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# GEVREY REGULARIZING EFFECT FOR THE INITIAL VALUE PROBLEM FOR A DISPERSIVE OPERATOR

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## Introduction

Gevrey regularizing effect or analytic regularizing effect for (nonlinear) Schrödinger equations is studied in [1], [2], [7], [10], and Gevrey regularizing effect for the equations of Schrödinger type is studied in [3], [4], [5], [6], [9]. Especially, concerning Gevrey regularizing effect for nonlinear Schrödinger equations

$$\begin{cases} i\partial_t u + \Delta u = f(u), \\ u(0,x) = u_0(x), \end{cases} \tag{1}$$

the following result is obtained: Assume that the initial data $u_0(x)$ belongs to a Gevrey class of order $s$. Then, the solution $u(t,x)$ belongs to a Gevrey class of order $\max(s/2,1)$ in $t \neq 0$ (in case $f(u)$ is a polynomial, for simplicity). A simple extension to a higher order of $\Delta$ is

$$\begin{cases} i\partial_t u + \Delta^l u = f(u), \\ u(0,x) = u_0(x). \end{cases} \tag{2}$$

In this case, we easily obtain that the solution $u(t,x)$ of (2) belongs to a Gevrey class of order $\max(s/(2l),1)$ in $t \neq 0$ if the initial data $u_0(x)$ belongs to a Gevrey class of order $s$. In the present paper, we replace $\Delta^l$ in (2) by a semi-elliptic operator $A$ and consider a dispersive equation

$$\begin{cases} Lu \equiv i\partial_t u + Au = f(u), \\ u(0,x) = u_0(x). \end{cases} \tag{3}$$

Let $\mathbf{m} = (m_1, m_2, \dots, m_n)$ be a vector of even numbers $m_k$ and set $|\alpha : \mathbf{m}| = \sum_{k=1}^n \alpha_k/m_k$ for a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$. Then, the operator $A$ is defined by

$$A = \sum_{|\alpha:\mathbf{m}|=1} a_\alpha D_x^\alpha,$$

where $a_\alpha$ are real constants and $D_x = (D_{x_1}, \dots, D_{x_n})$, $D_{x_k} = -i\partial/\partial x_k$. We assume

(A.0) $A$ is semi-elliptic,

that is, $a(\xi) = \sum_{|\alpha:\mathbf{m}|=1} a_\alpha \xi^\alpha$ is not zero for $\xi \neq 0$.

Now, we introduce some notations. For $\mathbf{m} = (m_1, m_2, \dots, m_n)$, we set $m = \max\{m_1, \dots, m_n\}$, $\rho = (\rho_1, \dots, \rho_n)$ with $\rho_k = m/m_k$ and $\rho'_k = 1/m_k$. Write $P = t\partial_t + \sum_{k=1}^n \rho'_k x_k \partial_{x_k}$. Then, we have $[L, P] = L$. Denote $\lambda(\xi) = \left(1 + \sum_{k=1}^n \xi_k^{m_k}\right)^{1/m}$, a basic weight function, and denote by $H_{\lambda,r} = \{u \in \mathcal{S}\,;\, \lambda(\xi)^r \hat{u}(\xi) \in L_2\}$ a Sobolev space with respect to $\lambda(\xi)$, whose norm $\|u\|_{\lambda,r}$ is $\|\lambda(\xi)^r \hat{u}(\xi)\|_{L_2}$. Let $S^\kappa_\lambda$ be a class of symbols $q(x,\xi)$ satisfying

$$|\partial_\xi^\alpha \partial_x^\beta q(x,\xi)| \leq C_{\alpha,\beta} \lambda(\xi)^{\kappa - |\alpha|}$$

and we also use same notations $S^\kappa_\lambda$ to a class of pseudo-differential operators $Q = q(X, D_x)$ with a symbol $q(x,\xi)$ in $S^\kappa_\lambda$. Then, $Q \in S^\kappa_\lambda$ maps $H_{\lambda,r}$ to $H_{\lambda,r-\kappa}$ (see [8]).

Now, we state our main results. First, we consider the case the non-linear term $f(u)$ is a polynomial of $u$ and $\bar{u}$. In this case we take $n_0$ such that $n_0$ satisfies $n_0 > \mu_0/2$ for $\mu_0 = \sum_{k=1}^n \rho_k \equiv \sum_{k=1}^n m/m_k$. In the following, we denote $P_0 = \sum_{k=1}^n \rho'_k x_k \partial_{x_k}$.

**Theorem 1.** *Assume* (A.0) *and that $f(u)$ is a polynomial of $u$ and $\bar{u}$ with $f(0) = 0$. Then, for the initial data $u_0(x)$ satisfying*

$$\|P_0^l u_0\|_{\lambda,n_0} \leq C M_1^l l!^s, \tag{4}$$

*there exists a positive constant $T$ such that the equation* (3) *has a unique solution $u(t,x)$ in $C^0([0,T]; H_{\lambda,n_0}) \cap C^1([0,T]; H_{\lambda,n_0-m})$ and it satisfies $P^l u(t,x) \in C^0([0,T]; H_{\lambda,n_0})$ with an estimate*

$$\sup_t \|P^l u(t,x)\|_{\lambda,n_0} \leq C M^l l!^s \tag{5}$$

*for any $l$.*

For the regularizing effect with respect to the space variables we have

**Theorem 2.** *Let $\sigma = \max(s/m, 1)$ and assume that the assumption in* Theorem 1 *are valid. Then, under the condition* (4) *the solution $u(t,x)$ of* (3) *satisfies the following property*: *For any $C^\infty$-function $\varphi(x)$ there exist constants $C = C_\varphi$ and $M = M_\varphi$ such that*

$$\|\varphi(x)\partial_x^\alpha u(t,x)\|_{\lambda,n_0} \leq C M^{|\alpha|} t^{-\nu} \alpha!^{\rho\sigma} \tag{6}$$

*holds with $\nu = [\rho \cdot \alpha]_*$.*

Here, $\alpha!^{\rho\sigma} = \alpha_1!^{\rho_1\sigma} \alpha_2!^{\rho_2\sigma} \cdots \alpha_n!^{\rho_n\sigma}$, $\rho \cdot \alpha = \rho_1\alpha_1 + \cdots + \rho_n\alpha_n$ and $[r]_*$ is the smallest integer $\nu$ such that $\nu \geq r$.

REMARK 1. When $s \le m$ and $m_k = m$, then $u(t,x)$ is locally analytic with respect to $x_k$.

In case that the nonlinear term $f(u)$ is a general function of $u$ and $\bar{u}$, we introduce the following assumption (depending on $s$).

$(\mathrm{A}.1)_s$ For every positive number $K$, there exist constants $C = C_K$ and $M_2 = M_{2,K}$ such that

$$|\partial_u^k \partial_{\bar{u}}^{k'} f(u)| \le C M_2^{k+k'} k!^s k'!^s \qquad \text{for } |u| \le K$$

holds, where $\partial_{\bar{u}}$ is the differentiation with respect to the complex conjugate of $u$.

In this case we assume that the constant $n_0$ satisfies $n_0 > \mu_0/2$ and $n_0 m_k/m$ are integers for any $k$.

**Theorem 3.** *Assume* (A.0) *and* $(\mathrm{A}.1)_s$ *with* $f(0) = 0$*. Then, for the initial data* $u_0(x)$ *satisfying* (4)*, there exists a positive constant* $T$ *such that the equation* (3) *has a unique solution* $u(t,x)$ *in* $C^0([0,T]; H_{\lambda,n_0}) \cap C^1([0,T]; H_{\lambda,n_0-m})$*, and it satisfies* $P^l u(t,x) \in C^0([0,T]; H_{\lambda,n_0})$ *with the estimate* (5) *for any* $l$.

**Theorem 4.** *Let* $\sigma$ *satisfy* $\max(s/m, 1) \le \sigma \le s$ *and assume that* (A.0) *and* $(\mathrm{A}.1)_\sigma$ *with* $f(0) = 0$*. Then, under the condition* (4) *the solution* $u(t,x)$ *of* (3) *satisfies the following property*: *for any* $C^\infty$*-function* $\varphi(x)$ *there exist constants* $C = C_\varphi$ *and* $M = M_\varphi$ *such that* (6) *holds with* $\nu = [\rho \cdot \alpha]_*$.

REMARK 2. In $m = 2$, the problem (3) coincides with the problem (1) and in this case Theorems 3 and 4 are already proved in [7]. So, in the following, we always assume $m \ge 4$.

REMARK 3. In Theorem 3 and Theorem 4 we assume $n_0 m_k/m$ are integers for any $k$. We conjecture that this assumption can be removed if we treat the idea of Bessov spaces.

The outline of the present paper is as follows. In Section 1 we give preliminaries. In Section 2 we show the existence of the solution of (3) by using the idea in [7]. Finally, in Section 3, we show the regularizing effect for the solution of (3). Since the operator $A$ is semi-elliptic, we use not homogeneous factorials with respect to the space variables. So, we emphasize that we have to pay attention to very carefully the power of constants corresponding to radius of convergence, the power of factorial and so on, in order to proceed the proof of Theorem 2 and Theorem 4.

## 1. Preliminary

As in Introduction, we always denote $\mu_0 = \sum_{j=1}^n \rho_j \equiv \sum_{j=1}^n m/m_j$.

**Lemma 1.1.** *Let $r$ satisfy $r > \mu_0/2$. Then, we have $\mathcal{B}^0 \subset H_{\lambda,r}$.*

Proof. Using polar coordinates, we know that $\lambda(\xi)^{-2r}$ is integrable in $\mathbb{R}^n$. Hence, we obtain for $u \in \mathcal{S}$

$$\begin{aligned}|u(x)| &= \left|\int \lambda(\xi)^{-r}\{\lambda(\xi)^r\hat{u}(\xi)\}\,d\!\!\!/\xi\right| \le \left\{\int \lambda(\xi)^{-2r}\,d\!\!\!/\xi\right\}^{1/2}\|u\|_{\lambda,r}\\ &\le C_1\|u\|_{\lambda,r}.\end{aligned}\tag{1.1}$$

This implies $\mathcal{B}^0 \subset H_{\lambda,r}$. □

In (1.1) and in what follows, we use $d\!\!\!/\xi = (2\pi)^{-n}\,d\xi$.

**Proposition 1.2.** *Let $n_0$ satisfy $n_0 > \mu_0/2$. Then, we have for $u$, $v \in H_{\lambda,n_0}$*

$$\|uv\|_{\lambda,n_0} \le C_2\|u\|_{\lambda,n_0}\|v\|_{\lambda,n_0}.\tag{1.2}$$

Proof. We may only prove (1.2) for $u$, $v \in \mathcal{S}$. Denote $|\eta|_{\mathbf{m}} = \sum_{j=1}^n |\eta_j|^{m_j/m}$. Then, we can find a constant $c_0$ such that

$$|\lambda(\xi-\eta)-\lambda(\xi)| \le \frac{1}{2}\lambda(\xi) \qquad \text{for} \qquad |\eta|_{\mathbf{m}} \le c_0\lambda(\xi).$$

This implies

$$\begin{aligned}&\int \frac{1}{\lambda(\xi-\eta)^{2n_0}\lambda(\eta)^{2n_0}}\,d\!\!\!/\eta\\ &\quad= \int_{|\eta|_{\mathbf{m}}\le c_0\lambda(\xi)} \frac{1}{\lambda(\xi-\eta)^{2n_0}\lambda(\eta)^{2n_0}}\,d\!\!\!/\eta + \int_{|\eta|_{\mathbf{m}}\ge c_0\lambda(\xi)} \frac{1}{\lambda(\xi-\eta)^{2n_0}\lambda(\eta)^{2n_0}}\,d\!\!\!/\eta\\ &\quad\le C_2'\lambda(\xi)^{-2n_0}\int_{|\eta|_{\mathbf{m}}\le c_0\lambda(\xi)} \frac{1}{\lambda(\eta)^{2n_0}}\,d\!\!\!/\eta + C_2'\lambda(\xi)^{-2n_0}\int_{|\eta|_{\mathbf{m}}\ge c_0\lambda(\xi)} \frac{1}{\lambda(\xi-\eta)^{2n_0}}\,d\!\!\!/\eta\\ &\quad\le C_2''\lambda(\xi)^{-2n_0}.\end{aligned}$$

Now, set $\hat{u}_{n_0}(\xi) = \lambda(\xi)^{n_0}\hat{u}(\xi)$ and $\hat{v}_{n_0}(\xi) = \lambda(\xi)^{n_0}\hat{v}(\xi)$ for $u$, $v \in \mathcal{S}$. Then, we have

$$\begin{aligned}|\widehat{uv}(\xi)|^2 &= \left|\int \lambda(\xi-\eta)^{-n_0}\lambda(\eta)^{-n_0}\hat{u}_{n_0}(\xi-\eta)\hat{v}_{n_0}(\eta)\,d\!\!\!/\eta\right|^2\\ &\le \int \lambda(\xi-\eta)^{-2n_0}\lambda(\eta)^{-2n_0}\,d\!\!\!/\eta \int |\hat{u}_{n_0}(\xi-\eta)|^2|\hat{v}_{n_0}(\eta)|^2\,d\!\!\!/\eta\\ &\le C_2''\lambda(\xi)^{-2n_0}\int |\hat{u}_{n_0}(\xi-\eta)|^2|\hat{v}_{n_0}(\eta)|^2\,d\!\!\!/\eta.\end{aligned}$$

Multiplying both sides by $\lambda(\xi)^{2n_0}$ and integrating with respect to $\xi$, we get (1.2). $\square$

**Lemma 1.3.** *Let $n_0$ satisfy $n_0 > \mu_0/2$ and $n_{0,k} \equiv n_0 m_k/m$ be integers for any $k$. Then, if $u \in L^2$ satisfies $D_{x_k}^{n_{0,k}} u \in L^2$ for $k = 1, \dots, n$, we have $u \in H_{\lambda,n_0}$ and*

$$\|u\|_{\lambda,n_0} \leq C_3 \Big\{ \|u\| + \sum_{k=1}^{n} \|D_{x_k}^{n_{0,k}} u\| \Big\}. \tag{1.3}$$

We note that, for $u \in H_{\lambda,n_0}$, we have $D_x^\alpha u \in L_2$ with $\alpha$ satisfying $\rho \cdot \alpha \leq n_0$, since $D_x^\alpha \in S_\lambda^{\rho\cdot\alpha}$. Hence, we get

**Corollary 1.4.** *Suppose that $n_0$ satisfies $n_0 > \mu_0/2$ and that $n_{0,k} \equiv n_0 m_k/m$ be integers for any $k$. Then, the norm $\|u\|_{\lambda,n_0}$ is equivalent to $\|u\| + \sum_{k=1}^{n} \|D_{x_k}^{n_{0,k}} u\|$.*

Proof of Lemma. Using Fourier transformation, we have

$$\begin{aligned}
\|u\|_{\lambda,n_0}^2 = \|\lambda(\xi)^{n_0} \hat{u}(\xi)\|^2 &= \int \Big(1 + \sum_{k=1}^{n} \xi_k^{m_k}\Big)^{2n_0/m} |\hat{u}(\xi)|^2 \, d\!\!\bar{}\,\xi \\
&\leq C_3' \int \Big(1 + \sum_{k=1}^{n} \xi_k^{2n_{0,k}}\Big) |\hat{u}(\xi)|^2 \, d\!\!\bar{}\,\xi = C_3' \Big\{ \|u\|^2 + \sum_{k=1}^{n} \|D_{x_k}^{n_{0,k}}\|^2 \Big\}.
\end{aligned}$$

Hence, we get (1.3). $\square$

**Lemma 1.5.** *Let $r$ satisfy $r > \mu_0$ and $q_1, q_2, \dots, q_{\nu+1}$ be nonnegative constants such that $q_1 + \cdots + q_{\nu+1} \leq r$. Then, we have*

$$\iint \frac{1}{\lambda(\xi - \eta^1)^{r-q_1}} \prod_{j=2}^{\nu+1} \frac{1}{\lambda(\eta^{j-1} - \eta^j)^{r-q_j}} \, d\!\!\bar{}\,\tilde{\eta}^\nu \leq C_4 \qquad (\eta^0 = \xi, \ \eta^{\nu+1} = 0), \tag{1.4}$$

*where $\tilde{\eta}^\nu = (\eta^1, \dots, \eta^\nu)$ and $d\!\!\bar{}\,\tilde{\eta}^\nu = d\!\!\bar{}\,\eta^1 \cdots d\!\!\bar{}\,\eta^\nu$.*

Proof. For $\nu = 2$ we have

$$\begin{aligned}
&\int \frac{1}{\lambda(\xi - \eta)^{r-q_1} \lambda(\eta)^{r-q_2}} \, d\!\!\bar{}\,\eta \\
&\quad \leq C_4' \int_{|\eta|_{\mathbf{m}} \leq \lambda(\xi-\eta)} \frac{1}{\lambda(\eta)^{2r-q_1-q_2}} \, d\!\!\bar{}\,\eta + C_4' \int_{|\eta|_{\mathbf{m}} \geq \lambda(\xi-\eta)} \frac{1}{\lambda(\xi - \eta)^{2r-q_1-q_2}} \, d\!\!\bar{}\,\eta \\
&\quad \leq C_4''.
\end{aligned}$$

Hence, we get (1.4) for $\nu = 2$. For $\nu \geq 3$ we get (1.4) by induction on $\nu$. $\square$

**Proposition 1.6.** *Let $n_0$ satisfy $n_0 > \mu_0/2$ and $n_0 m_k/m$ be an integer for each $k$. Suppose that $g(u) \in C^\infty(\mathbb{C};\mathbb{C})$ satisfies $|\partial_u^\nu \partial_{\bar u}^{\nu'} g(u)| \le M_K$ for $\nu + \nu' \le n_0$ and $|u| \le K$. Then, for $u$, $v \in H_{\lambda,n_0}$, we have $g(u)v \in H_{\lambda,n_0}$ and*

$$\|g(u)v\|_{\lambda,n_0} \le C_5 M_K G(\|u\|_{\lambda,n_0})\|v\|_{\lambda,n_0}, \tag{1.5}$$

*where $G(\cdot)$ is a polynomial of order $n_0$ and $C_5$ is a constant depending only on $n$ and $n_0$.*

Proof. From the differentiation of composite function we have

$$\begin{aligned} D_{x_k}^{n_{0,k}}\{g(u)v\} &= g(u)D_{x_k}^{n_{0,k}} v \\ &+ \sum_{l=1}^{n_{0,k}} \sum_{\nu=1}^{l} \sum_{\nu'+\nu''=\nu} \frac{n_{0,k}!}{(n_{0,k}-l)!\nu'!\nu''!} \partial_u^{\nu'} \partial_{\bar u}^{\nu''} g(u) \\ &\qquad \times \sum_{\substack{l_1+\cdots+l_\nu = l \\ l_j \ge 1}} \prod_{j=1}^{\nu'} \frac{1}{l_j!} D_{x_k}^{l_j} u \prod_{j=\nu'+1}^{\nu} \frac{1}{l_j!} D_{x_k}^{l_j} \bar u \cdot D_{x_k}^{n_{0,k}-l} v. \end{aligned}$$

Take a constant $K$ such that $|u(x)| \le K$, which is assured by Lemma 1.1. Then, we have

$$\begin{aligned} \|D_{x_k}^{n_{0,k}}\{g(u)v\}\| &\le C_5' M_K \|v\|_{\lambda,n_0} \\ &+ C_5' M_K \sum_{l=1}^{n_{0,k}} \sum_{\nu=1}^{l} \sum_{\substack{l_1+\cdots+l_\nu = l \\ l_j \ge 1}} \Big\| \prod_{j=1}^{\nu} D_{x_k}^{l_j} u \cdot D_{x_k}^{n_{0,k}-l} v \Big\|. \end{aligned} \tag{1.6}$$

In order to estimate $\|\prod_{j=1}^{\nu} D_{x_k}^{l_j} u \cdot D_{x_k}^{n_{0,k}-l} v\|$ we set for $l_1, \ldots, l_\nu$ satisfying $l_1 + \cdots + l_\nu = n_{0,k} - l$

$$\Lambda(\xi, \tilde\eta^\nu) = \prod_{j=1}^{\nu} \lambda(\eta^{j-1} - \eta^j)^{n_0 - \rho_k l_j} \cdot \lambda(\eta^\nu)^{n_0 - \rho_k (n_{0,k} - l)},$$

with $\tilde\eta^\nu = (\eta^1, \ldots, \eta^\nu)$, $\eta^0 = \xi$ and $\rho_k = m/n_{0,k}$. Then, we have

$$\begin{aligned} &\Big| \mathcal{F}\Big[ \prod_{j=1}^{\nu} D_{x_k}^{l_j} u \cdot D_{x_k}^{n_{0,k}-l} v \Big](\xi) \Big| \\ &= \Big| \iint \prod_{j=1}^{\nu} \Big\{ (\eta_k^{j-1} - \eta_k^j)^{l_j} \hat u(\eta^{j-1} - \eta^j) \Big\} \Big\{ (\eta_k^\nu)^{n_{0,k}-l} \hat v(\eta^\nu) \Big\} \,d\!\!\!{}^-\tilde\eta^\nu \Big| \\ &\le \Big| \iint \Lambda(\xi, \tilde\eta^\nu)^{-1} \prod_{j=1}^{\nu} \hat u_{n_0}(\eta^{j-1} - \eta^j) \cdot \hat v_{n_0}(\eta^\nu) \,d\!\!\!{}^-\tilde\eta^\nu \Big| \end{aligned} \tag{1.7}$$

$$\leq \left\{ \iint \Lambda(\xi, \tilde{\eta}^{\nu})^{-2}\, d\!\!\!/\tilde{\eta}^{\nu} \right\}^{1/2}$$

$$\times \left\{ \iint \prod_{j=1}^{\nu} |\hat{u}_{n_0}(\eta^{j-1} - \eta^{j})|^2 \cdot |\hat{v}_{n_0}(\eta^{\nu})|^2\, d\!\!\!/\tilde{\eta}^{\nu} \right\}^{1/2}.$$

Since an inequality

$$\int \Lambda(\xi, \tilde{\eta}^{\nu})^{-2}\, d\!\!\!/\tilde{\eta}^{\nu} \leq C_4$$

holds from Lemma 1.5, we get

$$\left\| \prod_{j=1}^{\nu} D_{x_k}^{l_j} u \cdot D_{x_k}^{n_{0,k}-l} v \right\|^2 \leq C_4 \|u\|_{\lambda,n_0}^{2\nu} \|v\|_{\lambda,n_0}^2$$

by squaring both sides of (1.7) and integrating with respect to $\xi$. This yields

$$\|D_{x_k}^{n_{0,k}} \{g(u)v\}\| \leq C_6 M_K G(\|u\|_{\lambda,n_0}) \|v\|_{\lambda,n_0}$$

with (1.6). Now, we use Corollary 1.4. Then, we have

$$\begin{aligned} \|g(u)v\|_{\lambda,n_0} &\leq C_7 \left\{ \|g(u)v\| + \sum_{k=1}^{n} \|D_{x_k}^{n_{0,k}} \{g(u)v\}\| \right\} \\ &\leq (n+1) C_7 C_6 M_K G(\|u\|_{\lambda,n_0}) \|v\|_{\lambda,n_0}. \end{aligned}$$

This proves Lemma 1.6. □

Finally, we quote two lemmas from [7].

**Lemma 1.7.** *There exists a constant $C_8$ without depending on $l$ such that*

$$\sum_{l'+l''=l} \frac{1}{(l'+1)^2 (l''+1)^2} \leq C_8 \frac{1}{(l+1)^2}. \tag{1.8}$$

**Lemma 1.8.** *For a multi-index $\beta$ and an integer $l$ we assume that the integers $\mu_j$ $(\geq 1)$ $(j = 1, \ldots, k)$ satisfy $\mu_1 + \cdots + \mu_k = |\beta| + l$. Then, we have*

$$\beta! l! \sum_{\substack{\beta^1 + \cdots + \beta^k = \beta \\ l_1 + \cdots + l_k = l \\ |\beta^j| + l_j = \mu_j}} \prod_{j=1}^{k} \frac{(|\beta^j| + l_j)!}{\beta^j! l_j!} = (|\beta| + l)!.$$

## 2. Existence of the solution

In this section we prove Theorem 3 by using Proposition 1.2 and Proposition 1.6. Then, we can prove Theorem 1 by the same method only by using Proposition 1.2. So, throughout this section we always assume that $n_0$ satisfies $n_0 > \mu_0/2$ and that $n_0 m_k/m$ are integers for any $k$. Since we can prove Theorem 3 by the same method of proving Theorem 1.1 in [7], we only give outline of the proof.

First, for a vector field $Q$ with analytic coefficients in $\mathbb{R}^n$ or $[0,T]\times\mathbb{R}^n$ we define a function space $G^s_R(Q; H_{\lambda,n_0})$ by

$$G^s_R(Q; H_{\lambda,n_0}) = \{u \in H_{\lambda,n_0}\,;\ \|u\|_{G^s_R(Q;H_{\lambda,n_0})} < \infty\}$$

or

$$G^s_R(Q; H_{\lambda,n_0}) = \{u = u(t) \in C^\infty([0,T]\,;\ H_{\lambda,n_0})\,;\\ \|u(t)\|_{G^s_R(Q;H_{\lambda,n_0})} < \infty \text{ for any } t\}$$

with

$$\|u\|_{G^s_R(Q;H_{\lambda,n_0})} = \|u\|_{\lambda,n_0} + \sum_{l=1}^{\infty} \frac{R^{l-1}\|Q^l u\|_{\lambda,n_0}}{l!(l-1)^{s-1}}.$$

Especially, for $P = t\partial_t + \sum_{k=1}^n \rho'_k x_k \partial_{x_k}$ we denote $G^s_R(P; H_{\lambda,n_0})$ by $\mathcal{G}^s(R)$, $\|u\|_{G^s_R(P;H_{\lambda,n_0})}$ by $\|u\|_{\mathcal{G}^s(R)}$ and set

$$|||u|||_{\mathcal{G}^s(R)} = \sup_{t\in[0,T]} \|u\|_{\mathcal{G}^s(R)}.$$

Next, we write $\|u\|'_{\mathcal{G}^s(R)} = \|u\|_{\mathcal{G}^s(R)} - \|u\|_{\lambda,n_0}$. Then, as in Lemma 2.3 in [7], we have

**Lemma 2.1.** *Let $f(u)$ be a $C^\infty$-function satisfying* (A.1)$_s$ *and suppose that $u \equiv u(t,x)$ belongs to $G^s_R(P; H_{\lambda,n_0})$ for a constant $R$. Then, there exists a constant $R_1$ depending on $R$ such that an inequality*

$$\|f(u)\|'_{\mathcal{G}^s(R_1)} \le \frac{C_9\|u\|'_{\mathcal{G}^s(R_1)}}{1 - C_{10}R_1\|u\|'_{\mathcal{G}^s(R_1)}}$$

*holds with constants $C_9$ and $C_{10}$ depending only $f$, $\|u\|_{\lambda,n_0}$, $n_0$ and $n$.*

Now, we give the outline of the proof of Theorem 3.

Proof of Theorem 3. Consider a linearized equation with respect to (3):

$$\begin{cases} Lu \equiv i\partial_t u + Au = f(v), \\ \quad u(0,x) = u_0(x). \end{cases} \tag{2.1}$$

Then, the solution of (2.1) is written as

$$u = e^{itA}u_0 + i^{-1}\int_0^t e^{i(t-s)A} f(v(s))\,ds$$

with the evolution operator $e^{itA}$ for $i\partial_t + A$. Since $A$ is defined by $A = \sum_{|\alpha:\mathbf{m}|=1} a_\alpha D_x^\alpha$ with real coeffients $a_\alpha$, an operator $e^{itA}$ maps $H_{\lambda,r}$ to itself for any $r$ with the estimate $\|e^{itA}u\|_{\lambda,r} = \|u\|_{\lambda,r}$. This implies

$$\|u\|_{\lambda,n_0} \le \|u_0\|_{\lambda,n_0} + T\|f(v)\|_{\lambda,n_0}. \tag{2.2}$$

Now, we use $[L, P] = L$. Then, from (2.1) we have with $P_0 = \sum_{k=1}^n \rho_k' x_k \partial_{x_k}$

$$\begin{cases} LP^l u = (P+1)^l f(v), \\ (P^l u)|_{t=0} = P_0^l u_0, \end{cases}$$

which yields

$$\|P^l u\|_{\lambda,n_0} \le \|P_0^l u_0\|_{\lambda,n_0} + T\|(P+1)^l\{f(v)\}\|_{\lambda,n_0}.$$

Combine this with (2.2) and Lemma 2.1. Then, for $u_0 \in G_R^s(P_0; H_{\lambda,n_0})$, there exists a constant $R_1$ $(< R)$ such that

$$\begin{aligned} \|u\|_{\mathcal{G}^s(R_1)} &\le \|u_0\|_{G^s_{R_1}(P_0;H_{\lambda,n_0})} + T\|f(v)\|_{G^s_{R_1}(P+1;H_{\lambda,n_0})} \\ &\le \|u_0\|_{G^s_R(P_0;H_{\lambda,n_0})} + e^{R_1}T\|f(v)\|_{\mathcal{G}^s(R_1)} \\ &\le \|u_0\|_{G^s_R(P_0;H_{\lambda,n_0})} + e^{R_1}T\{\|f(v)\|_{\lambda,n_0} + \|f(v)\|'_{\mathcal{G}^s(R_1)}\} \\ &\le \|u_0\|_{G^s_R(P_0;H_{\lambda,n_0})} + e^{R_1}T\left\{C_{11}\|v\|_{\lambda,n_0} + \frac{C_9\|v\|'_{\mathcal{G}^s(R_1)}}{1 - C_{10}R_1\|v\|'_{\mathcal{G}^s(R_1)}}\right\}. \end{aligned} \tag{2.3}$$

Set $K = 2\|u_0\|_{G^s_R(P_0;H_{\lambda,n_0})}$ and write $\mathcal{G}^s(R, K) = \{u \in C^\infty([0, T]\,; \mathcal{G}^s(R))\,; |||u|||_{\mathcal{G}^s(R)} \le K\}$. Then, denoting by $S$ the mapping which corresponds $v$ to the solution $u$ of (2.1), the inequality (2.3) shows that $S$ maps $\mathcal{G}^s(R_1, K)$ into $\mathcal{G}^s(R_1, K)$, if we take $T$ small enough. Moreover, using Lemma 2.1 again, we can prove that

$$|||Su - Sv|||_{\mathcal{G}^s(R_1)} \le C_{12}T|||u - v|||_{\mathcal{G}^s(R_1)} \qquad \text{for } u, v \in \mathcal{G}^s(R, K)$$

with a constant $C_{12}$ independent of $T$, $u$ and $v$. Hence, if we retake a constant $T$ such that $C_{12}T < 1$, $S$ is a contraction mapping from $\mathcal{G}^s(R_1, K)$ to $\mathcal{G}^s(R_1, K)$, and we get a solution of (3). This proves Theorem 3. □

## 3. Local Gevrey regularizing property

As in Section 2 we only prove Theorem 4. Then, we can prove Theorem 2 by the same method. So, throughout this section we always assume that $n_0$ satisfies $n_0 >$

$\mu_0/2$ and that $n_0 m_j/m$ are integers for any $j$. Now, let $u(t,x)$ be a solution constructed in Theorem 3. Then, since $[L,P]=L$, we have

$$LP^l u = (P+1)^l\{f(u)\},$$

which implies

$$\begin{aligned} AP^l u &= -i\partial_t P^l u + (P+1)^l\{f(u)\} \\ &= -i\frac{1}{t}P^{l+1}u + i\frac{1}{t}\sum_{k=1}^{n}\rho'_k x_k \partial_{x_k} P^l u + (P+1)^l\{f(u)\}, \end{aligned} \tag{3.1}$$

since $\partial_t = (1/t)P - (1/t)\sum_{k=1}^{n}\rho'_k x_k \partial_{x_k}$.

**Proposition 3.1.** *Let $u(t,x)$ be a solution of* (3). *Then, for an integer $N_0$ and a $C_0^\infty$-function $\varphi(x)$, there exist constants $C_{13}$ and $C_{14}$ such that*

$$\|\varphi(x)P^l u\|_{\lambda,n_0+1} \le C_{13} M^l t^{-1}(l+1)!\, l!^{s-1}\cdot\frac{1}{(l+1)^2}, \tag{3.2}$$

$$\|\varphi(x)P^l u\|_{\lambda,n_0+\nu} \le C_{14} M^{l+\nu-2} t^{1-\nu}(l+\nu-1)!\,(l+\nu-2)!^{s-1}\cdot\frac{1}{(l+1)^2} \tag{3.3}$$

*hold for $2\le\nu\le N_0 m$ in* (3.3) *and for all integer $l$ in* (3.2) *and* (3.3).

For the proof we prepare two lemmas.

**Lemma 3.2.** *Assume that* (3.2) *and* (3.3) *hold for $2\le\nu\le Nm$. Then, we have for a $C_0^\infty$-function $\chi(x)$ with $\chi\Subset\varphi$*

$$\begin{aligned} \|\chi(x)P^l\{f(u)\}\|_{\lambda,n_0+Nm} \le{}& C_{15} M^{l+Nm-1} t^{-Nm} \\ &\times (l+Nm)!\,(l+Nm-1)!^{s-1}\cdot\frac{1}{(l+1)^2} \end{aligned} \tag{3.4}$$

*with a constant $C_{15}$ depending on $\varphi(x)$, $\chi(x)$ and $N$ and independent on $l$.*

Here, for two $C_0^\infty$-functions $\varphi(x)$ and $\chi(x)$, $\chi\Subset\varphi$ means that $\varphi(x)=1$ on $\operatorname{supp}\chi$.

**Lemma 3.3.** *Assume that the following hold for $C_0^\infty$-functions $\psi(x)$ and $\varphi(x)$ with $\varphi\Subset\psi$*

$$\|\psi(x)P^l u\|_{\lambda,\mu} \le C_{16} M^{l+m'-1} t^{-j_1}(l+m')!\,(l+m'-1)!^{s-1}\cdot\frac{1}{(l+1)^2}, \tag{3.5}$$

$$\|\varphi(x)P^l\{f(u)\}\|_{\lambda,\mu} \le C_{17} M^{l+m''-1} t^{-j_2}(l+m'')!\,(l+m''-1)!^{s-1}\cdot\frac{1}{(l+1)^2}. \tag{3.6}$$

*Then, we have*

$$\|\varphi(x)^{\nu} P^l u\|_{\lambda,\mu+\nu} \leq C_{18,\nu} M^{l+\tilde{m}-1} t^{-j'} (l+\tilde{m})!\,(l+\tilde{m}-1)!^{s-1} \cdot \frac{1}{(l+1)^2} \quad \text{for} \quad \nu \leq m, \tag{3.7}$$

*with* $j' = (j_1+1) \vee j_2$ (*for* $\nu \leq m/2$), $j' = \{(j_1+1) \vee j_2\} + 1$ (*for* $\nu > m/2$) *and* $\tilde{m} = (m'+1) \vee m''$, *where* $j_1 \vee j_2 = \max(j_1, j_2)$.

In order to fix the constant $M$ in (3.2)–(3.7), we prepare

$$\|P^l u\|_{\lambda,n_0} \leq C_{19} M^{l-1} l!\,(l-1)!^{s-1} \cdot \frac{1}{(l+1)^2}, \tag{3.8}$$

$$\|P^l\{f(u)\}\|_{\lambda,n_0} \leq C_{20} M^{l-1} l!\,(l-1)!^{s-1} \cdot \frac{1}{(l+1)^2}, \tag{3.9}$$

$$|\partial_u^{k'} \partial_{\bar{u}}^{k''} f(u)| \leq C_{21} M_1^{k'+k''} k'!\,k''!\,(k'+k''-1)!^{s-1}, \tag{3.10}$$

which is assured by the assumption and the result in the previous section. Then, the constant $M$ in (3.2)–(3.7) is always taken such that (3.8), (3.9) and

$$M_1 C_2 C_8 C_{19} < M \tag{3.11}$$

hold with the constants $C_2$ and $C_8$ in (1.2) and (1.8). Now, we start to prove Proposition 3.1 and Lemmas 3.2–3.3. First, admitting Lemmas 3.2 and 3.3, we prove Proposition 3.1.

Proof of Proposition 3.1. From (3.8) and (3.9), we have (3.5) and (3.6) with $j_1 = j_2 = 0$, $m' = m'' = 0$ and $\mu = n_0$. In this case we can take $\psi(x) = 1$ instead of the function with compact support. Hence, applying Lemma 3.3, we get for any $C_0^\infty$-function $\varphi_1(x)$ and for any $\nu \leq m$

$$\|\varphi_1(x)^{\nu} P^l u\|_{\lambda,n_0+\nu} \leq C_{22,\nu} M^l t^{-j'} (l+1)!\,l!^{s-1} \cdot \frac{1}{(l+1)^2}$$

with $j' = 1$ ($\nu \leq m/2$) and $j' = 2$ ($\nu > m/2$). This implies (3.2) and (3.3) for $2 \leq \nu \leq m$, since $m \geq 4$. In order to prove (3.3) for $m < \nu \leq N_0 m$, we take $C_0^\infty$-functions $\varphi_N(x)$, $N = 2, \ldots, N_0$ satisfying

$$\varphi_1 \Supset \varphi_2 \Supset \cdots \Supset \varphi_{N_0}$$

and prove

$$\|\varphi_N(x) P^l u\|_{\lambda,n_0+\nu} \leq C_{23,N} M^{l+\nu-2} t^{1-\nu} (l+\nu-1)!\,(l+\nu-2)!^{s-1} \cdot \frac{1}{(l+1)^2} \quad \text{for} \quad 2 \leq \nu \leq Nm \tag{3.12}$$

by induction on $N$. Now, we assume (3.12). Then, from (3.2) and (3.12) we get (3.4) with $\chi(x) = \varphi_{N+1}(x)$ from Lemma 3.2. This implies (3.6) with $\varphi(x) = \varphi_{N+1}(x)$, $\mu = n_0 + Nm$, $j_2 = Nm$ and $m'' = Nm$. Moreover, from (3.12) with $\nu = Nm$, an inequality (3.5) holds with $\psi(x) = \varphi_N(x)$, $\mu = n_0 + Nm$, $j_1 = Nm - 1$ and $m' = Nm - 1$. Hence, from Lemma 3.3 we get

$$\|\varphi_{N+1}(x)^\nu P^l u\|_{\lambda,n_0+Nm+\nu} \le C_{24,\nu} M^{l+Nm-1} t^{-j'} (l + Nm)!\,(l + Nm - 1)!^{s-1} \cdot \frac{1}{(l+1)^2} \quad \text{for} \quad \nu \le m$$

with $j' = Nm$ ($\nu \le m/2$), $j' = Nm + 1$ ($\nu > m/2$). This proves that the inequality (3.12) holds for $Nm+1 \le \nu \le (N+1)m$ with $\varphi_N$ replaced by $\varphi_{N+1}$, and hence, we get (3.12) with $N$ replaced by $N+1$. Summing up, by the induction on $N$ we get (3.3) for $2 \le \nu \le N_0 m$ from (3.12). $\square$

Proof of Lemma 3.2. Using

$$\lambda(\xi)^{Nm} = \sum_{|\beta'| \le N} \frac{N!}{(N - |\beta'|)!\,\beta'!} \xi_1^{\beta'_1 m_1} \cdots \xi_n^{\beta'_n m_n},$$

we write

$$\begin{aligned} \|\chi(x) P^l\{f(u)\}\|_{\lambda,n_0+Nm} &\le C_{25} \sum_{|\beta:\mathbf{m}| \le N} \|D_x^\beta \chi P^l\{f(u)\}\|_{\lambda,n_0} \\ &\le C_{25} \sum_{|\beta:\mathbf{m}| \le N} \sum_{\gamma \le \beta} \binom{\beta}{\gamma} \|(D^{\beta-\gamma}\chi) D_x^\gamma P^l\{f(u)\}\|_{\lambda,n_0}. \end{aligned}$$

For convenience of the notation below, we set $\tilde\varphi_0(x) = 1$ and $\tilde\varphi_\gamma(x) = \varphi(x)$ for $\gamma \ne 0$. Then, we have from (3.8), (3.2) and (3.3) for $2 \le \nu \le Nm$

$$\begin{cases} \|\tilde\varphi_0(x) P^l u\|_{\lambda,n_0} \le C_{19} M^{l-1} l!\,(l-1)^{s-1} \cdot \dfrac{1}{(l+1)^2}, \\ \|\tilde\varphi_\gamma(x) P^l u\|_{\lambda,n_0+[\rho\cdot\gamma]_*} \le C_{26} M^{l+[\rho\cdot\gamma]-1} t^{-[\rho\cdot\gamma]} \\ \qquad \times (l + [\rho\cdot\gamma])!\,(l + [\rho\cdot\gamma] - 1)!^{s-1} \cdot \dfrac{1}{(l+1)^2} \\ \qquad\qquad \text{for } 0 < |\gamma| \le Nm. \end{cases}$$

Here, we used $[\rho\cdot\gamma]_* \le [\rho\cdot\gamma] + 1$ and $\rho\cdot\gamma = 1$ if $[\rho\cdot\gamma]_* = 1$. Hence, for $l + |\gamma| \ge 1$ we get from $\chi \Subset \tilde\varphi_\gamma$, Proposition 1.6 and (3.10)

$$\begin{aligned} &\|(D^{\beta-\gamma}\chi) D_x^\gamma P^l\{f(u)\}\|_{\lambda,n_0} \\ &\quad \le \sum_{k=1}^{l+|\gamma|} \sum_{k'+k''=k} \frac{\gamma!\,l!}{k'!\,k''!} \Big\|(D^{\beta-\gamma}\chi) \partial_u^{k'} \partial_{\bar u}^{k''} f(u) \end{aligned}$$

$$\times \sum_{\substack{l_1+\cdots+l_k=l\\ \gamma^1+\cdots+\gamma^k=\gamma\\ l_j+|\gamma^j|\geq 1}} \prod_{j=1}^{k'} \frac{1}{\gamma^j!\,l_j!} D_x^{\gamma^j} P^{l_j} u \prod_{j=k'+1}^{k} \frac{1}{\gamma^j!\,l_j!} D_x^{\gamma^j} P^{l_j} \overline{u} \Big\|_{\lambda,n_0}$$

$$= \sum_{k=1}^{l+|\gamma|} \sum_{k'+k''=k} \frac{\gamma!\,l!}{k'!\,k''!} \Big\| (D^{\beta-\gamma}\chi) \partial_u^{k'} \partial_{\overline{u}}^{k''} f(u)$$

$$\times \sum_{\substack{l_1+\cdots+l_k=l\\ \gamma^1+\cdots+\gamma^k=\gamma\\ l_j+|\gamma^j|\geq 1}} \prod_{j=1}^{k'} \frac{1}{\gamma^j!\,l_j!} D_x^{\gamma^j} \tilde{\varphi}_{\gamma^j}(x) P^{l_j} u \prod_{j=k'+1}^{k} \frac{1}{\gamma^j!\,l_j!} D_x^{\gamma^j} \tilde{\varphi}_{\gamma^j}(x) P^{l_j} \overline{u} \Big\|_{\lambda,n_0}$$

$$\leq C_{27} \sum_{k=1}^{l+|\gamma|} \gamma!\,l! \cdot C_{21} M_1^k C_2^{k-1} (k-1)!^{s-1}$$

$$\times \sum_{\substack{l_1+\cdots+l_k=l\\ \gamma^1+\cdots+\gamma^k=\gamma\\ l_j+|\gamma^j|\geq 1}} \prod_{j=1}^{k} \Big\| \frac{1}{\gamma^j!\,l_j!} \tilde{\varphi}_{\gamma^j}(x) P^{l_j} u \Big\|_{\lambda,n_0+[\rho\cdot\gamma^j]_*}$$

$$\leq C_{27} C_{21} \sum_{k=1}^{l+|\gamma|} l!\,(k-1)!^{s-1} M_1^k C_2^{k-1} C_{26}^{|\gamma|} C_{19}^{(k-|\gamma|)_+} \Big( \sum_{\gamma^1+\cdots+\gamma^k=\gamma} \frac{\gamma!}{\gamma^1!\cdots\gamma^k!} \Big)$$

$$\times \max_{\gamma^j} \sum_{l_1+\cdots+l_k=l} \prod_{j=1}^{k} \frac{(l_j+[\rho\cdot\gamma^j])!}{l_j!} M^{l_j+[\rho\cdot\gamma^j]-1} t^{-[\rho\cdot\gamma^j]}$$

$$\times (l_j+[\rho\cdot\gamma^j]-1)!^{s-1} \cdot \frac{1}{(l_j+1)^2}$$

$$\leq C'_{27} C_{21} C_2^{-1} (C_{26}/C_{19})^{|\gamma|} l!\, M^{l+Nm} t^{-Nm}$$

$$\times \sum_{k=1}^{l+|\gamma|} \left( \frac{M_1 C_2 C_{19}}{M} \right)^k k^{|\gamma|} \cdot \frac{(l+Nm)!}{l!}$$

$$\times (l+Nm-1)!^{s-1} \sum_{l_1+\cdots+l_k=l} \frac{1}{(l_j+1)^2}$$

$$\leq C_{28} M^{l+Nm} t^{-Nm} (l+Nm)!\,(l+Nm-1)!^{s-1} \cdot \frac{1}{(l+1)^2}$$

$$\times \sum_{k=1}^{l+|\gamma|} \left( \frac{M_1 C_2 C_8 C_{19}}{M} \right)^k k^{|\gamma|}.$$

This proves (3.4), since we have from (3.11)

$$\sum_{k=1}^{l+|\gamma|}\left(\frac{M_1C_2C_8C_{19}}{M}\right)^k k^{|\gamma|}\leq C_{29}M^{-1}. \qquad \square$$

Proof of Lemma 3.3. We prove (3.7) by the induction on $\nu$. For $\nu=0$ the inequality (3.7) holds if we regard $\varphi(x)^0$ as $\psi(x)$. So, we assume that (3.7) holds for $\nu'<\nu$ and prove (3.7). Since $A=\sum_{|\alpha:\mathbf{m}|=1}a_\alpha D_x^\alpha\in S_\lambda^m$ is semi-elliptic, we have

$$\begin{aligned}(3.13)\quad \|\varphi(x)^\nu P^l u\|_{\lambda,\mu+\nu} &\leq C_{30}\{\|A\varphi(x)^\nu P^l u\|_{\lambda,\mu+\nu-m}+\|\varphi(x)^\nu P^l u\|_{\lambda,\mu+\nu-m}\}\\ &\leq C_{30}\Big\{\|\varphi(x)^\nu AP^l u\|_{\lambda,\mu+\nu-m}+\sum_{|\alpha:\mathbf{m}|=1}\|[a_\alpha D^\alpha,\varphi(x)^\nu]P^l u\|_{\lambda,\mu+\nu-m}\\ &\qquad +\|\varphi(x)^\nu P^l u\|_{\lambda,\mu+\nu-m}\Big\}.\end{aligned}$$

Use $[a_\alpha D^\alpha,\varphi(x)^\nu]=-a_\alpha\sum_{0<\beta\leq\alpha}(-1)^{|\beta|}\binom{\alpha}{\beta}D_x^{\alpha-\beta}\{D_x^\beta\varphi(x)^\nu\}$ and

$$D_x^{\alpha-\beta}\in S_\lambda^{\rho\cdot(\alpha-\beta)}\subset S_\lambda^{m-[\rho\cdot\beta]}.$$

Then, we have

$$\|[a_\alpha D^\alpha,\varphi(x)^\nu]P^l u\|_{\lambda,\mu+\nu-m}\leq C_{31}\sum_{0<\beta\leq\alpha}\|\{D_x^\beta\varphi(x)^\nu\}P^l u\|_{\lambda,\mu+\nu-[\rho\cdot\beta]}$$

and, in case $\nu-[\rho\cdot\beta]\leq 0$, we have from $\varphi\Subset\psi$, $D_x^\beta\varphi(x)^\nu\in S_\lambda^0$ and (3.5)

$$\begin{aligned}\|\{D_x^\beta\varphi(x)^\nu\}P^l u\|_{\lambda,\mu+\nu-[\rho\cdot\beta]}&=\|\{D_x^\beta\varphi(x)^\nu\}\psi(x)P^l u\|_{\lambda,\mu+\nu-[\rho\cdot\beta]}\\ &\leq C_{32}\|\psi(x)P^l u\|_{\lambda,\mu}\\ &\leq C_{32}C_{16}M^{l+m'-1}t^{-j_1}(l+m')!\,(l+m'-1)!^{s-1}\cdot\frac{1}{(l+1)^2}.\end{aligned}$$

For the case $\nu-[\rho\cdot\beta]>0$, we write $D_x^\beta\varphi(x)^\nu=\varphi_{\nu,\beta}\varphi^{\nu-|\beta|}=\varphi_{\nu,\beta}\varphi^{[\rho\cdot\beta]-|\beta|}\varphi^{\nu'}$ with a function $\varphi_{\nu,\beta}(x)$ and $\nu'=\nu-[\rho\cdot\beta]$. Then, noting $|\beta|<\nu$ and $\varphi_{\nu,\beta}\varphi^{[\rho\cdot\beta]-|\beta|}\in S_\lambda^0$, an inequality

$$\begin{aligned}\|\{D_x^\beta\varphi(x)^\nu\}P^l u\|_{\lambda,\mu+\nu-[\rho\cdot\beta]}&\leq C_{33}\|\varphi(x)^{\nu'}P^l u\|_{\lambda,\mu+\nu'}\\ &\leq C_{33}C_{18,\nu'}M^{l+\tilde m-1}t^{-j'}(l+\tilde m)!\,(l+\tilde m-1)!^{s-1}\cdot\frac{1}{(l+1)^2}\end{aligned}$$

holds by the assumption of the induction. In order to estimate the last term of (3.13), we use $\varphi\Subset\psi$, $\varphi(x)^\nu\in S_\lambda^0$ and (3.5) again. Then, we have

$$\|\varphi(x)^\nu P^l u\|_{\lambda,\mu+\nu-m}\leq C_{34}C_{16}M^{l+m'-1}t^{-j_1}(l+m')!\,(l+m'-1)!^{s-1}\cdot\frac{1}{(l+1)^2}.$$

Summing up, we get

$$\|\varphi(x)^\nu P^l u\|_{\lambda,\mu+\nu} \le C_{30}\Big\{\|\varphi(x)^\nu AP^l u\|_{\lambda,\mu+\nu-m} + C_{35}M^{l+\tilde m-1}t^{-j'}(l+\tilde m)!\,(l+\tilde m-1)!^{s-1}\cdot\frac{1}{(l+1)^2}\Big\}. \tag{3.14}$$

Now, we use (3.1). Then, we have

$$\|\varphi(x)^\nu AP^l u\|_{\lambda,\mu+\nu-m} \le \frac{1}{t}\|\varphi(x)^\nu P^{l+1}u\|_{\lambda,\mu+\nu-m} + \sum_{k=1}^n \frac{1}{t}\|\varphi(x)^\nu x_k\partial_{x_k}P^l u\|_{\lambda,\mu+\nu-m} + \|\varphi(x)^\nu (P+1)^l\{f(u)\}\|_{\lambda,\mu+\nu-m}. \tag{3.15}$$

We estimate each term of (3.15). For the first term we get

$$\begin{aligned}\frac{1}{t}\|\varphi(x)^\nu P^{l+1}u\|_{\lambda,\mu+\nu-m} &= \frac{1}{t}\|\varphi(x)^\nu\psi(x)P^{l+1}u\|_{\lambda,\mu+\nu-m}\\ &\le C_{36}\frac{1}{t}\|\psi(x)P^{l+1}u\|_{\lambda,\mu}\\ &\le C_{36}C_{16}M^{l+m'}t^{-j_1-1}(l+m'+1)!\,(l+m')!^{s-1}\cdot\frac{1}{(l+1)^2}\end{aligned} \tag{3.16}$$

from (3.5). In order to estimate the second term of (3.15), we divide into two cases $\nu \le m/2$ and $\nu > m/2$. Then, for the case $\nu \le m/2$, we get from (3.5)

$$\begin{aligned}\frac{1}{t}\|\varphi(x)^\nu x_k\partial_{x_k}P^l u\|_{\lambda,\mu+\nu-m} &= \frac{1}{t}\|\varphi(x)^\nu x_k\partial_{x_k}\psi(x)P^l u\|_{\lambda,\mu+\nu-m}\\ &\le C_{37}t^{-1}\|\psi(x)P^l u\|_{\lambda,\mu}\\ &\le C_{37}C_{16}M^{l+m'-1}t^{-j_1-1}(l+m')!\,(l+m'-1)!^{s-1}\cdot\frac{1}{(l+1)^2},\end{aligned} \tag{3.17}$$

since we have $\partial_{x_k} \in S_\lambda^{\rho_k}$ and $\nu - m + \rho_k = \nu - m + m/m_k \le m/2 - m + m/2 = 0$. For the case of $\nu > m/2$, we use (3.7) with $\nu = m/2$, the assumption of the induction. Then, we have

$$\begin{aligned}&\frac{1}{t}\|\varphi(x)^\nu x_k\partial_{x_k}P^l u\|_{\lambda,\mu+\nu-m}\\ &\quad\le \frac{1}{t}\{\|x_k\partial_{x_k}\varphi(x)^\nu P^l u\|_{\lambda,\mu} + \|x_k\{\partial_{x_k}\varphi(x)^\nu\}P^l u\|_{\lambda,\mu}\}\\ &\quad\le C_{38}t^{-1}\|\varphi(x)^{m/2}P^l u\|_{\lambda,\mu+m/2}\\ &\quad\le C_{38}C_{18,m/2}M^{l+\tilde m-1}t^{-\{(j_1+1)\vee j_2\}-1}(l+\tilde m)!\,(l+\tilde m-1)!^{s-1}\cdot\frac{1}{(l+1)^2}.\end{aligned} \tag{3.18}$$

Finally, we get from (3.6)

$$\|\varphi(x)^\nu (P+1)^l\{f(u)\}\|_{\lambda,\mu+\nu-m} \le \sum_{l'=0}^{l} \binom{l}{l'} \|\varphi(x)^\nu P^{l'}\{f(u)\}\|_{\lambda,\mu} \tag{3.19}$$
$$\le C_{39}C_{17}M^{l+m''-1}t^{-j_2}(l+m'')!\,(l+m''-1)!^{s-1}\cdot\frac{1}{(l+1)^2}.$$

Summing up, we have proved (3.7) from (3.14)–(3.19). □

Now, we proceed to prove Theorem 4. From Proposition 3.1 we can estimate $\partial_x^\beta P^l u$ locally for $\beta$ satisfying $|\rho\cdot\beta| \le mn$ if we take $N_0 = n$ in Proposition 3.1. Especially, taking a $C_0^\infty$-function $\varphi(x)$ and a constant $M_0$ appropriately, the following inequality holds for $\beta$ satisfying $m \le [\rho\cdot\beta]_* \le mn$

$$\|\varphi(x)^{\nu-1}\partial_x^\beta P^l u\|_{\lambda,n_0} \le M_0^{\nu+l+1-m}t^{1-\nu-l}(\nu+l-m)!^\sigma\, l!^{s-\sigma} \qquad (\nu = [\rho\cdot\beta]_*). \tag{3.20}$$

In fact, taking another $C_0^\infty$-functions $\psi(x)$ with $\varphi \Subset \psi$ and using $\varphi(x)^{\nu-1}\partial_x^\beta \in S_\lambda^{\rho\cdot\beta}$ we get

$$\begin{aligned}
\|\varphi(x)^{\nu-1}\partial_x^\beta P^l u\|_{\lambda,n_0} &= \|\varphi(x)^{\nu-1}\partial_x^\beta \psi(x)P^l u\|_{\lambda,n_0} \le C_{40}\|\psi(x)P^l u\|_{\lambda,n_0+\nu}\\
&\le C_{40}C_{14}M^{l+\nu-2}t^{1-\nu}(l+\nu-1)!\,(l+\nu-2)!^{s-1}\cdot\frac{1}{(l+1)^2}\\
&\le C_{40}C_{14}T^l M^{\nu+l-2}t^{1-\nu-l}(\nu+l)!^\sigma\,(l+mn)!^{s-\sigma}\\
&\le C_{40}C_{14}T^l M^{\nu+l-2}t^{1-\nu-l}M_3^{\nu+l}(\nu+l-m)!^\sigma\, l!^{s-\sigma}\\
&= C_{40}C_{14}T^l M^{m-2}M_3^m (MM_3)^{\nu+l-m}t^{1-\nu-l}(\nu+l-m)!^\sigma\, l!^{s-\sigma}
\end{aligned}$$

from (3.3) with $\varphi(x)$ replaced by $\psi(x)$. Hence, taking $M_0$ such that $MM_3 \le M_0$ and $C_{40}C_{14}T^l M^{m-2}M_3^m \le M_0$, we get (3.20). In the following, we prove (3.20) for any $\nu = [\rho\cdot\beta]_*$. Then, as see below, the inequalities (3.20) and (3.2)–(3.3) implies (6). So, it remains to estimate (3.20) for $[\rho\cdot\beta]_* > mn$. In order to do so, we shall employ the induction on $\nu$.

Now, we assume $\nu = [\rho\cdot\beta]_* > mn$. Then, there exists $j$ such that $\beta_j \ge m_j$. Set $\gamma = m_j e_j$ with a unit vector $e_j$ with the $j$-th element = 1, and set $\beta' = \beta - \gamma$ and $\nu' = \rho\cdot\beta'$. Then, we have $\nu' = \nu - m$. Write

$$\varphi(x)^{\nu-1}\partial_x^\beta P^l u = \partial_x^\gamma \varphi(x)^{\nu-1}\partial_x^{\beta'} P^l u + [\varphi(x)^{\nu-1}, \partial_x^\gamma]\partial_x^{\beta'} P^l u. \tag{3.21}$$

and estimate each term of the right hand side of the above identity.

**Lemma 3.4.** *Assume that* (3.20) *holds for $\nu' < \nu$. Then, we have*

$$\|[\varphi(x)^{\nu-1}, \partial_x^\gamma]\partial_x^{\beta'} P^l u\|_{\lambda,n_0} \le C_{41}M_0^{\nu'+l}t^{1-\nu'-l-m}(\nu'+l)!^\sigma\, l!^{s-\sigma}. \tag{3.22}$$

Proof. Since

$$[\varphi(x)^{\nu-1}, \partial_x^\gamma] = -\sum_{\substack{\gamma'\le\gamma\\ \gamma'\neq 0}} \binom{\gamma}{\gamma'} \sum_{k=1}^{|\gamma'|} \frac{(\nu-1)!}{(\nu-1-k)!} \varphi_{\gamma',k}(x)\varphi(x)^{\nu-1-k}\partial_x^{\gamma-\gamma'}$$

holds with functions $\varphi_{\gamma',k}(x)$ independent of $\nu$, we have

$$\begin{aligned}(3.23)\qquad & \|[\varphi(x)^{\nu-1}, \partial_x^\gamma]\partial_x^{\beta'} P^l u\|_{\lambda,n_0}\\ &\le C'_{41} \sum_{\substack{\gamma'\le\gamma\\ \gamma'\neq 0}} \sum_{k=1}^{|\gamma'|} \frac{(\nu-1)!}{(\nu-1-k)!} \|\varphi_{\gamma',k}(x)\varphi(x)^{\nu-1-k}\partial_x^{\beta'+\gamma-\gamma'} P^l u\|_{\lambda,n_0}.\end{aligned}$$

Set $\mu = [\rho\cdot(\beta'+\gamma-\gamma')]_*$. Then, since $\mu \le \nu - |\gamma'| \le \nu' + m - k$, we have

$$\begin{aligned}&\frac{(\nu-1)!}{(\nu-1-k)!}\|\varphi_{\gamma',k}(x)\varphi(x)^{\nu-1-k}\partial_x^{\beta'+\gamma-\gamma'} P^l u\|_{\lambda,n_0}\\ &\quad= \frac{(\nu-1)!}{(\nu-1-k)!}\|\{\varphi_{\gamma',k}(x)\varphi(x)^{\nu-k-\mu}\}\varphi(x)^{\mu-1}\partial_x^{\beta'+\gamma-\gamma'} P^l u\|_{\lambda,n_0}\\ &\quad\le \frac{(\nu'+m-1)!}{(\nu'+m-1-k)!} C''_{41} M_0^{\mu+l+1-m} t^{1-\mu-l-m}(\mu+l-m)!^\sigma\, l!^{s-\sigma}\\ &\quad\le \{2^k(\nu'+l)\cdots(\nu'+l+1-k)\}\\ &\qquad\qquad \times C''_{41} M_0^{\nu'-k+l+1} t^{1-(\nu'+m-k)-l}(\nu'-k+l)!^\sigma\, l!^{s-\sigma}\\ &\quad\le 2^k C''_{41} M_0^{\nu'+l} t^{1-\nu'-l-m}(\nu'+l)!^\sigma\, l!^{s-\sigma}.\end{aligned}$$

Combining this with (3.23), we get (3.22). □

Next, we estimate the $H_{\lambda,n_0}$-norm of the first term of (3.21) by using the semi-ellipticity of $A$ and (3.1). Then, we have

$$\begin{aligned}(3.24)\qquad & \|\partial_x^\gamma\varphi(x)^{\nu-1}\partial_x^{\beta'} P^l u\|_{\lambda,n_0} \le \|\varphi(x)^{\nu-1}\partial_x^{\beta'} P^l u\|_{\lambda,n_0+m}\\ &\le C_{42}\{\|A\varphi(x)^{\nu-1}\partial_x^{\beta'} P^l u\|_{\lambda,n_0} + \|\varphi(x)^{\nu-1}\partial_x^{\beta'} P^l u\|_{\lambda,n_0}\}\\ &\le C_{42}\{\|\varphi(x)^{\nu-1}\partial_x^{\beta'} AP^l u\|_{\lambda,n_0} + \|[A, \varphi(x)^{\nu-1}]\partial_x^{\beta'} P^l u\|_{\lambda,n_0}\\ &\qquad + C_{43}\|\varphi(x)^{\nu'-1}\partial_x^{\beta'} P^l u\|_{\lambda,n_0}\}\\ &\le C_{42}\Big\{t^{-1}\|\varphi(x)^{\nu-1}\partial_x^{\beta'} P^{l+1} u\|_{\lambda,n_0} + t^{-1}\sum_{k=1}^n \|\varphi(x)^{\nu-1}\partial_x^{\beta'} x_k\partial_{x_k} P^l u\|_{\lambda,n_0}\\ &\qquad + \|\varphi(x)^{\nu-1}\partial_x^{\beta'}(P+1)^l\{f(u)\}\|_{\lambda,n_0}\\ &\qquad + \|[A, \varphi(x)^{\nu-1}]\partial_x^{\beta'} P^l u\|_{\lambda,n_0}\Big\}\end{aligned}$$

$$+ C_{42}C_{43}M_0^{\nu'+l+1-m}t^{1-\nu'-l}(\nu'+l-m)!^{\sigma}\, l!^{s-\sigma}.$$

We estimate each term in the last member of (3.24).

**Lemma 3.5.** *Assume that* (3.20) *holds for* $\nu' < \nu$. *Then, we have*

$$\|[A, \varphi(x)^{\nu-1}]\partial_x^{\beta'} P^l u\|_{\lambda,n_0} \leq C_{44}M_0^{\nu'+l}t^{1-\nu'-l-m}(\nu'+l)!^{\sigma}\, l!^{s-\sigma}. \tag{3.25}$$

We can prove this lemma by the same method as in the proof of Lemma 3.4.

**Lemma 3.6.** *Let* $\sigma$ *satisfy* $\sigma \geq s/m$. *Assume that* (3.20) *holds for* $\nu' < \nu$. *Then, the inequality*

$$\|\varphi(x)^{\nu-1}\partial_x^{\beta'} P^{l+1}u\|_{\lambda,n_0} \leq C_{45}M_0^{\nu'+l}t^{2-\nu'-l-m}(\nu'+l)!^{\sigma}\, l!^{s-\sigma} \tag{3.26}$$

*holds.*

Proof. Note $\nu' \geq m$. Then, since $\sigma \geq s/m$, we have

$$\begin{aligned}(l+1)^{s-\sigma} &\leq (l+1)^{(m-1)\sigma}\\ &\leq \prod_{k=1}^{m-1}(l+1+\nu'-m+k) = \prod_{k=2}^{m}(\nu'+l+k-m).\end{aligned}$$

Hence, we get from (3.20)

$$\begin{aligned}\|\varphi(x)^{\nu-1}\partial_x^{\beta'} P^{l+1}u\|_{\lambda,n_0} &\leq C'_{45}\|\varphi(x)^{\nu'-1}\partial_x^{\beta'} P^{l+1}u\|_{\lambda,n_0}\\ &\leq C'_{45}M_0^{\nu'+(l+1)+1-m}t^{1-\nu'-(l+1)}(\nu'+l+1-m)!^{\sigma}\,(l+1)!^{s-\sigma}\\ &\leq C'_{45}M_0^{\nu'+l}t^{-\nu'-l}(\nu'+l)!^{\sigma}\, l!^{s-\sigma}.\end{aligned}$$

This yields (3.26) from $m \geq 2$. □

**Lemma 3.7.** *Assume that* (3.20) *holds for* $\nu' < \nu$. *Then, we have*

$$\|\varphi(x)^{\nu-1}\partial_x^{\beta'} x_k\partial_{x_k} P^l u\|_{\lambda,n_0} \leq C_{46}M_0^{\nu'+l}t^{2-\nu'-l-m}(\nu'+l)!^{\sigma}\, l!^{s-\sigma}. \tag{3.27}$$

Proof. Write

$$\|\varphi(x)^{\nu-1}\partial_x^{\beta'} x_k\partial_{x_k} P^l u\|_{\lambda,n_0} \leq \|\varphi(x)^{\nu-1}x_k\partial_{x_k}\partial_x^{\beta'} P^l u\|_{\lambda,n_0} + \beta_k\|\varphi(x)^{\nu-1}\partial_x^{\beta'} P^l u\|_{\lambda,n_0}$$

and set $\mu = [\rho\cdot\beta' + \rho_k]_*$. Then, since $\rho_k = m/m_k$ and $m_k \geq 2$, we have $\mu \leq \nu' + m - 1$. Hence, we obtain from the boundedness of $\operatorname{supp}\varphi$

$$\|\varphi(x)^{\nu-1}\partial_x^{\beta'} x_k\partial_{x_k} P^l u\|_{\lambda,n_0}$$

$$\leq C'_{46}\{\|\varphi(x)^{\mu-1}\partial_{x_k}\partial_x^{\beta'}P^l u\|_{\lambda,n_0} + \beta_k\|\varphi(x)^{\nu'-1}\partial_x^{\beta'}P^l u\|_{\lambda,n_0}\}$$
$$\leq C'_{46}\{M_0^{\mu+l+1-m}t^{1-\mu-l}(\mu+l-m)!^\sigma\, l!^{s-\sigma}$$
$$+\beta_k M_0^{\nu'+l+1-m}t^{1-\nu'-l}(\nu'+l-m)!^\sigma\, l!^{s-\sigma}\}$$
$$\leq C''_{46}M_0^{\nu'+l}t^{2-\nu'-l-m}(\nu'+l)!^\sigma\, l!^{s-\sigma}.$$

This yields (3.27). □

**Lemma 3.8.** *Let $f(u)$ be a function satisfying* (A.1)$_\sigma$ *and assume that* (3.20) *holds for $\nu' < \nu$. Then, an inequality*

$$\|\varphi(x)^{\nu-1}\partial_x^{\beta'}(P+1)^l\{f(u)\}\|_{\lambda,n_0} \leq C_{47}M_0^{\nu'+l}t^{1-\nu'-l-m}(\nu'+l)!^\sigma\, l!^{s-\sigma} \tag{3.28}$$

*holds.*

Proof. Since we have

$$\|\varphi(x)^{\nu-1}\partial_x^{\beta'}(P+1)^l\{f(u)\}\|_{\lambda,n_0} \leq \sum_{l'=0}^{l}\binom{l}{l'}\|\varphi(x)^{\nu-1}\partial_x^{\beta'}P^{l'}\{f(u)\}\|_{\lambda,n_0},$$

we may only prove

$$\|\varphi(x)^{\nu-1}\partial_x^{\beta'}P^l\{f(u)\}\|_{\lambda,n_0} \leq C_{48}M_0^{\nu'+l}t^{1-\nu'-l-m}(\nu'+l)!^\sigma\, l!^{s-\sigma} \tag{3.29}$$

in order to prove (3.28). So, in the following, we prove (3.29). For $\beta''$ satisfying $m \leq \nu'' = [\rho\cdot\beta'']_* < \nu$ we have from (3.20), $m \geq 3$ and $\nu'' \leq [\rho\cdot\beta''] + 1$

$$\|\varphi(x)^{\nu''-1}\partial_x^{\beta''}P^l u\|_{\lambda,n_0} \leq M_0^{\nu''+l+1-m}t^{1-\nu''-l}(\nu''+l-m)!^\sigma\, l!^{s-\sigma}$$
$$\leq M_0^{[\rho\cdot\beta'']+l+2-m}t^{-[\rho\cdot\beta'']-l}([\rho\cdot\beta'']+1+l-m)!^\sigma\, l!^{s-\sigma}$$
$$\leq C_{49}M_0^{[\rho\cdot\beta'']+l-1}t^{-[\rho\cdot\beta'']-l}([\rho\cdot\beta'']+l)!\,([\rho\cdot\beta'']+l-1)!^{\sigma-1}\, l!^{s-\sigma}$$
$$\times\frac{1}{(|\beta''|+l+1)^2},$$

since $(|\beta''|+l+1)^2 \leq C_{49}([\rho\cdot\beta'']+l-1)([\rho\cdot\beta'']+l)$ holds for some constant $C_{49}$. This yields

$$\|\varphi(x)^{(\nu''-1)\vee 1}\partial_x^{\beta''}P^l u\|_{\lambda,n_0} \tag{3.30}$$
$$\leq C_{49}M_0^{[\rho\cdot\beta'']+l-1}t^{-[\rho\cdot\beta'']-l}([\rho\cdot\beta'']+l)!\,([\rho\cdot\beta'']+l-1)!^{\sigma-1}\, l!^{s-\sigma}$$
$$\times\frac{1}{(|\beta''|+l+1)^2},$$

for $\beta''$ satisfying $m \le \nu'' = [\rho\cdot\beta'']_* < \nu$. Moreover, from (3.2) and (3.3) we get (3.30) for $\beta''$ satisfying $1 \le \nu'' = [\rho\cdot\beta'']_* < m$ with an appropriate constant $M_0$. We also use

$$\|P^l u\|_{\lambda,n_0} \le C_{49} M_0^{l-1} t^{-l} l!\,(l-1)!^{\sigma-1}\, l!^{s-\sigma}\cdot \frac{1}{(|\beta''|+l+1)^2}, \tag{3.31}$$

which is guaranteed in the previous section.

Now, we use the differentiation of composite function. Then, writing $\tilde\varphi_0(x) = 1$ and $\tilde\varphi_\beta(x) = \varphi(x)$ for $\beta \ne 0$ we have

$$\begin{aligned}
&\|\varphi(x)^{\nu-1}\partial_x^{\beta'} P^l\{f(u)\}\|_{\lambda,n_0}\\
&\quad\le \sum_{k=1}^{|\beta'|+l}\sum_{k'+k''=k}\sum_{\substack{\beta^1+\cdots+\beta^k=\beta'\\ l_1+\cdots+l_k=l\\ |\beta^j|+l_j\ge1}} \frac{\beta'!\,l!}{k'!\,k''!}\Big\|\varphi(x)^{\nu-\bar\nu^k(\tilde\beta^k)-1}\partial_u^{k'}\partial_{\bar u}^{k''} f(u)\\
&\qquad\times\prod_{j=1}^{k'}\frac{1}{\beta^j!\,l_j!}\tilde\varphi_{\beta^j}(x)^{(\nu_j-1)\vee1}\partial_x^{\beta^j}P^{l_j}u\\
&\qquad\times\prod_{j=k'+1}^{k}\frac{1}{\beta^j!\,l_j!}\tilde\varphi_{\beta^j}(x)^{(\nu_j-1)\vee1}\partial_x^{\beta^j}P^{l_j}\bar u\Big\|_{\lambda,n_0}
\end{aligned}$$

with $\nu_j = [\rho\cdot\beta^j]_*$, and $\bar\nu^k(\tilde\beta^k) = \sum_{\{j\,;\,\beta^j\ne0\}}\{(\nu_j-1)\vee1\}$ for $\tilde\beta^k = (\beta^1,\dots,\beta^k)$. Since we can prove $0 \le \nu - \bar\nu^k(\tilde\beta^k) \le k-1$ there exists a constant $C_{50}$ such that $\|\varphi^{\nu-\bar\nu^k(\tilde\beta^k)-1}v\|_{\lambda,n_0} \le C_{50}^k\|v\|_{\lambda,n_0}$ for $v \in H_{\lambda,n_0}$. Hence, using Proposition 1.2 and Proposition 1.6 we have from (3.30) and (3.31)

$$\begin{aligned}
\|\varphi(x)^{\nu-1}\partial_x^{\beta'}P^l\{f(u)\}\|_{\lambda,n_0} &\le C_{51}\sum_{k=1}^{n} k\cdot\beta'!\,l!\,M_1^k k!^{\sigma-1}\\
&\quad\times C_{50}^k\sum_{\substack{\beta^1+\cdots+\beta^k=\beta'\\ l_1+\cdots+l_k=l\\ |\beta^j|+l_j\ge1}} C_2^{k-1}\prod_{j=1}^{k}\frac{1}{\beta^j!\,l_j!}C_{49}M_0^{[\rho\cdot\beta^j]+l_j-1}t^{-[\rho\cdot\beta^j]-l_j}([\rho\cdot\beta^j]+l_j)!\\
&\quad\times([\rho\cdot\beta^j]+l_j-1)!^{\sigma-1}\,l_j!^{s-\sigma}/(|\beta^j|+l_j+1)^2.
\end{aligned}$$

Note $[\rho\cdot\beta^1]+\cdots+[\rho\cdot\beta^k] \le [\rho\cdot\beta']_* = \nu'$. Then, we have from Lemma 1.8

$$\begin{aligned}
&\|\varphi(x)^{\nu-1}\partial_x^{\beta'}P^l\{f(u)\}\|_{\lambda,n_0}\\
&\quad\le C'_{50}C_2^{-1}M_0^{\nu'+l}t^{-\nu'-l}\sum_{k=1}^{l+|\beta'|} k\left(\frac{M_1C_2C_{49}C_{50}}{M_0}\right)^k(\nu'+l)!^{\sigma-1}\,l!^{s-\sigma}
\end{aligned}$$

$$
\begin{aligned}
&\times \sum_{\substack{\mu_1+\cdots+\mu_\nu=|\beta'|+l\\ \mu_j\ge 1}} \Bigl\{\beta'!\,l! \sum_{\substack{\beta^1+\cdots+\beta^k=\beta'\\ l_1+\cdots+l_k=l\\ |\beta^j|+l_j=\mu_j}} \prod_{j=1}^{k} \frac{(|\beta^j|+l_j)!}{l_j!\,\beta^j!}\\
&\qquad \times \prod_{j=1}^{k} \frac{([\rho\cdot\beta^j]+l_j)!}{(|\beta^j|+l_j)!}\,\frac{1}{(\mu_j+1)^2}\Bigr\},\\
&\le C'_{50}C_2^{-1}M_0^{\nu'+l}t^{-\nu'-l}(\nu'+l)!^{\sigma-1}\,l!^{s-\sigma}\sum_{k=1}^{l+|\beta'|} k\left(\frac{M_1C_2C_{49}C_{50}}{M_0}\right)^k\\
&\quad \times \sum_{\substack{\mu_1+\cdots+\mu_\nu=|\beta'|+l\\ \mu_j\ge 1}} (|\beta'|+l)!\,\frac{(\nu'+l)!}{(|\beta'|+l)!}\prod_{j=1}^{k}\frac{1}{(\mu_j+1)^2}\\
&\le C'_{50}C_2^{-1}C_8^{-1}M_0^{\nu'+l}t^{1-\nu'-l-m}(\nu'+l)!^{\sigma}\,l!^{s-\sigma}\\
&\qquad \times \sum_{k=1}^{l+|\beta'|} k\left(\frac{M_1C_2C_{49}C_{50}C_8}{M_0}\right)^k\cdot\frac{1}{(|\beta'|+l)^2}.
\end{aligned}
$$

This proves (3.29) and hence (3.28) if we take $M_0$ such that $M_1C_2C_{49}C_{50}C_8 < M_0$. □

Now, we are prepared to prove Theorem 4.

Proof of Theorem 4. Let $\varphi(x)$ be a $C_0^\infty$-function. Then, from (3.2) and (3.3) with $2 \le \nu \le mn$ we have for $\beta$ satisfying $|\rho\cdot\beta| \le mn$

(3.32) $$\|\varphi(x)\partial_x^\beta u\|_{\lambda,n_0} \le M_0^{\nu+1}t^{-\nu}\nu!^\sigma \qquad (\nu = [\rho\cdot\beta]_*).$$

Now, let $\nu > mn$. Then, if we assume (3.20) holds for $\beta$ satisfying $[\rho\cdot\beta]_* < \nu$, we have from (3.21), (3.24) and Lemmas 3.4–3.8

$$
\begin{aligned}
\|\varphi(x)^{\nu-1}\partial_x^\beta P^l u\|_{\lambda,n_0} &\le \{C_{41}+C_{42}(C_{45}+C_{46}+C_{47}+C_{44}+C_{43})\}\\
&\qquad \times M_0^{\nu'+l}t^{1-\nu'-l-m}(\nu'+l)!^\sigma l!^{s-\sigma}.
\end{aligned}
$$

with $\nu' = \nu-m = [\rho\cdot\beta]_*-m$. Retake the constant $M_0$ so large such that $C_{41}+C_{42}(C_{45}+C_{46}+C_{47}+C_{44}+C_{43}) \le M_0$. Then, we have (3.20) for $\beta$ satisfying $|\rho\cdot\beta| = \nu$, and hence, by the induction, we have (3.20) for any $\beta$ with $|\rho\cdot\beta| \ge mn$. This implies (3.32) for any $\beta$. Finally, we use $[\rho\cdot\beta]_*!^\sigma \le C_{52}^{|\beta|+1}\beta!^{\rho\sigma}$. Then, we get (6) from (3.32). □

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