A ring $R$ is called right bounded if every essential right ideal contains a nonzero two-sided ideal which is an essential ideal as a right ideal. If $R$ is both right and left bounded, then we call $R$ bounded. The boundedness is well studied for noetherian rings ([1]). In this paper, we study (Von Neumann) regular rings which are right bounded. This class includes all regular rings whose primitive factor rings are artinian. We know from S. Page [3] that a regular ring $R$ is a FPF-ring if and only if $R$ is a self-injective ring with bounded index of nilpotence. As a result, it follows that regular FPF-rings are bounded.

In Proposition 1, we shall show that right non-singular right FPF-rings are right bounded. In our main theorem (Theorem 1), we shall show that a regular right self-injective ring $R$ is right bounded if and only if $R \cong \prod M_{n(i)}(R_i) \times \prod T_s$, where each $R_i$ is an abelian regular self-injective ring and each $T_s$ is a right full linear ring. In Theorem 2, as an application of Theorem 1, we shall give a necessary and sufficient condition for the maximal right quotient ring $Q(R)$ of a regular ring $R$ to be Type $I_f$.

1. Preliminaries

Throughout this paper, $R$ denotes an associative ring with identity element and we assume that all $R$-modules are unitary. We freely use terminologies and the results in [2].

Let $R$ be a regular, right self-injective ring. $R$ is called Type I if it contains an idempotent $e$ such that $eR$ is faithful right $R$-module and $eRe$ is an abelian regular ring. $R$ is called Type II if it contains an idempotent $e$ such that $eR$ is faithful right $R$-module and $eRe$ is a directly finite regular ring, and $R$ contains no nonzero idempotent $f$ such that $fRf$ is an abelian regular ring. $R$ is called Type III if it contains no nonzero idempotent $e$ such that $eRe$ is directly finite.

Goodearl and Boyle have shown that if $R$ is a regular, right self-injective ring, then $R = R_1 \times R_2 \times R_3$, where $R_1$ is Type I, $R_2$ is Type II and $R_3$ is Type III (Theorem 10.13 [2]). Furthermore, they have shown that if $R$ is a regular, right
self-injective ring of Type $I_f$ (Type $I$ and directly finite), then $R \cong \prod M_{\alpha(i)}(R_i)$, where each $R_i$ is an abelian regular, self-injective ring (Theorem 10.24 [2]).

For any ring $R$, we use $B(R)$ to denote the set of all central idempotents of $R$. If $A$ and $B$ are modules, then the notation $A \subseteq B$ means that $A$ is an essential submodule of $B$.

Let $A$ be a nonzero right $R$-module over a regular, right self-injective ring $R$. Put $X = \{ e \in B(R) | A(1-e) \neq 0 \}$. Since $B(R)$ is a complete Boolean algebra by Proposition 9.9 [2], there exists an element $e$ in $B(R)$ such that $e = \bigwedge X$. We call $e$ the central cover of $A$ and denote it by $cc(A)$.

2. Structure theorem

A ring $R$ is said to be right bounded if any essential right ideal $I$ of $R$ contains a two-sided ideal $J$ such that $J \subseteq I$ as a right $R$-module. If $R$ is both right and left bounded, $R$ is simply called bounded. A ring $R$ is called right finitely pseudo-Frobenius (FPF) if every finitely generated faithful right $R$-module is a generator in the category of right $R$-modules.

S. Page has shown in [3] that, for a given regular ring $R$, the following conditions are equivalent: 1) $R$ is right FPF. 2) $R$ is left FPF. 3) $R$ is a right self-injective ring with bounded index of nilpotence. Therefore, combining this theorem with [2, Lemma 6.20 and Theorem 7.20] we see that regular right FPF-rings are bounded rings. This result suggests the following.

Proposition 1. If $R$ is a right non-singular right FPF-ring, then $R$ is a right bounded ring.

Proof. Let $I$ be an essential right ideal of $R$, and let $J$ be the right annihilator ideal of $R/I$. Assume that $J \neq 0$. Then $R/I$ is a generator, since $R$ is a right FPF-ring. Hence $R/I$ generates $R$. However this is impossible since $R$ is a non-singular right $R$-module and $R/I$ is a singular right $R$-module. Therefore $J = 0$. Now, by Proposition 3 [2], we can obtain a central idempotent $e$ of $R$ such that $J \subseteq eR$. We claim that $e = 1$. If $e \neq 1$, then $I \cap (1-e)R = 0$. Put $M = ((1-e)R/(I \cap (1-e)R)) \oplus eR$. Then it is easy to see that $M$ is a faithful right $R$-module. Thus $M$ is a generator by the assumption. Hence there exist a positive integer $n$ and an epimorphism $f$ from the direct sum $nM$ of $n$-copies of $M$ to $(1-e)R$. Since $n((1-e)R/(I \cap (1-e)R))$ is singular and $(1-e)R$ is non-singular, we see that the restriction of $f$ on $n(eR)$ is an epimorphism. But this contradicts that $eR(1-e) = 0$.

Now, we are concerned with regular, right self-injective rings which are right bounded. Let $R$ be a regular, right self-injective right bounded ring. By Theorem 10.21 [2], there exists a decomposition $R = R_1 \times R_2$ such that $R_1$ is directly finite and $R_2$ is purely infinite, where a regular, right self-injective
ring \( R \) is called purely infinite if it contains no nonzero central idempotent \( e \) such that \( eR \) is directly finite. Clearly each \( R_i \) is a regular, right self-injective and right bounded ring. So we may observe \( R \) by dividing into two cases; the directly finite case and the purely infinite one. First we mention the directly finite case.

**Proposition 2.** Let \( R \) be a directly finite regular, right self-injective ring. Then the following conditions are equivalent.

1. \( R \) is right bounded.
2. \( R \) is bounded.
3. \( R \) is Type \( I_f \).

**Proof.** (3) \( \Rightarrow \) (2). According to Theorem 10.24 [2], \( R \) is isomorphic to \( \prod M_{\alpha}(R_i) \), where each \( R_i \) is an abelian regular, self-injective ring. Since each \( M_{\alpha}(R_i) \) has bounded index of nilpotence, it is bounded. Thus \( R \) is bounded.

(2) \( \Rightarrow \) (1). This is clear.

(1) \( \Rightarrow \) (3). By Theorem 10.22 [2], there exists a decomposition \( R=R_1 \times R_2 \) such that \( R_1 \) is Type \( I_f \) and \( R_2 \) is Type \( II_f \). We shall show that \( R_2=0 \). Assume that \( R_2 \neq 0 \), and put \( R=R_2 \) for brevity. For any finitely generated projective right \( R \)-module \( P \), there exists a decomposition \( P=P_1 \oplus P_2 \) such that \( P_1 \cong P_2 \) by Lemma 11.19 [2]. We use repeatedly this result. First we shall construct orthogonal idempotents \( e_1, e_2, \ldots \) in \( R \) such that \( R \cong 2^n(e_nR) \) and \( (1-\sum_{i=1}^n e_iR) \cong R/(\oplus e_iR) \cong e_nR \) for all \( n \). In fact, we can take an idempotent \( e_1 \) such that \( e_1R=(1-e_1)R \). Put \( I_2=(1-e_1)R \). Since there exists a decomposition \( I_2=J_1 \oplus J_2 \) such that \( J_1 \cong J_2 \), there exist orthogonal idempotents \( e_{21}, e_{22} \) such that \( J_1=e_{21}R \) and \( J_2=e_{22}R \), and \( 1-e_1=e_{21}+e_{22} \) by Proposition 2.11 [2]. Put \( e_2=e_{21} \). Then \( e_1e_2=e_2e_1=0 \), \( R \cong 2^2(e_2R) \) and \( R/(e_2R \oplus e_2R) \cong e_2R \). Continuing this procedure, we have desired idempotents \( \{e_n\} \). Now we put \( I=\oplus e_nR \). First we shall show that \( I \) does not contain a nonzero central idempotent. Suppose that there exists a nonzero central idempotent \( e \) in \( I \). Then \( e \) is in \( \oplus e_nR \) for some positive integer \( t \). This shows that \( e \cdot e_n=0 \) for all \( n>t \). On the other hand, \( 2^t(e \cdot e_nR) \cong eR \) for all \( n \geq 1 \). Consequently, we obtain that \( e=0 \). Therefore \( I \) does not contain a nonzero central idempotent of \( R \). Next we shall show that \( I \) does not contain a nonzero two-sided ideal of \( R \). We assume that \( RxR \subseteq I \) for some \( x \neq 0 \). Since \( R \) satisfies the general comparability by Theorem 9.14 [2], there exist \( f_n \in B(R) \) such that \( f_ne_n \leq f_ne_nR \) and \( (1-f_n)xR \leq (1-f_n)e_nR \) for all \( n \). If \( f_n=0 \) for all \( n \), then \( n(xR) \subseteq R \) for all \( n \). By Corollary 9.23 [2], we have that \( xR=0 \). Hence there exists \( t \) such that \( f_t \neq 0 \). Since \( f_te_nR \leq f_ne_nR \leq xR \) and \( fxe_nR \cong 2^t(f_te_nR) \leq 2^t(xR) \), we obtain that
by Corollary 2.23 \[2\]. Hence \((0+)f \subseteq I\), a contradiction. Finally we shall show that \(I\) is an essential right ideal of \(R\). Suppose that \(I \cap fR = 0\) for some idempotent \(f\) of \(R\). Since \(fR \cap \bigoplus_{i=1}^{t} e_i R = 0\) for all \(t\), we obtain that \(fR \subseteq R/\bigoplus_{i=1}^{t} e_i R = e_i R\). Since \(t(fR) \leq 2'(fR) \leq 2'(e_i R) = R\) for all \(t\), we have that \(R = 0\) by Corollary 9.23 \[2\]. This is a contradiction. Therefore a regular, right self-injective ring of Type \(II_f\) is not right bounded. Thus \(R\) is Type \(I_f\).

We note that the general comparability is left-right symmetric, and moreover, for idempotents \(e, f\) of a regular ring \(R\), \(R f \leq Re\) implies \(fR \leq eR\). Therefore by Proposition 2 and it's proof, we have the following corollary.

**Corollary 1.** Let \(R\) be a regular, right self-injective ring of Type \(II_f\). Then \(R\) is neither right nor left bounded.

**Corollary 2.** Let \(R\) be a regular, right self-injective and right bounded ring. Then \(R\) is directly finite if and only if \(R\) is left self-injective.

Proof. If \(R\) is a directly finite, then \(R\) is Type \(I_f\) by Proposition 2. Hence \(R\) is left self-injective by Corollary 10.25 \[2\]. The converse is clear by Theorem 9.29 \[2\].

Now we consider the case that \(R\) is a purely infinite, right self-injective, regular ring. Let \(Z(R)\) be the center of \(R\) and \(S\) be the socle of \(Z(R)\). There exists a central idempotent \(e\) of \(R\) such that \(S \subseteq eZ(R)\). Therefore we have a decomposition \(R = R_1 \times R_2\) such that \(Z(R_1)\) has an essential socle and \(Z(R_2)\) has a zero socle.

**Proposition 3.** Let \(R\) be a purely infinite, regular, right self-injective ring whose socle of \(Z(R)\) is zero. Then \(R\) is neither right nor left bounded.

Proof. According to Theorems 9.7 and 10.16 \[2\], there exist orthogonal idempotents \(e_1, e_2, \ldots\) in \(R\) such that \(\bigoplus e_n R\) is an essential right ideal of \(R\) and \(e_n R \approx R_g\) for all \(n\). First we show that \(R\) is not left bounded. Assume that \(R\) is left bounded, then since \(Re_n \approx R\) for all \(n\), we set \(H = (\bigoplus Re_n) \oplus I\), where \(I\) is a complement left ideal of \(\bigoplus Re_n\) in \(R\), and take a nonzero two-sided ideal \(J\) in \(H\). Choose a nonzero element \(x\) in \(J\) and set \(c = cc(xR)\). Since \(Soc(Z(R)) = 0\), we can write \(c = \bigvee c_n\) for some orthogonal, countably infinite, nonzero central idempotents \(c_1, c_2, \ldots\) in \(R\). Since \(Re_n \approx R\), we have a nonzero \(\varphi_n \in Hom_R (Rxc_n, Re_n)\) for all \(n\). We define an \(R\)-homomorphism \(\varphi: \bigoplus Rxc_n \rightarrow \bigoplus Re_n\) by \(\varphi(\sum rxc_n) = \varphi_n(\sum r_nxc_n)\). Furthermore, each \(\varphi_n\) can be extended to an \(R\)-
homomorphism \( \varphi = (\varphi_a) : \prod R gösterim için \Pi R^e_{R^e} \) and there exists a natural isomorphism \( \theta : R \rightarrow \prod R a_n \) given by \( \theta(rc) = (rc_a) \) \[2, Proposition 9.10\]. Hence \( \varphi \) is given by the right multiplication by some element \( a \) of \( R \). Therefore we obtain that \( \varphi \) is the right multiplication by \( a \). On the other hand, since \( J \) is a two-sided ideal of \( R \), \( xa \in J \subseteq \oplus R e_{R^e} \oplus I \), so we can write that \( xa = \sum r_n e_{R^e} + r \) for some \( r_n \in R \) and \( r \in I \). But we observe that \( xa e_{R^e} \subseteq (Re_{R^e} \cap \sum \oplus R e_{R^e} \oplus I) (=0) \). This is a contradiction. Consequently \( H \) can not contain any nonzero two-sided ideal of \( R \). Therefore \( R \) is not left bounded. Similarly we can show that \( R \) is not right bounded. Now the proof is complete.

**Proposition 4.** Let \( R \) be a purely infinite, prime, regular right self-injective and right bounded ring. Then \( R \) is a right full linear ring.

**Proof.** Since \( R \) is a prime, regular, right self-injective ring, \( R \) is indecomposable as a ring. Hence \( \text{Soc}(R) \) is essential or zero. We claim that \( \text{Soc}(R) \) is essential. If not, then we can take a minimal ideal \( I \) of \( R \) by Theorem 12.23 \[2\]. Now there exists nonzero orthogonal, infinite, idempotents \( \{f_\beta\}_{\beta \in \kappa} \) in \( I \) such that \( \bigoplus \beta f_\beta R \subseteq \varepsilon I_R \) since \( \text{Soc}(R) = 0 \). If \( \bigoplus \beta f_\beta R = I \), then we can take countably, infinite orthogonal idempotents \( \{g_\beta n\} \) in \( f_\beta R \) for all \( \beta \), such that \( \bigoplus \beta g_\beta n R \subseteq f_R \) since \( \text{Soc}(R) = 0 \). So we may assume that \( \bigoplus \beta f_\beta R \neq I \). On the other hand, since \( I \) is a minimal ideal of \( R \), \( \bigoplus \beta f_\beta R \) can not contain nonzero ideal. This contradicts that \( R \) is right bounded. Therefore \( \text{Soc}(R) \) is an essential ideal of \( R \), as claimed. Then \[2, Theorem 9.12\] shows that \( R \) is a right full linear ring.

**Corollary.** Let \( R \) be a purely infinite, regular, right self-injective and right bounded such that \( \text{Soc}(Z(R)) \) is an essential ideal of \( Z(R) \). Then \( R \approx \Pi R \), where each \( R \) is a right full linear ring.

**Proof.** Let \( \{e_{\alpha n} : n \in I \} \) be all central primitive idempotents of \( \text{Soc}(Z(R)) \). Evidently, we obtain that \( \bigvee e_{\alpha} = 1 \) since \( \text{Soc}(Z(R)) \subseteq Z(R) \). Then by Proposition 9.10 \[2\], \( R \approx \Pi e_{\alpha} R \), where each \( e_{\alpha} R \) is a prime, regular, right self-injective and right bounded ring. Then Proposition 4 shows that each \( e_{\alpha} R \) is a right full linear ring.

Now we shall prove our main theorem.

**Theorem 1.** Let \( R \) be a regular, right self-injective ring. Then the following conditions are equivalent.

1. \( R \) is right bounded.
(2) \( R \approx \prod M_{\pi(R)}(R_t) \times \prod T_t \), where each \( R_t \) is an abelian regular, self-injective ring and each \( T_t \) is a right full linear ring.

Proof. \((1) \Rightarrow (2)\). We have a decomposition \( R = R_1 \times R_2 \) such that \( R_1 \) is directly finite and \( R_2 \) is purely infinite by Theorem 10.21 [2]. Since each \( R_t \) is also right bounded, \( R_1 \) is Type I by Proposition 2 and \( R_2 \) is isomorphic to a direct product of right full linear rings by Propositions 3 and 4. Furthermore, Theorem 10.24 [2] shows that \( R \approx \prod M_{\pi(R)}(R_t) \), where each \( R_t \) is an abelian regular, self-injective ring.

\((2) \Rightarrow (1)\). It is easily seen that a direct product of right bounded rings is also right bounded. Moreover, we have pointed out that a regular, right self-injective of bounded index of nilpotence is right bounded, and by Theorem 9.12 [2], we see that a right full linear ring is also right bounded. Therefore \( R \) is right bounded.

**Corollary.** Let \( R \) be a regular, right self-injective ring. Then if \( R \) is right bounded, then \( R \) is bounded.

Proof. This is clear by Theorem 1.

### 3. Application

In this section, we apply Theorem 1 to a right bounded regular ring which is not necessarily self-injective.

**Lemma 1.** Let \( R \) be a regular, right bounded ring such that every nonzero two-sided ideal of \( R \) contains a nonzero central idempotent. Then the maximal right quotient ring \( Q(R) \) of \( R \) is also right bounded.

Proof. Let \( I \) be an essential right ideal of \( Q(R) \). Then clearly \( I \cap R \subseteq R_2 \). Since \( R \) is right bounded, there exists a nonzero two-sided ideal \( J \) of \( R \) such that \( J \subseteq I \cap R \). From the assumption, \( J \) contains a nonzero central idempotent \( e \) of \( R \). Note that \( e \) is also central in \( Q(R) \). Thus \( I \) contains a nonzero central idempotent of \( Q(R) \). Let \( H \) be the ideal generated by all central idempotents in \( I \). We claim that \( H \subseteq eQ(R)Q(R) \). If not, then \( l_{Q(R)}(H) \), the left annihilator ideal of \( H \) in \( Q(R) \), is not zero and \( H \cap l_{Q(R)}(H) = 0 \) since \( Q(R) \) is semi-prime. Hence there exists a nonzero central idempotent \( f \) in \( J \cap l_{Q(R)}(H) \) since \( J \cap l_{Q(R)}(H) \) is a nonzero two-sided ideal of \( R \). On the other hand, \( f \) is in \( H \) since \( f \in J \subseteq I \). But this contradicts that \( H \cap l_{Q(R)}(H) = 0 \). Consequently, \( H \) is an essential right ideal of \( Q(R) \), as claimed. Therefore \( Q(R) \) is right bounded.

**Theorem 2.** Let \( R \) be a regular ring and \( Q \) be the maximal right quotient ring of \( R \). Then the following conditions are equivalent.

(1) \( Q \approx \prod M_{\pi(S)}(S_t) \), where each \( S_t \) is an abelian regular, self-injective ring.
(2) There exist orthogonal central idempotents \( \{e_i\} \) such that \( e_i R \cong M_{n(i)}(A_{i}) \), where each \( A_i \) is an abelian regular ring and \( \sum_{i; \sigma \in \mathcal{K}_i} e_i R \subseteq R_R. \)

(3) For any nonzero two-sided ideal \( J \) of \( R \), \( J \) contains a nonzero central idempotent \( e \) such that \( eR \) is isomorphic to a full matrix ring over an abelian regular ring.

(4) \( R \) is right bounded and every nonzero two-sided ideal of \( R \) contains a nonzero central idempotent of \( R \).

Proof. (1)\( \Rightarrow \) (2). Since \( Q \cong \prod M_{n(t)}(S_t) \), there exist orthogonal central idempotents \( f_1, f_2, \ldots \) in \( Q \) such that \( f_t Q = M_{n(t)}(S_t) \). Clearly, \( f_t Q \) has bounded index \( n(t) \), so \( R \cap f_t Q \) has also bounded index \( n(t) \) by Corollary 7.4 [2]. Thus in view of Theorem 7.2 and Lemma 7.17 [2], we have a central idempotent \( e_{i_0} \) in \( R \cap f_t Q \) such that \( e_{i_0} R \cong M_{n(t)}(A_{i_0}) \) for some abelian regular ring \( A_{i_0} \). It is easy to see that \( e_{i_0} \) is a central idempotent of \( R \). For a maximal family \( \{e_{i_0} \}_{\sigma \in \mathcal{K}_i} \) of orthogonal central idempotents of \( f_t Q \cap R \) as above, we note that \( \sum_{i; \sigma \in \mathcal{K}_i} e_{i_0} R \subseteq f_t Q \cap R_{f_t Q \cap R} \). Next we show that \( \sum_{i; \sigma \in \mathcal{K}_i} e_{i_0} R \subseteq R_R \). Let \( I \) be a right ideal of \( R \) such that \( \sum_{i; \sigma \in \mathcal{K}_i} e_{i_0} R \cap I = 0 \). Since \( \bigoplus f_t Q \) is an essential right ideal of \( Q \), if \( I \) is not zero, then \( f_t Q \cap I \neq 0 \) for some \( t \). But, then since \( \sum_{i; \sigma \in \mathcal{K}_i} e_{i_0} R \) is an essential \( R \)-submodule of \( f_t Q \cap R \), this is impossible by our assumption. Hence \( I \) must be zero, so \( \sum_{i; \sigma \in \mathcal{K}_i} e_{i_0} R \) is an essential right ideal of \( R \).

(2)\( \Rightarrow \) (3). Let \( J \) be a nonzero two-sided ideal of \( R \). Since \( \sum_{i; \sigma \in \mathcal{K}_i} e_{i_0} R \) is an essential right ideal of \( R \), there exists a positive integer \( t_a \) such that \( e_{i_0} R \cap J \neq 0 \). Now since \( e_{i_0} R \cong M_{n(t)}(A_{i_0}) \) for some abelian regular ring \( A_{i_0} \), Theorem 6.6 [2] shows that \( J \cap e_{i_0} R \) contains a nonzero central idempotent \( e \) of \( R \) such that \( eR \) is isomorphic to a full matrix ring over an abelian regular ring.

(3)\( \Rightarrow \) (4). We shall show that \( R \) is right bounded. Let \( I \) be an essential right ideal of \( R \). First we claim that \( I \) contains a nonzero central idempotent \( e \) of \( R \). Choose a nonzero element \( x \) of \( R \). Then \( RxR \) contains a nonzero central idempotent \( f \) of \( R \) such that \( fR \) is isomorphic to a full matrix ring over an abelian regular ring. Since \( I \) is essential, \( I \cap fR \subseteq fR \). Now since an essential right ideal of a full matrix ring over an abelian regular ring contains a central idempotent, it is easy to see that \( I \) contains a nonzero central idempotent \( e \) of \( fR \). Clearly, \( e \) is also central in \( R \), so \( I \) contains a nonzero central idempotent \( e \) of \( R \), as claimed. Let \( H \) be the ideal generated by all central idempotents in \( I \). We show that \( H \) is an essential right ideal of \( R \). If not, then \( l_R(H) \), the left annihilator ideal of \( H \) in \( R \), is nonzero and \( H \cap l_R(H) = 0 \). On the other hand, since \( I \) is an essential right ideal of \( R \), \( I \cap l_R(H) = 0 \), so \( I \cap l_R(H) \) contains a nonzero central idempotent \( h \) of \( R \). But this is a contradiction. Therefore \( R \)
is right bounded.

(4)\Rightarrow(1). By Lemma 1, \( Q \) is also right bounded. Hence we have that 
\( Q = Q_1 \times Q_2 \), where \( Q_1 \) is Type \( I_f \) and \( Q_2 \) is a direct product of right full linear rings by Theorem 1. By the assumption, every nonzero two-sided ideal of \( R \) contains a nonzero central idempotent of \( R \). Clearly, the same property holds for \( Q \). But it is easily seen that a right full linear ring does not satisfy this one. Hence we obtain that \( Q_2 = 0 \). Therefore \( Q \) is isomorphic to a direct product of full matrix rings over abelian regular, self-injective rings.

**Corollary.** Let \( R \) be a regular, right self-injective ring. Then \( R \) is isomorphic to a finite direct product of full matrix rings over abelian regular, self-injective rings if and only if

1. \( R \) is right bounded.
2. All prime ideals of \( R \) are maximal.

**Proof.** Since \( R \) is right self-injective, the condition (2) is equivalent to the condition that, for each \( x \in R \), the two-sided ideal \( RxR \) is generated by a central idempotent. Therefore Corollary is an immediate consequence of Theorems 1 and 2.

**Remark 1.** Without (1) or (2), Corollary can fail, as the following examples show.

There exists a prime regular, right self-injective ring which is right bounded, but not simple and not directly finite.

For example, choose a field \( F \), let \( V \) be a countable-infinite dimensional vector space over \( F \) and set \( R = \text{End}_F(V) \) and \( M = \{ x \in R \mid \dim_F(xV) < \infty \} \). Then clearly, \( R \) is a prime regular, right self-injective ring. Given any \( x \in R - M \), we have that \( \dim_F(xV) = \dim_F(V) \) and so \( xV \cong V \), whence \( xR \cong R_R \). This shows that \( R \) is not directly finite. On the other hand, \( R \) is right bounded by Theorem 1.

Furthermore, there exists a simple regular, right self-injective ring, which is not right bounded and not Type \( I_f \).

For example, choose fields \( F_1, F_2, \ldots \), set \( R_n = M_n(F_n) \) for all \( n = 1, 2, \ldots, \) and set \( R = \prod R_n \). Let \( M \) be a maximal ideal of \( R \) which contains \( \bigoplus R_n \). Then \( R/M \) be a simple right and left self-injective ring of Type \( II_f \) by Example 10.7 [2]. Therefore by Proposition 2, \( R/M \) is not right bounded.

**Remark 2.** Let \( R \) be a regular ring whose maximal right quotient ring \( Q(R) \) is right bounded. Then according to Theorem 1, \( R \) is also right bounded. But we don't know whether \( Q(R) \) is right bounded when \( R \) is right bounded.

**Acknowledgement.** The authors would like to express their sincere gratitude to Mr. J. Kado for his useful conversations.
References


Department of Mathematics
Osaka City University
Osaka, 558, Japan.