and } \xi_{A_1} \simeq \mathcal{O}_{A_1}(b, c) \text{ with } a > 0 \text{ and } b, c \geq 0. \] By the adjunction formula we have
\[
\mathcal{O}_{A_1}(-2, -4) \simeq K_{A_1} \simeq (K_X + A_1)_{A_1} \simeq [-5\xi + (\deg \mathcal{E} - 2)F + a_1\xi + b_1F]_{A_1} \\
= (a_1 - 5)\xi_{A_1} + (b_1 + \deg \mathcal{E} - 2)F_{A_1} \\
\simeq \mathcal{O}_{A_1}(b(a_1 - 5) + a(b_1 + \deg \mathcal{E} - 2), c(a_1 - 5)).
\]
This gives the following two equations:

\begin{align*}
(b) & \quad 4 = c(5 - a_1), \\
(z) & \quad 2 = b(5 - a_1) - a(b_1 + \deg \mathcal{E} - 2).
\end{align*}

Note that from \((b)\) it follows that \(1 \leq a_1 \leq 4\) and then
\[
4 \geq a_1 = [A_1]_F \cdot [\xi]_F^3 = F_{A_1} \cdot \xi_{A_1}^3 = \mathcal{O}_{A_1}(a, 0) \cdot \mathcal{O}_{A_1}(b, c)^3 = ac^3.
\]
Thus we deduce that \(c = 1\) and \(a_1 = a\). By \((b)\) we have also that \(a = a_1 = 1\) and the equation \((z)\) gives \(4b = b_1 + \deg \mathcal{E}\). Since
\[
\mathcal{O}_{A_1}(1, 1) \simeq [H_1]_{A_1} = [\alpha_1\xi + \beta_1F]_{A_1} = \mathcal{O}_{A_1}(\alpha_1b + \beta_1, \alpha_1),
\]
we see that \(\alpha_1 = 1\) and \(1 = b + \beta_1 \geq b + 1 \geq 1\), i.e. \(b = 0\) and \(\beta_1 = 1\). Therefore, by \((z)\) we get \(0 = 4b = b_1 + \deg \mathcal{E} \geq \deg \mathcal{E}\), i.e. \(a_1 = \ldots = a_4 = 0\) and \(b_1 = 0\). This shows that \(X \cong \mathbb{P}(\mathcal{O}_p^5) \cong \mathbb{P}^1 \times \mathbb{P}^4, A_1 \in |\mathcal{O}_X(0, 1)| \text{ and } H_1 \in |\mathcal{O}_X(1, 1)|\). Consider \(A_i \in |\mathcal{O}_X(d, e)| \text{ and } H_i \in |\mathcal{O}_X(t, s)|\) for some \(i = 2, \ldots, r\), where \(d, e \geq 0\) and \(t, s > 0\). Then we have
\[
4 \geq [H_i]_{A_i}^4 = \mathcal{O}_X(d, e) \cdot \mathcal{O}_X(t, s)^4 = s^3(ds + 4et) \geq s^3.
\]
This gives \(s = 1\) and \(0 \leq e \leq 1\). If \(e = 1\), then we get \(d = 0, t = s = e = 1\) and \((A_i, [H_i]_{A_i})\) is of the same type as \((A_1, [H_1]_{A_1})\). If \(e = 0\), then we see that \(1 \leq d \leq 4, t \geq 1\) and \(s = 1\). This shows that \(A_i \in |\mathcal{O}_X(d, 0)|\) and \(H_i \in |\mathcal{O}_X(t, 1)|\) with \(t \geq 1\) and \(1 \leq d \leq 4\), i.e. \(A_i \to \mathbb{P}^4\) is a \(d\)-cover of \(\mathbb{P}^4\) with \(1 \leq d \leq 4\). Note that in this case we have
\[
|\mathcal{O}_X(d, 0)| = \left| \mathcal{O}_X(1, 0) \right| + \cdots + \left| \mathcal{O}_X(1, 0) \right|.
\]
th-d-times

Thus, since \(A_i \in |\mathcal{O}_X(d, 0)|\) is irreducible and reduced, we conclude that \(d = 1\).

Finally, since the above argument works independently from the choice of the component \(A_1\) of \(A \in |L|\), we can assume, without loss of generality, that every component \(A_i\) of \(A\) is such that \(\text{Pic}(A_i) = \mathbb{Z}[\langle H_i \rangle_{A_i}]\). Then by Proposition 5.3 and Corollary 4.3 (III), we conclude that \(\text{Pic}(X) = \mathbb{Z}[\langle A \rangle]\) for some ample line bundle \(\Lambda\) on \(X\). Write
\[ A_i = a_i \Lambda \text{ and } H_i = b_i \hat{\Lambda} \] for suitable positive integers \( a_i \) and \( b_i \). Thus we obtain that
\[ a_i b_i^{n-1} \Lambda^n = (a_i \Lambda)(b_i \hat{\Lambda})^{n-1} = [H_i]^{n-1}_{\Lambda_i} \leq 4, \]
i.e. \( H_1 = \cdots = H_r = \Lambda \) and \( a_i \Lambda^n \leq 4 \). Consider the map \( \varphi \colon X \to \mathbb{P}^N \) associated to \( |\Lambda| \). Since \( \Lambda \) is ample and spanned, we have \( \Lambda^n = \deg \varphi \cdot \deg \varphi(X) \leq 4 \) and by arguing as in case \( r = 1 \), we obtain the statement.

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**References**


