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# ON REGULAR SELF-INJECTIVE RINGS

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A ring $R$ is called right bounded if every essential right ideal contains a nonzero two-sided ideal which is an essential ideal as a right ideal. If $R$ is both right and left bounded, then we call $R$ bounded. The boundedness is well studied for noetherian rings ([1]). In this paper, we study (Von Neumann) regular rings which are right bounded. This class includes all regular rings whose primitive factor rings are artinian. We know from S. Page [3] that a regular ring $R$ is a FPF-ring if and only if $R$ is a self-injective ring with bounded index of nilpotence. As a result, it follows that regular FPF-rings are bounded.

In Proposition 1, we shall show that right non-singular right FPF-rings are right bounded. In our main theorem (Theorem 1), we shall show that a regular right self-injective ring $R$ is right bounded if and only if $R\cong\prod M_{n(t)}(R_t)\times\prod T_s$, where each $R_t$ is an abelian regular self-injective ring and each $T_s$ is a right full linear ring. In Theorem 2, as an application of Theorem 1, we shall give a necessary and sufficient condition for the maximal right quotient ring $Q(R)$ of a regular ring $R$ to be Type $I_f$.

## 1. Preliminaries

Throughout this paper, $R$ denotes an associative ring with identity element and we assume that all $R$-modules are unitary. We freely use terminologies and the results in [2].

Let $R$ be a regular, right self-injective ring. $R$ is called Type $I$ if it contains an idempotent $e$ such that $eR$ is faithful right $R$-module and $eRe$ is an abelian regular ring. $R$ is called Type $II$ if it contains an idempotent $e$ such that $eR$ is faithful right $R$-module and $eRe$ is a directly finite regular ring, and $R$ contains no nonzero idempotent $f$ such that $fRf$ is an abelian regular ring. $R$ is called Type $III$ if it contains no nonzero idempotent $e$ such that $eRe$ is directly finite.

Goodearl and Boyle have shown that if $R$ is a regular, right self-injective ring, then $R=R_1\times R_2\times R_3$, where $R_1$ is Type $I$, $R_2$ is Type $II$ and $R_3$ is Type $III$ (Theorem 10.13 [2]). Furthermore, they have shown that if $R$ is a regular, right self-injective ring of Type $I_f$ (Type $I$ and directly finite), then $R \cong \prod M_{n(t)}(R_t)$, where each $R_t$ is an abelian regular, self-injective ring (Theorem 10.24 [2]).

For any ring $R$, we use $B(R)$ to denote the set of all central idempotents of $R$. If $A$ and $B$ are modules, then the notation $A \subseteq_e B$ means that $A$ is an essential submodule of $B$.

Let $A$ be a nonzero right $R$-module over a regular, right self-injective ring $R$. Put $X = \{e \in B(R) \mid A(1-e) = 0\}$. Since $B(R)$ is a complete Boolean algebra by Proposition 9.9 [2], there exists an element $e$ in $B(R)$ such that $e = \wedge X$. We call $e$ the central cover of $A$ and denote it by $cc(A)$.

## 2. Structure theorem

A ring $R$ is said to be *right bounded* if any essential right ideal $I$ of $R$ contains a two-sided ideal $J$ such that $J \subseteq_e I$ as a right $R$-module. If $R$ is both *right and left bounded*, $R$ is simply called *bounded*. A ring $R$ is called *right finitely pseudo-Frobenius* (FPF) if every finitely generated faithful right $R$-module is a *generator* in the category of right $R$-modules.

S. Page has shown in [3] that, for a given regular ring $R$, the following conditions are equivalent: 1) $R$ is right FPF. 2) $R$ is left FPF. 3) $R$ is a right self-injective ring with bounded index of nilpotence. Therefore, combining this theorem with [2, Lemma 6.20 and Theorem 7.20] we see that regular right FPF-rings are *bounded* rings. This result suggests the following.

**Proposition 1.** *If $R$ is a right non-singular right FPF-ring, then $R$ is a right bounded ring.*

Proof. Let $I$ be an essential right ideal of $R$, and let $J$ be the right annihilator ideal of $R/I$. Assume that $J=0$. Then $R/I$ is a *generator*, since $R$ is a right FPF-ring. Hence $R/I$ generates $R$. However this is impossible since $R$ is a non-singular right $R$-module and $R/I$ is a singular right $R$-module. Therefore $J \neq 0$. Now, by Proposition 3 [3], we can obtain a central idempotent $e$ of $R$ such that $J \subseteq_e eR_R$. We claim that $e=1$. If $e \neq 1$, then $I \cap (1-e)R \neq 0$. Put $M = ((1-e)R/(I \cap (1-e)R)) \oplus eR$. Then it is easy to see that $M$ is a faithfyl right $R$-module. Thus $M$ is a *generator* by the assumption. Hence there exist a positive integer $n$ and an epimorphism $f$ from the direct sum $nM$ of $n$-copies of $M$ to $(1-e)R$. Since $n((1-e)R/(I \cap (1-e)R))$ is singular and $(1-e)R$ is non-singular, we see that the restriction of $f$ on $n(eR)$ is an epimorphism. But this contradicts that $eR(1-e)=0$.

Now, we are concerned with regular, right self-injective rings which are right bounded. Let $R$ be a regular, right self-injective right bounded ring. By Theorem 10.21 [2], there exists a decomposition $R = R_1 \times R_2$ such that $R_1$ is directly finite and $R_2$ is purely infinite, where a regular, right self-injective ring $R$ is called *purely infinite* if it contains no nonzero central idempotent $e$ such that $eR$ is directly finite. Clearly each $R_i$ is a regular, right self-injective and right bounded ring. So we may observe $R$ by dividing into two cases; the directly finite case and the purely infinite one. First we mention the directly finite case.

**Proposition 2.** *Let $R$ be a directly finite regular, right self-injective ring. Then the following conditions are equivalent.*

(1) *$R$ is right bounded.*

(2) *$R$ is bounded.*

(3) *$R$ is Type $I_f$.*

Proof. (3)$\Rightarrow$(2). According to Theorem 10.24 [2], $R$ is isomorphic to $\prod M_{n(t)}(R_t)$, where each $R_t$ is an abelian regular, self-injective ring. Since each $M_{n(t)}(R_t)$ has bounded index of nilpotence, it is bounded. Thus $R$ is bounded.

(2)$\Rightarrow$(1). This is clear.

(1)$\Rightarrow$(3). By Theorem 10.22 [2], there exists a decomposition $R=R_1\times R_2$ such that $R_1$ is Type $I_f$ and $R_2$ is Type $II_f$. We shall show that $R_2=0$. Assume that $R_2\neq 0$, and put $R=R_2$ for brevity. For any finitely generated projective right $R$-module $P$, there exists a decomposition $P=P_2\oplus P_2$ such that $P_1\cong P_2$ by Lemma 11.19 [2]. We use repeatedly this result. First we shall construct orthogonal idempotents $e_1, e_2, \cdots$ in $R$ such that $R_R\cong 2^n(e_nR)$ and $(1-\sum_{i=1}^{n} e_iR)\cong R/\oplus e_iR\cong e_nR$ for all $n$. In fact, we can take an idempotent $e_1$ such that $e_1R\cong(1-e_1)R$. Put $I_2=(1-e_1)R$. Since there exists a decomposition $I_2=J_1\oplus J_2$ such that $J_1\cong J_2$, there exist orthogonal idempotents $e_{21}$, $e_{22}$ such that $J_1=e_{21}R$ and $J_2=e_{22}R$, and $1-e_1=e_{21}+e_{22}$ by Proposition 2.11 [2]. Put $e_2=e_{21}$. Then $e_1e_2=e_2e_1=0$, $R\cong 2^2(e_2R)$ and $R/(e_1R\oplus e_2R)\cong e_2R$. Continuing this procedure, we have desired idempotents $\{e_n\}$. Now we put $I=\oplus_n e_nR$. First we shall show that $I$ does not contain a nonzero central idempotent. Suppose that there exists a nonzero central idempotent $e$ in $I$. Then $e$ is in $\oplus_{n=1}^{t} e_nR$ for some positive integer $t$. This shows that $e\cdot e_n=0$ for all $n>t$. On the other hand, $2^n(e\cdot e_nR)\cong eR$ for all $n>1$. Consequently, we obtain that $e=0$. Therefore $I$ does not contain a nonzero central idempotent of $R$. Next we shall show that $I$ does not contain a nonzero two-sided ideal of $R$. We assume that $RxR\subseteq I$ for some $x\neq 0$. Since $R$ satisfies *the general comparability* by Theorem 9.14 [2], there exist $f_n\in B(R)$ such that $f_ne_n\lesssim f_nxR$ and $(1-f_n)xR\lesssim(1-f_n)e_nR$ for all $n$. If $f_n=0$ for all $n$, then $n(xR)\lesssim R$ for all $n$. By Corollary 9.23 [2], we have that $xR=0$. Hence there exists $t$ such that $f_t\neq 0$. Since $f_te_tR\lesssim f_txR\subseteq xR$ and $f_tR\cong 2^t(f_te_tR)\lesssim 2^t(xR)$, we obtain that

$f_tR \subseteq RxR$ by Corollary 2.23 [2]. Hence $(0\neq)f_t \in I$, a contradiction. Finally we shall show that $I$ is an essential right ideal of $R$. Suppose that $I \cap fR=0$ for some idempotent $f$ of $R$. Since $fR \cap \bigoplus_{n=1}^{t} e_nR=0$ for all $t$, we obtain that $fR \lesssim R/\bigoplus_{n=1}^{t} e_nR \cong e_tR$. Since $t(fR) \lesssim 2^t(fR) \lesssim 2^t(e_tR) \cong R$ for all $t$, we have that $R=0$ by Corollary 9.23 [2]. This is a contradiction. Therefore a regular, right self-injective ring of Type $II_f$ is not right bounded. Thus $R$ is Type $I_f$.

We note that *the general comparability* is left-right symmetric, and moreover, for idempotents $e, f$ of a regular ring $R$, $Rf \lesssim Re$ implies $fR \lesssim eR$. Therefore by Proposition 2 and it's proof, we have the following corollary.

**Corollary 1.** *Let $R$ be a regular, right self-injective ring of Type $II_f$. Then $R$ is neither right nor left bounded.*

**Corollary 2.** *Let $R$ be a regular, right slef-injective and right bounded ring. Then $R$ is directly finite if and only if $R$ is left self-injective.*

Proof. If $R$ is a directly finite, then $R$ is Type $I_f$ by Proposition 2. Hence $R$ is left self-injective by Corollary 10.25 [2]. The converse is clear by Theorem 9.29 [2].

Now we consider the case that $R$ is a purely infinite, right self-injective, regular ring. Let $Z(R)$ be *the center* of $R$ and $S$ be *the socle* of $Z(R)$. There exists a central idempotent $e$ of $R$ such that $S \subseteq_e eZ(R)_{Z(R)}$. Therefore we have a decomposition $R=R_1 \times R_2$ such that $Z(R_1)$ has an essential socle and $Z(R_2)$ has a zero socle.

**Proposition 3.** *Let $R$ be a purely infinite, regular, right self-injective ring whose socle of $Z(R)$ is zero. Then $R$ is neither right nor left bounded.*

Proof. According to Theorems 9.7 and 10.16 [2], there exist orthogonal idempotents $e_1, e_2, \cdots$ in $R$ such that $\bigoplus_n e_nR$ is an essential right ideal of $R$ and $e_nR \cong R_R$ for all $n$. First we show that $R$ is not left bounded. Assume that $R$ is left bounded, then since $Re_n \cong_R R$ for all $n$, we set $H=(\bigoplus Re_n)\bigoplus_n I$, where $I$ is a complement left ideal of $\bigoplus_n Re_n$ in $R$, and take a nonzero two-sided ideal $J$ in $H$. Choose a nonzero element $x$ in $J$ and set $c=cc(xR)$. Since $Soc(Z(R))=0$, we can write $c=\bigvee_n c_n$ for some orthogonal, countably infinite, nonzero central idempotents $c_1, c_2, \cdots$ in $R$. Since $Re_n \cong_R R$, we have a nonzero $\varphi_n \in Hom_R(Rxc_n, Re_n)$ for all $n$. We define an $R$-homomorphism $\varphi$: $\bigoplus_n Rxc_n \to \bigoplus_n Re_n$ by $\varphi(\sum_n rxc_n)=\varphi_n(\sum_n r_nxc_n)$. Furthermore, each $\varphi_n$ can be extended to an $R$-

homomorphism $\bar{\varphi}=(\bar{\varphi}_n)$: $\prod Rc_n \to \prod Re_n$, and there exists a natural isomorphism $\theta$: $Rc \to \prod Rc_n$ given by $\theta(rc)=(rc_n)$ [2, Proposition 9.10]. Hence $\bar{\varphi}$ is given by the right multiplication by some element $a$ of $R$. Therefore we obtain that $\varphi$ is the right multiplication by $a$. On the other hand, since $J$ is a two-sided ideal of $R$, $xa \in J \subseteq \bigoplus_n Re_n \oplus I$, so we can write that $xa=\sum_{n=t}^{s} r_n e_n + r$ for some $r_n \in R$ and $r \in I$. But we observe that $xae_{s+1} \in (Re_{s+1} \cap \sum_{n=t}^{s} Re_n \oplus I)$ $(=0)$. This is a contradiction. Consequently $H$ can not contain any nonzero two-sided ideal of $R$. Therefore $R$ is not left bounded. Similarly we can show that $R$ is not right bounded. Now the proof is complete.

**Proposition 4.** *Let $R$ be a purely infinite, prime, regular right self-injective and right bounded ring. Then $R$ is a right full linear ring.*

Proof. Since $R$ is a prime, regular, right self-injective ring, $R$ is indecomposable as a ring. Hence $Soc(R)$ is essential or zero. We claim that $Soc(R)$ is essential. If not, then we can take a minimal ideal $I$ of $R$ by Theorem 12.23 [2]. Now there exists nonzero orthogonal, infinite, idempotents $\{f_\beta\}_{\beta \in K}$ in $I$ such that $\bigoplus_\beta f_\beta R \subseteq_e I_R$ since $Soc(R)=0$. If $\bigoplus_\beta f_\beta R = I$, then we can take countably, infinite orthogonal idempotents $\{g_{\beta n}\}$ in $f_\beta R$ for all $\beta$, such that $\bigoplus g_{\beta n} R \not\subseteq_e fR$ since $Soc(R)=0$. So we may assume that $\bigoplus_\beta f_\beta R \neq I$. On the other hand, since $I$ is a minimal ideal of $R$, $\bigoplus_\beta f_\beta R$ can not contain nonzero ideal. This contradicts that $R$ is right bounded. Therefore $Soc(R)$ is an essential ideal of $R$, as claimed. Then [2, Theorem 9.12] shows that $R$ is a right full linear ring.

**Corollary.** *Let $R$ be a purely infinite, regular, right self-injective and right bounded such that $Soc(Z(R))$ is an essential ideal of $Z(R)$. Then $R \cong \prod R_\alpha$, where each $R_\alpha$ is a right full linear ring.*

Proof. Let $\{e_\alpha\}_{\alpha \in I}$ be all central primitive idempotents of $Soc(Z(R))$. Evidently, we obtain that $\bigvee_\alpha e_\alpha = 1$ since $Soc(Z(R)) \subseteq_e Z(R)$. Then by Proposition 9.10 [2], $R \cong \prod_\alpha e_\alpha R$, where each $e_\alpha R$ is a prime, regular, right self-injective and right bounded ring. Then Proposition 4 shows that each $e_\alpha R$ is a right full linear ring.

Now we shall prove our main theorem.

**Theorem 1.** *Let $R$ be a regular, right self-injective ring. Then the following conditions are equivalent.*

(1) *$R$ is right bounded.*

(2) $R \cong \prod M_{n(t)}(R_t) \times \prod T_s$, *where each* $R_t$ *is an abelian regular, self-injective ring and each* $T_s$ *is a right full linear ring.*

Proof. (1)⇒(2). We have a decomposition $R=R_1 \times R_2$ such that $R_1$ is directly finite and $R_2$ is purely infinite by Theorem 10.21 [2]. Since each $R_i$ is also right bounded, $R_1$ is Type $I_f$ by Proposition 2 and $R_2$ is isomorphic to a direct product of right full linear rings by Propositions 3 and 4. Furthermore, Theorem 10.24 [2] shows that $R \cong \prod M_{n(t)}(R_t)$, where each $R_t$ is an abelian regular, self-injective ring.
(2)⇒(1). It is easily seen that a direct product of right bounded rings is also right bounded. Moreover, we have pointed out that a regular, right self-injective of bounded index of nilpotence is right bounded, and by Theorem 9.12 [2], we see that a right full linear ring is also right bounded. Therefore $R$ is right bounded.

**Corollary.** *Let R be a regular, right self-injective ring. Then if R is right bounded, then R is bounded.*

Proof. This is clear by Theorem 1.

## 3. Application

In this section, we apply Theorem 1 to a right bounded regular ring which is not necessarily self-injective.

**Lemma 1.** *Let R be a regular, right bounded ring such that every nonzero two-sided ideal of R contains a nonzero central idempotent. Then the maximal right quotient ring Q(R) of R is also right bounded.*

Proof. Let $I$ be an essential right ideal of $Q(R)$. Then clearly $I \cap R \subseteq_e R_R$. Since $R$ is right bounded, there exists a nonzero two-sided ideal $J$ of $R$ such that $J \subseteq_e I \cap R$. From the assumption, $J$ contains a nonzero central idempotent $e$ of $R$. Note that $e$ is also central in $Q(R)$. Thus $I$ contains a nonzero central idempotent of $Q(R)$. Let $H$ be the ideal generated by all central idempotents in $I$. We claim that $H \subseteq_e Q(R)_{Q(R)}$. If not, then $l_{Q(R)}(H)$, *the left annihilator ideal* of $H$ in $Q(R)$, is not zero and $H \cap l_{Q(R)}(H)=0$ since $Q(R)$ is semi-prime. Hence there exists a nonzero central idempotent $f$ in $J \cap l_{Q(R)}(H)$ since $J \cap l_{Q(R)}(H)$ is a nonzero two-sided ideal of $R$. On the other hand, $f$ is in $H$ since $f \in J \subseteq I$. But this contradicts that $H \cap l_{Q(R)}(H)=0$. Consequently, $H$ is an essential right ideal of $Q(R)$, as claimed. Therefore $Q(R)$ is right bounded.

**Theorem 2.** *Let R be a regular ring and Q be the maximal right quotient ring of R. Then the following conditions are equivalent.*

(1) $Q \cong \prod M_{n(t)}(S_t)$, *where each* $S_t$ *is an abelian regular, self-injective ring.*

(2) *There exist orthogonal central idempotents* $\{e_{t\alpha}\}$ *such that* $e_{t\alpha}R \cong M_{n(t)}(A_{t\alpha})$, *where each* $A_t$ *is an abelian regular ring and* $\sum_{t\,;\,\alpha\in K_t} \oplus e_{t\alpha}R \subseteq_e R_R$.

(3) *For any nonzero two-sided ideal* $J$ *of* $R$, $J$ *contains a nonzero central idempotent* $e$ *such that* $eR$ *is isomorphic to a full matrix ring over an abelian regular ring.*

(4) $R$ *is right bounded and every nonzero two-sided ideal of* $R$ *contains a nonzero central idempotent of* $R$.

Proof. (1)$\Rightarrow$(2). Since $Q \cong \prod M_{n(t)}(S_t)$, there exist orthogonal central idempotents $f_1, f_2, \cdots$ in $Q$ such that $f_tQ = M_{n(t)}(S_t)$. Clearly, $f_tQ$ has bounded index $n(t)$, so $R \cap f_tQ$ has also bounded index $n(t)$ by Corollary 7.4 [2]. Thus in view of Theorem 7.2 and Lemma 7.17 [2], we have a central idempotent $e_{t\alpha}$ in $R \cap f_tQ$ such that $e_{t\alpha}R \cong M_{n(t)}(A_{t\alpha})$ for some abelian regular ring $A_{t\alpha}$. It is easy to see that $e_{t\alpha}$ is a central idempotent of $R$. For a maximal family $\{e_{t\alpha}\}_{\alpha\in K_t}$ of orthogonal central idempotents of $f_tQ \cap R$ as above, we note that $\sum_{\alpha\in K_t} \oplus e_{t\alpha}R \subseteq_e f_tQ \cap R_{(f_tQ\cap R)}$. Next we show that $\sum_{t\,;\,\alpha\in K_t} \oplus e_{t\alpha}R \subseteq_e R_R$. Let $I$ be a right ideal of $R$ such that $\sum_{t\,;\,\alpha\in K_t} \oplus e_{t\alpha}R \cap I = 0$. Since $\oplus f_tQ$ is an essential right $R$-submodule of $Q$, if $I$ is not zero, then $f_tQ \cap I \neq 0$ for some $t$. But, then since $\sum_{\alpha\in K_t} \oplus e_{t\alpha}R$ is an essential $R$-submodule of $f_tQ \cap R$, this is impossible by our assumption. Hence $I$ must be zero, so $\sum_{t\,;\,\alpha\in K_t} \oplus e_{t\alpha}R$ is an essential right ideal of $R$.

(2)$\Rightarrow$(3). Let $J$ be a nonzero two-sided ideal of $R$. Since $\sum_{t\,;\,\alpha\in K_t} \oplus e_{t\alpha}R$ is an essential right ideal of $R$, there exists a positive integer $t_\alpha$ such that $e_{t\alpha}R \cap J \neq 0$. Now since $e_{t\alpha}R \cong M_{n(t)}(A_{t\alpha})$ for some abelian regular ring $A_{t\alpha}$, Theorem 6.6 [2] shows that $J \cap e_{t\alpha}R$ contains a nonzero central idempotent $e$ of $R$ such that $eR$ is isomorphic to a full matrix ring over an abelian regular ring.

(3)$\Rightarrow$(4). We shall show that $R$ is right bounded. Let $I$ be an essential right ideal of $R$. First we claim that $I$ contains a nonzero central idempotent $e$ of $R$. Choose a nonzero element $x$ of $R$. Then $RxR$ contains a nonzero central idempotent $f$ of $R$ such that $fR$ is isomorphic to a full matrix ring over an abelian regular ring. Since $I$ is essential, $I \cap fR \subseteq_e fR_{fR}$. Now since an essential right ideal of a full matrix ring over an abelian regular ring contains a central idempotent, it is easy to see that $I$ contains a nonzero central idempotent $e$ of $fR$. Clearly, $e$ is also central in $R$, so $I$ contains a nonzero central idempotent $e$ of $R$, as claimed. Let $H$ be the ideal generated by all central idempotents in $I$. We show that $H$ is an essential right ideal of $R$. If not, then $l_R(H)$, the left annihilator ideal of $H$ in $R$, is nonzero and $H \cap l_R(H) = 0$. On the other hand, since $I$ is an essential right ideal of $R$, $I \cap l_R(H) \neq 0$, so $I \cap l_R(H)$ contains a nonzero central idempotent $h$ of $R$. But this is a contradiction. Therefore $R$

is right bounded.
(4)$\Rightarrow$(1). By Lemma 1, $Q$ is also right bounded. Hence we have that $Q=Q_1\times Q_2$, where $Q_1$ is Type $I_f$ and $Q_2$ is a direct product of right full linear rings by Theorem 1. By the assumption, every nonzero two-sided ideal of $R$ contains a nonzero central idempotent of $R$. Clearly, the same property holds for $Q$. But it is easily seen that a right full linear ring does not satisfy this one. Hence we obtain that $Q_2=0$. Therefore $Q$ is isomorphic to a direct product of full matrix rings over abelian regular, self-injective rings.

**Corollary.** *Let $R$ be a regular, right self-injective ring. Then $R$ is isomorphic to a finite direct product of full matrix rings over abelian regular, self-injective rings if and only if*
(1) *$R$ is right bounded.*
(2) *All prime ideals of $R$ are maximal.*

Proof. Since $R$ is right self-injective, the condition (2) is equivalent to the condition that, for each $x\in R$, the two-sided ideal $RxR$ is generated by a central idempotent. Therefore Corollary is an immediate consequence of Theorems 1 and 2.

REMARK 1. Without (1) or (2), Corollary can fail, as the following examples show.

There exists a prime regular, right self-injective ring which is right bounded, but not simple and not directly finite.

For example, choose a field $F$, let $V$ be a countable-infinite dimensional vector space over $F$ and set $R=End_R(V)$ and $M=\{x\in R\,|\,dim_F(xV)<\infty\}$. Then clearly, $R$ is a prime regular, right self-injective ring. Given any $x\in R-M$, we have that $dim_F(xV)=dim_F(V)$ and so $xV\cong V$, whence $xR\cong R_R$. This shows that $R$ is not directly finite. On the other hand, $R$ is right bounded by Theorem 1.

Furthermore, there exists a simple regular, right self-injective ring, which is not right bounded and not Type $I_f$.

For example, choose fields $F_1$, $F_1$, $\cdots$, set $R_n=M_n(F_n)$ for all $n=1, 2, \cdots$, and set $R=\prod R_n$. Let $M$ be a maximal ideal of $R$ which contains $\oplus R_n$. Then $R/M$ be a simple right and left self-injective ring of Type $II_f$ by Example 10.7 [2]. Therefore by Proposition 2, $R/M$ is not right bounded.

REMARK 2. Let $R$ be a regular ring whose maximal right quotient ring $Q(R)$ is right bounded. Then according to Theorem 1, $R$ is also right bounded. But we don't know whether $Q(R)$ is right bounded when $R$ is right bounded.

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