<table>
<thead>
<tr>
<th><strong>Title</strong></th>
<th>ZERO MEAN CURVATURE SURFACES IN LORENTZ-MINKOWSKI 3-SPACE WHICH CHANGE TYPE ACROSS A LIGHT-LIKE LINE</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Author(s)</strong></td>
<td>Fujimori, S; Kim, Y.W; Koh, S-E; Rossman, W; Shin, H; Umehara, M; Yamada, K; Yang, S-D</td>
</tr>
<tr>
<td><strong>Citation</strong></td>
<td>Osaka Journal of Mathematics. 52(1) P.285-P.297</td>
</tr>
<tr>
<td><strong>Issue Date</strong></td>
<td>2015-01</td>
</tr>
<tr>
<td><strong>Text Version</strong></td>
<td>publisher</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="https://doi.org/10.18910/57676">https://doi.org/10.18910/57676</a></td>
</tr>
<tr>
<td><strong>DOI</strong></td>
<td>10.18910/57676</td>
</tr>
<tr>
<td><strong>rights</strong></td>
<td></td>
</tr>
</tbody>
</table>
ZERO MEAN CURVATURE SURFACES IN LORENTZ–MINKOWSKI 3-SPACE WHICH CHANGE TYPE ACROSS A LIGHT-LIKE LINE


(Received October 2, 2013)

Abstract

It is well-known that space-like maximal surfaces and time-like minimal surfaces in Lorentz–Minkowski 3-space \( R_3^1 \) have singularities in general. They are both characterized as zero mean curvature surfaces. We are interested in the case where the singular set consists of a light-like line, since this case has not been analyzed before. As a continuation of a previous work by the authors, we give the first example of a family of such surfaces which change type across a light-like line. As a corollary, we also obtain a family of zero mean curvature hypersurfaces in \( R^{n+1} \) that change type across an \((n-1)\)-dimensional light-like plane.

Introduction

Many examples of space-like maximal surfaces containing singular curves in the Lorentz–Minkowski 3-space \( R_3^1; t, x, y \) of signature \((-++\)) have been constructed in [11], [1], [12], [8], [4] and [5].

In this paper, we are interested in the zero mean curvature surfaces in \( R_3^1 \) changing their causal type: Klyachin [10] showed under a sufficiently weak regularity assumption that a zero mean curvature surface in \( R_3^1 \) changes its causal type only on the following two subsets:

- null curves (i.e., regular curves whose velocity vector fields are light-like) which are non-degenerate (i.e., their projections into the \( xy \)-plane are locally convex plane curves), or
- light-like lines, which are degenerate everywhere.

Given a non-degenerate null curve \( \gamma \) in \( R_3^1 \), there exists a zero mean curvature surface which changes its causal type across this curve from a space-like maximal surface to

2010 Mathematics Subject Classification. Primary 53A10; Secondary 53B30, 35M10.

Kim was supported by NRF 2009-0086794, Koh by NRF 2009-0086794 and NRF 2011-0001565, Shin by NRF-2014R1A2A2A01007324 and Yang by NRF 2012R1A1A2042530. Fujimori was partially supported by the Grant-in-Aid for Young Scientists (B) No. 21740052, Rossman was supported by Grant-in-Aid for Scientific Research (B) No. 20340012, Umehara by (A) No. 22244006 and Yamada by (B) No. 21340016 from Japan Society for the Promotion of Science.
a time-like minimal surface (cf. [6], [10], [9] and [7]). This construction can be accomplished using the Björling formula for the Weierstrass-type representation formula of maximal surfaces. (The reference [3] is an expository article on this subject.) However, if \( \gamma \) is a light-like line, the aforementioned construction fails, since the isothermal coordinates break down at the light-like singular points. Locally, such a surface is the graph of a function \( t = f(x, y) \) satisfying

\[
(1 - f_y^2)f_{xx} + 2f_xf_yf_{xy} + (1 - f_x^2)f_{yy} = 0,
\]

where \( f_x = \partial f/\partial x, \ f_{xy} = \partial^2 f/\partial y \partial x, \) etc. We call this and its graph the zero mean curvature equation and a zero mean curvature surface, respectively. Until now, zero mean curvature surfaces which actually change type across a light-like line were unknown. As announced in [2], the main purpose of this paper is to construct such an example. In Section 1, we give a formal power series solution of the zero mean curvature equation describing all zero mean curvature surfaces which contain a light-like line. Using this, we give the precise statement of our main result and show how the statement can be reduced to a proposition (cf. Proposition 1.3). In Section 2, we then prove it. As a consequence, we obtain the first example of (a family of) zero mean curvature surfaces which change type across a light-like line.

1. The main theorem

We discuss solutions of the zero mean curvature equation (*) which have the following form

\[
f(x, y) = b_0(y) + \sum_{k=1}^{\infty} \frac{b_k(y)}{k} x^k,
\]

where \( b_k(y) (k = 1, 2, \ldots) \) are \( C^\infty \)-functions. When \( f \) contains a singular light-like line, we may assume without loss of generality that (cf. [2])

\[
b_0(y) = y, \quad b_1(y) = 0.
\]

As was pointed out in [2], there exists a real constant \( \mu \) called the characteristic of \( f \) such that \( b_2(y) \) satisfies the following equation

\[
b_2'(y) + b_2(y)^2 + \mu = 0 \quad (\gamma = d/dy).
\]

Now we derive the differential equations satisfied by \( b_k(y) \) for \( k \geq 3 \) assuming (1.2). If we set

\[
Y := f_y - 1 = \sum_{k=2}^{\infty} \frac{b_k(y)}{k} x^k
\]
and
\[ P := 2(Y f_{xx} - f_x f_{xy}), \quad Q := Y^2 f_{xx} - 2 f_x f_{xy} Y, \quad R := f_x^2 f_{yy}, \]
then, by straightforward calculations, we see that
\[ P = -b_2 b'_2 x^2 - \frac{4}{3} b_2 b'_2 x^3 - \sum_{k=4}^{\infty} \left( P_k + \frac{2(k-1)}{k} b_2 b'_k + (3-k) b'_k b_k \right) x^k, \]
\[ Q = -\sum_{k=4}^{\infty} Q_k x^k, \quad R = \sum_{k=4}^{\infty} R_k x^k, \]
where
\begin{align*}
P_k &:= \sum_{m=3}^{k-1} \frac{2(k-2m+3)}{k-m+2} b_m b'_{k-m+2}, \\
Q_k &:= \sum_{m=2}^{k-2} \sum_{n=2}^{k-m} \frac{3n-k+m-1}{mn} b'_m b'_{n} b_{k-m-n+2}, \\
R_k &:= \sum_{m=2}^{k-2} \sum_{n=2}^{k-m} \frac{b_m b_{n} b''_{k-m-n+2}}{k-m-n+2}
\end{align*}
(1.4)
for \( k \geq 4 \), and that the zero mean curvature equation \((*)\) reduces to
\[ \sum_{k=2}^{\infty} \frac{b''_k}{k} x^k = f_{yy} = P + Q + R. \]
It is now immediate, by comparing the coefficients of \( x^k \) from both sides, to see that each \( b_k \) \((k \geq 3)\) satisfies the following ordinary differential equation
\[ b''_k(y) + 2(k-1)b_2(y)b'_k(y) + k(3-k)b'_2(y)b_k(y) = -k(P_k + Q_k - R_k), \]
(1.5)
where \( P_3 = Q_3 = R_3 = 0 \) and \( P_k, Q_k \) and \( R_k \) are as in (1.4) for \( k \geq 4 \). Note that \( P_k, Q_k \) and \( R_k \) are written in terms of \( b_j \) \((j = 1, \ldots, k-1)\) and their derivatives.
Now, we consider the case that \( 1 - f_x^2 - f_y^2 \) changes sign across the light-like line \( \{t = y, \ x = 0\} \). This case occurs only when the characteristic \( \mu \) as in (1.3) of \( f \) vanishes [2]. If we set
\[ b_2(y) = 0 \quad (y \in \mathbb{R}), \]
then (1.3) holds for \( \mu = 0 \). So we assume
\[ b_0(y) = y, \quad b_1(y) = 0, \quad b_2(y) = 0, \quad b_3(y) = 3cy, \]
(1.6)
where \( c \) is a non-zero constant. Then \( f(x, y) \) in (1.1) can be rewritten as

\[
(1.7) \quad f(x, y) = y + cyx^3 + \sum_{k=4}^{\infty} \frac{b_k(y)}{k} x^k.
\]

In this situation, we will find a solution satisfying

\[
(1.8) \quad b_k(0) = b'_k(0) = 0 \quad (k \geq 4).
\]

Then (1.5) reduces to

\[
(1.9) \quad b''_k(y) = -k(P_k + Q_k - R_k), \quad b_k(0) = b'_k(0) = 0, \quad (k \geq 4),
\]

\[
(1.10) \quad P_k = \sum_{m=3}^{k-1} \frac{2(k - 2m + 3)}{k - m + 2} b_m(y)b'_{k-m+2}(y) \quad (k \geq 4),
\]

\[
(1.11) \quad Q_k = \sum_{m=3}^{k-1} \sum_{n=3}^{k-m-1} \frac{3n - k + m - 1}{mn} b_m(y)b'_n(y)b_{k-m-n+2}(y) \quad (k \geq 7),
\]

\[
(1.12) \quad R_k = \sum_{m=3}^{k-1} \sum_{n=3}^{k-m-1} \frac{b_m(y)b_n(y)b''_{k-m-n+2}(y)}{k - m - n + 2} \quad (k \geq 7),
\]

and \( Q_k = R_k = 0 \) for \( 4 \leq k \leq 6 \), where the fact that \( b_2(y) = 0 \) has been extensively used. For example,

\[
\begin{align*}
 b_0 &= y, \quad b_1 = b_2 = 0, \quad b_3 = 3cy, \quad b_4 = -4c^2y^3, \quad b_5 = 9c^3y^5, \\
 b_6 &= -24c^2y^7, \quad b_7 = 70c^5y^9 - 14c^3y^3, \ldots .
\end{align*}
\]

In this article, we show the following assertion:

**Theorem 1.1.** For each positive number \( c \), the formal power series solution \( f(x, y) \) uniquely determined by (1.9), (1.10), (1.11) and (1.12) gives a real analytic zero mean curvature surface on a neighborhood of \((x, y) = (0, 0)\). In particular, there exists a non-trivial 1-parameter family of real analytic zero mean curvature surfaces each of which changes type across a light-like line (see Fig. 1).

As a consequence, we get the following:

**Corollary 1.2.** There exists a family of zero-mean curvature hypersurfaces in Lorentz–Minkowski space \( \mathbb R^{n+1} \) each of which changes type across an \((n - 1)\)-dimensional light-like plane.

**Proof.** Let \( f \) be as in the theorem. The graph of the function defined by

\[
\mathbb R^n \ni (x_1, \ldots, x_n) \mapsto f(x_1, x_2) \in \mathbb R
\]
Fig. 1. The graph of $t = f(x, y)$ for $c = 1/2$ and $|x|, |y| < 0.8$
(The range of the graph is wider than the range used in our mathematical estimation. However, this figure still has a sufficiently small numerical error term in the Taylor expansion.)

gives the desired hypersurface. In this case, the zero mean curvature equation

$$
\left(1 - \sum_{j=1}^{n} f_{x_j}^2 \right) \sum_{i=1}^{n} f_{x_i} + \sum_{i,j=1}^{n} f_{x_i x_j} f_{x_i} f_{x_j} = 0 \quad (f_{x_i} := \partial f / \partial x_i, \ f_{x_i x_j} := \partial^2 f / \partial x_j \partial x_i)
$$

reduces to (*) in the introduction.

To prove Theorem 1.1, it is sufficient to show that for arbitrary positive constants $c > 0$ and $\delta > 0$ there exist positive constants $n_0$, $\theta_0$, and $C$ such that

$$
(1.13) \quad |b_k(y)| \leq \theta_0 C^k \quad (|y| \leq \delta)
$$

holds for $k \geq n_0$. In fact, if (1.13) holds, then the series (1.7) converges uniformly over the rectangle $[-C^{-1}, C^{-1}] \times [-\delta, \delta]$.

The key assertion to prove (1.13) is the following
Proposition 1.3. For each \( c > 0 \) and \( \delta > 0 \), we set
\[
M := 3 \max \{144c\tau |\delta|^{3/2}, \sqrt[3]{192c^2\tau}\},
\]
where \( \tau \) is the positive constant given by (A.3) in the appendix, such that
\[
t \int_1^{1-t} \frac{du}{u^2(1-u)^2} \leq \tau \quad (0 < t < 1/2).
\]
Then the function \( \{b_l(y)\}_{l \geq 3} \) formally determined by the recursive formulas (1.9)--(1.12) satisfies the inequalities
\[
|b_l''(y)| \leq c|y|^{l^*}M^{l-3},
\]
\[
|b_l'(y)| \leq \frac{3c|y|^{l^*+1}}{l^* + 2}M^{l-3},
\]
\[
|b_l(y)| \leq \frac{3c|y|^{l^*+2}}{(l^* + 2)^2}M^{l-3}
\]
for any
\[
y \in [-\delta, \delta],
\]
where
\[
l^* := \frac{1}{2}(l - 1) - 2 \quad (l \geq 3).
\]
Once this proposition is proven, (1.13) follows immediately. In fact, if we set
\[
\theta_0 = \frac{3}{c}(\delta M)^3, \quad C := \delta M
\]
and \( n_0 \geq 7 \), then \( 1 \leq l^* + 2 < l - 3 \) and (1.13) follows from
\[
\frac{3c|y|^{l^*+2}}{(l^* + 2)^2}M^{l-3} \leq \theta_0 C^l.
\]

2. Proof of Proposition 1.3

We prove the proposition using induction on the number \( l \geq 3 \). If \( l = 3 \), then
\[
|b_3''(y)| = 0 \leq \frac{c}{|y|} = c|y|^3M^0,
\]
\[
|b_3'(y)| = 3c = \frac{3c|y|^{3^*+1}}{3^* + 2}M^0,
\]
\[
|b_3(y)| = 3c|y| = \frac{3c|y|^{3^*+2}}{(3^* + 2)^2}M^0
\]
hold, using that \( b_3(y) = 3cy \), \( M^0 = 1 \) and \( 3^* = -1 \). So we prove the assertion for \( l \geq 4 \). Since (1.17), (1.18) follow from (1.16) by integration, it is sufficient to show that (1.16) holds for each \( l \geq 4 \). (In fact, the most delicate case is \( l = 4 \). In this case \( l^* = -1/2 \) and we can use the fact that \( \int_0^{\gamma_0} 1/\sqrt{y} \, dy \) for \( \gamma_0 > 0 \) converges.)

The inequality (1.16) follows if one shows that, for each \( k \geq 4 \)

\[
(2.1) \quad |kP_k(y)|, |kQ_k(y)|, |kR_k(y)| \leq \frac{c}{3} |y|^{k^*} M^{k-3} \quad (|y| \leq \delta)
\]

under the assumption that (1.16), (1.17) and (1.18) hold for all \( 3 \leq l \leq k - 1 \). In fact, if (2.1) holds, (1.16) for \( l = k \) follows immediately. Then by the initial condition (1.9) (cf. (1.8)), we have (1.17) and (1.18) for \( l = k \) by integration.

The estimation of \( |kP_k| \) for \( k \geq 4 \). By (1.10) and using the fact that (1.17), (1.18) hold for \( l \leq k - 1 \), we have for each \( |y| < \delta \) that

\[
|kP_k| \leq \sum_{m=3}^{k-1} \frac{2k|k-2m+3|}{k-m+2} |b_m(y)| |b_{k-m+2}(y)|
\]

\[
\leq \sum_{m=3}^{k-1} \frac{2k|k-2m+3|}{k-m+2} \left( \frac{3cM^{m-3}}{(m^* + 2)^m} \right) \left( \frac{3cM^{k-m+2-3}}{(k-m+2)^* + 2} \right)
\]

\[
= cM^{k-3} |y|^{k^*} \frac{144c}{M} \sum_{m=3}^{k-1} \frac{k|k-2m+3|}{(m-1)^2(k-m+1)(k-m+2)}
\]

\[
\leq cM^{k-3} |y|^{k^*} \frac{144c}{M} \sum_{m=3}^{k-1} \frac{k|k-2m+3|}{(m-1)^2(k-m+1)(k-m+2)}
\]

\[
\leq \frac{c}{3\tau} M^{k-3} |y|^{k^*} \sum_{m=3}^{k-1} \frac{k|k-2m+3|}{(m-1)^2(k-m+1)^2}.
\]

Here, we used (1.14). Since

\[
\max_{m=3, \ldots, k-1} |k-2m+3| = \max_{m=3, k-1} |k-2m+3| = \max\{|k-3|, |k+5|\},
\]

by setting \( q = m - 1 \), we have that

\[
|kP_k| \leq \frac{c}{3\tau} M^{k-3} |y|^{k^*} \sum_{m=3}^{k-1} \frac{k^2}{(m-1)^2(k-m+1)^2} = \frac{c}{3\tau} M^{k-3} |y|^{k^*} \frac{1}{k} \sum_{q=2}^{k-2} \frac{k^3}{q^2(q-k)^2}
\]

\[
\leq \frac{c}{3\tau} M^{k-3} |y|^{k^*} \frac{1}{k} \int_{1/k}^{1} \frac{du}{u^2(1-u)^2} \leq \frac{c}{3} M^{k-3} |y|^{k^*},
\]

where we applied Lemma A.1 and (1.15) at the last step of the estimations. Hence, we get (2.1) for \( kP_k \).
The estimation of $|kQ_k|$ for $k \geq 7$. By (1.11) and the induction assumption, we have that

$$|kQ_k| \leq \sum_{m=3}^{k-4} \sum_{n=3}^{k-m-1} k^{3n-k+m-1} \frac{b'_m(y) |b'_n(y) | b_{k-m-n+2}(y)}{mn} \leq \sum_{m=3}^{k-4} \sum_{n=3}^{k-m-1} k^{3n-k+m-1} \left(\frac{3cM^{m-3} |y|^{m+1}}{m^* + 2}\right) \times \left(\frac{3cM^{k-m-n+2-3} |y|^{(k-m-n+2)^* + 2}}{(k-m-n+2)^* + 2}\right)$$

$$= cM^{k-3} |y|^k \sum_{m=3}^{k-4} \sum_{n=3}^{k-m-1} \frac{k^{3n-k+m-1}}{(m-1)^2(n-1)^2(k-m-n+2)^2}.$$

Now we apply the inequalities

$$\max_{3 \leq m \leq k-4} \frac{3n-k+m-1}{3 \leq n \leq k-m-1} = \max_{(m,n)=(3,3),(3,k-4),(k-4,3)} [3n-k+m-1] = \max\{|-k+11|, 4, |2k-10|\} \leq 2k,$$

and also

$$\frac{432c^2}{M^4} \leq \frac{1}{36\tau},$$

which follows from (1.14). Setting $p := m-1$, $q = n-1$, we have that

$$|kQ_k| \leq \frac{c}{36\tau} M^{k-3} |y|^k \sum_{m=3}^{k-4} \sum_{n=3}^{k-m-1} \frac{2k^2}{(m-1)^2(n-1)^2(k-m-n+2)^2}$$

$$= \frac{c}{18\tau} M^{k-3} |y|^k \sum_{p=2}^{k-5} \sum_{q=2}^{k-p-2} \frac{k^2}{p^2q^2(k-p-q)^2}.$$

Now applying Lemma A.2, we have that

$$|kQ_k| \leq \frac{c}{18\tau} M^{k-3} |y|^k \times 6\tau \leq \frac{c}{3} M^{k-3} |y|^k,$$

which proves (2.1) for $kQ_k$. 
The estimation of $|kR_k|$ for $k \geq 7$. As in the case of $|kQ_k|$, we have that

$$|kR_k| \leq \sum_{m=3}^{k-4} \sum_{n=3}^{k-m-1} \frac{k|b_m(y)| |b_n(y)| |b''_{m-n+2}(y)|}{k - m - n + 2}$$

$$\leq \sum_{m=3}^{k-4} \sum_{n=3}^{k-m-1} \frac{k}{k - m - n + 2} \left( \frac{3cM^{m-3}|y|^{m+2}}{(m^*)^2} \right)$$

$$\times \left( \frac{3cM^{n-3}|y|^{n+2}}{(n^*)^2} \right) (cM^{k-m-n+2+3} |y|^{(k-m-n+2)^*})$$

$$= 144c^3M^{k-7} |y|^{k^*} \sum_{m=3}^{k-4} \sum_{n=3}^{k-m-1} \frac{k}{(k - m - n + 2)(m - 1)^2(n - 1)^2}$$

$$= 144c^3M^{k-7} |y|^{k^*} \frac{144c^2}{M^4} \sum_{m=3}^{k-4} \sum_{n=3}^{k-m-1} \frac{k^2}{(k - m - n + 2)(m - 1)^2(n - 1)^2}.$$ 

Now we set $p = m - 1$, $q = n - 1$, and using the inequality

$$3^4 \times 144c^2 \tau \leq 3^4 \times 192c^2 \tau \leq M^4,$$

we have that

$$|kR_k| \leq \frac{c}{3^4 \tau} M^{k-3} |y|^{k^*} \sum_{p=2}^{k-5} \sum_{q=2}^{k-p-2} \frac{k^2}{p^2 q^2 (k - p - q)^2}.$$ 

By applying Lemma A.2, we have that

$$|kR_k| \leq \frac{c}{3^4 \tau} M^{k-3} |y|^{k^*} \times 6\tau \leq \frac{c}{3} M^{k-3} |y|^{k^*},$$

which proves (2.1) for $kR_k$. This completes the proof of Proposition 1.3.

Appendix A. Inequalities used in the proof of Theorem 1.1

For $a > 0$, it holds that

$$\frac{1}{u^2(a-u)^2} = \frac{1}{a^3} \left( \frac{a}{u^2} + \frac{2}{u} + \frac{a}{(a-u)^2} + \frac{2}{a-u} \right).$$

Therefore,

$$\int_t^{a-t} \frac{du}{u^2(a-u)^2} = \frac{2}{a^3} \left( \frac{a(a-2t)}{t(a-t)} + 2 \log \frac{a-t}{t} \right) \quad (0 < t < a/2).$$
In particular, one can show that there exists a positive constant $\tau$ such that

$$t \int_1^{1-t} \frac{du}{u^2(1-u)^2} \leq \tau \quad (0 < t < 1/2).$$

The following assertion is needed to prove (2.1) for $kP_k(y)$:

**Lemma A.1.** Let $p$ be a non-negative integer and $k$ an integer satisfying $k \geq p + 4$. Then the inequality

$$\sum_{q=2}^{k-p-2} \frac{k^3}{q^2(k-p-q)^2} \leq \int_1^{a^{-1/k}} \frac{du}{u^2(a-u)^2} \quad (a := 1 - p/k)$$

holds.

**Proof.** In fact, if we set $a := 1 - p/k$, then (A.1) yields that

$$\frac{k^3}{q^2(k-p-q)^2} = \frac{1}{k} \frac{1}{(q/k)^2(a-q/k)^2} = \frac{1}{a^3} \left[ \frac{1}{k} \left( \frac{a}{(q/k)^2} + \frac{2}{q/k} \right) + \frac{1}{k} \left( \frac{a}{(a-q/k)^2} + \frac{2}{a-q/k} \right) \right].$$

Since $x \mapsto (a + 2x)/x^2$ is a monotone decreasing function and the function $x \mapsto (a + 2(a-x))/(a-x)^2$ is monotone increasing on the interval $(0, a/2)$, we have that

$$\frac{k^3}{q^2(k-p-q)^2} \leq \frac{1}{a^3} \left[ \int_{q-1/k}^{q/k} \left( \frac{a}{u^2} + \frac{2}{u} \right) du + \int_{q/k}^{(q+1)/k} \left( \frac{a}{(a-u)^2} + \frac{2}{a-u} \right) du \right],$$

which yields that

$$\sum_{q=2}^{k-p-2} \frac{a^3k^3}{q^2(k-p-q)^2} \leq \int_1^{a^{-1/k}} \left( \frac{a}{u^2} + \frac{2}{u} \right) du + \int_1^{a^{-1/k}} \left( \frac{a}{(a-u)^2} + \frac{2}{a-u} \right) du$$

$$\leq \int_{1/k}^{a^{-1/k}} \left( \frac{a}{u^2} + \frac{2}{u} \right) du + \int_{1/k}^{a^{-1/k}} \left( \frac{a}{(a-u)^2} + \frac{2}{a-u} \right) du$$

$$= \int_{1/k}^{a^{-1/k}} \frac{du}{u^2(a-u)^2}.$$

This proves the assertion. \(\square\)

The following assertion is needed to prove (2.1) for $kQ_k(y)$ and $kR_k(y)$:
Lemma A.2. For any integer $k \geq 7$, the following inequalities hold:

$$
\sum_{p=2}^{k-5} \sum_{q=2}^{k-p-2} \frac{k^2}{p^2q^2(k-p-q)^2} \leq \frac{6}{k} \int_{1/k}^{1-1/k} \frac{du}{u^2(1-u)^2} \leq 6\tau,
$$

where $\tau$ is a constant satisfying (A.3).

Proof. We set $a = a(p) := 1 - (p/k)$. Applying Lemma A.1 and the identity (A.2), we have that

$$
\sum_{p=2}^{k-5} \sum_{q=2}^{k-p-2} \frac{k^2}{p^2q^2(k-p-q)^2} = \sum_{p=2}^{k-5} \left[ \frac{1}{kp^2} \sum_{q=2}^{k-p-2} \frac{k^3}{q^2(k-p-q)^2} \right] \leq \sum_{p=2}^{k-5} \left[ \frac{1}{kp^2} \int_{1/k}^{a-1/k} \frac{du}{u^2(a-u)^2} \right] = \sum_{p=2}^{k-5} \left[ \frac{1}{2} \frac{a-2/k}{p^2} \left( a-1/k + 2 \frac{\log(ka-1)}{ka} \right) \right] \leq \sum_{p=2}^{k-5} \frac{6}{p^2a^2},
$$

where we used the fact that $(\log ka)/(ka) < 1$. By applying Lemma A.1 and by using the property (A.3) of the constant $\tau$, it holds that

$$
\sum_{p=2}^{k-5} \sum_{q=2}^{k-p-2} \frac{k^2}{p^2q^2(k-p-q)^2} \leq \frac{6}{k} \sum_{p=2}^{k-5} \frac{1}{p^2(1-p/k)^2} = \frac{6}{k} \sum_{p=2}^{k-5} \frac{k^3}{p^2(k-p)^2} \leq \frac{6}{k} \sum_{p=2}^{k-2} \frac{k^3}{p^2(k-p)^2} \leq \frac{6}{k} \int_{1/k}^{1-1/k} \frac{du}{u^2(1-u)^2} \leq 6\tau,
$$

which proves the assertion. \hfill \Box

References


Shoichi Fujimori  
Department of Mathematics, Faculty of Science  
Okayama University  
Okayama 700-8530  
Japan  
e-mail: fujimori@math.okayama-u.ac.jp

Young Wook Kim  
Department of Mathematics  
Korea University  
Seoul 136-701  
Korea  
e-mail: ywkim@korea.ac.kr

Sung-Eun Koh  
Department of Mathematics  
Konkuk University  
Seoul 143-701  
Korea  
e-mail: sekoh@konkuk.ac.kr

Wayne Rossman  
Department of Mathematics, Faculty of Science  
Kobe University  
Kobe 657-8501  
Japan  
e-mail: wayne@math.kobe-u.ac.jp

Heayong Shin  
Department of Mathematics  
Chung-Ang University  
Seoul 156-756  
Korea  
e-mail: hshin@cau.ac.kr

Masaaki Umehara  
Department of Mathematical and Computing Sciences  
Tokyo Institute of Technology  
Tokyo 152-8552  
Japan  
e-mail: umehara@is.titech.ac.jp

Kotaro Yamada  
Department of Mathematics  
Tokyo Institute of Technology  
Tokyo 152-8551  
Japan  
e-mail: kotaro@math.titech.ac.jp

Seong-Deog Yang  
Department of Mathematics  
Korea University  
Seoul 136-701  
Korea  
e-mail: sdyang@korea.ac.kr