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THE SPLITTING OF FINITELY GENERATED MODULES

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Throughout this paper rings are commutative with identity and modules are unital. The purpose of this paper is to study splitting properties of modules with respect to their torsion submodule, with special attention to the cases of integral domains and (von Neumann) regular rings.

The notion of torsion theory has been used in the literature to define the splitting properties we are concerned with. Recall that a (hereditary) torsion theory over the ring R is defined by a class of R-modules which is closed under the formation of submodules, homomorphic images, extensions and direct sums; this class is called the torsion class of the theory. Given such a class, every R-module has a largest submodule in the class, called the torsion submodule, with torsionfree factor module. A torsion theory is said to have the *finitely generated splitting property* (FGSP) if the torsion submodule of every finitely generated module splits off as a direct summand of the module. A weaker condition (but more susceptible to full analysis) is the *cyclic splitting property* (CSP): the torsion submodule of every cyclic module splits off.

The FGSP question was largely inspired by several now classical theorems of I. Kaplansky [9], [10], which can be paraphrased as this single statement: the usual torsion theory for an integral domain R has FGSP if and only if R is a Prufer ring. One of the principal results of this paper suitably extends this result to arbitrary torsion theories over an integral domain. Of course, Kaplansky's result follows as a special case. As a first step we analyze CSP over domains. In fact, FGSP over domains reduces to CSP plus an arithmetic property. We also examine CSP and FGSP over regular rings: for CSP our results are fairly complete. For FGSP our results cannot be internalized to the ring as well as the domain case. It turns out that torsion submodules of finitely generated modules must be finitely generated, which answers a question about FGSP for regular rings posed by K. Oshiro [12].

The cyclic splitting of special torsion theories, e.g., the Goldie and simple theories, have been extensively studied (see K. Goodearl [8], L. Koifman [11], K. Oshiro [12], J. Pakala [13] and M. Teply [16]). L. Koifman [11] studied CSP in more generality, but still imposed certain primary decomposability constraints on the theories he considered. We synthesize much of the work of these authors in section 2, where CSP is analyzed in full generality. These results are applied in section 3 to integral domains and regular rings. In section 4 we take up the issue of FGSP, again synthesizing the work of many authors, particularly F. Call [4], F. Call and T. Shores [5], K. Goodearl [8], L. Koifman [11], M. Teply and J. Fuelberth [15], and M. Teply [17]. In particular, we apply these results to integral domains and regular rings.

Portions of this paper appear in the dissertation of J. Pakala [13], which was written under the direction of T. Shores.

1. Preliminaries. As usual, if $\mathcal{T}$ is a torsion class and M an R-module, then $t(M)$ is the $\mathcal{T}$-torsion submodule of M; in this way, $t(\ )$ is a left exact subfunctor of the identity. Also, the class of all torsionfree modules M (i.e., $t(M)=0$) is denoted by $\mathcal{F}$. The pair $(\mathcal{T}, \mathcal{F})$ is a (hereditary) torsion theory for R. We suppress reference to $\mathcal{T}$ or $\mathcal{F}$ when these are clear from context. Also, a *$\mathcal{T}$-delineation* of the module M is an exact sequence $0 \to T \to M \to F \to 0$ where $T \in \mathcal{T}$ and $F \in \mathcal{F}$. Such a sequence is unique up to isomorphism.

If $\mathcal{T}$ is a class of R-modules, then we define a set of ideals of R by $F(\mathcal{T}) = \{I \mid R/I \in \mathcal{T}\}$. If $\mathcal{T}$ is a torsion class, then $F(\mathcal{T})$ is a Gabriel filter of ideals of R, and conversely, a Gabriel filter determines a torsion class. The correspondence is bijective. We refer the reader to [14] for details.

Here are some important torsion theories:

(1) The Goldie theory, $\mathcal{D}$: this torsion class is generated by the set of all modules of the form R/I, where I is an essential ideal of R. If $g(R)=0$, then $g(M)$ is just the singular torsion theory for R and $F(\mathcal{D})$ is exactly the set of essential ideals of R. In the case that R is a domain, $F(\mathcal{D})$ is just the set of nonzero ideals of R, and $\mathcal{D}$ defines the so-called *usual torsion theory* for R.

(2) The simple theory, $\mathcal{S}$: this torsion class is generated by the set of all modules of the form R/I, where I is a maximal ideal of R. Any torsion class $\mathcal{T}$ which contains the class $\mathcal{S}$ is said to be a *generalization of the simple theory*.

(3) Half centered theories: these are the theories whose defining filter is of the form F_X, the set of all ideals not contained in any member of a set X of prime ideals of R (see [2], [3] and [14]). We show in section 2 that every CSP theory over an integral domain is of this type.

The radical of an ideal I of R is the intersection of all primes of R containing I and denoted by $\mathrm{rad}(I)$. Thus $\mathrm{rad}(0)$ is just the nilradical of R. As usual, the residual $(I : J)$ is the ideal $\{r \in R \mid rJ \subseteq I\}$.

The spectrum of the ring R is the space of all primes with the Zariski topology, and is denoted by $\mathrm{spec}(R)$. Closed sets are of the form $V(I) = \{P \in \mathrm{spec}(R) \mid I \subseteq P\}$ and the complement of $V(I)$ is denoted by $D(I)$. The subspace of all maximal ideals is denoted by $\mathrm{maxspec}(R)$. If $\mathcal{T}$ is a torsion class, the subspace of primes P of R such that $R/P \in \mathcal{T}$ is denoted by spec$(\mathcal{T})$.

Let f: $R \to S$ be a ring homomorphism and $\mathcal{T}$ a torsion class of R-modules. Then there is induced via f a torsion class of S-modules, namely the class $\mathcal{T}_S$ of S-modules also in $\mathcal{T}$. As usual, if P is a prime ideal of R and f: $R \to R_P$ the canonical localization map, then the induced torsion class is denoted by $\mathcal{T}_P$. We need a few elementary facts about a change of ring map f and induced classes. The following two propositions are found in [5] and are stated here without proof:

Proposition 1.1. *Let X and Y be R-modules and B an S-module.* (i) *If S is R-flat, then* $\mathrm{Tor}_n^S(X \otimes_R S, Y \otimes_R S) = \mathrm{Tor}_n^R(X, Y) \otimes_R S$ *for $n > 0$.* (ii) *If f is an epimorphism and* $\mathrm{Tor}_n^S(X, S) = 0$, *then* $\mathrm{Tor}_1^S(X \otimes_R S, B) = \mathrm{Tor}_1^R(X, B)$. (iii) *If f is an epimorphism, then the canonical map $B \to B \otimes_R S$ is an isomorphism.* (iv) *If f is a flat epimorphism, then for any S-submodule E of $X \otimes_R S$, $E = (E \cap im(X))S$.*

Proposition 1.2. *The class $\mathcal{T}_S$ is a torsion class for S such that $\{f(I)S \mid I \in F(\mathcal{T})\}$ is a cofinal subset of $F(\mathcal{T}_S)$, and for each S-module E, $t_S(E) = t(E)$. Furthermore, if f is a surjection or a flat epimorphism then the preceding families of ideals coincide.*

The next result was proved by L. Koifman in [11, p. 155] for the case of localizations at multiplicative sets.

Proposition 1.3. *Let f: $R \to S$ be a flat epimorphism of rings, $\mathcal{T}$ a torsion class for R and $\mathcal{T}_S$ be the induced torsion class for S. If $\mathcal{T}$ has one of the properties CSP or FGSP then so does $\mathcal{T}_S$.*

Proof. Note that $\mathcal{T}_S$ is the class of S-modules which are torsion as R-modules. If N is a cyclic or finitely generated S-module with a generating set $\{x_1, \cdots, x_n\}$, then $M = x_1R + \cdots + x_nR$ is an R-module of the same type and $M \otimes_R S = N$ as S-modules. We show that CSP is preserved (the argument for FGSP is similar). Let N be a cyclic S-module and M chosen as above. If $t_S(N) = T$, then $t(M) = T \cap M$ and $M/t(M) = (M+T)/T$ is an R-submodule of N/T. By CSP of $\mathcal{T}$, $0 \to t(M) \to M \to M/t(M) \to 0$ is split exact. Tensor with S to obtain a split exact sequence. But N and N/T are canonically iso- morphic with $M \otimes_R S$ and $(M/t(M)) \otimes_R S$, respectively. It follows easily that the exact sequence $0 \to T \to N \to N/T \to 0$ also splits, as required.

Recall that a torsion theory $\mathcal{T}$ is *stable* if $\mathcal{T}$ is closed under the formation of injective envelopes. A useful property of such theories is the following fact, due to M. Teply and J. Fuelberth [15, Lemma 3.2]: if $\mathcal{T}$ is stable and M is a finitely generated module, then $t(M)$ is bounded, i.e. $t(M)I = 0$ for some $I \in F(\mathcal{T})$.

Proposition 1.4. *If $f: R\rightarrow S$ is a flat epimorphism of rings and $\mathcal{T}$ is a stable torsion class for R, then $\mathcal{T}_S$ is a stable torsion class for S.*

Proof. Let M belong to $\mathcal{T}_S$ and form the injective envelope E of M over R. Then $F=E\otimes_R S$ is a torsion S-module and, by Proposition 1.1 (iv), easily seen to be the injective envelope of M as an S-module. Clearly F is a $\mathcal{T}_S$ torsion module, whence the desired result follows.

2. Characterizations of CSP. In this section we first relate the CSP property to the vanishing of certain Tor groups, and then to more intrinsic conditions. Also, under certain conditions which are satisfied by classes such as the Goldie theory and generalizations of the simple theory, we show that CSP implies that principal ideals of R are projective (in this case R is called a *p.p. ring*).

Theorem 2.1. *The torsion theory $(\mathcal{T}, \mathcal{F})$ has CSP if and only if, for $I\in F(\mathcal{T})$ and $J\in F(\mathcal{F})$,* (a) *torsion submodules of cyclic modules are bounded, and* (b) *$R/(I+J)$ is a projective module over R/I and* $\mathrm{Tor}_1^R(R/I, R/J)=0$.

Proof. Suppose $(\mathcal{T}, \mathcal{F})$ has CSP and I, J are as in the statement of the Theorem. Then torsion submodules of cyclic modules (being direct summands) are cyclic and so bounded. Now $0\rightarrow J/IJ\rightarrow R/IJ\rightarrow R/J\rightarrow 0$ is a $\mathcal{T}$-delineation of the cyclic module R/IJ, so split exact. Tensor this with R/I to obtain the split exact $0\rightarrow J/IJ\rightarrow R/J\rightarrow R/I+J\rightarrow 0$. Hence $R/I+J$ is R/I projective. Tensor the exact $0\rightarrow J\rightarrow R\rightarrow R/J\rightarrow 0$ with R/I to obtain an exact sequence $0\rightarrow \mathrm{Tor}_1^R(R/J, R/I)\rightarrow J/IJ\rightarrow R/I\rightarrow R/I+J\rightarrow 0$. By the projectivity of $R/I+J$, $\mathrm{Tor}_1^R(R/I, R/J)=0$, as desired.

Conversely, suppose the conditions of the Theorem are satisfied and let $0\rightarrow T\rightarrow M\rightarrow R/J\rightarrow 0$ be a $\mathcal{T}$-delineation of a cyclic module M. Since T is bounded, $TI=0$ for some $I\in F(\mathcal{T})$. Tensor the exact sequence with R/I and use (b) to obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \rightarrow & T & \longrightarrow & M & \longrightarrow & R/J & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & T & \rightarrow & M/MI & \rightarrow & R/I+J & \rightarrow & 0 \end{array}$$

with exact rows and splitting bottom row. Hence the top row splits and CSP holds for $(\mathcal{T}, \mathcal{F})$.

An important consequence of the above result is a local condition which has been proved by Koifman [11, Corollary 1.8] as well as the Corollary to Theorem 2.3 [11, Corollary 1.8]. In what follows the local ring R with maximal ideal m need not be Noetherian.

Corollary 2.2. *A nontrivial torsion theory for a local ring (R, m) has CSP if and only if R is a domain and the torsion theory is the usual theory for R.*

Proof. Trivially, the usual theory for any domain has CSP. Conversely, suppose $(\mathcal{I}, \mathcal{F})$ has CSP and K is an ideal of R such that $t(R/K)=J/K$. By CSP, J/K is a direct summand of R/K, a local ring. Thus, either $K\in F(\mathcal{I})$ or $K\in F(\mathcal{F})$. Let xR be a nonzero proper ideal of R. If $xR\in F(\mathcal{F})$, then the Tor condition of Theorem 2.1 implies that $xR\cap m=xm$. And thus $x=xa$ for some $a\in m$, so that $x(1-a)=0$. Since R is local, x is 0. Therefore $xR\in F(\mathcal{I})$, which is then the set of nonzero ideals of R. But then if $x, y\in R$ and $xy=0$, it follows that $y\in t(R)=0$. Thus R is a domain and $(\mathcal{I}, \mathcal{F})$ is the usual theory.

This local condition yields another useful characterization of CSP.

Theorem 2.3. *A torsion theory $(\mathcal{I}, \mathcal{F})$ for R has CSP if and only if* (a) *torsion submodules of cyclic modules are cyclic, and* (b) *$J_m=0$ whenever $J\subseteq m$, $J\in F(\mathcal{F})$ and m is a maximal ideal of R in $F(\mathcal{I})$.*

Proof. Assume $(\mathcal{I}, \mathcal{F})$ has CSP, and note that (a) is immediate. And if J, m satisfy the hypotheses of (b), then by Corollary 2.2 either $\mathcal{I}_m$ is trivial and every ideal is in $F(\mathcal{I}_m)$ or R_m is a domain and $F(\mathcal{I}_m)$ is the set of nonzero ideals of R_m. In the first case, $I_m=0$ for some $I\in F(\mathcal{I})$ and, since projectives localize, it follows that $(R/I+J)_m$ is free over R_m. Thus $J_m=0$. In the latter case, the Tor condition of Theorem 2.1 localizes at m to yield that R_m/J_m is a cyclic flat module over the domain R_m and hence, is projective. Thus $J_m=0$.

Conversely, suppose (a) and (b) are satisfied. Then torsion submodules of cyclic modules are bounded by (a). Suppose R/I is torsion, R/J torsionfree. For a maximal ideal m of R we have that J_m is 0 or R_m. Hence $(\mathrm{Tor}_1^R(R/I, R/J))_m=0$ in both cases. Since m was arbitrary, the condition globalizes. Also, the sequence $0\to J/IJ\to R/IJ\to R/J\to 0$ is a $\mathcal{I}$-delineation of R/IJ, so J/IJ is cyclic by (a). Hence $R/I+J$ is a finitely presented R/I-module which is locally free. Therefore $R/I+J$ is projective (e.g., [1], Ch. II, Sec. 5, no. 2, Theorem 2.5). Theorem 2.1 now applies to yield CSP.

Corollary 2.4. *If $(\mathcal{I}, \mathcal{F})$ has CSP for R and $t(R)=0$, then R_P is a domain for every prime P in $F(\mathcal{F})$.*

Proof. If P is nontrivial, then R_P is a domain by Corollary 2.2. Suppose $\mathcal{I}_P$ is trivial and let x/s be a nonzero element of R_P with $x\in P$. Then $t(xR)\subseteq t(R)=0$, so $(0\colon x)\in F(\mathcal{I})$ and this ideal is contained in P. Hence $(0\colon x)_P=0$ by Theorem 2.3. It follows that $(0\colon x/s)=0$ and R_P is a domain.

We next consider the connection between CSP and p.p. rings. First we require a slight generalization of an idempotence condition found in [16] and

[17].

Lemma 2.5. *If $(\mathcal{T}, \mathcal{F})$ has CSP for R and all essential maximal ideals of R are in $F(\mathcal{T})$, then $J^2=J$ for all $J\in F(\mathcal{F})$.*

Proof. Let m be a maximal ideal in $F(\mathcal{T})$. Then J_m is 0 or R_m according as $J\subseteq m$ or not; in either case $(J_m)^2=J_m$. If m is not in $F(\mathcal{T})$, then m is no t essential, hence of the form eR for some idempotent e of R. Thus $(1-e)J=0$ and $J_m=0$. It follows that $J^2=J$ locally, hence globally, which completes the proof.

The hypotheses of the following lemma are satisfied by many torsion theories, e.g., the Goldie theory or any generalization of the simple theory.

Lemma 2.6. *Let $(\mathcal{T}, \mathcal{F})$ have CSP for R, where $t(R)=0$, and each finitely generated ideal in $F(\mathcal{F})$ is generated by an idempotent. Then* (i) *R is a p.p. ring, and* (ii) *if $K\in F(\mathcal{F})$ contains an essential ideal E which is a direct sum of principal ideals, then E is contained in an essential direct sum of orthogonal idempotents belonging to K.*

Proof. For (i), let $a\in R$ and decompose R/aR as $R/I\oplus R/J$, where R/I is torsion and R/J torsionfree. Since R/aR is finitely presented, I and J are finitely generated. By hypothesis, $J=eR$ for some idempotent e. One checks that $aR=eI$, so that $1-e\in(0\colon a)$. But if $ax=0$ then $0=xeI$, so $xe\in t(R)=0$. Thus $(0\colon a)=(1-e)R$ and R is p.p.

For (ii), let a, I, J, e be as above and note that aR is essential in eR. For otherwise, some nonzero ideal L is contained in eR with $0=aR\cap L$, whence $L\subseteq(0\colon a)=(1-e)R$, a contradiction. Suppose $E=\oplus x_iR$ and choose for each $a=x_i$ an idempotent e_i as above. Then $e\in K$, else $t(R/K)\neq 0$. And since $B_i=(x_iR\colon e_i)\in F(\mathcal{T})$, it follows that $e_ie_jB_iB_j\subseteq x_iRx_jR=0$ for distinct i and j. Thus $e_ie_j=0$. Certainly E is essential in $\oplus e_iR$, as required.

In general it is not the case that an ideal is in $F(\mathcal{T})$ or $F(\mathcal{F})$ if and only if its radical belongs to the corresponding filter. But torsion theories with CSP have this rather striking property. This fact is essential for our analysis of CSP.

Proposition 2.7. *Let $(\mathcal{T}, \mathcal{F})$ have CSP for the ring R and K be an ideal of R. Then $K\in F(\mathcal{T})$ or $F(\mathcal{F})$ if and only if rad(K) belongs to the same filter.*

Proof. Replace R/K by R and assume that $K=0$ and that $N=\text{rad}(K)$ is the prime radical of R. By CSP, $R/N=R/I\oplus R/J$ with $J/N=t(R/N)$ and R/J torsionfree. Also, $N=\text{rad}(I)\cap\text{rad}(J)$. If R is torsion then certainly R/N is torsion. Conversely, if R/N is torsion then $R/\text{rad}(J)$ must be torsion. But the induced torsion theory on R/J is then a generalization of the simple theory. By Lemma 2.6, R/J is p.p. Since p.p. rings are reduced, it follows that $J=\text{rad}(J)$; hence, R/J is torsion and torsionfree, so $J=R$ and R is torsion, as desired. Now suppose that R is torsionfree. Note that spec(R) is the disjoint union of $V(I)$ and $V(J)$, so these sets are clopen and there exists an idempotent e such that $\text{rad}(I)=\text{rad}(eR)$. By the first part, R/eR is torsion, so that $1-e\in t(R)=0$; so $I=R$ and $N\in F(\mathcal{F})$. Conversely, suppose R/N is torsionfree; we have that $t(R)=fR$ for some idempotent f, so that $R/N=R/\text{rad}(fR)\oplus R/\text{rad}((1-f)R)$. However, $(1-f)R\in F(\mathcal{T})$, so rad $((1-f)R)\in F(\mathcal{T})$, whence $f=0$. Thus we obtain $R\in\mathcal{F}$, as desired.

If $\mathcal{T}$ is a generalization of the simple theory and $t(R)=0$, Teply has shown [17, Theorem 1.5] that cyclic torsionfree modules are projective. We conclude this section with a slightly more general version of this fact which turns out to be particularly useful for regular rings.

Proposition 2.8. *Let $(\mathcal{T}, \mathcal{F})$ have CSP for R, where $t(R)=0$ and finitely generated ideals in $F(\mathcal{F})$ are idempotent. If, in addition, maxspec$(\mathcal{T})$ is dense in maxspec(R), then every ideal in $F(\mathcal{F})$ is idempotent generated and $\mathcal{T}=\mathcal{D}$.*

Proof. Suppose that R/K is torsionfree and K is not finitely generated. Choose a maximal direct sum of principal ideals contained in K. Such an ideal is necessarily essential in K and, in view of Lemma 2.6, K contains an essential ideal J generated by orthogonal idempotents e_i. Let $t(R/J)=K'/J$ and obtain by CSP that $K'=aR+J$ for some $a\in K'$ with $a-a^2\in J$. But then we may choose an idempotent f such that $aR\subseteq fR$ and $f\in K'$. Note that $K'=fR+J=fR\oplus(1-f)J=fR\oplus(\oplus e_i(1-f)R)$ and K' is essential in K. Now $e_i(1-f)\neq 0$ for infinitely many i; if not, K' is finitely generated and hence generated by an idempotent. However, K' is essential in K, so cannot be idempotent generated. Thus $(1-f)J=\oplus e_i(1-f)R$ is an infinite direct sum. Let $R^*=(1-f)R$ and $J^*=(1-f)J$. Then maxspec$(\mathcal{T}^*)=D(1-f)$ maxspec$(\mathcal{T})$ is dense in maxspec(R^*) and $R/K'=R^*/J^*$, so $J^*\in F(\mathcal{F})$. Since maxspec$(\mathcal{T})$ is dense in maxspec(R), we can choose a maximal ideal m_i in $D(e_i(1-f)m_i))$ with R/m_i torsion. But then if $L^*=\oplus e_i(1-f)m_i$, one checks that $t(R^*/L^*)=J^*/L^*=\oplus R/m_i$, an infinite direct sum. Such a module cannot be a direct summand of R^*/L^*, which violates CSP. This shows that each K in $F(\mathcal{F})$ must be finitely generated, and so generated by an idempotent. Finally, we observe that $t(R)=0$, so that ideals in $F(\mathcal{T})$ are essential, whence $F(\mathcal{T})\subseteq F(\mathcal{D})$. Conversely, if K is an essential ideal of R and $t(R/K)=J/K$, then J is also essential and by the above, $J=eR$. Hence $e=1$ and $K\in F(\mathcal{T})$. It follows that $\mathcal{T}=\mathcal{D}$.

3. CSP for special rings. Results of the previous section allow us to derive very satisfactory descriptions of CSP torsion theories for integral domains and regular rings. For domains the key idea is that of the half centered torsion theories with defining filters F_X as described in section 1. Recall that a subset X of spec(R) is *closed under generalization* if, for $P \in X$ and Q a prime contained in P, we have $Q \in X$.

Theorem 3.1. *Let R be an integral domain and X a subset of spec(R) closed under generalization and such that for every proper closed set C of spec(R), $C \cap X$ is clopen in C. Then F_X is a filter for a torsion theory on R with CSP and conversely, every torsion filter on R with CSP is so obtained.*

Proof. First assume the conditions on X. Let K be a nonzero proper ideal of R and use the hypothesis on X to express the set $C=V(K)$ as the disjoint union $V(I) \cup V(J)$, where $V(J)=V(K) \cap X$ and $V(I) \cap X=\phi$. Hence we can choose an idempotent e modulo K such that $J=eR+K$ and $I=(1-e)R+K$. Thus, we have $R/K=R/I \oplus R/J$ with $I \in F_X$. Then R/J is torsionfree; otherwise, pick an element a not in J such that $(J: a) \in F_X$, so $V(J: a) \cap X=\phi$. However, $V(J: a) \subseteq V(J) \subseteq X$ and so $(J: a)=R$, whence $a \in J$, contrary to out choice of a.

Conversely, suppose $(\mathcal{T}, \mathcal{F})$ is a nontrivial torsion theory with CSP. If J is a nonzero ideal in $F(\mathcal{F})$ and P is a prime containing it, then $P \in F(\mathcal{F})$, for if $P \in F(\mathcal{T})$, then $J_P=0$ by Theorem 2.3 and so $J=0$. Hence $V(J) \subseteq$ spec$(\mathcal{F})$ and so $F(\mathcal{T})=\{I \mid V(I) \subseteq \text{spec}(\mathcal{T})\}$. To see this, let $V(I) \subseteq$ spec$(\mathcal{T})$, where $t(R/I)=J/I$. If $J \neq R$, choose a prime P containing it; then P belongs to $V(I) \cap \text{spec}(\mathcal{F})=\phi$, a contradiction. It follows that $X=\text{spec}(\mathcal{F})$ is closed under generalization and $F(\mathcal{T})=F_X$. Now a closed subset C of spec(R) has the form $V(K)$ for some ideal K. But $R/K=R/I \oplus R/J$ with R/I torsion and R/K torsionfree. Thus $V(K)=V(I) \cup V(J)$ is a disjoint union, whence $V(J)=V(K) \cap \text{spec}(\mathcal{F})$ is clopen in $V(K)$ and the proof is complete.

We now turn our attention to the study of CSP in the case that R is a regular ring. First we observe that the condition $t(R)=0$ in Proposition 2.8 is used only to choose *orthogonal* idempotents e_i. For regular rings this is automatic, as is the requirement that finitely generated ideals be idempotent generated. Hence we have the following fact.

Lemma 3.2. *If $(\mathcal{T}, \mathcal{F})$ has CSP for the regular ring R and spec$(\mathcal{T})$ is dense in spec(R), then every cyclic torsionfree module is projective and spec$(\mathcal{T})=$spec(R). If, in addition, $t(R)=0$, then $\mathcal{T}=\mathcal{D}$.*

Corollary 3.3. *If $\mathcal{T}$ has CSP for the regular ring R, then spec$(\mathcal{T})$ is closed in spec(R).*

Proof. Let $V(K)$ be the closure of spec$(\mathcal{T})$ and note that spec$(\mathcal{T}_{R/K})=$ spec$(\mathcal{T})$ is dense in $V(K)=$spec(R/K). Hence the preceding result implies that spec$(\mathcal{T})=V(K)$.

Recall that a ring in which every annihilator ideal is idempotent generated is called a *Baer ring*. In such a ring, every nonsingular cyclic module is projective, hence $\mathcal{D}$ has CSP ([12], Lemma 1.5). In particular, a regular ring R is a Baer ring if and only if spec(R) is *extremely disconnected*, i.e. the closure of every open set is open [14].

Proposition 3.4. *If $(\mathcal{T}, \mathcal{F})$ is a generalization of the simple theory with CSP for the regular ring R and $t(R)=0$, then R is a Baer ring and $\mathcal{T}=\mathcal{D}$.*

Proof. From Lemma 3.2, $\mathcal{T}=\mathcal{D}$. Suppose that spec(R) is not extremely disconnected; then we can choose an open set $N=D(K)$ such that the closure N^- of N is not open. Let M denote the complement of N^-. Then $M^-\cup N^-=$ spec(R) and $M^-\cap N^-\neq\phi$. By CSP, $R/K=R/I\oplus R/J$ so that $D(K)\subseteq D(J)$ and the ideal J in $F(\mathcal{F})$ is generated by an idempotent e by Lemma 3.2; thus $D(K)\subseteq D(e)$. Suppose that $D(K)\subseteq D(f)$ for idempotent f. Then the image of J/K under the canonical surjection $R/K\rightarrow R/fR$ is 0, so $eR\subseteq fR$. Hence $D(e)$ is the smallest clopen set containing N and $N^-\subseteq D(e)$. Now suppose that there is a prime m in $D(e)$ but not N^-. Then some clopen neighborhood U of m is contained in M; this implies that $D(e)-U$ is a clopen set containing N, contrary to the minimality of $D(e)$. Hence $N^-=D(e)$ is open, a contradiction. Thus spec(R) is extremely disconnected and the proof is complete.

Since every module over a factor ring R/K of R is naturally an R-module, a torsion class $\mathcal{T}$ over R/K yields a class of R-modules. But $\mathcal{T}$ need not be a torsion class over R. For regular rings the situation is better, and this simple fact is a key in our classification of regular CSP theories.

Lemma 3.5. *If R is regular, K an ideal of R and $\mathcal{T}$ a torsion class over R/K, then $\mathcal{T}$ is a torsion class over R.*

Proof. As a class of R-modules, $\mathcal{T}$ is clearly closed under submodules, homomorphic images and direct sums. It remains to show that $\mathcal{T}$ is closed under extensions in the category of R-modules. Let $0\rightarrow A\rightarrow B\rightarrow C\rightarrow 0$ be an exact sequence of R-modules with A, $C\in\mathcal{T}$. Then $BK\subseteq A$, so $BK^2=0$. However, $K=K^2$ since R is regular. Hence $BK=0$ and B is an R/K-module. Since $\mathcal{T}$ is closed under extensions in the category of R/K-modules, it follows that $\mathcal{T}$ is a torsion class for the ring R.

In what follows if K is an ideal of R and $\mathcal{T}$ a torsion class for R, then $\mathcal{T}_{R/K}$ is the induced class via the natural map $R\rightarrow R/K$. Also, we say $\mathcal{T}$ is a *direct sum* of torsion classes $\mathcal{T}_1$ and $\mathcal{T}_2$ if for all R-modules M, $t(M)=t_1(M)\oplus t_2(M)$. Finally, note that there are two trivial torsion classes for R: the class of zero modules and the class of all R-modules. We call the latter the "big" trivial class. Our main theorem on CSP for regular rings follows.

Theorem 3.6. *Let $\mathcal{T}$ be a torsion class with CSP for the regular ring R; then there exist disjoint closed sets $V(I)$ and $V(J)$ in spec(R) such that* (a) *$V(J)$ is extremely disconnected,* (b) *$\mathcal{T}_{R/J}$ is the Goldie theory for R/J and $\mathcal{T}_{R/I}$ is the (big) trivial class for R/I, and* (c) *$\mathcal{T}$ is the direct sum of the torsion classes $\mathcal{T}_{R/I}$ and $\mathcal{T}_{R/J}$.*

Proof. Let $S=\bigcap \operatorname{spec}(\mathcal{T})$ and use CSP to write $R/S=R/I\oplus R/J$. Then $I+J=R$ and $V(S)=V(I)\cup V(J)$ is a disjoint union. With the usual identification of spec(R/J) with $V(J)$, one sees that $\operatorname{spec}(\mathcal{T}_{R/J})=V(J)\operatorname{spec}(\mathcal{T}_{R/J})^{-}=\operatorname{spec}(R/J)$. So $\operatorname{spec}(\mathcal{T}_{R/J})$ is dense in spec(R/J). However R/J is torsionfree and $\mathcal{T}_{R/I}$ has CSP. Hence $\mathcal{T}_{R/J}$ is the Goldie theory and has spectrum equal to that of R by Lemma 3.2. Also, this theory is certainly a generalization of the simple theory, so $V(J)$ is extremely disconnected by Proposition 3.4. Clearly $\mathcal{T}_{R/I}$ is the big trivial torsion theory on R/I.

Let F_1 and F_2 be filters of the respective classes $\mathcal{T}_{R/I}$ and $\mathcal{T}_{R/J}$ as torsion classes on the ring R (which makes sense by Lemma 3.5). Any ideal which belongs to both filters must contain both I and J, hence equals R since $I+J=R$. Clearly, both filters are contained in $F(\mathcal{T})$. Conversely, let $A\in F(\mathcal{T})$ and observe that $A+I\in F_1$, while $A+J\in F_2$. Also, we have that $(A+I)\cap(A+J)=A\cap(I+J)=A$ by distributivity of ideals in a regular ring. We have shown that every ideal in $F(\mathcal{T})$ can be written as $K\cap L$ with $K\in F_1$ and $L\in F_2$. It follows that if $M=R/A$ is a cyclic module in $\mathcal{T}$, then $R/A=R/(A+I)\oplus R/(A+J)$, where the first factor is $\mathcal{T}_{R/I}$-torsion, second is $\mathcal{T}_{R/J}$-torsion. It follows that $\mathcal{T}$ is the direct sum of the two torsion classes $\mathcal{T}_{R/I}$ and $\mathcal{T}_{R/J}$, which completes the proof of the Theorem.

4. Classification of FGSP. We begin with several observations about the homological implications of FGSP. K. Goodearl has shown in [8, p. 59] that the Goldie theory has FGSP if and only if $\operatorname{Ext}^1_R(A, T)=0$ for all nonsingular modules T and finitely generated nonsingular modules A. His proof remains valid exactly as is if torsion submodule replaces the singular submodule. We require this fact in the proof of (a)$\Rightarrow$(b) in the following characterization of FGSP.

Theorem 4.1. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory for the ring R. The following are equivalent:* (a) *$(\mathcal{T}, \mathcal{F})$ has FGSP,* (b) *torsion submodules of finitely generated modules are finitely generated, and $\operatorname{Tor}^R_1(A, R/I)=0$ for R/I torsion and A torsionfree,* (c) (i) *torsion submodules of cyclic modules are bounded, and* (ii) *if A is finitely generated torsionfree and R/I is torsion, then A/AI is a projective* *module over* R/I *and* $\mathrm{Tor}_1^R(A, R/I)=0$.

Proof. (a)⇒(b): the torsions submodule of a finitely generated module M is a direct summand of M by FGSP and hence finitely generated. Let A be torsionfree and R/I torsion, and note that $\mathrm{Hom}_R(R/I, E)$ is torsion for any injective module E. If also B is a finitely generated submodule of A, then one can use the standard identity from [6, Chapter VI, 5.1], the injectivity of E and the remark preceding this theorem to obtain that

$$0 = \mathrm{Ext}_R^1(B, \mathrm{Hom}_R(R/I, E)) = \mathrm{Hom}_R(\mathrm{Tor}_1^R(B, R/I), E)$$

Now choose E to be the injective envelope of $\mathrm{Tor}_1^R(B, R/I)$ to obtain that this module is 0. By taking direct limits over finitely generated submodules B of A, we obtain that $\mathrm{Tor}_1^R(A, R/I)=0$.

(b)⇒(c): If M is a cyclic module, then $t(M)$ is finitely generated and therefore bounded by the intersection I of the annihilators of elements of a finite generating set. Clearly $t(M)I=0$ and $I\in F(\mathcal{I})$, so $t(M)$ is bounded. Now let A be a torsionfree module and I any ideal in $F(\mathcal{I})$. Choose an exact sequence $0\to K\to M\to A\to 0$ with M finite free. Tensor the sequence with R/I and use the vanishing Tor condition of (b) to obtain the exact sequence $0\to K/KI\to M/MI\to A/AI\to 0$, where M/MI is a finite free R/I-module. Since $t(M/MI)=K/KI$ is finitely generated, A/AI is a finitely presented R/I-module. Now it is sufficient to show that A/AI is R/I-flat since finitely presented flat modules are projective [7]. Let L/I be an ideal of R/I, so that $L\in F(\mathcal{I})$, and as above, the sequence

$$0 \to K/KL \to M/ML \to A/AL \to 0 \tag{1}$$

is exact. Tensor over R/I with the module R/L to obtain the exact sequence

$$0 \to \mathrm{Tor}_1^{R/I}(A/AI, R/L) \to K/KL \to M/ML \to A/AL \to 0\,.$$

Comparing with sequence (1), we see that $\mathrm{Tor}_1^{R/I}(A/AI, R/L) = 0$. Hence A/AI is R/I-flat.

(c)⇒(a): Let $0\to T\to M\to A\to 0$ be a $\mathcal{I}$-delineation of a finitely generated module. Theorem 2.1 implies CSP for $\mathcal{I}$ and hence this torsion theory is stable. Lemma 3.2 of [15] says that for a stable torsion theory, torsion submodules of finitely generated modules are bounded. Hence $TI=0$ for some $I\in F(\mathcal{I})$. Tensor the delineation with R/I and use the vanishing Tor condition of (c) to obtain the exact sequence $0\to T\to M/MI\to A/AI\to 0$, which splits since A/AI is projective. Thus we have a commutative diagram

$$\begin{array}{ccccccccc} 0 \to & T & \longrightarrow & M & \longrightarrow & A & \longrightarrow 0 \\ & \| & & \downarrow & & \downarrow & \\ 0 \to & T & \to & M/MI & \to & A/AI & \to 0 \end{array}$$

with bottom row split exact. Therefore the top row splits and we have FGSP. This completes the proof.

K. Oshiro proved ([12, Theorem 3.1]) that under a certain additional condition, the Goldie theory for a regular ring has FGSP if (and only if) torsion submodules of finitely generated modules are again finitely generated, and asked whether this condition can be dropped. In view of Theorem 4.1 and the fact that every Tor module over a regular ring vanishes, we have the following affirmative answer to Oshiro's question.

Corollary 4.2. *The torsion theory* $(\mathcal{T}, \mathcal{F})$ *for a regular ring has FGSP if (and only if) torsion submodules of finitely generated modules are again finitely generated.*

The following Corollary has been shown by several authors for special torsion cases and, utilizing these results, by Koifman [11, p. 157] in full generality. We sketch a proof.

Corollary 4.3. *Suppose that* $\mathcal{T}$ *has FGSP and* $t(R)=0$; *then every ideal in* $F(\mathcal{T})$ *is flat and for every prime* P *in* $F(\mathcal{T})$ *the ring* R_P *is a valuation domain.*

Proof. Since $t(R)=0$, every ideal A of R is a torsionfree module. Now use Theorem 4.1 and the Shifting Lemma [6] to obtain that for each I in $F(\mathcal{T})$,

$$\mathrm{Tor}_1^R(I, R/A) = \mathrm{Tor}_2^R(R/I, R/A) = \mathrm{Tor}_1^R(A, R/I) = 0\,.$$

It follows that I is R-flat. Let P be a prime in $F(\mathcal{T})$; thence R_P is a domain by Corollary 2.4. and, in view of Corollary 2.2, every nonzero ideal of R_P is the localization of an ideal from $F(\mathcal{T})$, hence R_P-flat. It follows that R_P is a valuation domain.

We conclude with an analysis of FGSP for domains. Implicit in the following theorem is the fact that a torsion theory for a domain has FGSP if and only if it has CSP and is locally a valuation domain at primes in $F(\mathcal{T})$. In general, one cannot hope for the FGSP condition to decompose so nicely into CSP plus some local condition. Consider, for example, regular rings. Locally, such rings are fields, which are as well behaved rings as one could expect. Yet there are regular rings for which the Goldie theory has CSP but not FGSP (see [12, p. 379]).

Theorem 4.4. *Let* R *be an integral domain and* X *a subset of* $spec(R)$, *closed under generalization and such that* (a) *for every proper closed subset* C

in spec(R), $C\cap X$ is clopen in C, and (b) *for every prime P not in X, R_P is a valuation domain. Then F_X is a filter for a torsion theory with FGSP and conversely, every torsion filter on R with FGSP is so obtained.*

Proof. Let F be the filter of a torsion theory with FGSP. Then the theory has CSP and $t(R)=0$ unless the theory is trivial, in which case the theorem reduces to a triviality. Assume for the rest of this proof that the theory is nontrivial. Then F satisfies the hypotheses of Theorem 3.1 and Corollary 4.3. The desired conclusions follow immediately.

Conversely, let F satisfy the hypotheses of the Theorem. Then $\mathcal{I}_P$ is nontrivial at each prime $P\in F(\mathcal{I})$, and for $J\in F(\mathcal{F})$ and $P\in F(\mathcal{I})$, we have $J_P=0$ or R_P according as $J\subseteq P$ or not. Thus $(R/J)_P$ has zero torsion submodule. It follows that every torsionfree R-module localizes to a torsionfree R_P-module (with respect to the induced torsion). By CSP, torsion submodules of cyclic modules are cyclic, so bounded. Let I be in $F(\mathcal{I})$ and A a torsionfree R-module. Let P be any prime and set $S=R_P$. We claim that $M=\mathrm{Tor}_1^S(A_P, R_P/I_P)=0$. If I is not contained in P, then $I_P=R_P$ and equality is trivial. Otherwise $P\in F(\mathcal{I})$, so R_P is a valuation domain and A_P is a finitely generated torsionfree module over R_P, hence a finite free module. Hence again $M=0$, as desired. Since P was arbitrary, $\mathrm{Tor}_1^R(A, R/I)=0$. To complete the proof, we show that A/AI is R/I-projective and appeal to Theorem 4.1 to get FGSP. Let P and Q be any two primes containing I. Since R is an integral domain, the quotient field of any localization of R is the same as that of R; hence any set of elements of A which localizes to a linearly independent set over R_P must also localize to a linearly independent set over R_Q. Thus the rank of A_P as an R_P-module is independent of P. Consequently A/AI is a finitely generated locally free R/I-module of constant rank, and therefore projective by [1, Chapter II, sec. 5.2]. This completes the proof.

In the special case that X is empty we obtain the well-known result of Kaplansky ([9], [10]), namely

Corollary 4.5. *The usual theory for a domain R has FGSP if and only if R is a Prufer domain.*

It is easy to see with the aid of Theorem 4.4 that there can be many torsion theories other than the usual theory for the integral domain R which have FGSP. For example, when R is a one dimensional Prufer domain (e.g., the ring of rational integers) and X is a finite subset of spec(R) consisting of the zero prime and isolated points of spec$(R)-\{(0)\}$, then the filter F_X defines a torsion theory on R with FGSP.

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