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<th><strong>Title</strong></th>
<th>On homogeneous Kähler manifolds with non-degenerate canonical Hermitian form of signature $(2, 2(n-1))$</th>
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We denote by $M$ a connected homogeneous Kahler manifold of complex dimension $n$ on which a connected Lie group $G$ acts effectively as a group of holomorphic isometries, and by $K$ an isotropy subgroup of $G$ at a point $o$ of $M$. Let $v$ be the $G$-invariant volume element corresponding to the Kahler metric. In a local coordinate system $\{z_1, \cdots, z_n\}$, $v$ has an expression $v = i^n F dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n$. The $G$-invariant hermitian form $h = \sum_{i,j} \frac{\partial^2 \log F}{\partial z_i \partial \bar{z}_j} dz_i d\bar{z}_j$ is called the canonical hermitian form of $M=G/K$. It is known that the Ricci tensor of the Kahler manifold $M$ is equal to $-h$. The purpose of this paper is to prove the following:

**Theorem 1.** Let $M=G/K$ be a simply connected homogeneous Kahler manifold with non-degenerate canonical hermitian form $h$ of signature $(2, 2(n-1))$. Then, if either $G$ is semi-simple or $G$ contains a one parameter normal subgroup, $M=G/K$ is a holomorphic fibre bundle whose base space is the unit disk $\{x \in \mathbb{C}; |x| < 1\}$, and whose fibre is a homogeneous Kahler manifold of a compact simple Lie group.

In the case of $\dim \mathbb{C} G/K = 2$, the assumption of Theorem 1 is fulfilled and we have

**Theorem 2.** Let $M=G/K$ be a complex two dimensional homogeneous Kahler manifold with non-degenerate canonical hermitian form $h$ of signature $(2, 2)$. Then $G$ is semi-simple or $G$ contains a one parameter normal subgroup.

As an application of these Theorems, we obtain a classification of complex two dimensional homogeneous Kahler manifolds with non-degenerate canonical hermitian form.

1. Let $(I, g)$ be the $G$-invariant Kahler structure on $M$, i.e., $I$ is the $G$-invariant complex structure tensor on $M$ and $g$ is the $G$-invariant Kahler metric on $M$. Let $\mathfrak{g}$ be the Lie algebra of all left invariant vector fields on $G$ and let $\mathfrak{k}$ be the subalgebra of $\mathfrak{g}$ corresponding to $K$. We denote by $\pi$ the canonical projection
from $G$ onto $M = G/K$ and denote by $\pi_e$ the differential of $\pi$ at the identity $e$ of $G$. Let $X_e$, $I_e$ and $g_e$ be the values of $X$, $I$ and $g$ at $e$ and $\pi(e) = o$ respectively. Then there exist a linear endomorphism $J$ of $g$ and a skew symmetric bilinear form $\rho$ on $g$ such that

$$\pi_e(JX)_e = I_o(\pi_e X_e), \quad \rho(X, Y) = g_o(\pi_e X_e, \pi_e Y_e),$$

for $X, Y \in g$. Then $(g, \mathfrak{k}, J, \rho)$ satisfies the following properties [2], [3].

1. $J \mathfrak{k} \subset \mathfrak{k}$, $JX \equiv -X \pmod{\mathfrak{k}}$,
2. $[W, JX] \equiv J[W, X] \pmod{\mathfrak{k}}$,
3. $[JX, JY] \equiv J[JX, Y] + J[X, JY] + [X, Y] \pmod{\mathfrak{k}}$,
4. $\rho(W, X) = 0$,
5. $\rho(JX, JY) = \rho(X, Y)$,
6. $\rho(JX, X) > 0$, $X \in \mathfrak{t}$,
7. $\rho([X, X], Z) + \rho([Y, Z], X) + \rho([Z, X], Y) = 0$,

where $X, Y, Z \in g$, $W \in \mathfrak{k}$.

Then $(g, \mathfrak{k}, J, \rho)$ will be called the Kahler algebra of $M = G/K$.

Koszul proved that the canonical hermitian form $h$ of a homogeneous Kahler manifold $G/K$ has the following expression [3]. Put

$$\eta(X, Y) = h_o(\pi_e X_e, \pi_e Y_e), \quad \psi(X) = Tr_{g/o}(\text{ad} (JX) - J\text{ad} (X)),$$

it follows then

$$\eta(X, Y) = \frac{1}{2} \psi([JX, Y]),$$

for $X, Y \in g$. The form $\psi$ satisfies the following properties:

1. $\psi([W, X]) = 0$,
2. $\psi([JX, JY]) = \psi([X, Y])$, for $X, Y \in g$, $W \in \mathfrak{k}$.

Since $G$ acts effectively on $G/K$, $\mathfrak{k}$ contains no non-zero ideal of $g$ and there exists an $\text{ad}(\mathfrak{k})$-invariant inner product $(, )$ on $g$. Henceforth, we assume that the canonical hermitian form $h$ of $G/K$ is non-degenerate, which is equivalent to the following condition:

Let $X \in g$. If $\eta(X, Y) = 0$ for all $Y \in g$, then $X \in \mathfrak{k}$.

2. We shall now prepare a few lemmas for later use. The following lemma is due to [2].
**Lemma 2.1.** For \( E, X, Y \in \mathfrak{g} \),

\[
\frac{d}{dt} \rho(\exp t \text{ad}(JE)X, \exp t \text{ad}(JE)Y) = \rho(JE, \exp t \text{ad}(JE)[X, Y]).
\]

**Lemma 2.2.** The adjoint representation of \( \mathfrak{g} \) is faithful.

Proof. Let \( \alpha = \{X \in \mathfrak{g}; \text{ad}(X) = 0\} \), then \( \alpha \) is an ideal of \( \mathfrak{g} \). We have for \( X \in \mathfrak{a} \)

\[
2\eta(X, Y) = \psi([JX, Y])
\]

\[
= -\psi([X, JY]) = 0, \quad \text{for all } Y \in \mathfrak{g}.
\]

Since \( h \) is non-degenerate, we have \( X \in \mathfrak{t} \), and hence \( \alpha \subseteq \mathfrak{t} \). By the effectiveness, we have \( \alpha = \{0\} \). Q.E.D.

**Lemma 2.3.** Let \( \tau \) be a commutative ideal of \( \mathfrak{g} \). Then, \( \mathfrak{t} \cap \tau = \{0\}, \mathfrak{t} \cap J\tau = \{0\} \).

Proof. Let \( A \in \mathfrak{t} \cap \tau \). Since \( \tau \) is a commutative ideal, we have \( \text{ad}(A) = 0 \).

By the effectiveness, it follows that

\[
(\text{ad}(A)^2 X, X) + (\text{ad}(A) X, \text{ad}(A) X) = 0,
\]

\[
(\text{ad}(A) X, \text{ad}(A) X) = 0,
\]

for \( X \in \mathfrak{g} \), with respect to the \( \text{ad}(\mathfrak{t}) \)-invariant inner product \((\ , \)\) on \( \mathfrak{g} \). Hence \( \text{ad}(A) X = 0 \) for all \( X \in \mathfrak{g} \), and \( A = 0 \) by Lemma 2.2, which proves \( \mathfrak{t} \cap \tau = \{0\} \).

\( \mathfrak{t} \cap J\tau = \{0\} \) follows from \( \mathfrak{t} \cap \tau = \{0\} \). Q.E.D.

**Lemma 2.4.** Let \( \tau \) be a non-zero commutative ideal of \( \mathfrak{g} \). Then \( \psi \equiv 0 \) on \( \tau \).

Proof. Assume \( \psi = 0 \) on \( \mathfrak{t} \). For \( X \in \mathfrak{t} \), we have \( 2\eta(X, Y) = -\psi([JY, X]) = 0 \) for all \( Y \in \mathfrak{g} \). Since \( h \) is non-degenerate, we have \( X \in \mathfrak{t} \) and hence \( \tau \subseteq \mathfrak{t} \), which contradicts to Lemma 2.3. Q.E.D.

**Lemma 2.5.** \( \text{Tr}_{\mathfrak{g}/\text{rad}(W)} = 0 \), for \( W \in \mathfrak{t} \).

Proof. Using (1.4), (1.7), we have

\[
\rho(W, [X, Y]) + \rho(X, [Y, W]) + \rho(Y, [W, X]) = 0,
\]

\[
\rho(X, [Y, W]) + \rho(Y, [W, X]) = 0,
\]

for \( X, Y \in \mathfrak{g} \). Hence it follows that

\[
\rho(JX, [W, Y]) + \rho([W, JX], Y) = 0,
\]

\[
\rho(JX, [W, Y]) + \rho(J[W, X], Y) = 0,
\]

for \( X, Y \in \mathfrak{g} \). This implies that the endomorphism of \( \mathfrak{g}/\mathfrak{t} \) which is induced by
ad(W) is skew symmetric with respect to the inner product which is defined by \( \rho(JX, Y) \). Therefore \( \text{Tr}_{g} \text{ad}(W) = 0 \). Q.E.D.

**Lemma 2.6.** Let \( \{E\} \) be a one dimensional ideal of \( g \). Then \( [E, W] = 0 \), for \( W \in \mathfrak{t} \). Moreover, there exists an endomorphism \( J \) of \( g \) such that \( JX \equiv JX(\text{mod} \, \mathfrak{t}) \), \( [J, W] = 0 \), for \( X \in g, W \in \mathfrak{t} \).

Proof. Put \( [E, W] = \lambda E, \lambda \in \mathbb{R} \). Using (1.10), we have \( 0 = \psi([E, W]) = \lambda \psi(E) \). Since \( \psi(E) \neq 0 \) by Lemma 2.4, it follows \( \lambda = 0 \) and hence \( [E, W] = 0 \). Put \( \mathfrak{h} = \mathfrak{t} + \{JE\} \), then \( [\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h} \). Let \( \{L\} \) be the orthogonal complement of \( \mathfrak{t} \) in \( \mathfrak{h} \) with respect to the \( \text{ad}(\mathfrak{t}) \)-invariant inner product on \( g \). Then \( \{\mathfrak{t}, \{L\}\} \subset \{L\} \). We may assume that \( L = \mathfrak{W} + \mathfrak{J} \mathfrak{E} \) where \( \mathfrak{W} \in \mathfrak{t} \). Therefore we can choose a linear endomorphism \( J \) on \( g \) such that \( J = L, JX \equiv JX(\text{mod} \, \mathfrak{t}) \) for \( X \in g \). Then it follows

\[
[J, \mathfrak{t}] = [L, \mathfrak{t}] \subset \{L\},
\]

\[
[J, \mathfrak{t}] \subset [\mathfrak{h}, \mathfrak{t}] \subset \mathfrak{t}.
\]

This implies \( [J, W] = 0 \) for \( W \in \mathfrak{t} \). Q.E.D.

Therefore, for any one dimensional ideal \( \{E\} \) of \( g \), we may assume that \( [J, \mathfrak{t}] = \{0\} \).

3. We shall prove the following theorem.

**Theorem 1'.** Let \((g, \mathfrak{t}, J, \rho)\) be the Kähler algebra of a homogeneous Kähler manifold \( G/K \) with non-degenerate canonical hermitian form \( h \) of signature \((2, 2(n-1))\). If there exists a one dimensional ideal \( \tau \) of \( g \), then we have the following.

1) With suitable choice of \( \mathfrak{t} \), we have \( [J, E] = 0 \).

2) Put \( \mathfrak{p} = \{P \in g; [P, E] = [J, P, E] = 0\} \). Then we have the decomposition \( g = \{JE\} + \{E\} + \mathfrak{p} \) of \( g \) into the direct sum of vector spaces. We know also that \( \mathfrak{p} \) is a compact semi-simple \( J \)-invariant ideal of \( g \) and that the real parts of the eigenvalues of \( \text{ad}(J) \) on \( \mathfrak{p} \) are equal to 0.

The first part of the proof of Theorem 1' is nearly the same as the previous one [6]. But, for the sake of completeness we carry out the proof.

**Lemma 3.1.** Let \( \{E\} \) be a one dimensional ideal of \( g \) and put \( \mathfrak{p} = \{P \in g; [P, E] = [J, P, E] = 0\} \). Then we have

1) \( \mathfrak{t} \subset \mathfrak{p}, \)

2) \( J \mathfrak{p} \subset \mathfrak{p}, \text{ad}(J) \mathfrak{p} \subset \mathfrak{p}, \)

3) \( \text{ad}(J) \equiv J \text{ad}(J) \text{ (mod} \, \mathfrak{t}\text{)} \) on \( \mathfrak{p} \).

Proof. 1) follows from Lemma 2.6. For \( P \in \mathfrak{p} \), we have
\[
= J[JE, P] + W_0
\]
for some \(W_0 \in \mathfrak{f}\), and hence 3) is proved. For \(P \in \mathfrak{p}\), it follows that
\[
[[JE, P], E] = [[JE, E], P] + [JE, [P, E]] = 0,
\]
\[
[J[JE, P], E] = [[JE, JP] - W_0, E] = 0,
\]
where \(W_0 \in \mathfrak{f}\). Therefore \(\text{ad} (JE) P \in \mathfrak{p}\) for all \(P \in \mathfrak{p}\), which proves 2).

**Lemma 3.2.** Let \(\{E\}\) be a one dimensional ideal of \(\mathfrak{g}\). Then \([JE, E] = 0\), therefore with suitable choice of \(\mathfrak{e} \neq 0\), we have \([JE, E] = E\).

**Proof.** Assume that \([JE, E] = 0\). For \(X \in \mathfrak{g}\), we have \(J[JE, X] = [JE, JX] - J[E, JX] - [E, X] + W_0 = [JE, JX] - \lambda JE - \mu E + W_0\), where \(\lambda, \mu \in \mathbb{R}, \ W_0 \in \mathfrak{f}\). We have
\[
[[JE, X], E] = [[JE, E], X] + [JE, [X, E]] = 0,
\]
\[
[J[JE, X], E] = [[JE, JX], E] - [E, E] + [W_0, E] = 0,
\]
which implies that \([JE, X] \in \mathfrak{p}\), and hence we have
\[
(3.1) \quad \text{ad} (JE) \mathfrak{g} \subseteq \mathfrak{p}.
\]
Let \(P \in \mathfrak{p}\). We have
\[
\rho(JE, [JE, P]) = \rho(-E, J[JE, P])
= -\rho(E, [JE, JP])
= \rho(JE, [JP, E]) + \rho(JP, [E, JE])
= 0,
\]
and it follows that for \(X \in \mathfrak{g}\)
\[
(3.2) \quad \rho(JE, \text{ad} (JE)^a X) = 0.
\]
Applying Lemma 2.1, (3.2), we have for \(X, Y \in \mathfrak{g}\)
\[
\frac{d^2}{dt^2} \rho(\exp t \text{ad} (JE) X, \exp t \text{ad} (JE) Y)
= \frac{d^2}{dt^2} \rho(JE, \exp t \text{ad} (JE)[X, Y])
= \rho(JE, \text{ad} (JE)^a \exp t \text{ad} (JE)[X, Y])
= 0.
\]
Hence we may put
\[(3.3) \quad \rho(\exp t \text{ad}(JE)X, \exp t \text{ad}(JE)Y) = at^2 + bt + c\]

where \(a, b\) and \(c\) are real numbers not depending on \(t\). Since \(\text{ad}(JE)\mathfrak{p} \subset \mathfrak{p}\), \(\text{ad}(JE)\mathfrak{t} = \{0\}\) by Lemma 2.6, (3.1), \(\text{ad}(JE)\) induces a linear endomorphism \(\tilde{\text{ad}}(JE)\) on \(\mathfrak{p} / \mathfrak{t}\). Let \(\alpha + i\beta (\alpha, \beta \in \mathbb{R})\) be an eigenvalue of \(\tilde{\text{ad}}(JE)\). As \(\text{ad}(JE)J = J \text{ad}(JE) \pmod{\mathfrak{t}}\) on \(\mathfrak{p}\), there exists an element \(P \in \mathfrak{p}, P \notin \mathfrak{t}\) such that \([JE, P] \equiv (\alpha + \beta J)P \pmod{\mathfrak{t}}\), and hence \(\exp t \text{ad}(JE)P \equiv \exp t(\alpha + \beta J)P \pmod{\mathfrak{t}}\). Therefore we have by Lemma 3.1,

\[
\rho(\exp t \text{ad}(JE)JP, \exp t \text{ad}(JE)P) = \rho(\exp t(\alpha + \beta J)P, \exp t(\alpha + \beta J)P) \equiv e^{at^2 + bt + c} \rho(JP, P) \pmod{\mathfrak{t}}.
\]

From this and (3.3), we have

\[
e^{sat} \rho(JP, P) = at^2 + bt + c.
\]

Since \(P \notin \mathfrak{t}\), \(\rho(JP, P) > 0\) and hence \(\alpha = 0\). This fact and \(\text{ad}(JE)\mathfrak{t} = \{0\}\) show that the real parts of the eigenvalues of \(\text{ad}(JE)\) on \(\mathfrak{p}\) are equal to 0. Therefore we have

\[
\psi(E) = Tr_{\mathfrak{g}/\mathfrak{t}}(\text{ad}(JE) - J \text{ad}(E)) = Tr_{\mathfrak{g}}(\text{ad}(JE) - J \text{ad}(E)) - Tr(\text{ad}(JE) - J \text{ad}(E)) = 0.
\]

However this contradicts to \(\psi \neq 0\) on \(\{E\}\) by Lemma 2.4. Q.E.D.

**Lemma 3.3** Let \(\{E\}\) be a one dimensional ideal of \(\mathfrak{g}\). Then we get the decomposition

\[
\mathfrak{g} = \{JE\} + \{E\} + \mathfrak{p}
\]

of \(\mathfrak{g}\) into the direct sum of vector spaces with the following properties:

1) \([JE, E] = E\).
2) The factors of the decomposition are mutually orthogonal with respect to the form \(\eta\), and \(\eta\) is positive definite on \(\{JE\} + \{E\}\).
3) The real parts of the eigenvalues of \(\text{ad}(JE)\) on \(\mathfrak{p}\) are equal to 0 or \(1/2\).
4) \(\rho(JE, P) = 0\) for \(P \in \mathfrak{p}\).
Proof. By lemma 3.2, we may assume that $E$ satisfies the condition $[JE, E] = E$. Since $\{E\}$ is a one dimensional ideal of $g$, we get $[X, E] = \alpha(X)E$, $[JX, E] = \beta(X)E$, for $X \in g$, where $\alpha, \beta$ are linear functions on $g$. It is easily seen that $P = X - \alpha(X)JE - \beta(X)E$ belongs to $\mathfrak{p}$ for any $X \in g$. Therefore we have the decomposition $g = \{JE\} + \{E\} + \mathfrak{p}$. Now, by Lemma 2.1, we have for $P \in p$,

$$
\frac{d}{dt} \rho(\exp t \text{ad}(JE)E, \exp t \text{ad}(JE)P) = \rho(JE, \exp t \text{ad}(JE)[E, P]) = 0 .
$$

Since $\exp t \text{ad}(JE)E = e^t E$, we have

$$
\rho(E, \exp t \text{ad}(JE)P) = a'e^{-t}
$$

where $a'$ is a constant determined by $P$ and independent of $t$. We have then

$$
\rho(JE, \exp t \text{ad}(JE)P) = -\rho(E, J \exp t \text{ad}(JE)P) = -\rho(E, \exp t \text{ad}(JE)P) = ae^{-t}
$$

where $a$ is a constant determined by $JP$. Let $X = \lambda JE + \mu E + P \in g$, where $\lambda, \mu \in \mathbb{R}$, $P \in \mathfrak{p}$. Then we have

$$
\rho(JE, \exp t \text{ad}(JE)X) = \rho(JE, \lambda JE + \mu e^t E + \exp t \text{ad}(JE)P)
= \mu \rho(JE, E)e^t + \rho(JE, \exp t \text{ad}(JE)P)
= ae^{-t} + be^t
$$

where $a, b$ are constants independent of $t$. This fact and Lemma 2.1 show that for $X, Y \in g$

$$
\frac{d}{dt} \rho(\exp t \text{ad}(JE)X, \exp t \text{ad}(JE)Y) = \rho(JE, \exp t \text{ad}(JE)[X, Y]) = ae^{-t} + be^t .
$$

Hence we obtain

$$
\rho(\exp t \text{ad}(JE)X, \exp t \text{ad}(JE)Y) = ae^{-t} + be^t + c ,
$$

where $a, b$ and $c$ are constants independent of $t$. Let $\alpha + i\beta$ be an eigenvalue of $\text{ad}(JE)$ on $\mathfrak{p}/\mathfrak{t}$. As $\text{ad}(JE)J \equiv J \text{ad}(JE) \text{ (mod } \mathfrak{t})$ on $\mathfrak{p}$, there exists an element $P \in \mathfrak{p}$, $P \in \mathfrak{t}$ such that $\text{ad}(JE)P \equiv (\alpha + \beta J)P \text{ (mod } \mathfrak{t})$. Hence we have
\[
\rho(\exp t \text{ad}(JE)JP, \exp t \text{ad}(JE)P) \\
= \rho(J \exp t \text{ad}(JE)P, \exp t \text{ad}(JE)P) \\
= \rho(J \exp t(\alpha+\beta J)P, \exp t(\alpha+\beta J)P) \\
= \rho(\exp t(\alpha+\beta J)JP, \exp t(\alpha+\beta J)P) \\
= e^{t(\alpha+i\beta)\frac{1}{2}J} \rho(JP, P) \\
= e^{2\alpha t} \rho(JP, P).
\]

Therefore

(3.4) \quad e^{2\alpha t} \rho(JP, P) = ae^{-t} + be^{t} + c.

Since \( P \in \mathfrak{f} \) and \( \rho(JP, P) > 0 \), we have \( \alpha = 0 \) or \( 1/2 \) or \( -1/2 \). Let \( \tilde{J} \) be the linear endomorphism of \( \mathfrak{p} = \mathfrak{p}/\mathfrak{f} \) which is induced by \( J \) and put for \( \alpha, \beta \in \mathbb{R} \):

\[
\bar{\mathfrak{p}}_{(\alpha+i\beta)} = \{ \tilde{P} \in \mathfrak{p}; (\tilde{\text{ad}}(JE) - (\alpha+\beta \tilde{J}))^m \tilde{P} = 0 \},
\]

\[
\bar{\mathfrak{p}}_{\alpha} = \sum_{\beta} \bar{\mathfrak{p}}_{(\alpha+i\beta)}.
\]

Then we have

\[
\mathfrak{p} = \sum_{\alpha+\beta} \bar{\mathfrak{p}}_{(\alpha+i\beta)}
\]

where \( \alpha = 0 \) or \( 1/2 \) or \( -1/2 \).

Let \( \tilde{P} \in \mathfrak{p}_{(\alpha+i\beta)} \) and let \( P \in \mathfrak{p} \) be a representation of \( \tilde{P} \). Then there exists a positive integer \( m \) such that \( (\tilde{\text{ad}}(JE) - (\alpha+\beta \tilde{J}))^m \tilde{P} = 0 \). Therefore we have

\[
\exp t \tilde{\text{ad}}(JE)\tilde{P} = \exp t(\alpha+\beta \tilde{J}) \sum_{i=0}^{m-1} \frac{t^i}{i!} (\tilde{\text{ad}}(JE) - (\alpha+\beta \tilde{J}))^i \tilde{P}
\]

\[
= e^{t \alpha} \{ \cos \beta t \sum_{i=0}^{m-1} \frac{t^i}{i!} (\tilde{\text{ad}}(JE) - (\alpha+\beta \tilde{J}))^i \tilde{P}
\]

\[+ \sin \beta t \sum_{i=0}^{m-1} \frac{t^i}{i!} (\tilde{\text{ad}}(JE) - (\alpha+\beta \tilde{J}))^i \tilde{J} \tilde{P} \} \mod \mathfrak{f}
\]

This shows that

\[
\exp t \text{ad}(JE)P = e^{t \alpha} \{ \cos \beta t \sum_{i=0}^{m-1} \frac{t^i}{i!} (\text{ad}(JE) - (\alpha+\beta J))^i P
\]

\[+ \sin \beta t \sum_{i=0}^{m-1} \frac{t^i}{i!} (\text{ad}(JE) - (\alpha+\beta J))^i JP \} \mod \mathfrak{f}
\]

Hence we have

\[
\rho(JE, \exp t \text{ad}(JE)P)
\]

\[
= e^{t \alpha} \{ \cos \beta t \sum_{i=0}^{m-1} \frac{1}{i!} \rho(JE, (\text{ad}(JE) - (\alpha+\beta J))^i P) t^i
\]

\[+ \sin \beta t \sum_{i=0}^{m-1} \frac{1}{i!} \rho(JE, (\text{ad}(JE) - (\alpha+\beta J))^i JP) t^i \}.
\]
Put
\[
h(t) = \sum_{l=1}^{m-1} \frac{1}{l!} \rho(JE, (\text{ad}(JE) - (\alpha + \beta J))^{l} P)t^{l},
\]
\[
k(t) = \sum_{l=1}^{m-1} \frac{1}{l!} \rho(JE, (\text{ad}(JE) - (\alpha + \beta J))^{l} J P)t^{l}.
\]

Then \( h(t) \) and \( k(t) \) are polynomials of degree \( \leq m-1 \). We have then
\[
h(t) \cos \beta t + k(t) \sin \beta t = a e^{-\alpha t},
\]
Assume that \( a \neq 0 \). Since \( 1 + \alpha > 0 \) and since \( h(t) \) and \( k(t) \) are polynomials of degree \( \leq m-1 \), the left side of the above formula approaches to 0 and the right side to \( \infty \), when \( t \to -\infty \). This is a contradiction, and we get \( a = 0 \), which implies that
\[
\rho(JE, \exp t \text{ad}(JE)P) = 0
\]
where \( P \) is a representative of \( \tilde{P} \in \mathfrak{P}(\alpha + i \beta) \). Thus we have
\[
\rho(JE, \exp t \text{ad}(JE)P) = 0, \quad \text{for all } P \in \mathfrak{p},
\]
and hence
\[
\rho(JE, P) = 0, \quad \text{for all } P \in \mathfrak{p}.
\]
Therefore 4) is proved. Moreover the formula (3.4) is reduced to
\[
(3.4)' \quad e^{\alpha t} \rho(JP, P) = b e^{t} + c.
\]
This implies that \( \alpha = 0 \) or \( 1/2 \). Therefore we know that the real parts of the eigenvalues of \( \text{ad}(JE) \) on \( \mathfrak{p} \) are equal to 0 or \( 1/2 \). Thus the assertion 3) is proved. Now we shall show 2). The assertion that the decomposition \( g = \{JE\} + \{E\} + \mathfrak{p} \) is an orthogonal decomposition is clear. Put \( f = \text{ad}(JE) - J \text{ad}(E) \). Then we have \( f(W) = 0 \) for \( W \in \mathfrak{t}, f(JE) = JE, f(E) = E \) and \( f(P) = [JE, P] \) for \( P \in \mathfrak{p} \). Hence it follows that
\[
\psi(E) = Tr_{\mathfrak{g}/\mathfrak{t}}(\text{ad}(JE) - J \text{ad}(E))
\]
\[
= Tr_{\mathfrak{g}}(\text{ad}(JE) - J \text{ad}(E))
\]
\[
= 2 + Tr_{\mathfrak{g}} \text{ad}(JE)
\]
\[
> 0.
\]
Therefore \( 2\eta(JE, JE) = 2\eta(E, E) = \psi(E) > 0 \). Q.E.D.

For \( \alpha, \beta \in \mathbb{R} \), put
\[ \mathfrak{p}_{(a+i\beta)} = \{ P \in \mathfrak{p}; \ (\text{ad}(JE)-(\alpha + i\beta))^m P = 0 \} , \]
\[ \mathfrak{p}_a = \sum_{\beta} \mathfrak{p}_{(a+i\beta)} , \]

and let \( \pi' \) be the canonical projection from \( \mathfrak{g} \) onto \( \mathfrak{g}/\mathfrak{l} \). Then we have
\[ \tilde{\mathfrak{p}}_{(a+i\beta)} = \pi'(\mathfrak{p}_{(a+i\beta)}) , \quad \tilde{\mathfrak{p}}_a = \pi'(\mathfrak{p}_a) , \]
\[ \mathfrak{p} = \mathfrak{p}_0 + \mathfrak{p}_1 , \]
\[ J\mathfrak{p}_a \subset \mathfrak{l} + \mathfrak{p}_a , \]
\[ \text{ad}(JE)\mathfrak{p}_a \subset \mathfrak{p}_a . \]

**Lemma 3.4.** The form \( \eta \) is positive definite on \( \mathfrak{p}_1 \).

Proof. We shall first prove that the decomposition \( \mathfrak{p}_1 = \sum_{\beta} \mathfrak{p}_{(1+i\beta)} \) is an orthogonal decomposition with respect to \( \eta \). Let \( P \in \mathfrak{p}_{(1+i\beta)} , \ Q \in \mathfrak{p}_{(1+i\beta')}, \) and assume \( \beta \neq \beta' \). Then we have
\[ \exp t \text{ad}(JE)P \equiv \exp t(1/2+\beta) \sum \frac{t^l}{l!} (\text{ad}(JE)-(1/2+\beta))^l P \pmod{\mathfrak{l}} , \]
\[ \exp t \text{ad}(JE)Q \equiv \exp t(1/2+\beta') \sum \frac{t^l}{l!} (\text{ad}(JE)-(1/2+\beta'))^l Q \pmod{\mathfrak{l}} . \]

By Lemma 2.1, we have
\[ \frac{d}{dt} \rho(\exp t \text{ad}(JE)J\mathfrak{p}_1 , \ \exp t \text{ad}(JE)Q) = \rho(JE, \ \exp t \text{ad}(JE)[J\mathfrak{p}_1 , \ Q]) . \]

The left side of this equation is equal to
\[ \frac{d}{dt} \rho(J \exp t \text{ad}(JE)P , \ \exp t \text{ad}(JE)Q) = \frac{d}{dt} \rho(\exp t(1/2+\beta) \sum \frac{t^l}{l!} (\text{ad}(JE)-(1/2+\beta))^l P , \ \exp t(1/2+\beta') \sum \frac{t^l}{l!} (\text{ad}(JE)-(1/2+\beta'))^l Q) \]
\[ = \frac{d}{dt} \rho(\exp \beta t \sum \frac{t^l}{l!} (\text{ad}(JE)-(1/2+\beta))J\mathfrak{p}_1 , \ \exp \beta' t \sum \frac{t^l}{l!} (\text{ad}(JE)-(1/2+\beta'))^l Q) \]
\[ = \frac{d}{dt} \rho(\{\cos \beta t+(\sin \beta t)J\} u(t) , \ \{\cos \beta' t+(\sin \beta' t)J\} v(t)) \]
$$\frac{d}{dt} \varepsilon \{ (\cos \beta t \cos \beta' t + \sin \beta t \sin \beta' t) \rho(u(t), v(t)) + (\sin \beta t \cos \beta' t - \cos \beta t \sin \beta' t) \rho(Ju(t), v(t)) \}$$

$$= \frac{d}{dt} \varepsilon \{ h(t) \cos (\beta - \beta') t + k(t) \sin (\beta - \beta') t \}$$

$$= \varepsilon \{ a(t) \cos (\beta - \beta') t + b(t) \sin (\beta - \beta') t \},$$

where

$$u(t) = \sum_{l=0}^{\infty} \frac{t^l}{l!} (\text{ad}(JE) - (1/2 + \beta J') P),$$

$$v(t) = \sum_{l=0}^{\infty} \frac{t^l}{l!} (\text{ad}(JE) - (1/2 + \beta' J')) Q,$$

$$h(t) = \rho(u(t), v(t)), k(t) = \rho(Ju(t), v(t)),$$

$$a(t) = h(t) + h'(t) + (\beta - \beta') k(t),$$

$$b(t) = h(t) + k'(t) - (\beta - \beta') h(t).$$

Hence $a(t)$ and $b(t)$ are polynomials. Since $[JP, Q] \in \{ l + \lambda, \psi \} \subset \{ E \} + \psi$, we put $[JP, Q] = \lambda E + P'$, where $\lambda \in R, P' \in \psi$. Using Lemma 3.3; 4), the right side of the equation (3.5) is equal to

$$\rho(JE, \exp t \text{ad}(JE)[JP, Q]) = \rho(JE, \exp t \text{ad}(JE)(\lambda E + P'))$$

$$= \varepsilon \lambda \rho(JE, E) + \rho(JE, \exp t \text{ad}(JE) P')$$

$$= \varepsilon \lambda \rho(JE, E).$$

Therefore we have

$$a(t) \cos (\beta - \beta') t + b(t) \sin (\beta - \beta') t = \lambda \rho(JE, E).$$

Since $a(t) - \lambda \rho(JE, E)$ is a polynomial and since $a(t_n) - \lambda \rho(JE, E) = 0$ for $t_n = \frac{2n\pi}{\beta - \beta'}$, where $n$ integer, it follows that $a(t)$ is a constant $a$. Similarly $b(t)$ is a constant $b$. Hence we have

$$a \cos (\beta - \beta') t + b \sin (\beta - \beta') t = \lambda \rho(JE, E).$$

By this formula, we have $(\beta - \beta')^k \lambda \rho(JE, E) = 0$. Since $\beta - \beta' \neq 0$ and $\rho(JE, E) > 0$, we get $\lambda = 0$. Moreover $\text{ad}(JE)$ is non-singular on $\psi$ and so there exists an element $P'' \in \psi$ such that $P' = [JE, P'']$. Thus we have
This shows that $p_{4+iβ}$ and $p_{3+iβ'}$ are mutually orthogonal with respect to $η$. Now, let $P \neq 0 \in p_{4+iβ}$. Then we have

$$\exp t \text{ad}(JE)P \equiv \exp t(1/2+βJ)u(t) \pmod{1},$$

where $u(t)=\sum_{l=0}^{n} \frac{t^l}{l!} (\text{ad}(JE)-(1/2+βJ))^l P$. By Lemma 2.1, it follows that

$$\frac{d}{dt} \rho(\exp t \text{ad}(JE)P, \exp t \text{ad}(JE)P) = \rho(\exp t \text{ad}(JE)[JP, P])$$

The left side of the equation (3.6) is equal to

$$\frac{d}{dt} \rho(J \exp t \text{ad}(JE)P, \exp t \text{ad}(JE)P)$$

$$= \frac{d}{dt} \rho(J \exp t(1/2+βJ)u(t), \exp t(1/2+βJ)u(t))$$

$$= \frac{d}{dt} \rho(\exp t(1/2+βJ)Ju(t), \exp t(1/2+βJ)u(t))$$

$$= \frac{d}{dt} e^{\frac{3}{4}+iβ^2} e^{\frac{3}{2}+iβ^2} \rho(Ju(t), u(t))$$

$$= \frac{d}{dt} e^t \rho(Ju(t), u(t))$$

$$= e^t (h'(t)+h(t))$$

where $h(t)=\rho(Ju(t), u(t))$, and $h(t)$ is a polynomial of degree $\leq 2m-2$. Since $[JP, P]=λE+P'$, where $λ \in R, P' \in p_{1}$, the right side of the equation (3.6) is equal to

$$\rho(JE, \exp t \text{ad}(JE)(λE+P'))$$

$$= e^t \lambda \rho(JE, E)+\rho(JE, \exp t \text{ad}(JE)P')$$

$$= e^t \lambda \rho(JE, E).$$

Hence we have

$$h'(t)+h(t) = λ \rho(JE, E).$$

The solution of this equation is $h(t)=ce^{-t}+λ \rho(JE, E)$, where $c$ is an arbitrary
constant. However, \( h(t) \) is a polynomial and so \( c=0 \). Hence we have

\[
h(t) = \lambda \rho(JE, E),
\]

and hence it follows that

\[
\lambda = \frac{h(t)}{\rho(JE, E)} = \frac{h(0)}{\rho(JE, E)} = \frac{\rho(JP, P)}{\rho(JE, E)} > 0.
\]

Therefore we have

\[
2\eta(P, P) = \psi([JP, P]) = \lambda \psi(E) + \psi(P') = \lambda \psi(E) > 0.
\]

This shows that \( \eta \) is positive definite on \( \mathfrak{p}_{(\mathfrak{h}+\mathfrak{m})} \) and hence on \( \mathfrak{p} = \sum_{\mathfrak{p}} \mathfrak{p}_{(\mathfrak{h}+\mathfrak{m})} \).

Q.E.D.

Proof of Theorem 1'. Since \( \eta \) is positive definite on \( \{JE\} + \{E\} + \mathfrak{p} \) and since the signature of \( h \) is \((2, 2(n-1))\), we have \( \mathfrak{p}_1 = \{0\} \), and hence \( \mathfrak{p} = \mathfrak{p}_1 \). Let \( P, Q \in \mathfrak{p} \). Since \([P, Q] \in \mathfrak{g}_0, \mathfrak{g}_3] \subseteq \mathfrak{g}_0 \), where \( \mathfrak{g}_0 = \{JE\} + \mathfrak{p} \), we put \([P, Q] = \lambda JE + P' \), where \( \lambda \in \mathfrak{r}, P' \in \mathfrak{p} \). It follows that \([E, [P, Q]] = [E, \lambda JE + P'] = -\lambda E \) and \([E, [P, Q]] = [[E, P], Q] + [P, [E, Q]] = 0 \). This implies that \( \lambda = 0 \) and \([P, Q] \in \mathfrak{p} \). Therefore \( \mathfrak{p} \) is a subalgebra of \( \mathfrak{g} \) and also an ideal of \( \mathfrak{g} \). Moreover we see easily that \( (\mathfrak{p}, \mathfrak{f}, J, \rho) \) is an effective Kähler algebra. Since the decomposition \( \mathfrak{g} = \{JE\} + \{E\} + \mathfrak{p} \) is orthogonal with respect to \( \eta \) and \( \eta \) is positive definite on \( \{JE\} + \{E\} \) and since the signature of \( h \) is \((2, 2(n-1)) \), we know that \( \eta(P, P) < 0 \), for \( P \in \mathfrak{p}, P \notin \mathfrak{f} \). Now, for \( P, Q \in \mathfrak{p}, P \notin \mathfrak{f} \), put

\[
\psi'(P) = Tr_{\mathfrak{p}/\mathfrak{f}}(ad(JP) - J ad(P)),
\]

\[
2\gamma(P, Q) = \psi'([JP, Q])
\]

For \( P \in \mathfrak{p}, P \notin \mathfrak{f} \), we have \((ad(JP) - J ad(P))E = 0\), \((ad(JP) - J ad(P))JE \equiv 0 \) (mod \( \mathfrak{f} \)) and hence \( \psi(P) = \psi'(P) \). This implies that

\[
2\gamma(P, P) = \psi'([JP, P]) = \psi([JP, P]) = 2\eta(P, P) < 0,
\]

which proves that the canonical hermitian form of \( (\mathfrak{p}, \mathfrak{f}, J, \rho) \) is negative definite. Therefore we know that \( \mathfrak{p} \) is a compact semi-simple subalgebra of \( \mathfrak{g} \) [5].

Q.E.D.

Proof of Theorem 1. When \( G \) is a semi-simple Lie group, our assertion follows from the results of Borel [1] and Koszul [3]. We shall show the case where \( G \) contains a one parameter normal subgroup of \( G \). Let \( \{E\} \) be the ideal
of $\mathfrak{g}$ corresponding to the one parameter subgroup. With appropriate choice of $J$, we may assume that $JE=-E$. Put $\mathfrak{g'}=\{JE\}+\{E\}$. Then $(\mathfrak{g'}, J, \rho)$ is a Kähler algebra of the unit disk $\{z \in \mathbb{C}; |z| < 1\}$. Now, for $X, Y \in \mathfrak{g}$, we define
\[ \rho(X, Y) = \rho(X', Y') \]
where $X', Y'$ are the $\mathfrak{g}'$-components of $X, Y$ with respect to the decomposition $\mathfrak{g}=\mathfrak{g}'+\mathfrak{p}$ respectively. Then $(\mathfrak{g}, \mathfrak{p}, J, \rho)$ is a Kähler algebra. We denote by $G'$ (resp. $P$) the connected subgroup of $G$ corresponding to $\mathfrak{g}'$ (resp. $\mathfrak{p}$). Since $(\mathfrak{g}, \mathfrak{p}, J, \rho)$ is a Kähler algebra, $G/P$ admits an invariant Kähler structure and is holomorphically isomorphic to the $G'$-orbit passing through the origin $o$. We know by Theorem 1' that $G/K$ is a holomorphic fibre bundle whose base space is $G/P \cong \{z \in \mathbb{C}; |z| < 1\}$, and whose fibre is $P/K$. Q.E.D.

4. Proof of Theorem 2

Let $(\mathfrak{g}, \mathfrak{p}, J, \rho)$ be the Kähler algebra of $G/K$. We show that, if $\mathfrak{g}$ is not semi-simple, then there exists a one dimensional ideal of $\mathfrak{g}$. Assume that $\mathfrak{g}$ is not semi-simple. Then there exists a non-zero commutative ideal $\mathfrak{t}$ of $\mathfrak{g}$. Consider a $J$-invariant subalgebra $\mathfrak{g}'=\mathfrak{tl}+\mathfrak{t}J+\mathfrak{t}$. Then we have

**Lemma 4.1.** $\dim_{\mathbb{C}} \mathfrak{g}'/\mathfrak{t}=1$.

Since $\mathfrak{t} \cap \mathfrak{t} = \{0\}$ by Lemma 2.3 and $\dim_{\mathbb{C}} \mathfrak{g}/\mathfrak{t}=2$, $\dim_{\mathbb{C}} \mathfrak{g}'/\mathfrak{t}=1$ or 2. Suppose that $\dim_{\mathbb{C}} \mathfrak{g}'/\mathfrak{t}=2$. Then $\dim_{\mathbb{C}} \mathfrak{g}/\mathfrak{t}=\dim_{\mathbb{C}} \mathfrak{g}'/\mathfrak{t}$, $\dim \mathfrak{g}=\dim \mathfrak{g}'$ and hence $\mathfrak{g}=\mathfrak{tl}+\mathfrak{t}J+\mathfrak{t}$. Since $\dim \mathfrak{g}'/\mathfrak{t}=4$ and $\mathfrak{t} \cap \mathfrak{t} = \{0\}$, we have $2 \leq \dim \mathfrak{t} \leq 4$. Let $\pi'$ be the projection from $\mathfrak{g}$ onto $\mathfrak{g}/\mathfrak{t}$. Then it follows that
\[
\dim \pi'(\mathfrak{t}J) \cap \pi'(\mathfrak{t}) = \dim \pi'(\mathfrak{t}J) + \dim \pi'(\mathfrak{t}) - \dim \pi'(\mathfrak{g}) = 2 \dim \mathfrak{t} - 4.
\]
First, we shall show $\dim \mathfrak{t} \neq 3, 4$. Suppose $\dim \mathfrak{t} = 3$ or 4. Then $\dim \pi'(\mathfrak{t}J) \cap \pi'(\mathfrak{t}) \geq 0$, and so there exist $A \neq 0, B \neq 0 \in \mathfrak{t}$ and $W \in \mathfrak{t}$ such that $JA=B+W$. Therefore we have $2\eta(A, C)=\psi([JA, C])=\psi([B+W, C])=0$ and $2\eta(A, JC)=\psi([JA, JC])=\psi([A, C])=0$ for all $C \in \mathfrak{t}$. Since $\mathfrak{g}=\mathfrak{tl}+\mathfrak{t}J+\mathfrak{t}$, we know $\eta(A, X)=0$ for all $X \in \mathfrak{g}$. This implies $A \in \mathfrak{t}$, which is a contradiction to Lemma 2.3. Next, we shall prove $\dim \mathfrak{t} \neq 2$. Suppose $\dim \mathfrak{t} = 2$. Then $\dim \pi'(\mathfrak{t}J) \cap \pi'(\mathfrak{t}) = 0$, and hence $\mathfrak{g}=\mathfrak{tl}+\mathfrak{t}J+\mathfrak{t}$ is a direct sum as vector spaces. Let $A$ be an element in $\mathfrak{t}$ such that $\eta(A, B)=0$ for all $B \in \mathfrak{t}$. Since $\mathfrak{g}=\mathfrak{tl}+\mathfrak{t}J+\mathfrak{t}$ and $2\eta(A, JB)=\psi([JA, JB])=\psi([A, B])=0$, we have $\eta(A, X)=0$ for any $X \in \mathfrak{g}$, which implies $A \in \mathfrak{t}$, and hence $A=0$ by Lemma 2.3. This shows that $\eta$ is non-degenerate on $\mathfrak{t}$. Therefore there exists a unique non-zero element $E \in \mathfrak{t}$ such that $2\eta(E, A)=\psi(A)$ for all $A \in \mathfrak{t}$. We have then
Indeed, for \( A \in \mathfrak{r} \) we have

\[
2\eta([JE, E], A) = \psi([J[JE, E], A]) = -\psi([[JE, E], JA]) = \psi([[E, JA], JE]) + \psi([[JA, JE], E]) = -\psi(A) + \psi([JA, E]) + \psi([A, JE]) = \psi(A).
\]

This shows that \([JE, E] = E\). Let \( F \) be an element in \( \mathfrak{r} \) independent of \( E \). Put \([JE, F] = \lambda E + \mu F\), where \( \lambda, \mu \in \mathbb{R} \). Then \( \psi(F) = Tr_{\mathfrak{g}/\mathfrak{r}}(ad(JE) - J ad(E)) = 2(1 + \mu) \). We shall show \( \psi(F) \neq 0 \). Suppose \( \psi(F) = 0 \). Then \( \mu = -1 \) and \( \psi(F) = 2\eta(E, F) = \psi([JE, F]) = \lambda \psi(E) - \psi(F) = -\psi(F) \). Therefore \( \psi(F) = 0 \), and hence \( \psi = 0 \) on \( \mathfrak{r} \), which is a contradiction to Lemma 2.4.

(4.2) There exists an element \( F \) in \( \mathfrak{r} \) independent of \( E \) such that

\[
[JF, E] = \beta F, \quad [JF, F] = -E, \quad \psi(F) = 0,
\]

where \( \alpha, \beta \in \mathbb{R} \).

Proof. By \( 2\eta(E, E) = \psi(E) \neq 0 \), there exists \( \tilde{F} \neq 0 \) such that \( 2\eta(E, \tilde{F}) = \psi(\tilde{F}) = 0 \). Since \( \eta \) is non-degenerate on \( \mathfrak{r} \) and the signature is \((1,1)\), we have \( \eta(E, E) \eta(\tilde{F}, \tilde{F}) < 0 \). Put \([JE, \tilde{F}] = \alpha \tilde{F} + \alpha' E\), where \( \alpha, \alpha' \in \mathbb{R} \). We have then \( 0 = \psi(\tilde{F}) = 2\eta(E, \tilde{F}) = \psi([JE, \tilde{F}]) = \alpha \psi(\tilde{F}) + \alpha' \psi(E) = \alpha' \psi(E) \), and hence \( \alpha' = 0 \), and \([JE, F] = \alpha F\). Similarly we have \([J \tilde{F}, E] = \beta \tilde{F}\). Now, we put \([J \tilde{F}, \tilde{F}] = \gamma E + \delta \tilde{F}\), where \( \gamma, \delta \in \mathbb{R} \). Then we have \( 0 = \psi(\tilde{F}) = Tr_{\mathfrak{g}/\mathfrak{r}}(ad(J \tilde{F}) - J ad(\tilde{F})) = 28 \) and so \([J \tilde{F}, \tilde{F}] = \gamma E\). Since \( 2\eta(\tilde{F}, \tilde{F}) = \psi([J \tilde{F}, \tilde{F}]) = \gamma \psi(E) = 2\gamma \eta(E, E) \), it follows \( \gamma = \frac{\eta(\tilde{F}, \tilde{F})}{\eta(E, E)} < 0 \). Putting \( F = \frac{1}{\sqrt{-\gamma}} \tilde{F} \), we have \([JF, F] = -E\).

Q.E.D.

(4.3) \( \mathfrak{f} = \{0\} \).

Proof. For \( W \in \mathfrak{f} \), put \([W, E] = \lambda E + \mu F\). Since \( 0 = \psi([W, E]) = \lambda \psi(E) + \mu \psi(F) = \lambda \psi(E) \), \( \lambda = 0 \) and hence \([W, E] = \mu F\). We have \( \psi([[JF, [W, E]]]) = -\mu \psi(E) \) and \( \psi([[JF, [W, E]]]) = \psi([[JF, W], E]) + \psi([W, [JF, E]]) = \psi([J[F, W], E]) = \psi([JE, [F, W]]) = \psi([F, W]) = 0 \). Thus \( \mu = 0 \) and \([W, E] = 0\). Now, put \([W, F] = \lambda E + \mu F\). By \( 0 = \psi([W, F]) = \lambda \psi(E) + \mu \psi(F) = \lambda \psi(E) \),
we have \( \lambda = 0 \) and \([W, F] = \mu F\). Hence it follows \( \psi([JF, [W, F]]) = -\mu \psi(E) \).

On the other hand we have \( \psi([JF, [W, F]]) = \psi([[JF, W], F]) = -\mu \psi(E) \).

Therefore \( 2\mu \psi(E) = 0 \), and hence \( \mu = 0 \), \([W, F] = 0\). Thus \([\mathfrak{I}, \mathfrak{I}] = 0\). Since \([\mathfrak{I}, J\mathfrak{I}] \subset \mathfrak{I}, [\mathfrak{I}, \mathfrak{I}] \subset \mathfrak{I}\) and \(g = \mathfrak{I} + J\mathfrak{I} + \mathfrak{I}\), we know that \(\mathfrak{I}\) is an ideal of \(g\). By the effectiveness, we have \(\mathfrak{I} = \{0\}\) .

Q.E.D.

(4.4)

\[ 2\alpha = \beta + 1. \]

Proof. Using Jacobi identity and (1.3), we have

\[ 0 = [[JE, JF], F] + [[JF, F], JE] + [[F, JE], JF] \]
\[ = [[JE, F] + [JF, F], JF] + [[F, JE], JF] = \alpha - \beta \] \[= (\alpha - \beta)(JF, E)^{-}E - \alpha(F, JF) \]
\[= (\alpha - \beta - \beta + 1)E. \]

Hence it follows \(2\alpha = \beta + 1\).

Q.E.D.

By (1.7), (4.2) and (4.4), we have

\[ 0 = \rho([JF, F], JF) + \rho([F, JF], JE) + \rho([JE, F], F) \]
\[= \alpha \rho(F, JF) + \rho(E, JE) - (\alpha - \beta) \rho(JF, F) \]
\[= (-2\alpha + \beta) \rho(JF, F) - \rho(JE, E) \]
\[= -\rho(JF, F) - \rho(JE, E). \]

This contradicts to \(\rho(JF, E) > 0, \rho(JF, F) > 0\). Therefore \(\dim \mathfrak{r} = 2\) and hence \(\dim_c g' \mathfrak{I} = 2\). Thus we have proved \(\dim_c g' \mathfrak{I} = 1\), this completes the proof of Lemma 4.1.

Q.E.D.

Let \(\mathfrak{r} \neq \{0\}\) be a commutative ideal of \(g\). Since \(\dim g' \mathfrak{I} = 2\) by Lemma 4.1 and \(\mathfrak{I} \cap \mathfrak{r} = \{0\}\), it follows that \(\dim \mathfrak{r} = 1\) or 2. Assume \(\dim \mathfrak{r} = 2\). Then we have

\[ \dim \pi'(J\mathfrak{r}) \cap \pi'(\mathfrak{r}) = \dim \pi'(J\mathfrak{r}) + \dim \pi'(\mathfrak{r}) - \dim (\pi'(J\mathfrak{r}) + \pi'(\mathfrak{r})) \]
\[= 2 \dim \mathfrak{r} - 2 \]
\[= 2. \]

This implies \(\pi'(J\mathfrak{r}) = \pi'(\mathfrak{r})\) and hence \(J\mathfrak{r} \subset \mathfrak{I} + \mathfrak{r}\). For any \(A \in \mathfrak{r}\), we have \(JA = A' + W\), where \(A' \in \mathfrak{r}, W \in \mathfrak{I}\). It follows then

\[ \psi(A) = Tr g' \mathfrak{I}(ad(JA) - J ad(A)) \]
\[= Tr g' \mathfrak{I}(ad(A') - J ad(A)) + Tr g' \mathfrak{I}(ad(W)) \]
\[= Tr \mathfrak{I} + \mathfrak{r}(ad(A') - J ad(A)) \]
\[= 0. \]
Hence $\psi = 0$ on $\tau$, which is a contradiction to Lemma 2.4. Thus $\tau$ is a one-dimensional ideal of $g$. Therefore Theorem 2 is proved. Q.E.D.

5. We shall classify two dimensional connected simply connected homogeneous Kähler manifolds with non-degenerate canonical hermitian form $h$. The signature of $h$ is $(4, 0)$ or $(2, 2)$ or $(0, 4)$.

(i) The case $(4, 0)$. Since $h$ is positive definite, $G/K$ is isomorphic to a homogeneous bounded domain. Hence $G/K$ is either $\{z \in \mathbb{C}; |z| < 1\} \times \{z \in \mathbb{C}; |z| < 1\}$ or $\{(z_1, z_2) \in \mathbb{C}^2; |z_1|^2 + |z_2|^2 < 1\}$.

(ii) The case $(0, 4)$. Since $h$ is negative definite, $G$ is a compact semi-simple Lie group by [5]. By a theorem in [4], $G/K$ is a hermitian symmetric space. Hence $G/K$ is either $P_n(\mathbb{C}) \times P_1(\mathbb{C})$ or $P_2(\mathbb{C})$, where $P_n(\mathbb{C})$ is a complex $n$-dimensional projective space.

(iii) The case $(2, 2)$. Applying Theorem 1 and 2, we obtain that $G/K$ is a holomorphic fibre bundle whose base space is the unit disk $\{z \in \mathbb{C}; |z| < 1\}$, and whose fibre is $P_1(\mathbb{C})$.

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References


