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# ZERO MEAN CURVATURE SURFACES IN LORENTZ–MINKOWSKI 3-SPACE WHICH CHANGE TYPE ACROSS A LIGHT-LIKE LINE

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## Abstract

It is well-known that space-like maximal surfaces and time-like minimal surfaces in Lorentz–Minkowski 3-space  $\mathbf{R}_1^3$  have singularities in general. They are both characterized as zero mean curvature surfaces. We are interested in the case where the singular set consists of a light-like line, since this case has not been analyzed before. As a continuation of a previous work by the authors, we give the first example of a family of such surfaces which change type across a light-like line. As a corollary, we also obtain a family of zero mean curvature hypersurfaces in  $\mathbf{R}_1^{n+1}$  that change type across an  $(n - 1)$ -dimensional light-like plane.

## Introduction

Many examples of space-like maximal surfaces containing singular curves in the Lorentz–Minkowski 3-space  $(\mathbf{R}_1^3; t, x, y)$  of signature  $(- + +)$  have been constructed in [11], [1], [12], [8], [4] and [5].

In this paper, we are interested in the zero mean curvature surfaces in  $\mathbf{R}_1^3$  changing their causal type: Klyachin [10] showed under a sufficiently weak regularity assumption that a zero mean curvature surface in  $\mathbf{R}_1^3$  changes its causal type only on the following two subsets:

- null curves (i.e., regular curves whose velocity vector fields are light-like) which are non-degenerate (i.e., their projections into the  $xy$ -plane are locally convex plane curves), or
- light-like lines, which are degenerate everywhere.

Given a non-degenerate null curve  $\gamma$  in  $\mathbf{R}_1^3$ , there exists a zero mean curvature surface which changes its causal type across this curve from a space-like maximal surface to

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a time-like minimal surface (cf. [6], [10], [9] and [7]). This construction can be accomplished using the Björling formula for the Weierstrass-type representation formula of maximal surfaces. (The reference [3] is an expository article on this subject.) However, if  $\gamma$  is a light-like line, the aforementioned construction fails, since the isothermal coordinates break down at the light-like singular points. Locally, such a surface is the graph of a function  $t = f(x, y)$  satisfying

$$(*) \quad (1 - f_y^2)f_{xx} + 2f_x f_y f_{xy} + (1 - f_x^2)f_{yy} = 0,$$

where  $f_x = \partial f / \partial x$ ,  $f_{xy} = \partial^2 f / \partial y \partial x$ , etc. We call this and its graph the *zero mean curvature equation* and a *zero mean curvature surface*, respectively. Until now, zero mean curvature surfaces which actually change type across a light-like line were unknown. As announced in [2], the main purpose of this paper is to construct such an example. In Section 1, we give a formal power series solution of the zero mean curvature equation describing all zero mean curvature surfaces which contain a light-like line. Using this, we give the precise statement of our main result and show how the statement can be reduced to a proposition (cf. Proposition 1.3). In Section 2, we then prove it. As a consequence, we obtain the first example of (a family of) zero mean curvature surfaces which change type across a light-like line.

## 1. The main theorem

We discuss solutions of the zero mean curvature equation (\*) which have the following form

$$(1.1) \quad f(x, y) = b_0(y) + \sum_{k=1}^{\infty} \frac{b_k(y)}{k} x^k,$$

where  $b_k(y)$  ( $k = 1, 2, \dots$ ) are  $C^\infty$ -functions. When  $f$  contains a singular light-like line, we may assume without loss of generality that (cf. [2])

$$(1.2) \quad b_0(y) = y, \quad b_1(y) = 0.$$

As was pointed out in [2], there exists a real constant  $\mu$  called the *characteristic* of  $f$  such that  $b_2(y)$  satisfies the following equation

$$(1.3) \quad b_2'(y) + b_2(y)^2 + \mu = 0 \quad (' = d/dy).$$

Now we derive the differential equations satisfied by  $b_k(y)$  for  $k \geq 3$  assuming (1.2). If we set

$$Y := f_y - 1 = \sum_{k=2}^{\infty} \frac{b_k'(y)}{k} x^k$$

and

$$P := 2(Yf_{xx} - f_x f_{xy}), \quad Q := Y^2 f_{xx} - 2f_x f_{xy} Y, \quad R := f_x^2 f_{yy},$$

then, by straightforward calculations, we see that

$$P = -b_2 b_2' x^2 - \frac{4}{3} b_2 b_3' x^3 - \sum_{k=4}^{\infty} \left( P_k + \frac{2(k-1)}{k} b_2 b_k' + (3-k) b_2' b_k \right) x^k,$$

$$Q = - \sum_{k=4}^{\infty} Q_k x^k, \quad R = \sum_{k=4}^{\infty} R_k x^k,$$

where

$$(1.4) \quad \begin{aligned} P_k &:= \sum_{m=3}^{k-1} \frac{2(k-2m+3)}{k-m+2} b_m b_{k-m+2}', \\ Q_k &:= \sum_{m=2}^{k-2} \sum_{n=2}^{k-m} \frac{3n-k+m-1}{mn} b_m' b_n' b_{k-m-n+2}, \\ R_k &:= \sum_{m=2}^{k-2} \sum_{n=2}^{k-m} \frac{b_m b_n b_{k-m-n+2}''}{k-m-n+2} \end{aligned}$$

for  $k \geq 4$ , and that the zero mean curvature equation (\*) reduces to

$$\sum_{k=2}^{\infty} \frac{b_k''}{k} x^k = f_{yy} = P + Q + R.$$

It is now immediate, by comparing the coefficients of  $x^k$  from both sides, to see that each  $b_k$  ( $k \geq 3$ ) satisfies the following ordinary differential equation

$$(1.5) \quad b_k''(y) + 2(k-1)b_2(y)b_k'(y) + k(3-k)b_2'(y)b_k(y) = -k(P_k + Q_k - R_k),$$

where  $P_3 = Q_3 = R_3 = 0$  and  $P_k$ ,  $Q_k$  and  $R_k$  are as in (1.4) for  $k \geq 4$ . Note that  $P_k$ ,  $Q_k$  and  $R_k$  are written in terms of  $b_j$  ( $j = 1, \dots, k-1$ ) and their derivatives.

Now, we consider the case that  $1 - f_x^2 - f_y^2$  changes sign across the light-like line  $\{t = y, x = 0\}$ . This case occurs only when the characteristic  $\mu$  as in (1.3) of  $f$  vanishes [2]. If we set

$$b_2(y) = 0 \quad (y \in \mathbf{R}),$$

then (1.3) holds for  $\mu = 0$ . So we assume

$$(1.6) \quad b_0(y) = y, \quad b_1(y) = 0, \quad b_2(y) = 0, \quad b_3(y) = 3cy,$$

where  $c$  is a non-zero constant. Then  $f(x, y)$  in (1.1) can be rewritten as

$$(1.7) \quad f(x, y) = y + cyx^3 + \sum_{k=4}^{\infty} \frac{b_k(y)}{k} x^k.$$

In this situation, we will find a solution satisfying

$$(1.8) \quad b_k(0) = b'_k(0) = 0 \quad (k \geq 4).$$

Then (1.5) reduces to

$$(1.9) \quad b''_k(y) = -k(P_k + Q_k - R_k), \quad b_k(0) = b'_k(0) = 0, \quad (k \geq 4),$$

$$(1.10) \quad P_k = \sum_{m=3}^{k-1} \frac{2(k-2m+3)}{k-m+2} b_m(y) b'_{k-m+2}(y) \quad (k \geq 4),$$

$$(1.11) \quad Q_k = \sum_{m=3}^{k-4} \sum_{n=3}^{k-m-1} \frac{3n-k+m-1}{mn} b'_m(y) b'_n(y) b_{k-m-n+2}(y) \quad (k \geq 7),$$

$$(1.12) \quad R_k = \sum_{m=3}^{k-4} \sum_{n=3}^{k-m-1} \frac{b_m(y) b_n(y) b''_{k-m-n+2}(y)}{k-m-n+2} \quad (k \geq 7),$$

and  $Q_k = R_k = 0$  for  $4 \leq k \leq 6$ , where the fact that  $b_2(y) = 0$  has been extensively used. For example,

$$\begin{aligned} b_0 &= y, \quad b_1 = b_2 = 0, \quad b_3 = 3cy, \quad b_4 = -4c^2y^3, \quad b_5 = 9c^3y^5, \\ b_6 &= -24c^2y^7, \quad b_7 = 70c^5y^9 - 14c^3y^3, \dots \end{aligned}$$

In this article, we show the following assertion:

**Theorem 1.1.** *For each positive number  $c$ , the formal power series solution  $f(x, y)$  uniquely determined by (1.9), (1.10), (1.11) and (1.12) gives a real analytic zero mean curvature surface on a neighborhood of  $(x, y) = (0, 0)$ . In particular, there exists a non-trivial 1-parameter family of real analytic zero mean curvature surfaces each of which changes type across a light-like line (see Fig. 1).*

As a consequence, we get the following:

**Corollary 1.2.** *There exists a family of zero-mean curvature hypersurfaces in Lorentz–Minkowski space  $\mathbf{R}_1^{n+1}$  each of which changes type across an  $(n-1)$ -dimensional light-like plane.*

*Proof.* Let  $f$  be as in the theorem. The graph of the function defined by

$$\mathbf{R}^n \ni (x_1, \dots, x_n) \mapsto f(x_1, x_2) \in \mathbf{R}$$

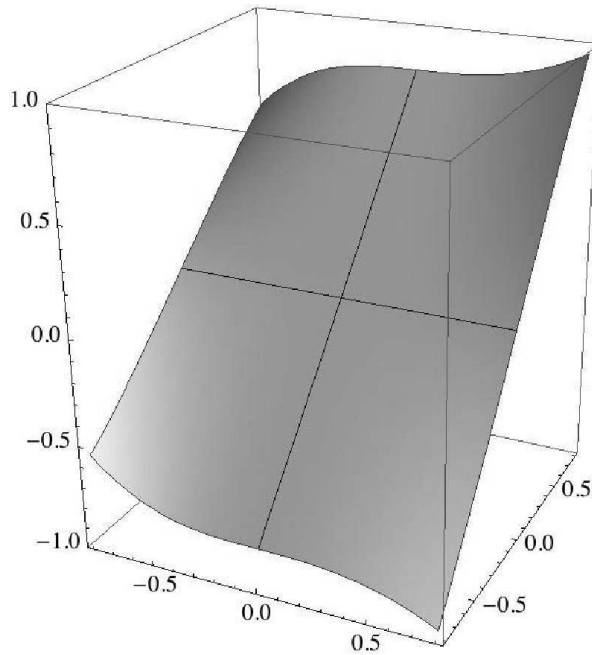


Fig. 1. The graph of  $t = f(x, y)$  for  $c = 1/2$  and  $|x|, |y| < 0.8$  (The range of the graph is wider than the range used in our mathematical estimation. However, this figure still has a sufficiently small numerical error term in the Taylor expansion.)

gives the desired hypersurface. In this case, the zero mean curvature equation

$$\left(1 - \sum_{j=1}^n f_{x_j}^2\right) \sum_{i=1}^n f_{x_i x_i} + \sum_{i,j=1}^n f_{x_i x_j} f_{x_i} f_{x_j} = 0 \quad (f_{x_i} := \partial f / \partial x_i, f_{x_i x_j} := \partial^2 f / \partial x_j \partial x_i)$$

reduces to (\*) in the introduction. □

To prove Theorem 1.1, it is sufficient to show that for arbitrary positive constants  $c > 0$  and  $\delta > 0$  there exist positive constants  $n_0$ ,  $\theta_0$ , and  $C$  such that

$$(1.13) \quad |b_k(y)| \leq \theta_0 C^k \quad (|y| \leq \delta)$$

holds for  $k \geq n_0$ . In fact, if (1.13) holds, then the series (1.7) converges uniformly over the rectangle  $[-C^{-1}, C^{-1}] \times [-\delta, \delta]$ .

The key assertion to prove (1.13) is the following

**Proposition 1.3.** *For each  $c > 0$  and  $\delta > 0$ , we set*

$$(1.14) \quad M := 3 \max\{144c\tau|\delta|^{3/2}, \sqrt[4]{192c^2\tau}\},$$

*where  $\tau$  is the positive constant given by (A.3) in the appendix, such that*

$$(1.15) \quad t \int_t^{1-t} \frac{du}{u^2(1-u)^2} \leq \tau \quad (0 < t < 1/2).$$

*Then the function  $\{b_l(y)\}_{l \geq 3}$  formally determined by the recursive formulas (1.9)–(1.12) satisfies the inequalities*

$$(1.16) \quad |b_l''(y)| \leq c|y|^{l^*} M^{l-3},$$

$$(1.17) \quad |b_l'(y)| \leq \frac{3c|y|^{l^*+1}}{l^*+2} M^{l-3},$$

$$(1.18) \quad |b_l(y)| \leq \frac{3c|y|^{l^*+2}}{(l^*+2)^2} M^{l-3}$$

*for any*

$$(1.19) \quad y \in [-\delta, \delta],$$

*where*

$$(1.20) \quad l^* := \frac{1}{2}(l-1) - 2 \quad (l \geq 3).$$

Once this proposition is proven, (1.13) follows immediately. In fact, if we set

$$\theta_0 = \frac{3}{c}(\delta M)^3, \quad C := \delta M$$

and  $n_0 \geq 7$ , then  $1 \leq l^* + 2 < l - 3$  and (1.13) follows from

$$\frac{3c|y|^{l^*+2}}{(l^*+2)^2} M^{l-3} \leq \theta_0 C^l.$$

## 2. Proof of Proposition 1.3

We prove the proposition using induction on the number  $l \geq 3$ . If  $l = 3$ , then

$$|b_3''(y)| = 0 \leq \frac{c}{|y|} = c|y|^{3^*} M^0,$$

$$|b_3'(y)| = 3c = \frac{3c|y|^{3^*+1}}{3^*+2} M^0,$$

$$|b_3(y)| = 3c|y| = \frac{3c|y|^{3^*+2}}{(3^*+2)^2} M^0$$

hold, using that  $b_3(y) = 3cy$ ,  $M^0 = 1$  and  $3^* = -1$ . So we prove the assertion for  $l \geq 4$ . Since (1.17), (1.18) follow from (1.16) by integration, it is sufficient to show that (1.16) holds for each  $l \geq 4$ . (In fact, the most delicate case is  $l = 4$ . In this case  $l^* = -1/2$  and we can use the fact that  $\int_0^{y_0} 1/\sqrt{y} dy$  for  $y_0 > 0$  converges.)

The inequality (1.16) follows if one shows that, for each  $k \geq 4$

$$(2.1) \quad |kP_k(y)|, |kQ_k(y)|, |kR_k(y)| \leq \frac{c}{3} |y|^{k^*} M^{k-3} \quad (|y| \leq \delta)$$

under the assumption that (1.16), (1.17) and (1.18) hold for all  $3 \leq l \leq k-1$ . In fact, if (2.1) holds, (1.16) for  $l = k$  follows immediately. Then by the initial condition (1.9) (cf. (1.8)), we have (1.17) and (1.18) for  $l = k$  by integration.

**The estimation of  $|kP_k|$  for  $k \geq 4$ .** By (1.10) and using the fact that (1.17), (1.18) hold for  $l \leq k-1$ , we have for each  $|y| < \delta$  that

$$\begin{aligned} |kP_k| &\leq \sum_{m=3}^{k-1} \frac{2k|k-2m+3|}{k-m+2} |b_m(y)| |b'_{k-m+2}(y)| \\ &\leq \sum_{m=3}^{k-1} \frac{2k|k-2m+3|}{k-m+2} \left( \frac{3cM^{m-3}|y|^{m^*+2}}{(m^*+2)^2} \right) \left( \frac{3cM^{k-m+2-3}|y|^{(k-m+2)^*+1}}{(k-m+2)^*+2} \right) \\ &= cM^{k-3}|y|^{k^*} \frac{144c|y|^{3/2}}{M} \sum_{m=3}^{k-1} \frac{k|k-2m+3|}{(m-1)^2(k-m+1)(k-m+2)} \\ &\leq cM^{k-3}|y|^{k^*} \frac{144c|\delta|^{3/2}}{M} \sum_{m=3}^{k-1} \frac{k|k-2m+3|}{(m-1)^2(k-m+1)(k-m+2)} \\ &\leq \frac{c}{3\tau} M^{k-3}|y|^{k^*} \sum_{m=3}^{k-1} \frac{k|k-2m+3|}{(m-1)^2(k-m+1)^2}. \end{aligned}$$

Here, we used (1.14). Since

$$\max_{m=3, \dots, k-1} |k-2m+3| = \max_{m=3, k-1} |k-2m+3| = \max\{|k-3|, |-k+5|\},$$

by setting  $q = m-1$ , we have that

$$\begin{aligned} |kP_k| &\leq \frac{c}{3\tau} M^{k-3}|y|^{k^*} \sum_{m=3}^{k-1} \frac{k^2}{(m-1)^2(k-m+1)^2} = \frac{c}{3\tau} M^{k-3}|y|^{k^*} \frac{1}{k} \sum_{q=2}^{k-2} \frac{k^3}{q^2(k-q)^2} \\ &\leq \frac{c}{3\tau} M^{k-3}|y|^{k^*} \frac{1}{k} \int_{1/k}^{1-1/k} \frac{du}{u^2(1-u)^2} \leq \frac{c}{3} M^{k-3}|y|^{k^*}, \end{aligned}$$

where we applied Lemma A.1 and (1.15) at the last step of the estimations. Hence, we get (2.1) for  $kP_k$ .



**The estimation of  $|kQ_k|$  for  $k \geq 7$ .** By (1.11) and the induction assumption, we have that

$$\begin{aligned}
 |kQ_k| &\leq \sum_{m=3}^{k-4} \sum_{n=3}^{k-m-1} \frac{k|3n-k+m-1|}{mn} |b'_m(y)| |b'_n(y)| |b_{k-m-n+2}(y)| \\
 &\leq \sum_{m=3}^{k-4} \sum_{n=3}^{k-m-1} \frac{k|3n-k+m-1|}{mn} \left( \frac{3cM^{m-3}|y|^{m^*+1}}{m^*+2} \right) \\
 &\quad \times \left( \frac{3cM^{n-3}|y|^{n^*+1}}{n^*+2} \right) \left( \frac{3cM^{k-m-n+2-3}|y|^{(k-m-n+2)^*+2}}{((k-m-n+2)^*+2)^2} \right) \\
 &= cM^{k-3}|y|^{k^*} \frac{432c^2}{M^4} \sum_{m=3}^{k-4} \sum_{n=3}^{k-m-1} \frac{k|3n-k+m-1|}{(m-1)^2(n-1)^2(k-m-n+2)^2}.
 \end{aligned}$$

Now we apply the inequalities

$$\begin{aligned}
 \max_{\substack{3 \leq m \leq k-4 \\ 3 \leq n \leq k-m-1}} |3n-k+m-1| &= \max_{(m,n)=(3,3),(3,k-4),(k-4,3)} |3n-k+m-1| \\
 &= \max\{|-k+11|, 4, |2k-10|\} \leq 2k,
 \end{aligned}$$

and also

$$\frac{432c^2}{M^4} \leq \frac{1}{36\tau},$$

which follows from (1.14). Setting  $p := m-1$ ,  $q := n-1$ , we have that

$$\begin{aligned}
 |kQ_k| &\leq \frac{c}{36\tau} M^{k-3}|y|^{k^*} \sum_{m=3}^{k-4} \sum_{n=3}^{k-m-1} \frac{2k^2}{(m-1)^2(n-1)^2(k-m-n+2)^2} \\
 &= \frac{c}{18\tau} M^{k-3}|y|^{k^*} \sum_{p=2}^{k-5} \sum_{q=2}^{k-p-2} \frac{k^2}{p^2q^2(k-p-q)^2}.
 \end{aligned}$$

Now applying Lemma A.2, we have that

$$|kQ_k| \leq \frac{c}{18\tau} M^{k-3}|y|^{k^*} \times 6\tau \leq \frac{c}{3} M^{k-3}|y|^{k^*},$$

which proves (2.1) for  $kQ_k$ .

**The estimation of  $|kR_k|$  for  $k \geq 7$ .** As in the case of  $|kQ_k|$ , we have that

$$\begin{aligned}
 |kR_k| &\leq \sum_{m=3}^{k-4} \sum_{n=3}^{k-m-1} \frac{k|b_m(y)| |b_n(y)| |b''_{k-m-n+2}(y)|}{k-m-n+2} \\
 &\leq \sum_{m=3}^{k-4} \sum_{n=3}^{k-m-1} \frac{k}{k-m-n+2} \left( \frac{3cM^{m-3}|y|^{m^*+2}}{(m^*+2)^2} \right) \\
 &\quad \times \left( \frac{3cM^{n-3}|y|^{n^*+2}}{(n^*+2)^2} \right) (cM^{k-m-n+2-3}|y|^{(k-m-n+2)^*}) \\
 &= 144c^3 M^{k-7} |y|^{k^*} \sum_{m=3}^{k-4} \sum_{n=3}^{k-m-1} \frac{k}{(k-m-n+2)(m-1)^2(n-1)^2} \\
 &= cM^{k-3} |y|^{k^*} \frac{144c^2}{M^4} \sum_{m=3}^{k-4} \sum_{n=3}^{k-m-1} \frac{k^2}{(k-m-n+2)^2(m-1)^2(n-1)^2}.
 \end{aligned}$$

Now we set  $p = m - 1$ ,  $q = n - 1$ , and using the inequality

$$3^4 \times 144c^2\tau \leq 3^4 \times 192c^2\tau < M^4,$$

we have that

$$|kR_k| \leq \frac{c}{3^4\tau} M^{k-3} |y|^{k^*} \sum_{p=2}^{k-5} \sum_{q=2}^{k-p-2} \frac{k^2}{p^2 q^2 (k-p-q)^2}.$$

By applying Lemma A.2, we have that

$$|kR_k| \leq \frac{c}{3^4\tau} M^{k-3} |y|^{k^*} \times 6\tau < \frac{c}{3} M^{k-3} |y|^{k^*},$$

which proves (2.1) for  $kR_k$ . This completes the proof of Proposition 1.3.

## Appendix A. Inequalities used in the proof of Theorem 1.1

For  $a > 0$ , it holds that

$$(A.1) \quad \frac{1}{u^2(a-u)^2} = \frac{1}{a^3} \left( \frac{a}{u^2} + \frac{2}{u} + \frac{a}{(a-u)^2} + \frac{2}{a-u} \right).$$

Therefore,

$$(A.2) \quad \int_t^{a-t} \frac{du}{u^2(a-u)^2} = \frac{2}{a^3} \left( \frac{a(a-2t)}{t(a-t)} + 2 \log \frac{a-t}{t} \right) \quad (0 < t < a/2).$$

In particular, one can show that there exists a positive constant  $\tau$  such that

$$(A.3) \quad t \int_t^{1-t} \frac{du}{u^2(1-u)^2} \leq \tau \quad (0 < t < 1/2).$$

The following assertion is needed to prove (2.1) for  $kP_k(y)$ :

**Lemma A.1.** *Let  $p$  be a non-negative integer and  $k$  an integer satisfying  $k \geq p + 4$ . Then the inequality*

$$\sum_{q=2}^{k-p-2} \frac{k^3}{q^2(k-p-q)^2} \leq \int_{1/k}^{a-1/k} \frac{du}{u^2(a-u)^2} \quad (a := 1 - p/k)$$

holds.

*Proof.* In fact, if we set  $a := 1 - p/k$ , then (A.1) yields that

$$\begin{aligned} \frac{k^3}{q^2(k-p-q)^2} &= \frac{1}{k} \frac{1}{(q/k)^2(a-q/k)^2} \\ &= \frac{1}{a^3} \left[ \frac{1}{k} \left( \frac{a}{(q/k)^2} + \frac{2}{q/k} \right) + \frac{1}{k} \left( \frac{a}{(a-q/k)^2} + \frac{2}{a-q/k} \right) \right]. \end{aligned}$$

Since  $x \mapsto (a+2x)/x^2$  is a monotone decreasing function and the function  $x \mapsto (a+2(a-x))/(a-x)^2$  is monotone increasing on the interval  $(0, a/2)$ , we have that

$$\frac{k^3}{q^2(k-p-q)^2} \leq \frac{1}{a^3} \left[ \int_{(q-1)/k}^{q/k} \left( \frac{a}{u^2} + \frac{2}{u} \right) du + \int_{q/k}^{(q+1)/k} \left( \frac{a}{(a-u)^2} + \frac{2}{a-u} \right) du \right],$$

which yields that

$$\begin{aligned} \sum_{q=2}^{k-p-2} \frac{a^3 k^3}{q^2(k-p-q)^2} &\leq \int_{1/k}^{a-2/k} \left( \frac{a}{u^2} + \frac{2}{u} \right) du + \int_{2/k}^{a-1/k} \left( \frac{a}{(a-u)^2} + \frac{2}{a-u} \right) du \\ &\leq \int_{1/k}^{a-1/k} \left( \frac{a}{u^2} + \frac{2}{u} \right) du + \int_{1/k}^{a-1/k} \left( \frac{a}{(a-u)^2} + \frac{2}{a-u} \right) du \\ &\leq \int_{1/k}^{a-1/k} \left( \frac{a}{u^2} + \frac{2}{u} + \frac{a}{(a-u)^2} + \frac{2}{a-u} \right) du \\ &= \int_{1/k}^{a-1/k} \frac{du}{u^2(a-u)^2}. \end{aligned}$$

This proves the assertion. □

The following assertion is needed to prove (2.1) for  $kQ_k(y)$  and  $kR_k(y)$ :

**Lemma A.2.** *For any integer  $k \geq 7$ , the following inequalities holds:*

$$\sum_{p=2}^{k-5} \sum_{q=2}^{k-p-2} \frac{k^2}{p^2 q^2 (k-p-q)^2} \leq \frac{6}{k} \int_{1/k}^{1-1/k} \frac{du}{u^2(1-u)^2} \leq 6\tau,$$

where  $\tau$  is a constant satisfying (A.3).

**Proof.** We set  $a = a(p) := 1 - (p/k)$ . Applying Lemma A.1 and the identity (A.2), we have that

$$\begin{aligned} \sum_{p=2}^{k-5} \sum_{q=2}^{k-p-2} \frac{k^2}{p^2 q^2 (k-p-q)^2} &= \sum_{p=2}^{k-5} \left[ \frac{1}{kp^2} \sum_{q=2}^{k-p-2} \frac{k^3}{q^2 (k-p-q)^2} \right] \\ &\leq \sum_{p=2}^{k-5} \left[ \frac{1}{kp^2} \int_{1/k}^{a-1/k} \frac{du}{u^2(a-u)^2} \right] = \sum_{p=2}^{k-5} \left[ \frac{1}{p^2} \frac{2}{a^2} \left( \frac{a-2/k}{a-1/k} + 2 \frac{\log(ka-1)}{ka} \right) \right] \\ &\leq \sum_{p=2}^{k-5} \left[ \frac{2}{p^2 a^2} \left( 1 + 2 \frac{\log ka}{ka} \right) \right] \leq \sum_{p=2}^{k-5} \frac{6}{p^2 a^2}, \end{aligned}$$

where we used the fact that  $(\log ka)/(ka) < 1$ . By applying Lemma A.1 and by using the property (A.3) of the constant  $\tau$ , it holds that

$$\begin{aligned} \sum_{p=2}^{k-5} \sum_{q=2}^{k-p-2} \frac{k^2}{p^2 q^2 (k-p-q)^2} &\leq 6 \sum_{p=2}^{k-5} \frac{1}{p^2 (1-p/k)^2} = \frac{6}{k} \sum_{p=2}^{k-5} \frac{k^3}{p^2 (k-p)^2} \\ &\leq \frac{6}{k} \sum_{p=2}^{k-2} \frac{k^3}{p^2 (k-p)^2} \leq \frac{6}{k} \int_{1/k}^{1-1/k} \frac{du}{u^2(1-u)^2} < 6\tau, \end{aligned}$$

which proves the assertion.  $\square$

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