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## COHOMOLOGY OF VECTOR BUNDLES FROM A DOUBLE COVER OF THE PROJECTIVE PLANE

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### Abstract

The paper deals with locally free sheaves  $\mathcal{F}_{p,q}$  on  $\mathbb{P}^2$  obtained from a morphism  $\mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^2$ . Bases of  $H^i(\mathbb{P}^2, \mathcal{F}_{p,q})$  are explicitly given in terms of elements of certain local cohomology modules, which built up canonically a complex for computing cohomology modules of locally free sheaves on  $\mathbb{P}^2$ .

### 1. Introduction

Let  $\mathbb{P}^n = \text{Proj } \kappa[X_0, X_1, \dots, X_n]$  be the projective  $n$ -space over a field  $\kappa$  and  $\mathcal{F}$  be a locally free sheaf of finite rank on  $\mathbb{P}^n$ . In [4], a new method is introduced to compute cohomology modules of  $\mathcal{F}$ . The method involves a complex of  $\kappa$ -vector spaces

$$0 \rightarrow \mathcal{F}^{(0)} \xrightarrow{d^{(0)}} \mathcal{F}^{(1)} \xrightarrow{d^{(1)}} \mathcal{F}^{(2)} \rightarrow \dots \rightarrow \mathcal{F}^{(n)} \rightarrow 0,$$

in which  $\mathcal{F}^{(i)}$  depends only on the rank of  $\mathcal{F}$  and  $d^{(i)}$  is determined by the transition functions of  $\mathcal{F}$ . It is shown that the  $i$ th cohomology of the complex  $\mathcal{F}^{(\bullet)}$  is isomorphic to the  $i$ th cohomology of  $\mathcal{F}$ . With computations of kernels and quotients of  $d^{(i)}$ , the problem of algebraic geometry on computing cohomology becomes a problem of linear algebra. In terms of elements of  $\mathcal{F}^{(i)}$ , one may ask *what a basis of the  $\kappa$ -vector space  $H^i(\mathbb{P}^n, \mathcal{F})$  looks like*. For twisted differentials  $\Omega_{\mathbb{P}^n/\kappa}^p(m)$ , this project is carried out [4]. A basis of the  $\kappa$ -vector space  $H^q(\mathbb{P}^n, \Omega_{\mathbb{P}^n/\kappa}^p(m))$  is exhibited, from which the Bott formula

$$\dim_{\kappa} H^q(\mathbb{P}^n, \Omega_{\mathbb{P}^n/\kappa}^p(m)) = \begin{cases} \binom{m-1}{p} \binom{m+n-p}{m}, & \text{for } q=0, 0 \leq p \leq n, p < m; \\ 1, & \text{for } m=0, 0 \leq p=q \leq n; \\ \binom{-m-1}{n-p} \binom{-m+p}{-m}, & \text{for } q=n, 0 \leq p \leq n, m < p-n; \\ 0, & \text{otherwise} \end{cases}$$

is recovered by counting the cardinality of the basis. Invoking elaborated computations, our approach to the Bott formula interprets the combinatorial numbers in the formula.

In this paper, we work on the project for some rank two locally free sheaves of modules on the projective plane  $\mathbb{P}^2$ . Let  $Q$  be the quadric surface in  $\mathbb{P}^3$  defined by the equation  $X_0X_1 - X_2X_3$ . Via the Segre embedding,  $Q$  identifies with  $\mathbb{P}^1 \times \mathbb{P}^1$ , whose invertible sheaves are classified as  $\mathcal{L}_{p,q}$ ,  $p, q \in \mathbb{Z}$ . We consider a projection from a point of  $\mathbb{P}^3$  to a plane, whose restriction to  $Q$  is denoted by  $\pi$ . It is known that

$$\dim_{\kappa} H^r(\mathbb{P}^2, \pi_*\mathcal{L}_{p,q}) = (-1)^r(p+1)(q+1)$$

if  $r = 0$  and  $p, q \geq 0$ ; or if  $r = 1$  and  $p \geq 0, q < 0$  or  $p < 0, q \geq 0$ ; or if  $r = 2$  and  $p, q < 0$ ; and is zero otherwise [6, Proposition 12]. The module structure of injective complexes defining sheaf cohomology is subtle. Our goal is to analyze  $H^r(\mathbb{P}^2, \pi_*\mathcal{L}_{p,q})$  in terms of elements of  $(\pi_*\mathcal{L}_{p,q})^{(r)}$  to reveal its combinatorial nature.

Usually, the word ‘‘basis’’ stands for a minimal generating set of a free module. However, a set may have different module structures. To avoid confusion, we reserve the term only for a minimal generating set of a  $\kappa$ -vector space in this paper.

This paper is organized as follows.

- Section 2 recalls the construction of  $\mathcal{F}^{(\bullet)}$  for a locally free sheaf  $\mathcal{F}$  on the projective plane.
- Section 3 describes locally free sheaves  $\mathcal{F}_{p,q}$  obtained from a double cover of the projective plane.
- Section 4 applies the construction of Section 2 to  $\mathcal{F}_{p,q}$ .
- Section 5 analyzes the module structure of  $\mathcal{F}_{p,q}^{(2)}$ .
- Section 6 gives bases of  $H^i(\mathbb{P}^2, \mathcal{F}_{p,q})$ .

## 2. Complex for computing cohomology

Let  $\mathcal{F}$  be a locally free sheaf of finite rank on  $\mathbb{P}^n$ . We recall the construction of the complex  $\mathcal{F}^{(\bullet)}$  for the case  $n = 2$ . Given  $\mathfrak{p} \in \mathbb{P}^2 = \text{Proj}(\kappa[T_0, T_1, T_2])$ , the local cohomology module

$$M(\mathfrak{p}) := H_{\mathfrak{m}_{\mathfrak{p}}}^{\text{ht } \mathfrak{p}} \left( \bigwedge^2 \Omega_{\mathcal{O}_{\mathbb{P}^2, \mathfrak{p}}/\kappa} \right)$$

supported at the maximal ideal  $\mathfrak{m}_{\mathfrak{p}}$  of  $\mathcal{O}_{\mathbb{P}^2, \mathfrak{p}}$  is an injective hull of the residue field  $\kappa_{\mathfrak{p}}$  of  $\mathcal{O}_{\mathbb{P}^2, \mathfrak{p}}$ . Elements of  $M(\mathfrak{p})$  can be written as generalized fractions, which we referred to [2, Chapter 2] or [5, §7]. We recall three special cases of  $M(\mathfrak{p})$  needed for defining  $\mathcal{F}^{(i)}$ .

### EXAMPLE 1.

- If  $\mathfrak{p}$  is the generic point of  $\mathbb{P}^2$ , we write  $M(\mathbb{P}^2)$  for  $M(\mathfrak{p})$ . Elements of  $M(\mathbb{P}^2)$  are of the form

$$\frac{f}{g} d \frac{T_0}{T_2} d \frac{T_1}{T_2},$$

where  $f \in \kappa[T_0/T_2, T_1/T_2]$  and  $g \in \kappa[T_0/T_2, T_1/T_2] \setminus (0)$ .

- If  $\mathfrak{p}$  is the generic point of the line  $T_2 = 0$ , we write  $M(\mathbb{P}^1)$  for  $M(\mathfrak{p})$ . Elements of  $M(\mathbb{P}^1)$  are of the form

$$(1) \quad \begin{bmatrix} \frac{f}{g} d \frac{T_2}{T_1} d \frac{T_0}{T_1} \\ \left(\frac{T_2}{T_1}\right)^i \end{bmatrix},$$

where  $f \in \kappa[T_2/T_1, T_0/T_1]$  and  $g \in \kappa[T_2/T_1, T_0/T_1] \setminus (T_2/T_1)$ .

- If  $\mathfrak{p}$  is the closed point  $T_2 = T_1 = 0$ , we write  $M(\mathbb{P}^0)$  for  $M(\mathfrak{p})$ . Elements of  $M(\mathbb{P}^0)$  are of the form

$$(2) \quad \begin{bmatrix} \frac{f}{g} d \frac{T_1}{T_0} d \frac{T_2}{T_0} \\ \left(\frac{T_1}{T_0}\right)^i, \left(\frac{T_2}{T_0}\right)^j \end{bmatrix},$$

where  $f \in \kappa[T_1/T_0, T_2/T_0]$  and  $g \in \kappa[T_1/T_0, T_2/T_0] \setminus (T_1/T_0, T_2/T_0)$ .

$M(\mathfrak{p})$ , being an injective hull of  $\kappa_{\mathfrak{p}}$ , is also a module over the completion  $\mathcal{O}_{\mathbb{P}^2, \mathfrak{p}}^\wedge$  of  $\mathcal{O}_{\mathbb{P}^2, \mathfrak{p}}$ . This can be seen from the following properties of generalized fractions.

**Proposition 2** (Linearity Law).

$$\begin{aligned} \begin{bmatrix} \left(\frac{f_1}{g_1} + \frac{f_2}{g_2}\right) d \frac{T_2}{T_1} d \frac{T_0}{T_1} \\ \left(\frac{T_2}{T_1}\right)^i \end{bmatrix} &= \begin{bmatrix} \frac{f_1}{g_1} d \frac{T_2}{T_1} d \frac{T_0}{T_1} \\ \left(\frac{T_2}{T_1}\right)^i \end{bmatrix} + \begin{bmatrix} \frac{f_2}{g_2} d \frac{T_2}{T_1} d \frac{T_0}{T_1} \\ \left(\frac{T_2}{T_1}\right)^i \end{bmatrix}, \\ \begin{bmatrix} \left(\frac{f_1}{g_1} + \frac{f_2}{g_2}\right) d \frac{T_1}{T_0} d \frac{T_2}{T_0} \\ \left(\frac{T_1}{T_0}\right)^i, \left(\frac{T_2}{T_0}\right)^j \end{bmatrix} &= \begin{bmatrix} \frac{f_1}{g_1} d \frac{T_1}{T_0} d \frac{T_2}{T_0} \\ \left(\frac{T_1}{T_0}\right)^i, \left(\frac{T_2}{T_0}\right)^j \end{bmatrix} + \begin{bmatrix} \frac{f_2}{g_2} d \frac{T_1}{T_0} d \frac{T_2}{T_0} \\ \left(\frac{T_1}{T_0}\right)^i, \left(\frac{T_2}{T_0}\right)^j \end{bmatrix}. \end{aligned}$$

**Proposition 3** (Vanishing Law). *If  $f \in (T_2/T_1)^i$ ,*

$$\begin{bmatrix} \frac{f}{g} d \frac{T_2}{T_1} d \frac{T_0}{T_1} \\ \left(\frac{T_2}{T_1}\right)^i \end{bmatrix} = 0.$$

If  $f$  is contained in the ideal generated by  $(T_1/T_0)^i$  and  $(T_2/T_0)^j$ , then

$$\left[ \begin{array}{c} \frac{f}{g} d \frac{T_1}{T_0} d \frac{T_2}{T_0} \\ \left(\frac{T_1}{T_0}\right)^i, \left(\frac{T_2}{T_0}\right)^j \end{array} \right] = 0.$$

Denominators of generalized fractions  $((T_2/T_1)^i$  in (1) and  $(T_1/T_0)^i, (T_2/T_0)^j$  in (2)) can be any system of parameters of  $\mathcal{O}_{\mathbb{P}^2, \mathfrak{p}}$ . The relations of generalized fractions in different system of parameters are given by the transformation law, which we refer to [2, Lemma 2.3.ii] or [5, Lemma 7.2.b]. Elements of  $M(\mathfrak{p})$  represented by generalized fractions are convenient to handle.

EXAMPLE 4. Elements of  $M(\mathbb{P}^0)$  can be written as

$$\left[ \begin{array}{c} h d \frac{T_1}{T_0} d \frac{T_2}{T_0} \\ \left(\frac{T_1}{T_0}\right)^i, \left(\frac{T_2}{T_0}\right)^j \end{array} \right],$$

where  $h \in \kappa[T_1/T_0, T_2/T_0]$ .

Proof. Write  $f/g$  in (2) as  $f_0/(1 - g_0)$ , where  $f_0 \in \kappa[T_1/T_0, T_2/T_0]$  and  $g_0 \in (T_1/T_0, T_2/T_0)$ .

$$\frac{1}{1 - g_0} = (1 + g_0 + g_0^2 + \cdots + g_0^{i+j-2})$$

is contained in the ideal generated by  $(T_1/T_0)^i$  and  $(T_2/T_0)^j$ . Let

$$h = f_0(1 + g_0 + g_0^2 + \cdots + g_0^{i+j-2}).$$

By the linearity law and the vanishing law,

$$\begin{aligned} \left[ \begin{array}{c} \frac{f}{g} d \frac{T_1}{T_0} d \frac{T_2}{T_0} \\ \left(\frac{T_1}{T_0}\right)^i, \left(\frac{T_2}{T_0}\right)^j \end{array} \right] &= \left[ \begin{array}{c} h d \frac{T_1}{T_0} d \frac{T_2}{T_0} \\ \left(\frac{T_1}{T_0}\right)^i, \left(\frac{T_2}{T_0}\right)^j \end{array} \right] + \left[ \begin{array}{c} \left(\frac{f_0}{1 - g_0} - h\right) d \frac{T_1}{T_0} d \frac{T_2}{T_0} \\ \left(\frac{T_1}{T_0}\right)^i, \left(\frac{T_2}{T_0}\right)^j \end{array} \right] \\ &= \left[ \begin{array}{c} h d \frac{T_1}{T_0} d \frac{T_2}{T_0} \\ \left(\frac{T_1}{T_0}\right)^i, \left(\frac{T_2}{T_0}\right)^j \end{array} \right]. \end{aligned}$$

□

Let  $J(\mathfrak{p})$  be the quasi-coherent  $\mathcal{O}_{\mathbb{P}^2}$ -module which is the constant sheaf  $M(\mathfrak{p})$  on  $\{\mathfrak{p}\}^-$ , and zero elsewhere. We write  $J(\mathbb{P}^2)$  (resp.  $J(C)$ ) for  $J(\mathfrak{p})$  if  $\mathfrak{p}$  is the generic point of  $\mathbb{P}^2$  (resp. a curve  $C$ ). In [3, 4], a residual complex

$$(3) \quad J(\mathbb{P}^2) \rightarrow \bigoplus_{\text{curves}} J(C) \rightarrow \bigoplus_{\text{closed points}} J(\mathfrak{m}) \rightarrow 0$$

on  $\mathbb{P}^2$  is described. (3) is an injective resolution of  $\mathcal{O}_{\mathbb{P}^2}(-3)$ . Tensoring with  $\mathcal{F}$  and  $\mathcal{O}_{\mathbb{P}^2}(3)$ , we get an injective resolution

$$\mathcal{F} \otimes J(\mathbb{P}^2)(3) \rightarrow \mathcal{F} \otimes \left( \bigoplus J(C) \right) (3) \rightarrow \mathcal{F} \otimes \left( \bigoplus J(\mathfrak{m}) \right) (3) \rightarrow 0$$

of  $\mathcal{F}$ . By definition, the cohomology of the complex

$$(4) \quad \begin{aligned} \Gamma(\mathbb{P}^2, \mathcal{F} \otimes J(\mathbb{P}^2)(3)) &\rightarrow \Gamma(\mathbb{P}^2, \mathcal{F} \otimes \left( \bigoplus J(C) \right) (3)) \\ &\rightarrow \Gamma(\mathbb{P}^2, \mathcal{F} \otimes \left( \bigoplus J(\mathfrak{m}) \right) (3)) \rightarrow 0 \end{aligned}$$

is the cohomology of  $\mathcal{F}$ . It was observed in [4] that a subcomplex  $\mathcal{F}^{(\bullet)}$  of (4) is quasi-isomorphic to (4).

DEFINITION 5. Let  $\{u_i\}$  (resp.  $\{v_i\}$  and  $\{w_i\}$ ) be a minimal generating set for the free module  $\mathcal{F}(D_+(T_2))$  (resp.  $\mathcal{F}(D_+(T_1))$  and  $\mathcal{F}(D_+(T_0))$ ) over  $\mathcal{O}_{\mathbb{P}^2}(D_+(T_2))$  (resp.  $\mathcal{O}_{\mathbb{P}^2}(D_+(T_1))$  and  $\mathcal{O}_{\mathbb{P}^2}(D_+(T_0))$ ). We define  $\mathcal{F}^{(0)}$  to be the submodule of  $\Gamma(\mathbb{P}^2, \mathcal{F} \otimes J(\mathbb{P}^2)(3)) = \Gamma(D_+(T_2), \mathcal{F} \otimes J(\mathbb{P}^2)(3))$  generated by

$$u_i \otimes d \frac{T_0}{T_2} d \frac{T_1}{T_2} \otimes T_2^3.$$

We define  $\mathcal{F}^{(1)}$  to be the submodule of  $\Gamma(\mathbb{P}^2, \mathcal{F} \otimes J(\mathbb{P}^1)(3)) = \Gamma(D_+(T_1), \mathcal{F} \otimes J(\mathbb{P}^1)(3))$  generated by

$$(5) \quad v_i \otimes \left[ \begin{array}{c} d \frac{T_2}{T_1} d \frac{T_0}{T_1} \\ \left( \frac{T_2}{T_1} \right)^j \end{array} \right] \otimes T_1^3 \quad (j \in \mathbb{N}).$$

We define  $\mathcal{F}^{(2)}$  to be the submodule of  $\Gamma(\mathbb{P}^2, \mathcal{F} \otimes J(\mathbb{P}^0)(3)) = \Gamma(D_+(T_0), \mathcal{F} \otimes J(\mathbb{P}^0)(3))$  generated by

$$w_i \otimes \left[ \begin{array}{c} d \frac{T_1}{T_0} d \frac{T_2}{T_0} \\ \left( \frac{T_1}{T_0} \right)^j, \left( \frac{T_2}{T_0} \right)^k \end{array} \right] \otimes T_0^3 \quad (j, k \in \mathbb{N}).$$

Assume that  $\mathcal{F}$  has rank  $n$ . Then  $\mathcal{F}^{(i)}$  is isomorphic to  $n$  copies of  $\mathcal{O}_{\mathbb{P}^2}^{(i)}$ . As  $\kappa$ -vector spaces,  $\mathcal{F}^{(0)}$  has a basis

$$(6) \quad \left\{ u_i \otimes \left( \frac{T_0}{T_2} \right)^j \left( \frac{T_1}{T_2} \right)^k d \frac{T_0}{T_2} d \frac{T_1}{T_2} \otimes T_2^3 \mid 1 \leq i \leq n \text{ and } 0 \leq j, k \right\},$$

and  $\mathcal{F}^{(1)}$  has a basis

$$(7) \quad \left\{ v_i \otimes \left[ \begin{array}{c} \left( \frac{T_0}{T_1} \right)^j d \frac{T_2}{T_1} d \frac{T_0}{T_1} \\ \left( \frac{T_2}{T_1} \right)^k \end{array} \right] \otimes T_1^3 \mid 1 \leq i \leq n, 0 \leq j \text{ and } 0 < k \right\},$$

and  $\mathcal{F}^{(2)}$  has a basis

$$(8) \quad \left\{ w_i \otimes \left[ \begin{array}{c} d \frac{T_1}{T_0} d \frac{T_2}{T_0} \\ \left( \frac{T_1}{T_0} \right)^j, \left( \frac{T_2}{T_0} \right)^k \end{array} \right] \otimes T_0^3 \mid 1 \leq i \leq n \text{ and } 0 < j, k \right\}.$$

The coboundary maps of the residual complex (3) are decomposed into

$$\delta_{p,q} : J(\mathfrak{p}) \rightarrow J(\mathfrak{q})$$

for  $\mathfrak{p}, \mathfrak{q} \in \mathbb{P}^2$ . We recall two special cases of  $\delta_{p,q}$  needed for defining the coboundary maps of  $\mathcal{F}^{(\bullet)}$ .

EXAMPLE 6.

- Let  $\mathfrak{p}$  be the generic point of  $\mathbb{P}^2$  and  $\mathfrak{q}$  be the generic point of the line  $T_2 = 0$ .  $\delta_{\mathbb{P}^2, \mathbb{P}^1} := \delta_{p,q}$  is determined by the map  $M(\mathbb{P}^2) \rightarrow M(\mathbb{P}^1)$  satisfying

$$(9) \quad \frac{f}{g} d \frac{T_1}{T_0} d \frac{T_2}{T_0} \mapsto \left[ \begin{array}{c} f d \frac{T_1}{T_0} d \frac{T_2}{T_0} \\ g \end{array} \right],$$

where  $f \in \kappa[T_1/T_0, T_2/T_0]$  and  $g \in \kappa[T_1/T_0, T_2/T_0] \setminus (0)$ .

- Let  $\mathfrak{p}$  be the generic point of the line  $T_2 = 0$  and  $\mathfrak{q}$  be the closed point  $T_2 = T_1 = 0$ .  $\delta_{\mathbb{P}^1, \mathbb{P}^0} := \delta_{p,q}$  is determined by the map  $M(\mathbb{P}^1) \rightarrow M(\mathbb{P}^0)$  satisfying

$$(10) \quad \left[ \begin{array}{c} \frac{f}{g} d \frac{T_1}{T_0} d \frac{T_2}{T_0} \\ \left( \frac{T_2}{T_0} \right)^i \end{array} \right] \mapsto \left[ \begin{array}{c} f d \frac{T_1}{T_0} d \frac{T_2}{T_0} \\ g, \left( \frac{T_2}{T_0} \right)^i \end{array} \right]$$

where  $f \in \kappa[T_1/T_0, T_2/T_0]$  and  $g \in \kappa[T_1/T_0, T_2/T_0] \setminus (T_2/T_0)$ .

In Example 1, elements of  $M(\mathbb{P}^2)$  (resp.  $M(\mathbb{P}^1)$ ) are represented in terms of  $T_0/T_2$  and  $T_1/T_2$  (resp.  $T_2/T_1$  and  $T_0/T_1$ ). We may use the formula

$$(11) \quad \begin{aligned} T_2^3 d \frac{T_0}{T_2} d \frac{T_1}{T_2} &= T_0^3 d \frac{T_1}{T_0} d \frac{T_2}{T_0}, \\ \begin{bmatrix} d \frac{T_2}{T_1} d \frac{T_0}{T_1} \\ \left(\frac{T_2}{T_1}\right)^3 \end{bmatrix} &= \begin{bmatrix} d \frac{T_1}{T_0} d \frac{T_2}{T_0} \\ \left(\frac{T_2}{T_0}\right)^3 \end{bmatrix} \end{aligned}$$

to rewrite elements of  $M(\mathbb{P}^2)$  and  $M(\mathbb{P}^1)$  before applying (9) and (10).

For  $i = 0, 1$ , the image of  $\mathcal{F}^{(1-i)}$  under the map

$$(\text{id}_{\mathcal{F}} \otimes \delta_{\mathbb{P}^{i+1}, \mathbb{P}^i} \otimes \text{id}_{\mathcal{O}_{\mathbb{P}^2}(3)}) (\mathbb{P}^2) : \Gamma(\mathbb{P}^2, \mathcal{F} \otimes J(\mathbb{P}^{i+1})(3)) \rightarrow \Gamma(\mathbb{P}^2, \mathcal{F} \otimes J(\mathbb{P}^i)(3))$$

is contained in  $\mathcal{F}^{(2-i)}$ .

DEFINITION 7. For  $i = 0, 1$ , let  $d^{(1-i)} : \mathcal{F}^{(1-i)} \rightarrow \mathcal{F}^{(2-i)}$  be the restriction of  $(\text{id}_{\mathcal{F}} \otimes \delta_{\mathbb{P}^{i+1}, \mathbb{P}^i} \otimes \text{id}_{\mathcal{O}_{\mathbb{P}^2}(3)}) (\mathbb{P}^2)$  on  $\mathcal{F}^{(1-i)}$ .

To make  $d^{(1-i)}$  explicit, we consider  $\text{id}_{\mathcal{F}} \otimes \delta_{\mathbb{P}^{i+1}, \mathbb{P}^i} \otimes \text{id}_{\mathcal{O}_{\mathbb{P}^2}(3)}$  on  $D_+(T_0)$ . Restricted to  $D_+(T_2) \cap D_+(T_0)$ ,

$$u_i = \sum_j \frac{f_{ij}}{(T_2/T_0)^{n_{ij}}} w_i$$

for some  $f_{ij} \in \kappa[T_1/T_0, T_2/T_0]$  and  $n_{ij} \geq 0$ . In terms of these transition functions,

$$\begin{aligned} d^0 \left( u_i \otimes d \frac{T_0}{T_2} d \frac{T_1}{T_2} \otimes T_2^3 \right) &= d^0 \left( \sum_j w_i \otimes \frac{f_{ij}}{(T_2/T_0)^{n_{ij}}} d \frac{T_1}{T_0} d \frac{T_2}{T_0} \otimes T_0^3 \right) \\ &= \sum_j w_i \otimes \begin{bmatrix} f_{ij} d \frac{T_1}{T_0} d \frac{T_2}{T_0} \\ \left(\frac{T_2}{T_0}\right)^{n_{ij}} \end{bmatrix} \otimes T_0^3. \end{aligned}$$

We may use (11) to write the image of  $d^{(0)}$  in terms of the generators (5) of  $\mathcal{F}^{(1)}$ . Restricted to  $D_+(T_1) \cap D_+(T_0)$ ,

$$v_i = \sum_j \frac{h_{ij}}{(T_1/T_0)^{n_{ij}}} w_i$$



for some  $n_{ij} \geq 0$  and  $h_{ij} \in \kappa[T_1/T_0, T_2/T_0]$ . In terms of these transition functions,

$$\begin{aligned} & d^{(1)} \left( v_i \otimes \left[ \begin{array}{c} d \frac{T_2}{T_1} d \frac{T_0}{T_1} \\ \left( \frac{T_2}{T_1} \right)^l \end{array} \right] \otimes T_1^3 \right) \\ &= d^{(1)} \left( \sum_j w_i \otimes \left[ \begin{array}{c} h_{ij} \left( \frac{T_1}{T_0} \right)^{l-n_{ij}} d \frac{T_1}{T_0} d \frac{T_2}{T_0} \\ \left( \frac{T_2}{T_0} \right)^l \end{array} \right] \otimes T_0^3 \right) \\ &= \sum_j w_i \otimes \left[ \begin{array}{c} h_{ij} d \frac{T_1}{T_0} d \frac{T_2}{T_0} \\ \left( \frac{T_1}{T_0} \right)^{n_{ij}-l}, \left( \frac{T_2}{T_0} \right)^l \end{array} \right] \otimes T_0^3. \end{aligned}$$

The following is our main tool.

**Theorem 8** ([4, Theorem 3.2]). *The  $i$ -th cohomology of  $\mathcal{F}^{(\bullet)}$  is isomorphic to  $H^i(\mathbb{P}^2, \mathcal{F})$ .*

**3. Vector bundles  $\mathcal{F}_{p,q}$**

Let  $S$  be the graded ring  $\kappa[X_0, X_1, X_2, X_3]/(X_0X_1 - X_2X_3)$  over a field  $\kappa$ . Denote by  $x_i$  the image of  $X_i$  under the canonical map  $\kappa[X_0, X_1, X_2, X_3] \rightarrow S$ . So, as a  $\kappa$ -algebra,  $S$  is generated by  $x_0, x_1, x_2, x_3$  with a relation  $x_0x_1 = x_2x_3$ .  $\text{Proj}(S)$  is a hypersurface of  $\mathbb{P}^3$  covered by three affine open sets:

$$\text{Proj}(S) = D_+(x_3) \cup D_+(x_2) \cup D_+(x_1 - x_0).$$

On  $D_+(x_3)$  and  $D_+(x_2)$ , the regular functions of  $\text{Proj}(S)$  form polynomial rings  $\kappa[x_0/x_3, x_1/x_3]$  and  $\kappa[x_0/x_2, x_1/x_2]$ , respectively. On  $D_+(x_1 - x_0)$ , its regular functions are

$$\kappa \left[ \frac{x_1}{x_1 - x_0}, \frac{x_2}{x_1 - x_0}, \frac{x_3}{x_1 - x_0} \right] / \left( \left( \frac{x_1}{x_1 - x_0} \right)^2 - \frac{x_1}{x_1 - x_0} - \frac{x_2}{x_1 - x_0} \frac{x_3}{x_1 - x_0} \right).$$

We identify  $\text{Proj}(S)$  with the fiber product of two projective lines, which can be described using a Cartesian product (that is, the scheme  $\text{Proj}(\kappa[Y_0, Y_1] \times_{\kappa} \kappa[Z_0, Z_1])$ ). The identification is given by the homomorphism of  $\kappa$ -algebras

$$\begin{aligned} \kappa[x_0, x_1, x_2, x_3] &\rightarrow \kappa[Y_0, Y_1] \times_{\kappa} \kappa[Z_0, Z_1], \\ x_0 &\mapsto Y_0Z_0, \\ x_1 &\mapsto Y_1Z_1, \end{aligned}$$

$$\begin{aligned} x_2 &\mapsto Y_1 Z_0, \\ x_3 &\mapsto Y_0 Z_1. \end{aligned}$$

Let  $\pi_1$  and  $\pi_2$  be the two projections from  $\text{Proj}(S)$  to  $\mathbb{P}^1$ . For  $p, q \in \mathbb{Z}$ ,

$$\mathcal{L}_{p,q} := \pi_1^* \mathcal{O}(p) \otimes \pi_2^* \mathcal{O}(q)$$

is an invertible sheaf on  $\text{Proj}(S)$ , which is the sheaf associated to the graded module

$$\kappa[Y_0, Y_1](p) \times_{\kappa} \kappa[Z_0, Z_1](q).$$

On  $D_+(x_3)$ ,  $\mathcal{L}_{p,q}$  is generated by  $Y_0^p Z_1^q$ . On  $D_+(x_2)$ , it is generated by  $Y_1^p Z_0^q$ .

**Proposition 9.** *Let  $\epsilon \geq \max\{0, -p, -q\}$ .  $\mathcal{L}_{p,q}(D_+(x_1 - x_0))$  is generated by  $Y_0^{\epsilon+p} Z_0^{\epsilon+q} / (x_1 - x_0)^\epsilon$  and  $Y_1^{\epsilon+p} Z_1^{\epsilon+q} / (x_1 - x_0)^\epsilon$ .*

*Proof.*  $\mathcal{L}_{p,q}(D_+(x_1 - x_0))$  is generated by  $Y_0^i Y_1^j Z_0^k Z_1^l / (x_1 - x_0)^n$ , where the indices  $i, j, k, l, n \geq 0$  satisfy  $i + j = n + p$  and  $k + l = n + q$ . Restricting to  $D_+(x_1) \cap D_+(x_1 - x_0)$ ,

$$\frac{Y_0^i Y_1^j Z_0^k Z_1^l}{(x_1 - x_0)^n} = \left(\frac{x_1}{x_1 - x_0}\right)^{n-i-k-\epsilon} \left(\frac{x_2}{x_1 - x_0}\right)^k \left(\frac{x_3}{x_1 - x_0}\right)^i \frac{Y_1^{\epsilon+p} Z_1^{\epsilon+q}}{(x_1 - x_0)^\epsilon}.$$

Restricting to  $D_+(x_0) \cap D_+(x_1 - x_0)$ ,

$$\frac{Y_0^i Y_1^j Z_0^k Z_1^l}{(x_1 - x_0)^n} = \left(\frac{x_0}{x_1 - x_0}\right)^{n-j-l-\epsilon} \left(\frac{x_2}{x_1 - x_0}\right)^j \left(\frac{x_3}{x_1 - x_0}\right)^l \frac{Y_0^{\epsilon+p} Z_0^{\epsilon+q}}{(x_1 - x_0)^\epsilon}.$$

Since  $D_+(x_1 - x_0)$  is covered by the subsets  $D_+(x_1) \cap D_+(x_1 - x_0)$  and  $D_+(x_0) \cap D_+(x_1 - x_0)$ ,  $\mathcal{L}_{p,q}(D_+(x_1 - x_0))$  is generated by  $Y_0^{\epsilon+p} Z_0^{\epsilon+q} / (x_1 - x_0)^\epsilon$  and  $Y_1^{\epsilon+p} Z_1^{\epsilon+q} / (x_1 - x_0)^\epsilon$ .  $\square$

Let  $\mathcal{O}$  be the point of  $\mathbb{P}^3 \setminus \text{Proj}(S)$  with homogeneous coordinate  $[1, 1, 0, 0]$ . Let

$$\pi : \text{Proj}(S) \rightarrow \mathbb{P}^2$$

be the double cover of  $\mathbb{P}^2$  defined by the immersion  $\text{Proj}(S) \rightarrow \mathbb{P}^3 \setminus \{\mathcal{O}\}$  followed by the projection from  $\mathcal{O}$  to the plane  $X_0 = 0$ , which is identified with  $\mathbb{P}^2 = \text{Proj}(\kappa[T_0, T_1, T_2])$ . The morphism  $\pi$  is determined by the graded homomorphism

$$\kappa[T_0, T_1, T_2] \rightarrow \kappa[x_0, x_1, x_2, x_3]$$

given by

$$T_0 \mapsto x_1 - x_0,$$

$$\begin{aligned} T_1 &\mapsto x_2, \\ T_2 &\mapsto x_3. \end{aligned}$$

We consider the locally free sheaf of modules

$$\mathcal{F}_{p,q} := \pi_* \mathcal{L}_{p,q}$$

on  $\mathbb{P}^2$ , which has rank 2. On  $D_+(T_2)$ ,  $\mathcal{F}_{p,q}$  is generated by  $(x_0/x_3)Y_0^p Z_1^q$  and  $Y_0^p Z_1^q$ . On  $D_+(T_1)$ , it is generated by  $(x_0/x_2)Y_1^p Z_0^q$  and  $Y_1^p Z_0^q$ .

**Proposition 10.** *Let  $\epsilon = \max\{0, -p, -q\}$ .  $\mathcal{F}_{p,q}(D_+(T_0))$  is generated by  $Y_0 Z_0/T_0^{\epsilon+1}$  and  $Y_1 Z_1/T_0^{\epsilon+1}$  if  $p = q \leq 0$ , otherwise by  $Y_0^{\epsilon+p} Z_0^{\epsilon+q}/T_0^\epsilon$  and  $Y_1^{\epsilon+p} Z_1^{\epsilon+q}/T_0^\epsilon$ .*

Proof.  $\mathcal{F}_{p,q}(D_+(T_0))$  is generated by  $Y_0^i Y_1^j Z_0^k Z_1^l/T_0^n$ , where  $i, j, k, l, n \geq 0$  satisfy  $i + j = n + p$  and  $k + l = n + q$ . Note that, if  $j$  and  $k$  are both positive, then

$$(12) \quad \frac{Y_0^i Y_1^j Z_0^k Z_1^l}{T_0^n} = \frac{T_1}{T_0} \frac{Y_0^i Y_1^j Z_0^{k-1} Z_1^{l+1}}{T_0^n} - \frac{T_1}{T_0} \frac{Y_0^{i+1} Y_1^{j-1} Z_0^k Z_1^l}{T_0^n},$$

if  $i$  and  $l$  are both positive, then

$$(13) \quad \frac{Y_0^i Y_1^j Z_0^k Z_1^l}{T_0^n} = \frac{T_2}{T_0} \frac{Y_0^{i-1} Y_1^{j+1} Z_0^k Z_1^l}{T_0^n} - \frac{T_2}{T_0} \frac{Y_0^i Y_1^j Z_0^{k+1} Z_1^{l-1}}{T_0^n}.$$

Assume that  $n > \epsilon$ . Then  $n, n + p, n + q > 0$  and

$$(14) \quad \frac{Y_0^i Y_1^j Z_0^k Z_1^l}{T_0^n} = \begin{cases} \frac{T_1}{T_0} \frac{Y_0^i Y_1^{j-1} Z_0^{k-1} Z_1^l}{T_0^{n-1}}, & \text{if } j, k > 0; \\ \frac{T_2}{T_0} \frac{Y_0^{i-1} Y_1^j Z_0^k Z_1^{l-1}}{T_0^{n-1}}, & \text{if } i, l > 0, \end{cases}$$

$$(15) \quad \frac{Y_0^{n+p} Z_0^{n+q}}{T_0^n} = \frac{Y_0^{n+p-1} Y_1 Z_0^{n+q-1} Z_1}{T_0^n} - \frac{Y_0^{n+p-1} Z_0^{n+q-1}}{T_0^{n-1}},$$

$$(16) \quad \frac{Y_1^{n+p} Z_1^{n+q}}{T_0^n} = \frac{Y_0 Y_1^{n+p-1} Z_0 Z_1^{n+q-1}}{T_0^n} + \frac{Y_1^{n+p-1} Z_1^{n+q-1}}{T_0^{n-1}}.$$

We consider first the case  $p \neq q$  or  $p = q > 0$ , in which either  $n + p - 1 > 0$  or  $n + q - 1 > 0$ . Using (14), (15) and (16), induction on  $n$  shows that  $\mathcal{F}_{p,q}(D_+(T_0))$  is generated by  $Y_0^i Y_1^j Z_0^k Z_1^l/T_0^\epsilon$ , where  $i, j, k, l \geq 0$  satisfy  $i + j = \epsilon + p$  and  $k + l = \epsilon + q$ . Applying (12) and (13) with  $n = \epsilon$ , we see that  $\mathcal{F}_{p,q}(D_+(T_0))$  is generated by  $Y_0^{\epsilon+p} Z_0^{\epsilon+q}/T_0^\epsilon$  and  $Y_1^{\epsilon+p} Z_1^{\epsilon+q}/T_0^\epsilon$ .

Now we consider the case  $p = q \leq 0$ . Assume that  $n > \epsilon + 1$ . In this case,  $n + p - 1 = n + q - 1 > 0$ . Using (14), (15) and (16), induction on  $n$  shows that  $\mathcal{F}_{p,q}(D_+(T_0))$

is generated by  $1/T_0^\epsilon$  and  $Y_0^i Y_1^j Z_0^k Z_1^l / T_0^{\epsilon+1}$ , where  $i, j, k, l \geq 0$  satisfy  $i + j = \epsilon + p + 1$  and  $k + l = \epsilon + q + 1$ . Applying (12) and (13) with  $n = \epsilon + 1$ , we see that  $\mathcal{F}_{p,q}(D_+(T_0))$  is generated by  $Y_0 Z_0 / T_0^{\epsilon+1}$ ,  $Y_1 Z_1 / T_0^{\epsilon+1}$  and  $1/T_0^\epsilon$ . The proposition follows from the identity

$$\frac{Y_1 Z_1}{T_0^{\epsilon+1}} - \frac{Y_0 Z_0}{T_0^{\epsilon+1}} = \frac{1}{T_0^\epsilon}. \quad \square$$

**4. Complexes  $\mathcal{F}_{p,q}^{(\bullet)}$**

From now on, we always assume that  $\epsilon = \max\{0, -p, -q\}$ . First we would like to write down bases of the  $\kappa$ -vector spaces  $\mathcal{F}_{p,q}^{(i)}$  explicitly.

DEFINITION 11. For  $i, j \geq 0$ , we define

$$\mathbf{u}^{ij} := \left(\frac{x_0}{x_3}\right)^i \left(\frac{x_1}{x_3}\right)^j Y_0^p Z_1^q \otimes d\frac{T_0}{T_2} d\frac{T_1}{T_2} \otimes T_2^3 \in \Gamma(\mathbb{P}^2, \mathcal{F}_{p,q} \otimes J(\mathbb{P}^2)(3)).$$

For  $i, j, m, n \in \mathbb{Z}$ , we choose  $\delta \geq \max\{-i, -j\}$  and define

$$\mathbf{v}_n^{ij} := \left(\frac{x_0}{x_2}\right)^{\delta+i} \left(\frac{x_1}{x_2}\right)^{\delta+j} Y_1^p Z_0^q \otimes \begin{bmatrix} d\frac{T_2}{T_1} d\frac{T_0}{T_1} \\ \left(\frac{T_2}{T_1}\right)^{\delta+n} \end{bmatrix} \otimes T_1^3$$

in  $\Gamma(\mathbb{P}^2, \mathcal{F}_{p,q} \otimes J(\mathbb{P}^1)(3))$  and

$$\mathbf{w}_{mn}^{ij} := \left(\frac{x_0}{x_1 - x_0}\right)^{\delta+i} \left(\frac{x_1}{x_1 - x_0}\right)^{\delta+j} \frac{Y_0^{\epsilon+p} Z_1^{\epsilon+q}}{(x_1 - x_0)^\epsilon} \otimes \begin{bmatrix} d\frac{T_1}{T_0} d\frac{T_2}{T_0} \\ \left(\frac{T_1}{T_0}\right)^{\delta+m}, \left(\frac{T_2}{T_0}\right)^{\delta+n} \end{bmatrix} \otimes T_0^3$$

in  $\Gamma(\mathbb{P}^2, \mathcal{F}_{p,q} \otimes J(\mathbb{P}^0)(3))$ .

The definitions of  $\mathbf{v}_n^{ij}$  and  $\mathbf{w}_{mn}^{ij}$  are independent of the choice of  $\delta$ . By Proposition 3,  $\mathbf{v}_0^{ij} = 0$  if  $i, j \geq 0$  and  $\mathbf{w}_{m(n+\epsilon+q)}^{(\epsilon+q)0} = \mathbf{w}_{m(n+\epsilon+p)}^{0(\epsilon+p)} = 0$  if  $m \leq 0$  or  $n \leq 0$ . Sometimes,  $\mathbf{w}_{mn}^{ij}$  are treated differently according to the values of  $p$  and  $q$ . The following notations are handy.

$$\mathbf{w}_{mn}^{\geq} := \begin{cases} \mathbf{w}_{m(n+\epsilon+q)}^{10} & \text{if } p = q \leq 0; \\ \mathbf{w}_{m(n+\epsilon+q)}^{(\epsilon+q)0} & \text{otherwise.} \end{cases}$$

$$\mathbf{w}_{mn}^{\leq} := \begin{cases} \mathbf{w}_{m(n+\epsilon+p)}^{01} & \text{if } p = q \leq 0; \\ \mathbf{w}_{m(n+\epsilon+p)}^{0(\epsilon+p)} & \text{otherwise.} \end{cases}$$

**Proposition 12.**

- The elements  $\mathbf{u}^{ij}$ , where  $i, j \geq 0$ , form a basis of  $\mathcal{F}_{p,q}^{(0)}$ .
- The elements  $\mathbf{v}_0^{ij}$ , where  $i < 0$  or  $j < 0$ , form a basis of  $\mathcal{F}_{p,q}^{(1)}$ .
- The elements  $\mathbf{w}_{mn}^{\geq}$  and  $\mathbf{w}_{mn}^{\leq}$ , where  $m, n > 0$ , form a basis of  $\mathcal{F}_{p,q}^{(2)}$ .

Proof. As an  $\mathcal{O}_{\mathbb{P}^2}(D_+(T_2))$ -module,  $\mathcal{F}_{p,q}(D_+(T_2))$  has a minimal generating set  $\{Y_0^p Z_1^q, (x_0/x_3)Y_0^p Z_1^q\}$ . Indicated in (6), as a  $\kappa$ -vector space,  $\mathcal{F}_{p,q}^{(0)}$  has a basis consisting of

$$Y_0^p Z_1^q \otimes \left(\frac{T_0}{T_2}\right)^i \left(\frac{T_1}{T_2}\right)^j d\frac{T_0}{T_2} d\frac{T_1}{T_2} \otimes T_2^3 \quad \text{and} \quad \frac{x_0}{x_3} Y_0^p Z_1^q \otimes \left(\frac{T_0}{T_2}\right)^i \left(\frac{T_1}{T_2}\right)^j d\frac{T_0}{T_2} d\frac{T_1}{T_2} \otimes T_2^3,$$

where  $i, j \geq 0$ . Since  $\kappa[x_0/x_3, x_1/x_3]$  is freely generated by 1 and  $x_0/x_3$  as a  $\kappa[T_0/T_2, T_1/T_2]$ -module, these elements are exactly  $u_{ij}$ , where  $i, j \geq 0$ .

For the second statement of the proposition, we use the fact that

$$\mathbf{v}_{n+1}^{(i+1)(j+1)} = \mathbf{v}_n^{ij}$$

for any  $i, j$  and  $n$ . Since  $\mathcal{F}_{p,q}^{(1)}$  is generated by all  $\mathbf{v}_n^{ij}$ , it is also generated by those  $\mathbf{v}_n^{00}, \mathbf{v}_n^{i0}$  and  $\mathbf{v}_n^{0j}$  with  $i, j > 0$  and  $n \in \mathbb{Z}$ . Note that  $\mathbf{v}_n^{00} = \mathbf{v}_n^{i0} = \mathbf{v}_n^{0j} = 0$  if  $i, j > 0$  and  $n \leq 0$  by Proposition 3. The generating set  $\{\mathbf{v}_n^{00}, \mathbf{v}_n^{i0}, \mathbf{v}_n^{0j} \mid i, j, n > 0\}$  for  $\mathcal{F}_{p,q}^{(1)}$  is exactly  $\{\mathbf{v}_0^{ij} \mid i < 0 \text{ or } j < 0\}$ . To prove that they are linearly independent, we recall (7) that

$$\left\{ \left(\frac{T_0}{T_1}\right)^j \mathbf{v}_n^{00}, \left(\frac{T_0}{T_1}\right)^j \mathbf{v}_n^{i0} \mid n > 0, j \geq 0 \right\}$$

is a basis of  $\mathcal{F}_{p,q}^{(1)}$ . For  $i, j, n > 0$ ,

$$\mathbf{v}_n^{0j} - \left(\frac{T_0}{T_1}\right)^j \mathbf{v}_n^{00} - \left(\frac{T_0}{T_1}\right)^{j-1} \mathbf{v}_n^{i0} \quad \text{and} \quad \mathbf{v}_n^{i0} - (-1)^{i-1} \left(\frac{T_0}{T_1}\right)^{i-1} \mathbf{v}_n^{10}$$

are contained in the subspace generated by those  $\mathbf{v}_m^{ij}$  with  $m < n$  and  $i, j \geq 0$ . This implies that  $\mathbf{v}_n^{00}, \mathbf{v}_n^{i0}$  and  $\mathbf{v}_n^{0j}$  are linearly independent.

For the last statement of the proposition, there are two cases. If  $p = q \leq 0$ , the elements

$$\frac{Y_0 Z_0}{T_0^{\epsilon+1}} \otimes \left[ \begin{array}{c} d\frac{T_1}{T_0} d\frac{T_2}{T_0} \\ \left(\frac{T_1}{T_0}\right)^m, \left(\frac{T_2}{T_0}\right)^n \end{array} \right] \otimes T_0^3 \quad \text{and} \quad \frac{Y_1 Z_1}{T_0^{\epsilon+1}} \otimes \left[ \begin{array}{c} d\frac{T_1}{T_0} d\frac{T_2}{T_0} \\ \left(\frac{T_1}{T_0}\right)^m, \left(\frac{T_2}{T_0}\right)^n \end{array} \right] \otimes T_0^3,$$

where  $m, n > 0$ , form a basis of  $\mathcal{F}_{p,q}^{(2)}$ . These elements are exactly  $\mathbf{w}_{m(n+\epsilon+q)}^{10}$  and  $\mathbf{w}_{m(n+\epsilon+p)}^{01}$ . If  $p \neq q$  or  $p = q > 0$ , the elements

$$\frac{Y_0^{\epsilon+p} Z_0^{\epsilon+q}}{(x_1 - x_0)^\epsilon} \otimes \left[ \begin{array}{c} d \frac{T_1}{T_0} d \frac{T_2}{T_0} \\ \left( \frac{T_1}{T_0} \right)^m, \left( \frac{T_2}{T_0} \right)^n \end{array} \right] \otimes T_0^3$$

and

$$\frac{Y_1^{\epsilon+p} Z_1^{\epsilon+q}}{(x_1 - x_0)^\epsilon} \otimes \left[ \begin{array}{c} d \frac{T_1}{T_0} d \frac{T_2}{T_0} \\ \left( \frac{T_1}{T_0} \right)^m, \left( \frac{T_2}{T_0} \right)^n \end{array} \right] \otimes T_0^3,$$

where  $m, n > 0$ , form a basis of  $\mathcal{F}_{p,q}^{(2)}$ . These elements are exactly  $\mathbf{w}_{m(n+\epsilon+q)}^{(\epsilon+q)0}$  and  $\mathbf{w}_{m(n+\epsilon+p)}^{0(\epsilon+p)}$  as seen from the computation:

$$\begin{aligned} & \frac{Y_0^{\epsilon+p} Z_0^{\epsilon+q}}{(x_1 - x_0)^\epsilon} \otimes \left[ \begin{array}{c} d \frac{T_1}{T_0} d \frac{T_2}{T_0} \\ \left( \frac{T_1}{T_0} \right)^m, \left( \frac{T_2}{T_0} \right)^n \end{array} \right] \otimes T_0^3 \\ &= \left( \frac{x_0}{x_1 - x_0} \right)^{\epsilon+q} \frac{Y_0^{\epsilon+p} Z_1^{\epsilon+q}}{(x_1 - x_0)^\epsilon} \otimes \left[ \begin{array}{c} d \frac{T_1}{T_0} d \frac{T_2}{T_0} \\ \left( \frac{T_1}{T_0} \right)^m, \left( \frac{T_2}{T_0} \right)^{n+\epsilon+q} \end{array} \right] \otimes T_0^3 = \mathbf{w}_{m(n+\epsilon+q)}^{(\epsilon+q)0} \end{aligned}$$

and

$$\begin{aligned} & \frac{Y_1^{\epsilon+p} Z_1^{\epsilon+q}}{(x_1 - x_0)^\epsilon} \otimes \left[ \begin{array}{c} d \frac{T_1}{T_0} d \frac{T_2}{T_0} \\ \left( \frac{T_1}{T_0} \right)^m, \left( \frac{T_2}{T_0} \right)^n \end{array} \right] \otimes T_0^3 \\ &= \left( \frac{x_1}{x_1 - x_0} \right)^{\epsilon+p} \frac{Y_0^{\epsilon+p} Z_1^{\epsilon+q}}{(x_1 - x_0)^\epsilon} \otimes \left[ \begin{array}{c} d \frac{T_1}{T_0} d \frac{T_2}{T_0} \\ \left( \frac{T_1}{T_0} \right)^m, \left( \frac{T_2}{T_0} \right)^{n+\epsilon+p} \end{array} \right] \otimes T_0^3 = \mathbf{w}_{m(n+\epsilon+p)}^{0(\epsilon+p)}. \quad \square \end{aligned}$$

The coboundary maps of  $\mathcal{F}_{p,q}^{(\bullet)}$  have easy descriptions.

**Proposition 13.**

$$d^{(0)}\mathbf{u}^{ij} = \mathbf{v}_{i+j}^{(p+i)(q+j)},$$

$$d^{(1)}\mathbf{v}_n^{ij} = \mathbf{w}_{(i+j-n)(n+\epsilon+p+q)}^{(i+q)(j+p)}.$$

Proof. The proposition follows from direct computations:

$$\begin{aligned} \mathbf{u}^{ij} &= \left(\frac{x_0}{x_1-x_0}\right)^i \left(\frac{x_1}{x_1-x_0}\right)^j \frac{Y_0^{\epsilon+p} Z_1^{\epsilon+q}}{(x_1-x_0)^\epsilon} \otimes \left(\frac{T_2}{T_0}\right)^{-i-j-\epsilon} d\frac{T_1}{T_0} d\frac{T_2}{T_0} \otimes T_0^3 \\ &\mapsto \left(\frac{x_0}{x_1-x_0}\right)^i \left(\frac{x_1}{x_1-x_0}\right)^j \frac{Y_0^{\epsilon+p} Z_1^{\epsilon+q}}{(x_1-x_0)^\epsilon} \otimes \left[ \begin{array}{c} d\frac{T_1}{T_0} d\frac{T_2}{T_0} \\ \left(\frac{T_2}{T_0}\right)^{i+j+\epsilon} \end{array} \right] \otimes T_0^3 \\ &= \left(\frac{x_0}{x_2}\right)^{i+\epsilon+p} \left(\frac{x_1}{x_2}\right)^{j+\epsilon+q} Y_1^p Z_0^q \otimes \left[ \begin{array}{c} d\frac{T_2}{T_1} d\frac{T_0}{T_1} \\ \left(\frac{T_2}{T_1}\right)^{i+j+\epsilon} \end{array} \right] \otimes T_1^3 \\ &= \mathbf{v}_{i+j}^{(p+i)(q+j)}, \\ \mathbf{v}_n^{ij} &= \left(\frac{x_0}{x_1-x_0}\right)^{\delta+i} \left(\frac{x_1}{x_1-x_0}\right)^{\delta+j} \frac{Y_1^{\epsilon+p} Z_0^{\epsilon+q}}{(x_1-x_0)^\epsilon} \otimes \left[ \begin{array}{c} \left(\frac{T_1}{T_0}\right)^{n-\delta-i-j-\epsilon} d\frac{T_1}{T_0} d\frac{T_2}{T_0} \\ \left(\frac{T_2}{T_0}\right)^{\delta+n} \end{array} \right] \otimes T_0^3 \\ &\mapsto \left(\frac{x_0}{x_1-x_0}\right)^{\delta+i} \left(\frac{x_1}{x_1-x_0}\right)^{\delta+j} \frac{Y_1^{\epsilon+p} Z_0^{\epsilon+q}}{(x_1-x_0)^\epsilon} \otimes \left[ \begin{array}{c} \left(\frac{T_1}{T_0}\right)^n d\frac{T_1}{T_0} d\frac{T_2}{T_0} \\ \left(\frac{T_1}{T_0}\right)^{\delta+i+j+\epsilon}, \left(\frac{T_2}{T_0}\right)^{\delta+n} \end{array} \right] \otimes T_0^3 \\ &= \left(\frac{x_0}{x_1-x_0}\right)^{\delta+i+\epsilon+q} \left(\frac{x_1}{x_1-x_0}\right)^{\delta+j+\epsilon+p} \frac{Y_0^{\epsilon+p} Z_1^{\epsilon+q}}{(x_1-x_0)^\epsilon} \\ &\quad \otimes \left[ \begin{array}{c} \left(\frac{T_1}{T_0}\right)^n d\frac{T_1}{T_0} d\frac{T_2}{T_0} \\ \left(\frac{T_1}{T_0}\right)^{\delta+i+j+\epsilon}, \left(\frac{T_2}{T_0}\right)^{\delta+n+2\epsilon+p+q} \end{array} \right] \otimes T_0^3 \\ &= \mathbf{w}_{(i+j-n)(n+\epsilon+p+q)}^{(i+q)(j+p)}. \quad \square \end{aligned}$$

### 5. Module structure of $\mathcal{F}_{p,q}^{(2)}$

We need polynomials  $f_i$  and  $g_i$  with integer coefficients which are defined inductively:

$$f_1 = g_1 = 0$$

and

$$\begin{aligned} f_{n+1} &= f_n + g_n, \\ g_{n+1} &= X + Xf_n \end{aligned}$$

for  $n \geq 1$ . Induction on  $n$ , it is easy to see that

$$(17) \quad g_n(1 + f_{n+1}) - g_{n+1}(1 + f_n) = (-X)^n.$$

If  $a$  and  $b$  are elements in a commutative ring satisfying  $b^2 = b + a$ , then

$$b^n = (1 + f_n(a))b + g_n(a).$$

$f_n$  and  $g_n$  are divisible by  $X$ . With  $f = f_n/X$  and  $g = g_n/X$ ,

$$b^n - b = a(f(a)b + g(a)).$$

This is a special case of the following lemma.

**Lemma 14.** *Let  $a$  and  $b$  be elements in a commutative ring satisfying  $b^2 = b + a$ . Then, for any  $n_0, n_1, l > 0$  and  $n_2 \geq 0$ , there exist  $f, g \in \mathbb{Z}[X]$  and  $h \in \mathbb{Z}[X, Y]$  such that*

$$b^{n_0} = (1 + af(a))b^{n_1} + ag(a)(1 - b)^{n_2} + a^l h(a, b).$$

*Proof.* We consider first the case that  $n_2 > 0$ . Choose  $h_{01}, h_{02}, h_{11}, h_{12} \in \mathbb{Z}[X]$  such that

$$(18) \quad \begin{aligned} b^{n_0} - b &= a(h_{01}(a)b + h_{02}(a)), \\ b^{n_1} - b &= a(h_{11}(a)b + h_{12}(a)). \end{aligned}$$

With  $h_0 = h_{01} - h_{11} + h_{02} - h_{12}$  and  $g = h_{02} - h_{12}$ , we have

$$b^{n_0} = b^{n_1} + ag(a)(1 - b) + ah_0(a)b.$$

Note that  $1 - b$  also satisfies the condition  $(1 - b)^2 = (1 - b) + a$ . Choose  $h_{21}, h_{22} \in \mathbb{Z}[X]$  such that

$$(19) \quad (1 - b)^{n_2} - (1 - b) = a(h_{21}(a)(1 - b) + h_{22}(a)).$$



With  $h_1 = h_0 - agh_{22}$  and  $h_2 = -ag(h_{21} + h_{22})$ , we have

$$b^{n_0} = b^{n_1} + ag(a)(1-b)^{n_2} + abh_1(a) + a(1-b)h_2(a).$$

Fix  $n_0, n_1, n_2$ . Assume that for an  $l > 1$ , there exist  $f, g, h_1, h_2 \in \mathbb{Z}[X]$  such that

$$b^{n_0} = (1 + af(a))b^{n_1} + ag(a)(1-b)^{n_2} + a^l bh_1(a) + a^l(1-b)h_2(a).$$

Choose  $h_{11}, h_{12}, h_{21}, h_{22} \in \mathbb{Z}[X]$  such that (18) and (19) hold. Then

$$\begin{aligned} b^{n_0} &= (1 + af(a) + a^l h_1(a))b^{n_1} \\ &\quad + a(g(a) + a^{l-1} h_2(a))(1-b)^{n_2} \\ &\quad + a^{l+1} b(-h_1(a)h_{11}(a) - h_1(a)h_{12}(a) - h_2(a)h_{22}(a)) \\ &\quad + a^{l+1}(1-b)(-h_2(a)h_{21}(a) - h_2(a)h_{22}(a) - h_1(a)h_{12}(a)). \end{aligned}$$

This induction process on  $l$  proves the lemma for the case  $n_2 > 0$ .

Now we consider the case that  $n_2 = 0$ . Choose  $f, g, h_{11}, h_{12} \in \mathbb{Z}[X]$  and  $h \in \mathbb{Z}[X, Y]$  such that (18) and

$$b^{n_0} = (1 + af(a))b^{n_1} + ag(a)(1-b) + a^l h(a, b)$$

hold. Denote

$$\frac{1}{1 + ah_{11}(a)} := 1 - ah_{11}(a) + (ah_{11}(a))^2 - (ah_{11}(a))^3 + \cdots + (-ah_{11}(a))^{l-1}$$

by abusing the notation. Then

$$\begin{aligned} b^{n_0} &= \left(1 + af(a) - \frac{ag(a)}{1 + ah_{11}(a)}\right) b^{n_1} \\ &\quad + ag(a) \left(1 + \frac{ah_{12}(a)}{1 + ah_{11}(a)}\right) + a^l (h(a, b) - ag(a)(-h_{11}(a))^l b). \quad \square \end{aligned}$$

For the rest of this paper, we consider elements

$$\begin{aligned} a &:= \frac{x_2 x_3}{(x_1 - x_0)^2}, \\ b &:= \frac{x_1}{x_1 - x_0} \end{aligned}$$

in the ring  $\Gamma(D_+(x_1 - x_0), \text{Proj}(S))$ , which satisfy the condition  $b^2 = b + a$ . The multi-  
plications of elements in  $\mathcal{F}_{p,q}^{(2)}$  by  $a$  and  $b$  are easy to describe:

$$a \mathbf{w}_{mn}^{ij} = \mathbf{w}_{mn}^{(i+1)(j+1)},$$

$$b\mathbf{w}_{mn}^{ij} = \mathbf{w}_{mn}^{i(j+1)}.$$

The condition  $(1 - b)^2 = (1 - b) + a$  also holds. The multiplication by  $1 - b$  gives rise to a negative sign:

$$(1 - b)^l \mathbf{w}_{mn}^{ij} = (-1)^l \mathbf{w}_{mn}^{(i+l)j}.$$

This is the reason that we include the condition “sum” in the following definition.

**DEFINITION 15.** An element  $\mathbf{w} \in \mathcal{F}_{p,q}^{(2)}$  is approximated by  $\mathbf{w}_{mn}^{\leq}$  (resp.  $\mathbf{w}_{mn}^{\geq}$ ), denoted by  $\mathbf{w} \approx \mathbf{w}_{mn}^{\leq}$  (resp.  $\mathbf{w} \approx \mathbf{w}_{mn}^{\geq}$ ), if their difference or sum  $\mathbf{w} \pm \mathbf{w}_{mn}^{\leq}$  (resp.  $\mathbf{w} \pm \mathbf{w}_{mn}^{\geq}$ ) is contained in the  $\kappa$ -vector subspace generated by the elements  $\mathbf{w}_{ij}^{\leq}$  and  $\mathbf{w}_{ij}^{\geq}$  with  $i < m$ .

**Proposition 16.** Let  $i, m > 0$  and  $n \in \mathbb{Z}$ . If  $p = q \leq 0$ ,

$$(20) \quad \mathbf{w}_{m(n+\epsilon+p)}^{0i} \approx \mathbf{w}_{mn}^{\leq},$$

$$(21) \quad \mathbf{w}_{m(n+\epsilon+q)}^{i0} \approx \mathbf{w}_{mn}^{\geq}.$$

If  $p \neq q$  or  $p = q > 0$ , the approximation (20) holds for  $\epsilon + p > 0$  and the approximation (21) holds for  $\epsilon + q > 0$ .

*Proof.* We prove only (20) and leave (21) to the reader. So we have the assumption  $\epsilon + p > 0$  if  $p \neq q$  or  $p = q > 0$ . We choose  $f, g \in \mathbb{Z}[X]$  and  $h \in \mathbb{Z}[X, Y]$  such that

$$b^i - a^m h(a, b) = \begin{cases} (1 + af(a))b + ag(a)(b - 1), & \text{if } p = q \leq 0; \\ (1 + af(a))b^{\epsilon+p} + ag(a)(b - 1)^{\epsilon+q}, & \text{otherwise.} \end{cases}$$

Then

$$\mathbf{w}_{m(n+\epsilon+p)}^{0i} - \mathbf{w}_{mn}^{\leq} = b^i \mathbf{w}_{m(n+\epsilon+p)}^{00} - \mathbf{w}_{mn}^{\leq} = af(a)\mathbf{w}_{m(n+p-q)}^{\leq} + ag(a)\mathbf{w}_{mn}^{\geq},$$

from which we get the required approximation (20). □

If  $\mathbf{w}'_{mn}, \mathbf{w}''_{mn} \in \mathcal{F}_{p,q}^{(2)}$  satisfy  $\mathbf{w}'_{mn} \approx \mathbf{w}_{mn}^{\leq}$  and  $\mathbf{w}''_{mn} \approx \mathbf{w}_{mn}^{\geq}$  for all positive  $m$  and  $n$ , then  $\{\mathbf{w}'_{mn}, \mathbf{w}''_{mn}\}_{m,n>0}$  is a basis of  $\mathcal{F}_{p,q}^{(2)}$ . More generally, if  $\mathbf{w}'_{mn} \approx \mathbf{w}_{mn}^{\leq}$  and  $\mathbf{w}''_{mn} - \mathbf{w}'''_{mn} \approx \mathbf{w}_{mn}^{\geq}$  for some  $\mathbf{w}'''_{mn}$  contained in the subspace generated by  $\mathbf{w}_{ij}^{\leq}$  with  $i < m + l$  for a fixed  $l$  independent of  $m$  and  $n$ , then  $\{\mathbf{w}'_{mn}, \mathbf{w}''_{mn}\}_{m,n>0}$  is still a basis of  $\mathcal{F}_{p,q}^{(2)}$ . This observation is useful accompanied with the following fact.

**Proposition 17.** Let  $i, m > 0$  and  $n \in \mathbb{Z}$ . Assume that  $p \neq q$  or  $p = q > 0$ .

$$\mathbf{w}_{m(n+\epsilon+p)}^{0i} \pm \mathbf{w}_{m(n+p-q)}^{\geq} \approx \mathbf{w}_{mn}^{\leq}, \quad \text{if } \epsilon + p = 0.$$

$$\mathbf{w}_{m(n+\epsilon+q)}^{i0} \pm \mathbf{w}_{m(n+q-p)}^{\leq} \approx \mathbf{w}_{mn}^{\geq}, \quad \text{if } \epsilon + q = 0.$$

Proof. We prove only the first approximation and leave the second to the reader. So we have the conditions  $\epsilon + p = 0$  and  $\epsilon + q > 0$ . We choose  $f, g \in \mathbb{Z}[X]$  and  $h \in \mathbb{Z}[X, Y]$  such that

$$\begin{aligned} b^i &= (1 + af(a))b + ag(a)(b - 1)^{\epsilon+q} + a^m h(a, b) \\ &= (1 + af(a)) + (1 + af(a))(b - 1) + ag(a)(b - 1)^{\epsilon+q} + a^m h(a, b). \end{aligned}$$

Since  $\epsilon + q > 0$ , we may also choose  $f', g' \in \mathbb{Z}[X]$  and  $h' \in \mathbb{Z}[X, Y]$  such that

$$1 - b = (1 + af'(a))(1 - b)^{\epsilon+q} + ag'(a) + a^m h'(a, b).$$

Then

$$\begin{aligned} &\mathbf{w}_{m(n+\epsilon+p)}^{0i} \\ &= (1 + af(a))\mathbf{w}_{m(n+\epsilon+p)}^{00} + (1 + af(a))\mathbf{w}_{m(n+\epsilon+p)}^{10} + ag(a)\mathbf{w}_{m(n+\epsilon+p)}^{(\epsilon+q)0} \\ &= (1 + af(a))\mathbf{w}_{m(n+\epsilon+p)}^{00} - ag'(a)(1 + af(a))\mathbf{w}_{m(n+\epsilon+p)}^{00} \\ &\quad - (-1)^{\epsilon+q}(1 + af(a))(1 + af'(a))\mathbf{w}_{m(n+\epsilon+p)}^{(\epsilon+q)0} + ag(a)\mathbf{w}_{m(n+\epsilon+p)}^{(\epsilon+q)0}. \end{aligned}$$

From the equality

$$\begin{aligned} &\mathbf{w}_{m(n+\epsilon+p)}^{0i} + (-1)^{\epsilon+q}\mathbf{w}_{m(n+p-q)}^{\geq} - \mathbf{w}_{mn}^{\leq} \\ &= a(f(a) - g'(a) - af(a)g'(a))\mathbf{w}_{mn}^{\leq} \\ &\quad - (-1)^{\epsilon+q}a(f(a) + f'(a) + af(a)f'(a))\mathbf{w}_{m(n+p-q)}^{\geq} + ag(a)\mathbf{w}_{m(n+p-q)}^{\geq}, \end{aligned}$$

we get the required approximation. □

**Corollary 18.** Let  $\mathbf{w}'_{mn}, \mathbf{w}''_{mn} \in \mathcal{F}_{p,q}^{(2)}$ . Assume that, for each  $m$  and  $n$ ,

$$\begin{aligned} \mathbf{w}'_{mn} &= \mathbf{w}_{m(n+\epsilon+p)}^{0i}, \\ \mathbf{w}''_{mn} &= \mathbf{w}_{m(n+\epsilon+q)}^{j0} \end{aligned}$$

for some positive  $i$  and  $j$ . Then  $\{\mathbf{w}'_{mn}, \mathbf{w}''_{mn}\}_{m,n>0}$  is a basis of  $\mathcal{F}_{p,q}^{(2)}$ .

Proof. If  $\epsilon + p$  and  $\epsilon + q$  are both zero, then  $p = q \leq 0$ . Proposition 16 proves the corollary. If  $\epsilon + p > 0$  or  $\epsilon + q > 0$ , the corollary follows from Proposition 16 and Proposition 17. □

In Section 6, we need also the following approximations.

**Proposition 19.** *Let  $i, m > 0$  and  $n \in \mathbb{Z}$ . Assume that  $p \neq q$  or  $p = q > 0$ . There exist  $g_1, g_2 \in \mathbb{Z}[X]$  such that*

$$\begin{aligned} \mathbf{w}_{mn}^{0(i-\epsilon-q)} \pm ag_1(a)\mathbf{w}_{(m+\epsilon+q)n}^{\geq} &\approx \mathbf{w}_{mn}^{\leq}, & \text{if } \epsilon + p = 0; \\ \mathbf{w}_{mn}^{(i-\epsilon-p)0} \pm ag_2(a)\mathbf{w}_{(m+\epsilon+p)n}^{\leq} &\approx \mathbf{w}_{mn}^{\geq}, & \text{if } \epsilon + q = 0. \end{aligned}$$

*Proof.* We prove the second approximation and leave the first to the reader. So we have the conditions  $\epsilon + p > 0$  and  $\epsilon + q = 0$ . Choose  $f_2, g_2 \in \mathbb{Z}[X]$  and  $h_2 \in \mathbb{Z}[X, Y]$  such that

$$(1 - b)^i = (1 + af_2(a))(1 - b)^{\epsilon+p} + ag_2(a) + a^{m+\epsilon+p}h_2(a, b).$$

Then

$$\begin{aligned} \mathbf{w}_{mn}^{(i-\epsilon-p)0} &= \mathbf{w}_{(m+\epsilon+p)(n+\epsilon+p)}^{i(\epsilon+p)} \\ &= (-1)^{i+\epsilon+p}(1 + af_2(a))\mathbf{w}_{mn}^{00} + (-1)^i ag_2(a)\mathbf{w}_{(m+\epsilon+p)(n+\epsilon+p)}^{0(\epsilon+p)}, \end{aligned}$$

from which we get the required approximation. □

**6. Cohomology of  $\mathcal{F}_{p,q}$**

**Proposition 20.** *Let  $p, q \geq 0$ .  $H^1(\mathbb{P}^2, \mathcal{F}_{p,q}) = H^2(\mathbb{P}^2, \mathcal{F}_{p,q}) = 0$ . The elements  $\mathbf{u}^{ij}$ , where  $0 \leq i \leq q$  and  $0 \leq j \leq p$ , form a basis of  $H^0(\mathbb{P}^2, \mathcal{F}_{p,q})$ .*

*Proof.* In this proposition,  $\epsilon = 0$ .  $d^{(0)}\mathbf{u}^{ij} = \mathbf{v}_0^{(p-j)(q-i)} = 0$  if and only if  $i \leq q$  and  $j \leq p$ . Those non-zero  $d^{(0)}\mathbf{u}^{ij}$  are linearly independent. Therefore the elements  $\mathbf{u}^{ij}$ , where  $0 \leq i \leq q$  and  $0 \leq j \leq p$ , form a basis of  $H^0(\mathbb{P}^2, \mathcal{F}_{p,q})$ .

Now we compute the images of  $\mathbf{v}_0^{ij}$ .

- For indices  $i \leq p$  and  $j \leq q$ ,  $\mathbf{v}_0^{ij}$  is the image of  $\mathbf{u}^{(q-j)(p-i)}$ . Therefore  $d^{(1)}\mathbf{v}_0^{ij} = 0$ .
- For indices  $i < 0$  and  $j > q$ ,

$$d^{(1)}\mathbf{v}_0^{ij} = \mathbf{w}_{(j-q)(p-i)}^{0(j-q+p-i)},$$

where the index  $j - q + p - i$  is positive.

- For indices  $j < 0$  and  $i > p$ ,

$$d^{(1)}\mathbf{v}_0^{ij} = \mathbf{w}_{(i-p)(q-j)}^{(i-p+q-j)0},$$

where the index  $i - p + q - j$  is also positive.

As noted in Corollary 18, except those  $\mathbf{v}_0^{ij}$  being images of  $d^{(0)}$ , images of other  $\mathbf{v}_0^{ij}$  form a basis of  $\mathcal{F}_{p,q}^{(2)}$ . This implies  $H^1(\mathbb{P}^2, \mathcal{F}_{p,q}) = H^2(\mathbb{P}^2, \mathcal{F}_{p,q}) = 0$ . □

**Proposition 21.** *Let  $q < 0 \leq p$ .  $H^0(\mathbb{P}^2, \mathcal{F}_{p,q}) = H^2(\mathbb{P}^2, \mathcal{F}_{p,q}) = 0$ . The elements  $\mathbf{v}_0^{ij}$ , where  $0 \leq i \leq p$  and  $q < j < 0$ , form a basis of  $H^1(\mathbb{P}^2, \mathcal{F}_{p,q})$ .*

Proof. In this proposition  $\epsilon = -q$ . The condition  $\epsilon + p > 0$  holds. The images of  $\mathbf{u}^{ij}$  are linearly independent. Therefore  $H^0(\mathbb{P}^2, \mathcal{F}_{p,q}) = 0$ . Other assertions of the proposition follows from the computations of the images of  $\mathbf{v}_0^{ij}$ :

- For indices  $i \leq p$  and  $j \leq q$ ,  $\mathbf{v}_0^{ij}$  is the image of  $\mathbf{u}^{(q-j)(p-i)}$ . Therefore  $d^{(1)}\mathbf{v}_0^{ij} = 0$ .
- For indices  $i < 0$  and  $j > q$ ,

$$d^{(1)}\mathbf{v}_0^{ij} = \mathbf{w}_{(j-q)(-i+p-q)}^{0(j-q+p-i)} \approx \mathbf{w}_{(j-q)(-i)}^{\leq}$$

by Proposition 16. The latter elements are exactly those  $\mathbf{w}_{mn}^{\leq}$  with positive indices  $m$  and  $n$ .

- For indices  $j < 0$  and  $i > p$ , by Proposition 19, there exists  $g_2 \in \mathbb{Z}[X]$  such that

$$d^{(1)}\mathbf{v}_0^{ij} \pm ag_2(a)\mathbf{w}_{(i-q)(-j)}^{\leq} \approx \mathbf{w}_{(i-p)(-j)}^{\geq}.$$

The latter elements are exactly those  $\mathbf{w}_{mn}^{\geq}$  with positive indices  $m$  and  $n$ .

- For indices  $0 \leq i \leq p$  and  $q < j < 0$ , we write  $m = p - i$  and  $n = j - q$ . With the polynomials  $f_i$  and  $g_i$  defined in the beginning of Section 5,

$$(22) \quad d^{(1)}\mathbf{v}_0^{ij} = \mathbf{w}_{n(m-q)}^{0(m+n)} = (1 + f_{m+n}(a))\mathbf{w}_{n(m-q)}^{01} + g_{m+n}(a)\mathbf{w}_{n(m-q)}^{00}.$$

As  $m - p = -i \leq 0$ ,

$$\mathbf{w}_{n(m-q)}^{0(p-q)} = \mathbf{w}_{n(m-p)}^{\leq} = 0.$$

Apply the relation

$$\begin{aligned} \mathbf{w}_{n(m-q)}^{01} &= \mathbf{w}_{n(m-q)}^{0(p-q)} - f_{p-q}(a)\mathbf{w}_{n(m-q)}^{01} - g_{p-q}(a)\mathbf{w}_{n(m-q)}^{00} \\ &= -f_{p-q}(a)\mathbf{w}_{n(m-q)}^{01} - g_{p-q}(a)\mathbf{w}_{n(m-q)}^{00} \end{aligned}$$

repeatedly  $l$  times to (22), we get

$$\begin{aligned} d^{(1)}\mathbf{v}_0^{ij} &= (1 + f_{m+n}(a))(-f_{p-q}(a))^l \mathbf{w}_{n(m-q)}^{01} \\ &\quad - (1 + f_{m+n}(a))(1 - f_{p-q}(a) + (f_{p-q}(a))^2 - \dots)g_{p-q}(a)\mathbf{w}_{n(m-q)}^{00} \\ &\quad + g_{m+n}(a)\mathbf{w}_{n(m-q)}^{00}. \end{aligned}$$

For  $l \geq n$ ,

$$(f_{p-q}(a))^l \mathbf{w}_{n(m-q)}^{00} = 0 = (f_{p-q}(a))^l \mathbf{w}_{n(m-q)}^{01}.$$

Without ambiguity, we may write

$$d^{(1)}\mathbf{v}_0^{ij} = (1 + f_{m+n}(a)) \left( \frac{g_{m+n}(a)}{1 + f_{m+n}(a)} - \frac{g_{p-q}(a)}{1 + f_{p-q}(a)} \right) \mathbf{w}_{n(m-q)}^{00}$$

$$= (1 + f_{m+n}(a)) \sum_{i=m+n}^{p-q-1} \left( \frac{g_i(a)}{1 + f_i(a)} - \frac{g_{i+1}(a)}{1 + f_{i+1}(a)} \right) \mathbf{w}_{n(m-q)}^{00}.$$

By (17),

$$d^{(1)}\mathbf{v}_0^{ij} = (1 + f_{m+n}(a)) \sum_{i=m+n}^{p-q-1} \frac{(-a)^i}{(1 + f_i(a))(1 + f_{i+1}(a))} \mathbf{w}_{n(m-q)}^{00}.$$

Since  $a^{m+n}\mathbf{w}_{n(m-q)}^{00} = 0$ ,  $d^{(1)}\mathbf{v}_0^{ij} = 0$  for  $0 \leq i \leq p$  and  $q < j < 0$ . □

Similarly, we have the following proposition.

**Proposition 22.** *Let  $p < 0 \leq q$ .  $H^0(\mathbb{P}^2, \mathcal{F}_{p,q}) = H^2(\mathbb{P}^2, \mathcal{F}_{p,q}) = 0$ . The elements  $\mathbf{v}_0^{ij}$ , where  $0 \leq j \leq q$  and  $p < i < 0$ , form a basis of  $H^1(\mathbb{P}^2, \mathcal{F}_{p,q})$ .*

**Proposition 23.** *Let  $q \leq p < 0$ .  $H^0(\mathbb{P}^2, \mathcal{F}_{p,q}) = H^1(\mathbb{P}^2, \mathcal{F}_{p,q}) = 0$ . The elements  $\mathbf{w}_{mn}^{\geq}$ , where  $0 < m < -p$  and  $0 < n < -q - m$ , together with the elements  $\mathbf{w}_{mn}^{\leq}$ , where  $m > 0$ ,  $n > 0$  and  $m + n \leq -p$ , form a basis of  $H^2(\mathbb{P}^2, \mathcal{F}_{p,q})$ .*

*Proof.* In this proposition,  $\epsilon = -q$ . The images of  $\mathbf{u}^{ij}$  are exactly those  $\mathbf{v}_0^{ij}$  with indices  $i \leq p$  and  $j \leq q$ . They are linearly independent. Therefore  $H^0(\mathbb{P}^2, \mathcal{F}_{p,q}) = 0$ .

Now we compute the images of  $\mathbf{v}_0^{ij}$ .

- For indices  $i \leq p$  and  $j \leq q$ ,  $\mathbf{v}_0^{ij}$  is the image of  $\mathbf{u}^{(q-j)(p-i)}$ . Therefore  $d^{(1)}\mathbf{v}_0^{ij} = 0$ .
- For indices  $i < 0$  and  $j > q$  satisfying  $p - i > q - j$ ,

$$d^{(1)}\mathbf{v}_0^{ij} = \mathbf{w}_{(j-q)(-i-q+p)}^{0(j-q+p-i)} \approx \mathbf{w}_{(j-q)(-i)}^{\leq}$$

by Proposition 16. The latter elements are exactly those  $\mathbf{w}_{mn}^{\leq}$  with positive indices  $m$  and  $n$  satisfying  $m + n + p > 0$ .

- For indices  $j < 0$  and  $i > p$  satisfying  $q - j > p - i$ ,

$$\begin{aligned} d^{(1)}\mathbf{v}_0^{ij} &= \mathbf{w}_{(i-p)(-j)}^{(i-p+q-j)0} \approx \mathbf{w}_{(i-p)(-j)}^{\geq}, \quad \text{if } p = q, \\ d^{(1)}\mathbf{v}_0^{ij} \pm \mathbf{w}_{(i-p)(-j+q-p)}^{\leq} &\approx \mathbf{w}_{(i-p)(-j)}^{\geq}, \quad \text{if } p > q \end{aligned}$$

by Proposition 16 and Proposition 17. The latter elements are exactly those  $\mathbf{w}_{mn}^{\geq}$  with positive indices  $m$  and  $n$  satisfying  $m + n + q > 0$ .

- For indices  $j < 0$  and  $i > p$  satisfying  $q - j = p - i$ ,

$$d^{(1)}\mathbf{v}_0^{ij} - \mathbf{w}_{(i-p)(-j)}^{01} = -\mathbf{w}_{(i-p)(-j)}^{10}.$$

If  $p = q$ ,

$$(23) \quad d^{(1)}\mathbf{v}_0^{ij} - \mathbf{w}_{(i-p)(-j)}^{\leq} = -\mathbf{w}_{(i-p)(-j)}^{\geq} \approx \mathbf{w}_{(i-p)(-j)}^{\geq}.$$

If  $p > q$ , by Proposition 16 and Proposition 17, there are approximations

$$\begin{aligned} \mathbf{w}_{(i-p)(-j)}^{01} &\approx \mathbf{w}_{(i-p)(-j+q-p)}^{\leq}, \\ \mathbf{w}_{(i-p)(-j)}^{10} \pm \mathbf{w}_{(i-p)(-j+q-p)}^{\leq} &\approx \mathbf{w}_{(i-p)(-j)}^{\geq}, \end{aligned}$$

that is, their differences or sums are contained in the subspace generated by the elements  $\mathbf{w}_{mn}^{\leq}$  and  $\mathbf{w}_{mn}^{\geq}$  with  $m < i - p$ . For suitable negative signs and an integer  $l$ ,

$$\begin{aligned} &d^{(1)}\mathbf{v}_0^{ij} + l\mathbf{w}_{(i-p)(-j+q-p)}^{\leq} \pm \mathbf{w}_{(i-p)(-j)}^{\geq} \\ &= (\mathbf{w}_{(i-p)(-j)}^{01} \pm \mathbf{w}_{(i-p)(-j+q-p)}^{\leq}) \\ &\quad - (\mathbf{w}_{(i-p)(-j)}^{10} \pm \mathbf{w}_{(i-p)(-j+q-p)}^{\leq} \pm \mathbf{w}_{(i-p)(-j)}^{\geq}). \end{aligned}$$

Therefore

$$(24) \quad d^{(1)}\mathbf{v}_0^{ij} + l\mathbf{w}_{(i-p)(-j+q-p)}^{\leq} \approx \mathbf{w}_{(i-p)(-j)}^{\geq}.$$

The latter elements of (23) or (24) are exactly those  $\mathbf{w}_{mn}^{\geq}$  with positive indices  $m$  and  $n$  satisfying  $m + n + q = 0$ .

- In order to have indices  $i \geq 0$  and  $j < 0$  satisfying  $p - i > q - j$ , the condition  $p > q$  has to be satisfied. With this condition, by Proposition 19, there exist  $g_2 \in \mathbb{Z}[X]$  such that

$$d^{(1)}\mathbf{v}_0^{ij} \pm ag_2(a)\mathbf{w}_{(i-q)(-j)}^{\leq} \approx \mathbf{w}_{(i-p)(-j)}^{\geq}.$$

The latter elements are exactly those  $\mathbf{w}_{mn}^{\geq}$  with indices  $m \geq -p$  and  $n > 0$  satisfying  $m + n + q < 0$ .

These computations show that the non-zero images of  $\mathbf{v}_0^{ij}$  together with the elements  $\mathbf{w}_{mn}^{\leq}$ , where  $m > 0$ ,  $n > 0$  and  $m+n \leq -p$ , and the elements  $\mathbf{w}_{mn}^{\geq}$ , where  $0 < m < -p$  and  $0 < n < -q - m$ , form a basis of  $\mathcal{F}_{p,q}^{(2)}$ . This concludes the proposition.  $\square$

Similarly, we have the following proposition.

**Proposition 24.** *Let  $p < q < 0$ .  $H^0(\mathbb{P}^2, \mathcal{F}_{p,q}) = H^1(\mathbb{P}^2, \mathcal{F}_{p,q}) = 0$ . The elements  $\mathbf{w}_{mn}^{\leq}$ , where  $0 < m < -q$  and  $0 < n < -p - m$ , together with the elements  $\mathbf{w}_{mn}^{\geq}$ , where  $m > 0$ ,  $n > 0$  and  $m + n \leq -q$ , form a basis of  $H^2(\mathbb{P}^2, \mathcal{F}_{p,q})$ .*

Counting the cardinality of the bases of  $H^r(\mathbb{P}^2, \mathcal{F}_{p,q})$  given in previous propositions, we recover the following.

**Corollary 25** ([6, Proposition 12]).

$$\dim_{\mathbb{K}} H^r(\mathbb{P}^2, \mathcal{F}_{p,q}) = (-1)^r (p + 1)(q + 1)$$

if  $r = 0$  and  $p, q \geq 0$ ; or if  $r = 1$  and  $p \geq 0, q < 0$  or  $p < 0, q \geq 0$ ; or if  $r = 2$  and  $p, q < 0$ ; and is zero otherwise.

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