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ON REGULARITY OF SOLUTIONS TO N-ORDER DIFFERENTIAL EQUATIONS OF PARABOLIC TYPE IN BANACH SPACES

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1. Introduction

The present paper is concerned with the following initial value problem for linear parabolic differential equations of higher order in the time variable in some Banach space X:

$$\text{(P)}\quad \begin{cases} u^{(n)}(t)+A_{n-1}u^{(n-1)}(t)+\cdots+A_1u'(t)+A_0u(t)=f(t), & 0\leq t\leq T<\infty, \\ u^{(k)}(0)=u_k, \ k=0,\cdots,n-1. \end{cases}$$

There is an extensive literature on this subject and many authors have defined and studied some types of parabolicity for the problem (P). We want to quote M.K. Balaev [3], A Favini & E. Obrecht [9], A. Favini & H. Tanabe [10], S.G. Krein [12], E. Obrecht [15],[16],[17], H. Tanabe [22], S. Ya. Yakubov [24],[25] and the references in [9].

There are also at least two recent monographs to be mentioned on polynomial operator pencils and their applications, that is A.S. Markus [14] and L. Rodman [18, pp. 83–88].

In [24] Yakubov devised a method of transforming a second order equation to a first order system so that the new system is of parabolic type and known results on parabolic evolution equations are applicable to it (see also Krein [12], Chapter III, Theorem 3.3). In [3] Balaev, using a method analogous to the one introduced by Yakubov [24], has given some conditions on the operators $A_j, j=0,1,\cdots,n-1$, so that after an introduction of suitable new unknown functions the problem (P) produces a first order system of parabolic type.

In this paper using a device similar to the one described in [3], to be compred with the methods discussed in [18] for n-order differential equations with coefficients of bounded linear operators, we transform (P) to a first order system of parabolic type in the product space $X\times X\times\cdots\times X$. To this new system we apply the existence, uniqueness and maximal regularity theorems established in many papers. In this way we extend the results of [3] in several directions, namely we allow A_j to have a non dense domain, make weaker assumptions on the links between the operators A_j (no fractional power are used), and discuss regularity properties for the solution. As references we indicate [1],[2],[19] since they contain a rather complete and definitive description of these results. Their affirmations are translated in our situation and involve the introduction of the real interpolation spaces.

We note here that Balaev [3] treats also nonlinear equations. He relies on a perturbation argument, and the essential role is played by the linear part.

On the other hand we show that also another approach works, i.e. we adapt the technique in [9] to deal with the problem (P) directly without reducing it to a first order system. In fact we prove that under our basic assumptions the operator pencil $P(z)=\sum_{j=0}^{n} z^j A_j$, z a complex variable, $A_n=I=$ the identity operator in X, is parabolic in the sense of [9],[15],[16]. In this manner we extend some results in [9] on second order equations to equations of arbitrary order using the method of [15],[16] relative to classical or strict solutions of (P). In the particular case $n=2$, we show that even a weaker assumption on the domains of A_j, $j=0,1$, implies the parabolicity of the pencil, provided that A_1 is multiplied by a sufficiently large number ρ.

We show in section 2 the existence, uniqueness and regularity of solutions of the problem (P) by transforming it to a first order system. In section 3 we show the parabolicity of the associated operator pencil $P(z)$ so that we may solve (P) directly.

Section 4 contains a number of examples and applications to partial differential equations, both in L^p spaces and in spaces of continuous functions. In this second case, we use the well known results by H.B. Stewart [20], G.Da Prato and P. Grisvard [6] and others.

We are also able to treat in the space L^p, $p\neq 2$, some problems for second order equations of the type considered by S. Chen and R. Triaggiani [4],[5]. Further, we indicate by means of a simple example how to solve some initial-boundary-value problems, more general than those treated by [4],[5] in the setup of continuous functions.

To finish with, we mention that in the paper [10] the case $n=3$ is studied even with operators suitably depending on t too, and some different applications are given.

Notation. We denote by $\|.;X\|$ the norm in the Banach space X. If Z is a linear subspace of X, Z^a denotes its closure in X. If Y is another Banach space $\mathscr{L}(X,Y)$ is the space of all bounded linear operators from

X into Y and we use $\mathscr{L}(X)$ for $\mathscr{L}(X,X)$.

For a compact interval I of R, $\mathscr{C}(I;\ X)$, $\mathscr{C}^{\sigma}(I;\ X)$, $0<\sigma<1$, $\mathscr{C}^{m}(I;\ X)$, $m\in N$, denote respectively the space of all X-valued functions on I which are continuous, which are σ-Hölder continuous, and which are m-times continuously differentiable.

$\mathscr{C}^{1,\sigma}(I;\ X)$, $0<\sigma<1$, is the space of all continuously differentiable X-valued functions on I whose derivative belongs to $\mathscr{C}^{\sigma}(I;\ X)$. Further, for $T>0$, $\mathscr{C}^{\sigma}((0,T];\ X)=\bigcap_{0<\varepsilon\leq T}\mathscr{C}^{\sigma}([\varepsilon,T];\ X)$ and $\mathscr{C}^{1,\sigma}((0,T];\ X)$ is analogously defined. For $T>0$, $B(0,T;\ X)$ denotes the space of all bounded and strongly measurable functions from $(0,T]$ into X.

For the definition and the main properties of the real interpolation spaces $(X,Y)_{\theta,\infty}$, $0<\theta<1$, we refer to [23]. In particular, we shall be interested in the space $(X,\mathscr{D}(A))_{\theta,\infty}=D_{-A}(\theta,\infty)$, when $-A$ generates an analytic semigroup in X with its domain $\mathscr{D}(A)$ not necessarily everywhere dense in X.

2. Reduction to a first order equation.

In the sequel we shall always suppose that $A_j, j=0,\cdots,n-1$, are closed linear operators in the complex Banach space X, with domain $\mathscr{D}(A_j)$, u_j, $j=0,\cdots,n-1$, are given elements of X and f is a continuous function from $[0,T]$ into X.

We then specify what we mean by a solution to (P) and we give the definitions as follows.

DEFINITION 1. We say that $u\in\mathscr{C}([0,T];\ X)$ is a *classical* solution to problem (P) if $u\in\mathscr{C}^{n}((0,T];\ X)$, $u^{(j)}(t)\in\mathscr{D}(A_j)$, $j=0,\cdots,n-1$, $0<t\leq T$, $u^{(j)}\in\mathscr{C}((0,T];\ \mathscr{D}(A_j))$ $j=0,\cdots,n-1$, and (P) is satisfied.

DEFINITION 2. A function $u\in\mathscr{C}([0,T];\ X)$ is a *strict* solution to (P) if $u\in\mathscr{C}^{n}([0,T];\ X)$, $u^{(j)}(t)\in\mathscr{D}(A_j)$, $j=0,\cdots,n-1$, $0\leq t\leq T$, $u^{(j)}\in\mathscr{C}([0,T];\ \mathscr{D}(A_j))$, $j=0,\cdots,n-1$ and (P) is true in the sense

$$\begin{cases} u^{(n)}(t)+\sum_{j=0}^{n-1} A_j u^{(j)}(t)=f(t)\ ,0\leq t\leq T, \\ u^{(j)}(0)=u_j,\ j=0,1,\cdots,n-1. \end{cases}$$

We transform (P) into the first order Cauchy problem

$$(1)\qquad \begin{cases} V'(t)=\mathscr{A}V(t)+F(t), & 0\le t\le T, \\ V(0)=V_0 \end{cases}$$

in the Banach space $E=X\times\cdots\times X$, n times, $V={}^t(v_0,v_1,\cdots,v_{n-1})$, $V_0={}^t(u_0,u_1,\cdots,u_{n-1})$, $F(t)={}^t(0,0,\cdots,0,f(t))$,

$$\mathscr{A}=\begin{bmatrix} 0 & 1 & 0 & \cdots\cdots & 0 \\ 0 & 0 & 1 & \cdots\cdots & 0 \\ \hdashline \\ 0 & 0 & 0 & \cdots\cdots & 1 \\ -A_0 & -A_1 & -A_2 & \cdots\cdots & -A_{n-1} \end{bmatrix}$$

As it is well known, the operator $\mathscr{A}$ not necessarily generates an analytic semigroup in E or even a $\mathscr{C}_0$ semigroup.

We are interested in the first case and we shall use a trick similar to the one in [3]; see [12,24] for $n=2$. It consists in observing that if we formally apply the operator K, defined by

$$K=\begin{bmatrix} A_1A_2A_3\cdots\cdots A_{n-1}I \\ 0\ A_2A_3\cdots\cdots A_{n-1}I \\ 0\ \ 0\ A_3\cdots\cdots A_{n-1}I \\ \hdashline \\ 0\ \ 0\ \ 0\ \cdots\cdots A_{n-1}I \\ 0\ \ 0\ \ 0\ \cdots\cdots\ 0\ \ I \end{bmatrix},$$

to the equation (1), we obtain the new problem

$$(2)\qquad \begin{cases} KV'(t)=K\mathscr{C}V(t)+KF(t), & 0<t\le T, \\ KV(0)=KV_0. \end{cases}$$

Call $KV(t)=Z(t)$, and assume that K have a bounded inverse. This hypothesis is not restrictive and is ensured by the existence of $A_j^{-1}\in\mathscr{L}(X)$, $j=0,1,\cdots,n-1$, as we shall always suppose. Then (2) becomes

$$(3)\qquad \begin{cases} Z'(t)=K\mathscr{A}K^{-1}Z(t)+KF(t), & 0<t\le T, \\ Z(0)=KV_0. \end{cases}$$

Our goal is to have some conditions on the operators A_j so that $K\mathscr{A}K^{-1}$ generates an analytic semigroup in E and thus we can apply to (3) known existence and maximal regularity results to be translated to problem (P) by means of $V=K^{-1}Z$.

The conditions on A_j read as follows.

HYPOTHESIS (H): There are n closed linear operators B_j, $j=1,\cdots,n$, in the space X such that

$$A_k=B_{n-k}\cdots B_2B_1,\ k=0,1,\cdots,n-1.$$

HYPOTHESIS (K): $-B_k$ for $k=1,\cdots,n$, generates an analytic semigroup in X, $\mathscr{D}(B_k)\subseteq\mathscr{D}(B_{k+1})$, $k=1,\cdots,n-1$, and there are α, $C>0$ such that

$$\|B_{k+1}(B_k+z)^{-1};\ \mathscr{L}(X)\|\leq C|z|^{-\alpha},\ k=1,\cdots,n-1,$$

for all $z\in\Sigma=\{z\in\boldsymbol{C};\ \mathrm{Re}z\geq -a_0|\mathrm{Imz}|\}$, where a_0 is another positive constant.

It is easily verified that $K\mathscr{A}K^{-1}$ coincides with $\mathscr{A}_1+\mathscr{A}_2$, where $\mathscr{D}(\mathscr{A}_1)=\mathscr{D}(B_n)\times\mathscr{D}(B_{n-1})\times\cdots\times\mathscr{D}(B_1)$, $\mathscr{D}(\mathscr{A}_2)=X\times\mathscr{D}(B_n)\times\cdots\times\mathscr{D}(B_2)$, and

$$\mathscr{A}_1=\begin{bmatrix} -B_n & 0 & 0 & 0 & \cdots\cdots & 0 & 0 \\ -B_n & -B_{n-1} & 0 & 0 & \cdots\cdots & 0 & 0 \\ -B_n & -B_{n-1} & -B_{n-2} & 0 & \cdots\cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -B_n & -B_{n-1} & -B_{n-2} & -B_{n-3} & \cdots\cdots & -B_2 & -B_1 \end{bmatrix},$$

$$\mathscr{A}_2=\begin{bmatrix} 0 & B_n & 0 & 0 & \cdots\cdots & 0 & 0 \\ 0 & B_n & B_{n-1} & 0 & \cdots\cdots & 0 & 0 \\ 0 & B_n & B_{n-1} & B_{n-2} & \cdots\cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & B_n & B_{n-1} & B_{n-2} & \cdots\cdots & B_3 & 0 \\ 0 & B_n & B_{n-1} & B_{n-2} & \cdots\cdots & B_3 & B_2 \\ 0 & B_n & B_{n-1} & B_{n-2} & \cdots\cdots & B_3 & B_2 \end{bmatrix}.$$

Assume (H),(K) and let $z\in\boldsymbol{C}$, $\mathrm{Re}z>0$, $|z|$ large. Then the equation

$$(-\mathscr{A}_1+z)x=F,$$

where $x={}^t(x_1,\cdots,x_n)$, $F={}^t(f_1,\cdots,f_n)\in E$, has the unique solution x for any $f_j\in X$, with x_k given by

$$x_k=-\sum_{j=1}^{k-1} z^{k-1-j}(B_{n-k+1}+z)^{-1}(B_{n-k+2}+z)^{-1}\cdots(B_{n-k+k-j}+z)^{-1}$$
$$B_{n-j+1}(B_{n-j+1}+z)^{-1}f_j+$$
$$+(B_{n-k+1}+z)^{-1}f_k;$$

furthermore,

$$\|x;E\|\leq C|z|^{-1}\|f;E\|.$$

On the other hand, $K\mathscr{A}K^{-1}-z=\mathscr{A}_1-z+\mathscr{A}_2=(I-\mathscr{A}_2(z-\mathscr{A}_1)^{-1})$ $(\mathscr{A}_1-z)$ and then all is reduced to show that $\mathscr{A}_2$ is $\mathscr{A}_1$-bounded, with $\mathscr{A}_1$-bound equal to zero, so that perturbation theory can be invoked [11]. Now this follows using some lengthy calculations from the fact that all the entries in the matrix $\mathscr{A}_2(z-\mathscr{A}_1)^{-1}$ are sums of

$$z^jB_{k+1}(B_k+z)^{-1}(B_{k+1}+z)^{-1}\cdots(B_{k+j}+z)^{-1}B_{k+j+1}(B_{k+j+1}+z)^{-1}.$$

We then apply our hypotheses (H) and (K) to check that the norm of this term in the space $\mathscr{L}(X)$ is bounded by $C|z|^{-\alpha}$, $|z|$ sufficiently large.

We have proved

Theorem 1. *Under* (H), (K), *the operator* $K\mathscr{A}K^{-1}$ *generates an analytic semigroup in the space* E, *with* (*possibly non dense*) *domain* $\mathscr{D}(B_n)\times\mathscr{D}(B_{n-1})\times\cdots\times\mathscr{D}(B_1)$.

REMARK 1. In view of [7], the assumptions (H), (K) are fulfilled if $A_{n-1}=B^{k_1}$, where $k_1\in \boldsymbol{N}$, $-B$ generates a bounded analytic semigroup of angle $\pi/2$ in the Banach space X, and $A_{n-j}=B^{k_1+\cdots+k_j}$ with $k_j\in\boldsymbol{N}$, $0<k_n<k_{n-1}<\cdots<k_1$.

REMARK 2. Assumption (K) can be weakened to suppose B_{k+1} to be B_k-bounded and to have zero as its B_k-bound.

We refer to [8].

We now describe some consequences of Theorem 1 relative to the solvability of problem (P).

Theorem 2. *Assume* (H), (K) *and let* $0<\sigma<1$. *If* $f\in C^{\sigma}((0,T];X)\cap C([0,T];X)$ *and* $u_k\in\mathscr{D}(A_{k+1})=\mathscr{D}(B_{n-k-1}\cdots B_2B_1)$, *with* $B_{n-k-1}\cdots B_1u_k$ $(=A_{k+1}u_k)\in\mathscr{D}(B_{n-k})^a$, *then* (P) *has a unique classial solution u such that* $u^{(n)}\in C^{\sigma}((0,T];X)$.

Proof. We apply [1, Theorem 4.1, p. 40] to problem [3]. We then know that if $KV_0\in\mathscr{D}(K\mathscr{A}K^{-1})^a$ and $KF\in C^{\sigma}((0,T],E)\cap C([0,T];E)$, then (3) has a unique classical solution $Z\in C^{1,\sigma}((0,T];E)$. Since $K^{-1}\in\mathscr{L}(E)$ and $V=K^{-1}Z$, V is differentiable and $KV'=(KV)'=Z$. Recall that $V={}^t(u,u',\cdots,u^{(n-1)})$, $V'={}^t(u',u^{(2)},\cdots,u^{(n)})$.

One also observes that $\mathscr{D}(K\mathscr{A}K^{-1})^a=\mathscr{D}(B_n)^a\times\cdots\times\mathscr{D}(B_1)^a$,

$$KV_0={}^t\Big(\sum_{j=1}^{n-1}A_ju_{j-1}+u_{n-1},\ \sum_{j=2}^{n-1}A_ju_{j-1}+u_{n-1},\cdots,A_{n-1}u_{n-2}+u_{n-1},u_{n-1}\Big),$$

$$KF(t)={}^t(f(t),\cdots,f(t)),$$

and the conditions above give $u_{n-1}\in\mathscr{D}(B_1)^a=\mathscr{D}(A_{n-1})^a, B_1u_{n-2}+u_{n-1}\in\mathscr{D}(B_2)^a,\cdots,A_2u_1+A_3u_2+\cdots+A_{n-1}u_{n-2}+u_{n-1}\in\mathscr{D}(B_{n-1})^a$, $A_1u_0+A_2u_1+\cdots+A_{n-1}u_{n-2}+u_{n-1}\in\mathscr{D}(B_n)^a$, that is, $u_{n-1}\in\mathscr{D}(B_1)^a$, $B_1u_{n-2}\in\mathscr{D}(B_2)^a$, $B_2B_1u_{n-3}\in\mathscr{D}(B_3)^a,\cdots,B_{n-2}\cdots B_2B_1u_1\in\mathscr{D}(B_{n-1})^a$, $B_{n-1}B_{n-2}\cdots B_2B_1u_0\in\mathscr{D}(B_n)^a$.#

Theorem 3. *Assume Hypotheses* (H), (K) *and let* $0<\sigma<1$. *If* $f\in C^{\sigma}([0,T];X)$ *and* $u_j\in\mathscr{D}(A_j)$, $j=0,1,\cdots,n-1$, with

$$f(0)-\sum_{j=0}^{k}A_ju_j\in\mathscr{D}(B_{n-k})^a,\ k=0,1,\cdots,n-1,$$

then problem (P) *has a unique strict solution u such that* $u^{(n)}\in C^{\sigma}((0,T];X)$.

Proof. We consider problem [3] and apply [1, Theorem 5.1, p.42]. In this case their condition

$$KF(0)+K\mathscr{A}K^{-1}KV_0\in\mathscr{D}(K\mathscr{A}K^{-1})^a$$

translates into $u_j\in\mathscr{D}(A_j)$ and $f(0)-A_0u_0\in\mathscr{D}(B_n)^a$, $f(0)-A_0u_0-A_1u_1\in\mathscr{D}(B_{n-1})^a,\cdots,f(0)-\sum_{k=0}^{n-1}A_ku_k\in\mathscr{D}(B_1)^a$; and this is precisely our assumption.#

Corollary 1. *Under* (H), (K), *if* $\mathscr{D}(B_j)$ *is everywhere dense in X, then problem* (P) *has one strict solution u with* $u^{(n)}\in C^{\sigma}((0,T];X)$, *provided* *that* $f^{(n)}\in C^{\sigma}([0,T];\ X)$ *and* $u_j\in\mathscr{D}(A_j)=\mathscr{D}(B_{n-j}\cdots B_2B_1)$.

Theorem 4. *If the assumptions* (H), (K) *hold*, $f\in C^{\sigma}([0,T];\ X)$, $0<\sigma<1$, and $u_j\in\mathscr{D}(A_j)=\mathscr{D}(B_{n-j}\cdots B_2B_1)$, *with*

$$f(0)-\sum_{j=0}^{k} A_ju_j\in D_{-B_{n-k}}(\sigma,\infty)=(X,\mathscr{D}(B_{n-k}))_{\sigma,\infty},$$

then the corresponding solution u to problem (P) *has the maximal regularity property* $u^{(n)}\in C^{\sigma}([0,T];\ X)$.

Proof. In view of [19, Proposition 1.15], since $\mathscr{D}(K\mathscr{A}K^{-1})=\mathscr{D}(B_n)\times\cdots\times\mathscr{D}(B_1)=\mathscr{D}(\mathscr{A}_0)$, where $\mathscr{A}_0$ is defined by

$$\mathscr{A}_0({}^t(x_1,\cdots,x_n))={}^t(-B_nx_1,-B_{n-1}x_2,\cdots,-B_1x_n)),\ x={}^t(x_1,\cdots x_n)\in\mathscr{D}(\mathscr{A}_0),$$

the space $D_{K\mathscr{A}K^{-1}}(\sigma,\infty)$ and $D_{\mathscr{A}_0}(\sigma,\infty)$ coincide.

By Theorem 5.3 in [1,p.45], the derivative Z' of the solution Z to problem (3) belongs to $C^{\sigma}([0,T];E)$ if and only if

$$KF(0)+K\mathscr{A}K^{-1}KV_0\in D_{\mathscr{A}_0}(\sigma,\infty)=D_{-B_n}(\sigma,\infty)\times D_{-B_{n-1}}(\sigma,\infty)\times\cdots\times D_{-B_1}(\sigma,\infty).$$

Now $V=K^{-1}Z$ and hence $V'={}^t(u',u^{(2)},\cdots u^{(n)})\in C^{\sigma}([0,T];\ E)$ if the above condition is verified. On the other hand, this is equivalent to what we have established, as one can easily see.#

Next theorems shall study what happens when the data are more regular with respect to the "space".

Theorem 5. *Under assumptions* (H), (K), *if* $f\in\mathscr{C}([0,T];\ X)\cap B(0,T,D_{-B_1}(\sigma,\infty))$, $0<\sigma<1$, *and* $A_{j+1}u_j=B_{n-j-1}\cdots B_2B_1u_j\in\mathscr{D}(B_{n-j})^a$, *then there is one classical solution u to* (P).

Proof. In view of Sinestrari's paper [19, Theorem 5.4, p. 59], if $KF\in\mathscr{C}([0,T];\ E)\cap B(0,T;D_{K\mathscr{A}K^{-1}}(\sigma,\infty))$ and $KV_0\in\mathscr{D}(B_n)^a\times\cdots\times\mathscr{D}(B_1)^a$, problem (3) has a unique classical solution Z. Hence $V=K^{-1}Z$ satisfies (2) and (1), too. The condition on KV_0 gives $u_{n-1}\in\mathscr{D}(B_1)^a$, $u_{n-1}+A_{n-1}u_{n-2}\in\mathscr{D}(B_2)^a,\cdots,u_{n-1}+A_{n-1}u_{n-2}+\cdots+A_1u_0\in\mathscr{D}(B_n)^a$, that are precisely the hypotheses that we have done on the initial values.#

Theorem 6. *Under hypotheses* (H), (K), *if* $f\in C([0,T];\ X)\cap$

$B(0,T,D_{-B_1}(\sigma,\infty)), 0<\sigma<1$, *and* $u_j\in\mathscr{D}(A_j)=\mathscr{D}(B_{n-j}\cdots B_2B_1)$, *with* $\sum_{j=0}^{k} B_{n-j}\cdots B_2B_1u_j\in\mathscr{D}(B_{n-k})^a$, *then problem* (P) *has one strict solution.*

Proof. In view of [19, Theorem 5.5, p. 60], we need only repeat the same argument as in Theorem 5, becasue $K\mathscr{A}V_0\in\mathscr{D}(B_n)^a\times\cdots\times\mathscr{D}(B_1)^a$ if

$$A_0u_0\in\mathscr{D}(B_n)^a,\ A_0u_0+A_1u_1\in\mathscr{D}(B_{n-1})^a,\cdots,\ \sum_{k=0}^{n-1}A_ku_k\in\mathscr{D}(B_1)^a.\#$$

Theorem 7. *Suppose* (H), (K) *and* $0<\sigma<1$, *if* $f\in C([0,T];\ X)\cap B(0,T;\ D_{-B_1}(\sigma,\infty))$ *and* $\sum_{j=0}^{k}B_{n-j}\cdots B_2B_1u_j\in D_{-B_{n-k}}(\sigma,\infty)$, $k=0,1,\cdots,n-1$, *then the strict solution* u *to* (P) *has the regularity*

$$A_ju^{(j)}\in C^{\sigma}([0,T];\ X),\ j=0,1,\cdots,n-1,$$
$$A_ju^{(j)}\in B(0,T;D_{-B_{n-j+1}}(\sigma,\infty)),\ j=1,\cdots,n,$$

(*where* $A_n=I$), *and* $\sum_{j=0}^{k}A_ju^{(j)}\in B(0,T;D_{-B_{n-k}}(\sigma,\infty))$, $k=0,1,\cdots,n-1$.

Proof. According to [19, Theorem 5.5], our assumptions imply that

$(KV)'\in B(0,T;D_{K\mathscr{A}K^{-1}}(\sigma,\infty))$ and $K\mathscr{A}V\in C^{\sigma}([0,T];\ E)\cap B(0,T;\ D_{K\mathscr{A}K^{-1}}(\sigma,\infty))$.

Theorem 7 then translates this statement in the present situation.#

3. A direct approach to problem (P).

In recent years some authors have studied higher order abstract differential equations of parabolic type by a direct approach, without any reduction to a system. We quote here mainly the papers [9,15,16,17].

We recall that according to these authors, the operator pencil P defined by

$$P:\ \mathscr{D}(P)(=\bigcap_{j=0}^{n-1}\mathscr{D}(A_j))\to X,\ P(z)x=\sum_{j=0}^{n}z^jA_jx,\ x\in\mathscr{D}(P),\ A_n=I,$$

is parabolic if

a) there exist $K\in R^+, \theta\in(\pi/2,\pi)$, such that

$$\Sigma=\{z\in \boldsymbol{C};\ |z|\geq K,\ |\arg z|\leq\theta_0\}\subset\rho(P),$$

where $\rho(P)=\{w\in \boldsymbol{C};\exists P(w)^{-1}\in\mathscr{L}(X)\}$.

b) $\|z^jA_jP(z)^{-1};\mathscr{L}(X)\|\leq M$, for all $z\in\Sigma, j=0,1,\cdots n$,

where M is a suitable positive constant.

In what follows we shall show that if $A_j=B_{n-j}\cdots B_2B_1$, $j=0,\cdots,n-1$, and Hypotheses (H), (K) hold, then the operator pencil $\sum_{j=0}^{n} z^jA_j$, $A_n=I$, is in fact parabolic, that is,

Theorem 8. *If Hypotheses* (H), (K) *are satisfied, then the pencil*

$$\sum_{j=0}^{n} z^jA_j,\ A_j=B_{n-j}\cdots B_2B_1,\ j=0,1,\cdots,n-1,\ A_n=I,$$

is parabolic.

Proof. We observe that for all $z\in \boldsymbol{C}$,

$$P(z)=z^n+z^{n-1}B_1+z^{n-2}B_2B_1+\cdots+B_nB_{n-1}\cdots B_2B_1=$$
$$=z^n+(z^{n-1}+z^{n-2}B_2+\cdots+B_nB_{n-1}\cdots B_2)B_1=$$
$$=(z^{n-1}+z^{n-2}B_2+\cdots+B_nB_{n-1}\cdots B_2)(z+B_1)-z(z^{n-2}+z^{n-3}B_3+\cdots$$
$$+B_nB_{n-1}\cdots B_3)B_2.$$

Let us define

$$P_j(z)=z^j+z^{j-1}B_{n-j+1}+z^{j-2}B_{n-(j-1)+1}B_{n-(j-1)}+z^{j-3}B_{n-j+3}B_{n-j+2}$$
$$B_{n-j+1}+\cdots+B_nB_{n-1}\cdots B_{n-j+1},\ j=2,\cdots,n,$$

so that

$$P(z)=P_n(z)=P_{n-1}(z)(I-zP_{n-1}(z)^{-1}P_{n-2}(z)B_2(z+B_1)^{-1})(B_1+z),$$

and, in this same way, provided that the inverses exist,

$$P_{n-1}(z)^{-1}=(B_2+z)^{-1}(I-zP_{n-2}(z)^{-1}P_{n-3}(z)B_3(B_2+z)^{-1})^{-1}$$
$$P_{n-2}(z)^{-1}$$

implies that

$$P_n(z)=P_{n-1}(z)(I-z(B_2+z)^{-1}(I-zP_{n-2}(z)^{-1}P_{n-3}(z)B_3(B_2+z)^{-1})^{-1}$$
$$B_2(B_1+z)^{-1})(B_1+z).$$

Consider

$$I-zP_{n-2}(z)^{-1}P_{n-3}(z)B_3(B_2+z)^{-1}=$$
$$=I-z(B_3+z)^{-1}(I-zP_{n-3}(z)^{-1}P_{n-4}(z)B_4(B_3+z)^{-1})^{-1}B_3(B_2+z)^{-1}.$$

If we continue this process, we verify that all is reduced to check the invertibility of

$$I-zP_3(z)^{-1}P_2(z)B_{n-2}(B_{n-3}+z)^{-1}.$$

Now,

$$P_3(z)=P_2(z)(I-zP_2(z)^{-1}(z+B_n)B_{n-1}(B_{n-2}+z)^{-1})(B_{n-2}+z)$$

and

$$P_2(z)=(z+B_n)(I-z(z+B_n)^{-1}B_n(B_{n-1}+z)^{-1})(z+B_{n-1}).$$

In view of (H) and (K), $P_2(z)$ is in fact invertible for all $z\in\Sigma$, $|z|$ sufficiently large. Futhermore, there exists $P_3(z)^{-1}\in\mathscr{L}(X)$ and

$$P_3(z)^{-1}=(B_{n-2}+z)^{-1}(I-z(z+B_{n-1})^{-1}(I-z(z+B^n)^{-1}$$
$$B_n(z+B_{n-1})^{-1})^{-1}B_{n-1}(B_{n-2}+z)^{-1})^{-1}P_2(z)^{-1}.$$

By induction, we obtain

$$P_n(z)^{-1}=(B_1+z)^{-1}R(z)P_{n-1}(z)^{-1},$$

where

$$R(z)=(I-zP_{n-1}(z)^{-1}P_{n-2}(z)B_2(B_1+z)^{-1})^{-1},$$

and also

$$\|P_n(z)^{-1};\ \mathscr{L}(X)\|\le C|z|^{-n},\quad z\in\Sigma,\ |z| \text{ large},$$
$$\|B_1P_n(z)^{-1};\ \mathscr{L}(X)\|\le C|z|^{-(n-1)},\quad z\in\Sigma,\ |z| \text{ large}.$$

On the other hand, if

$$Q_j(z)=z^j+z^{j-1}B_1+z^{j-2}B_2B_1+\cdots+B_jB_{j-1}\cdots B_1,\ j=1,\cdots,n,$$

then

$$P_n(z)=zQ_{n-1}(z)+B_nB_{n-1}\cdots B_1=$$
$$=(z+B_n-zB_n(z^{n-2}+z^{n-3}B_1+\cdots+B_{n-2}\cdots B_1)Q_{n-1}(z)^{-1})Q_{n-1}(z)$$
$$=(z+B_n)(I-z(z+B_n)^{-1}B_nQ_{n-2}(z)Q_{n-1}(z)^{-1})Q_{n-1}(z).$$

We deduce that

$$Q_{n-1}(z)=(z+B_{n-1})(I-z(z+B_{n-1})^{-1}B_{n-1}Q_{n-3}(z)Q_{n-2}(z)^{-1})Q_{n-2}(z)$$

and at last

$$Q_3(z)=(z+B_3)(I-z(z+B_3)^{-1}B_3Q_1(z)Q_2(z)^{-1})Q_2(z).$$

Now,

$$Q_2(z)=(z+B_2)(z+B_1)-zB_2$$
$$=(I-zB_2(z+B_1)^{-1}(z+B_2)^{-1})(z+B_2)(z+B_1)$$

and hence

$$Q_1(z)Q_2(z)^{-1}=(z+B_2)^{-1}(I-zB_2(z+B_1)^{-1}(z+B_2)^{-1})^{-1},$$
$$Q_3(z)=(z+B_3)(I-z(z+B_3)^{-1}B_3(z+B_2)^{-1}(I-zB_2(z+B_1)^{-1}$$
$$(z+B_2)^{-1})^{-1})Q_2(z).$$

Assumptions (H), (K) ensure that $Q_2(z)$ and $Q_1(z)$ have bounded inverses for any $z\in\Sigma$, $|z|$ large, and

$$\|B_2B_1Q_2(z)^{-1};\ \mathscr{L}(X)\|\le \text{Const},\ \|z^3Q_3(z)^{-1};\ \mathscr{L}(X)\|\le \text{Const},$$
$$\|B_3B_2B_1Q_3(z)^{-1};\ \mathscr{L}(X)\|\le \text{Const},\ \|zB_2B_1Q_3(z)^{-1};\ \mathscr{L}(X)\|\le \text{Const},$$
$$z\in\Sigma,\ |z|\ge K_0.$$

By induction on $n\in\boldsymbol{N}$, we see that

$$P(z)^{-1}=Q_{n-1}(z)^{-1}H(z)(z+B_n)^{-1},$$

where

$$H(z)=(I-z(z+B_n)^{-1}B_nQ_{n-2}(z)Q_{n-1}(z)^{-1})^{-1}$$

and $\|H(z);\mathscr{L}(X)\|\le \text{Const}$.

Hence, since $Q_j(z)$ has the same structure as $P(z)$, we conclude that

$$\|zB_{n-1}\cdots B_1P(z)^{-1};\mathscr{L}(X)\|\le \text{Const},\ \|z^2B_{n-2}\cdots B_1P(z)^{-1};\ \mathscr{L}(X)\|$$
$$\le \text{Const},\cdots,$$
$$\|z^{n-2}B_2B_1P(z)^{-1};\mathscr{L}(X)\|=\|z\cdot z^{n-3}B_2B_1Q_{n-1}(z)^{-1}H(z)(z+B_n)^{-1};$$
$$\mathscr{L}(X)\|\le\ \text{Const},\ z\in\Sigma,\ |z|\ \text{sufficiently large.}$$

The estimate

$$\|B_nB_{n-1}\cdots B_1P(z)^{-1};\mathscr{L}(X)\|\le \text{Const}$$

follows by the preceding ones and $A_0P(z)^{-1}=I-\sum_{j=1}^{n} z^jA_jP(z)^{-1}$.

Alternatively, we could prove by induction that

$$\|B_{j+2}Q_j(z)Q_{j+1}(z)^{-1};\mathscr{L}(X)\|\le C|z|^{-\alpha},\ j=1,\cdots,n-2,\ \text{if}\ z\in\Sigma,\ |z|\ \text{large},$$

noting that

$$Q_j(z)Q_{j+1}(z)^{-1}$$
$$=(I-z(z+B_{j+1})^{-1}B_{j+1}Q_{j-1}(z)Q_j(z)^{-1})^{-1}(z+B_{j+1})^{-1}$$
$$=(z+B_{j+1})^{-1}+z(z+B_{j+1})^{-1}B_{j+1}Q_{j-1}(z)Q_j(z)^{-1}Q_j(z)Q_{j+1}(z)^{-1},$$

where we used the equality $(I-S)^{-1}=I+S(I-S)^{-1}$. Hence, we have for $z\in\Sigma$, $|z|$ large,

$$Q_{j+1}(z)^{-1}=Q_j(z)^{-1}(I-z(z+B_{j+1})^{-1}B_{j+1}Q_{j-1}(z)Q_j(z)^{-1})^{-1}(z+B_{j+1})^{-1}.$$

With the aid of this we can show by induction

$$\|z^{j-k}B_k\cdots B_1Q_j(z)^{-1};\mathscr{L}(X)\|\le C,\ k=0,\cdots,j,\ \text{if}\ z\in\Sigma,\ |z|\ \ \text{large},\ j=1,\cdots,n.$$

In particular,

$$\|z^{n-k}B_k\cdots B_1P(z)^{-1};\mathscr{L}(X)\|\leq C,\ k=0,\cdots,n,\ \text{if } z\in\Sigma,\ |z|\ \text{large.}\ \sharp$$

If we carefully watch out the demonstration given above to Theorem 8, we realize that, at least in the case $n=2$, the strong connetion between the domains of B_1 and B_2 contained in the assumptions (H), (K) can be weakened to $\mathscr{D}(B_1)=\mathscr{D}(B_2)$, on condition that B_1 is multiplied by a sufficiently large constant ρ.

Notice that S. Chen and R. Triggiani [4,5] have studied, via reduction to a first order system in a suitable product space, parabolicity of (P), with $n=2$ and in a Hibert space, when A_0 is a positive selfadjoint operator A and A_1 is comparable to $A^{1/2}$.

We have

Theorem 9. *Let $-B_i$, $i=1,2$ be the generator of an analytic semigroup in X, with $\mathscr{D}(B_1)=\mathscr{D}(B_2)$. If $\rho>0$ is sufficiently large, then the operator pencil $P_\rho(z)=z^2+z\rho B_1+B_2B_1$ is parabolic.*

Proof. We write

$$P_\rho(z)=(z+\rho^{-1}B_2)(z+\rho B_1)-z\rho^{-1}B_2=(z+\rho^{-1}B_2)Q(z)(z+\rho B_1),$$

where

$$Q(z)=I-z\rho^{-1}(z+\rho^{-1}B_2)^{-1}B_2(z+\rho B_1)^{-1},\ z\in\Sigma.$$

On the other hand, if $z\in\Sigma$ and $|z|\geq\rho_0>0$ suitably chosen,

$$\|(z+\rho B_1)^{-1};\ \mathscr{L}(X)\|\leq M|z|^{-1},\ \|(z+\rho^{-1}B_2)^{-1};\ \mathscr{L}(X)\|\leq M|z|^{-1},$$
$$\|B_2(z+\rho B_1)^{-1};\ \mathscr{L}(X)\|\leq M\rho^{-1}.$$

The conclusion is then obvious.$\sharp$

4. Applications and examples

APPLICATION 1. We want to recall some basic results for which we refer to [23, pp. 333–334]. Let Ω be a bounded C^∞ domain in $\boldsymbol{R}^q$, $q\in\boldsymbol{N}$, and let

$$A(D)u=\sum_{|\alpha|=2m}c_\alpha D^\alpha u,\ c_\alpha\in\boldsymbol{C},$$

$m\in\boldsymbol{N}$, be a differential operator. We are also given a normal system $\{\boldsymbol{B}_j,\ j=1,\cdots m\}$ of boundary operators complemented with respect to $A(D)$

so that Theorem 4.9.1 in [23, p. 334] is satisfied. Notice that then

$$\boldsymbol{B}_j u(x)=\sum_{|\alpha|\leq k_j} b_{j\alpha}(x)D^\alpha u(x),\ b_{j\alpha}\in C^\infty(\partial\Omega),$$

$j=1,\cdots,m$, and $0\leq k_1<k_2<\cdots<k_m<2m$. (In particular, these assumptions hold when $\boldsymbol{B}_j u=\dfrac{\partial^{k+j-1}u}{\partial\nu^{k+j-1}}$, $j=1,\cdots,m$, where $k\in\{0,1,\cdots,m\}$ is fixed).

A remarkable result by R. Seeley characterizes the complex interpolation spaces between $L^p(\Omega)$, $1<p<\infty$, and

$$H^{2m}_{p,\{\boldsymbol{B}_j\}}=\{u\in H^{2m}_p(\Omega);\ \boldsymbol{B}_j u\,|_{\partial\Omega}=0,\ j=1,\cdots m\}.$$

Precisely, [23, p. 321], if there does not exist a number k_j, $j=1,\cdots m$, suh that $2m\theta-1/p=k_j$, then

$$\begin{aligned}[L^p(\Omega),\ H^{2m}_{p,\{\boldsymbol{B}_j\}}(\Omega)]_\theta&=H^{2m\theta}_{p,\{\boldsymbol{B}_j\}}(\Omega)\\&=\{u\in H^{2m\theta}_p(\Omega);\ \boldsymbol{B}_j u\,|_{\partial\Omega}=0\ \text{for}\ k_j<2m\theta-1/p\}.\end{aligned}$$

Furthermore, the operator $-A$, where

$$\mathscr{D}(A)=H^{2m}_{p,\{\boldsymbol{B}_j\}}(\Omega),\ (Au)(x)=A(D)u(x),$$

generates an analytic semigroup in $L^p(\Omega)$ and the imaginary powers A^{it} of A, t real, are bounded linear operators in $L^p(\Omega)$ such that $\|A^{it};\mathscr{L}(L^p(\Omega))\|\leq Ce^{\gamma|t|}$, for certain $C>0$ and $\gamma\geq 0$, [23, p. 334]. Therefore, if $2m\theta=h$ is a positive integer $<2m$, in view of [23, p. 103],

$$[L^p(\Omega),\ H^{2m}_{p,\{\boldsymbol{B}_j\}}(\Omega)]_{h/2m}=\{u\in H^h_p(\Omega)=W^{h,p}(\Omega);\ \boldsymbol{B}_j u\,|_{\partial\Omega}=0\ \text{for}\ k_j<h-1/p\}=\mathscr{D}(A^{h/2m}).$$

(It is not too restrictive to suppose A to be a positive operator in $L^p(\Omega)$).

These results indicate a possible choice of the operators $B_1,\cdots,B_n$ appearing in assumptions (H), (K) above. We start with $B_1=A$ and take as B_2 another operator of the same type as A, of lower order and such that $\mathscr{D}(B_2)\supseteq\mathscr{D}(A^{h/2m})$. Then we repeat this procedure to $B_3,\cdots,B_n$.

To clarify the idea, let $n=2$,

$$\mathscr{D}(B_1)=\{u\in H^4_p(\Omega);u\,|_{\partial\Omega}=\left.\frac{\partial u}{\partial\nu}\right|_{\partial\Omega}=0\},\ B_1u=\Delta^2u,\ u\in\mathscr{D}(B_1),$$

$$\mathscr{D}(B_2)=\{u\in H^2_p(\Omega);\ \left.\frac{\partial u}{\partial\nu}\right|_{\partial\Omega}=0\}\ ,B_2u=-\Delta u,\ u\in\mathscr{D}(B_2).$$

Since $-B_1, -B_2$ generate analytic semigroups in $L^p(\Omega)$, $1<p<\infty$ and

$$[L^p(\Omega),\mathscr{D}(B_1)]_{1/2}=\{u\in H_p^2(\Omega);\ u|_{\partial\Omega}=\left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega}=0\}\subset\mathscr{D}(B_2),$$

we deduce that (H) and (K) hold and thus we can treat the problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}+\Delta^2\dfrac{\partial u}{\partial t}-\Delta^3 u=f(t,x),\ 0<t\leq T,\ x\in\Omega, \\ u(0,x)=u_0(x),\ \dfrac{\partial u}{\partial t}(0,x)=u_1(x),\ x\in\Omega, \\ \dfrac{\partial u}{\partial t}(t,\cdot)\in\mathscr{D}(B_1),\ u(t,\cdot)\in\mathscr{D}(B_1)\cap H_p^6(\Omega), \\ \left.\dfrac{\partial\Delta^2 u(t,\cdot)}{\partial\nu}\right|_{\partial\Omega}=0,\ t>0. \end{cases}$$

Of course, one could perturb the differential operators by means of derivatives of lower order and to this purpose we refer to [9]. Henceforth we describe other examples that could be generalized in various directions, too.

EXAMPLE 1. Referring to Remark 1, let B be the operator in $L^p(\Omega)=X$, $1<p<\infty$, defined by

$$\mathscr{D}(B)=W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega),\ Bu=-\Delta u,\ u\in\mathscr{D}(B).$$

Then $-B$ generates an analytic semigroup in X satisfying the spectral properties in [7], so that for all $q\in\boldsymbol{N}$ the operator $-B_q$ given by

$$\mathscr{D}(B_q)=\{u\in W^{2q,p}(\Omega);\ u|_{\partial\Omega}=\Delta u|_{\partial\Omega}=\cdots=\Delta^{q-1}u|_{\partial\Omega}=0\},\ B^q=(-\Delta)^q$$

generates a holomorphic semigroup in X, too and one can apply Remark 1. This allows to treat the corresponding initial-boundary-value problem relative to the equation

$$\frac{\partial^n u}{\partial t^n}+c_1(-\Delta)^{q_1}\frac{\partial^{n-1}u}{\partial t^{n-1}}+\cdots+c_n(-\Delta)^{q_1+\cdots+q_n}u=f(t,x),$$

where $c_j>0$ and $q_1>q_2>\cdots>q_n>0$.

In our results the initial conditions u_j shall belong to some interpolation spaces that have been characterized in several papers by A. Lunardi [13], P. Acquistapace and B. Terreni [1,2] and others.

On the other hand, also the spaces $(X,\mathscr{D}(B_j))_{\theta,q}$, $1<q<\infty$, have a complete description ([23]) and $(X,\mathscr{D}(B_j))_{\theta,q}\subseteq(X,\mathscr{D}(B_j))_{\theta,\infty}$. In this way we have a wide class of initial values to which our results apply directly.

EXAMPLE 2. Let Ω be a bounded domain in R^q, $q\in\boldsymbol{N}$, with a smooth boundary $\partial\Omega$, $T>0$, $\Sigma=(0,T]\times\partial\Omega$. Consider the problem

$$(4)\qquad \begin{cases} u_{tt}+\Delta^2 u_t-\Delta^3 u=f(x,t),\ 0\le t\le T,\ x\in\Omega,\\ u(0,x)=u_0(x),\ u_t(0,x)=u_1(x),\ x\in\Omega,\\ u|_\Sigma=\left.\dfrac{\partial u}{\partial\nu}\right|_\Sigma=\Delta^2 u|_\Sigma=0 \text{ or } u|_\Sigma=\left.\dfrac{\partial u}{\partial\nu}\right|_\Sigma=\left.\dfrac{\partial\Delta^2 u}{\partial\nu}\right|_\Sigma=0,\end{cases}$$

This time we take $X=L^2(\Omega)$,

$$\mathscr{D}(B_1)=\{u\in H^4(\Omega);\ u|_{\partial\Omega}=\left.\frac{\partial u}{\partial\nu}\right|_{\partial\Omega}=0\},\ B_1u=\Delta^2u,\ u\in\mathscr{D}(B_1),$$

$$\mathscr{D}(B_2)=\{u\in H^2(\Omega);\ u|_{\partial\Omega}=0\} \text{ or } \{u\in H^2(\Omega);\ \left.\frac{\partial u}{\partial\nu}\right|_{\partial\Omega}=0\},$$

$$B_2u=-\Delta u,\ u\in\mathscr{D}(B_2).$$

Then $\mathscr{D}(B_2)\supset\mathscr{D}(B_1^{1/2})=[X,\mathscr{D}(B_1)]_{1/2}=H_0^2(\Omega)$, and therefore all our general results can be applied.

Of course, different exponents in the powers of the laplacian occuring in (4) are possible, provided that the assumptions (H) and (K) remain satisfied.

EXAMPLE 3. We take the same equation as in (4) but with other boundary conditions. Precisely, we assume $X=L^p(\Omega)$, $1<p<\infty$,

$$\mathscr{D}(B_1)=\{u\in W^{4,p}(\Omega);\ u|_{\partial\Omega}=\left.\frac{\partial u}{\partial\nu}\right|_{\partial\Omega}=0\},\ B_1u=\Delta^2u,\ u\in\mathscr{D}(B_1),$$

$$\mathscr{D}(B_2)=W^{2,p}(\Omega)\bigcap W_0^{1,p}(\Omega),\ B_2u=-\Delta u.$$

Since $\mathscr{D}(B_1^{1/2})=W_0^{2,p}(\Omega)\subset\mathscr{D}(B_2)$, the moment inequality (or interpolation theory) guarantees that

$$\|B_2(z+B_1)^{-1};\ \mathscr{L}(X)\|\le M|z|^{-1/2},\ z\in\rho(-B_1).$$

EXAMPLE 4. Let $a_0,a_1,a_2>0$. Consider in the space $X=L^p(\Omega)$, $1<p<\infty$, the equation

$$\frac{\partial^3 u}{\partial t^3}-a_2\Delta^3\frac{\partial^2 u}{\partial t^2}-a_1\Delta^5\frac{\partial u}{\partial t}+a_0\Delta^6 u=f(t,x),\ 0\le t\le T,\ x\in\Omega.$$

Let us define the operator B_1 by

$$\mathscr{D}(B_1)=W^{6,p}(\Omega)\cap W_0^{3,p}(\Omega),\ B_1u=-a_2\Delta^3 u,\ u\in\mathscr{D}(B_1),\ 1<p<\infty,$$

so that

$$\mathscr{D}(B_1^{2/3})=[X,\mathscr{D}(B_1)]_{2/3}=\{u\in W^{4,p}(\Omega);\ u|_{\partial\Omega}=\frac{\partial u}{\partial \nu}\bigg|_{\partial\Omega}=\frac{\partial^2 u}{\partial \nu^2}\bigg|_{\partial\Omega}=0\}.$$

Let us choose B_2 with

$$\mathscr{D}(B_2)\supseteqq\mathscr{D}(B_1^{2/3}),\ B_2u=(a_1/a_2)\Delta^2 u,\ u\in\mathscr{D}(B_2),$$

such that $-B_2$ generates an analytic semigroup in X (e.g., $\mathscr{D}(B_2)=W^{4,p}(\Omega)\cap W_0^{2,p}(\Omega)$, or else). Finally define B_3 so that we have

$$\mathscr{D}(B_2^{1/2})\subset\mathscr{D}(B_3)\subset W^{2,p}(\Omega),\ B_3u=-(a_0/a_1)\Delta u,\ u\in\mathscr{D}(B_3),$$

and $-B_3$ generates an analytic semigroup in X.

Of course, if $\mathscr{D}(B_2)=W^{4,p}(\Omega)\cap W_0^{2,p}(\Omega)$, then $\mathscr{D}(B_2^{1/2})=\{u\in W^{2,p}(\Omega);$ $u|_{\partial\Omega}=\frac{\partial u}{\partial\nu}|_{\partial\Omega}=0\}=W_0^{2,p}(\Omega)$, and hence the elements of $\mathscr{D}(B_3)$ could satisfy either Dirichlet or Neumann boundary conditions.

EXAMPLE 5. Consider the initial-boundary-value problem

$$(5)\qquad \begin{cases} u_{tt}-\rho\Delta u_t+\Delta^2 u=f=f(t,x),\ 0\le t\le T,\ x\in\Omega,\\ u(0,x)=u_0(x),\ u_t(0,x)=u_1(x),\ x\in\Omega,\\ u(t,\cdot)|_{\partial\Omega}=u_t(t,\cdot)|_{\partial\Omega}=0,\ 0<t\le T,\\ \Delta u(t,\cdot)|_{\partial\Omega}=0,\ 0<t\le T,\end{cases}$$

where Ω is a bounded domain in R^q, $q\in\boldsymbol{N}$, with a smooth boundary $\partial\Omega$. Now we take $X=L^p(\Omega)$, $1<p<\infty$, and define B_1,B_2 by

$$\mathscr{D}(B_1)=W^{2,p}(\Omega)\bigcap W_0^{1,p}(\Omega),\ B_1u=-\Delta u,\ u\in\mathscr{D}(B_1),\ 1<p<\infty,\ B_2=B_1.$$

Then Theorem 9 permits to handle (5) in the space X for any $\rho>0$

sufficiently large. In this way we extend to $p\in(1,\infty)$ the treatment in $L^2(\Omega)$ given in the paper [5]. Notice that other boundary conditions can be associated to the laplacian to generate well posed problems.

APPLICATION 2. Let Ω be a bounded open subset of R^q, $q\in N$, whose boundary $\partial\Omega$ is sufficiently smooth (see [20]), even if one could also treat the case Ω unbounded without too many changes. Let

$$A(x,D_x)=\sum_{|\alpha|=2m}a_\alpha(x)D^\alpha,\ D^\alpha=\frac{\partial^{\alpha_1}\cdots\partial^{\alpha_q}}{\partial x_1^{\alpha_1}\cdots\partial x_q^{\alpha_q}},\ \alpha=(\alpha_1,\cdots,\alpha_q),$$

be a uniformly strongly elliptic operator satisfying the roots condition, a_α being a real-valued continuous function on $\bar{\Omega}$. $\mathscr{C}(\bar{\Omega})$ shall denote the space of all complex-valued functions continuous on $\bar{\Omega}$. If one defines the operator A by means of

$$\mathscr{D}(A)=\{u\in\mathscr{C}(\bar{\Omega})\cap W^{2m,s}_{\rm loc}(\Omega);\ A(\cdot,D)u\in\mathscr{C}(\bar{\Omega}),\ D^\beta u=0\ \text{on}\ \partial\Omega,\ 0\le|\beta|<m\},$$

where $A(\cdot,D)u$ is intended in the sense of distributions, $Au=A(\cdot,D)u$, $u\in\mathscr{D}(A)$, with $s>q$, then ([20, p. 152]) $-A$ generates an analytic semigroup in $\mathscr{C}(\bar{\Omega})$, with non dense domain.

Further, it is known [20] that there exist $M>0$, $\lambda_0\ge 0$, $\varepsilon>0$ such that

$$\|u;\ \mathscr{C}(\bar{\Omega})\|+\sum_{0<|\beta|<2m}|z|^{-|\beta|/2m}\|D^\beta u;\ \mathscr{C}(\bar{\Omega})\|\le M|z|^{-1}\|(A+z)u;\ \mathscr{C}(\bar{\Omega})\|,$$

for all complex numbers z, $|z|\ge\lambda_0$, $|\arg z|\le\pi/2+\varepsilon$. For sake of brevity, let $\mathscr{C}(\bar{\Omega})=X$. Then we infer that

$$\|D^\beta u;\ X\|\le M|z|^{-(1-|\beta|/2m)}\ \|Au;\ X\|+M'|z|^{|\beta|/2m}\|u;\ X\|$$

and hence, taking $M|z|^{-(1-|\beta|/2m)}=\eta$, there is $C(\eta)$ such that

$$\|D^\beta u;X\|\le\eta\|Au;X\|+C(\eta)\|u;X\|,\ 0\le\beta<2m,\ \text{for any}\ u\in\mathscr{D}(A).$$

It follows that a differential operator of order $<2m$ and with its coefficients continuous on $\bar{\Omega}$ defines an operator A-bounded and with A-bound equal to 0.

We continue observing that if $k\in N$, $0<k<m$, and $b_\beta(\cdot)$ is continuous on $\bar{\Omega}$, $|\beta|\le 2k$, then the operator B given by

$$\mathscr{D}(B)=\{u\in X;\ u\in W^{2k,s}_{\rm loc}(\bar{\Omega});\ B(\cdot,D)u\in X,\ D^\beta u=0\ \text{on}\ \partial\Omega,\ 0\le|\beta|<k\},$$

$$Bu=\sum_{|\beta|\le k}b(\cdot)D^\beta u,\ u\in\mathscr{D}(B),$$

has properties analogous to the ones of A, and it is A-bounded; moreover its A-bound is zero. These remarks indicate how it is possible to treat by our methods the problem

$$\begin{cases} u_{tt}+A(x,D)u_t+B(x,D)u=f(t,x), \ 0\le t\le T, \ x\in\Omega, \\ u(0,\cdot)=u_0, \ u_t(0,\cdot)=u_1, \\ + \text{ boundary conditions} \end{cases}$$

in the space of continuous functions, as well as its generalization to equations of order n in time.

Example 6. If k is a positive integer, let us denote by $\mathscr{C}^k([0,1])=\{u\in\mathscr{C}([0,1]);\ u$ is k-times continuously differentiable on $[0,1]\}$. We also recall the following results by [2].

Let $\alpha_i,\beta_i\ge 0$, $\alpha_i+\beta_i>0$, $i=0,1$, $a,b,c\in\mathscr{C}([0,1];\ R)$, $\inf_{x\in[0,1]} a(x)>0$.

Define an operator K by

$$\mathscr{D}(K)=\{u\in\mathscr{C}^2([0,1]);\ \alpha_0u(0)-\beta_0u'(0)=\alpha_1u(1)+\beta_1u'(1)=0\},$$
$$Ku=a(\cdot)u''+b(\cdot)u'+c(\cdot)u-\omega_0u,\ \omega_0\in R,\ u\in\mathscr{D}(K).$$

If $\omega_0>\max_x |c(x)|$, then the spectrum of K is contained into $(-\infty,0)$ and K generates an analytic semigroup in $X=\mathscr{C}([0,1])$, endowed with the sup-norm; of course, $\mathscr{D}(K)^a\neq X$. Furthermore, if $0<h<1$, $\theta\neq 1/2$, the real interpolation space $(X,\mathscr{D}(K))_{\theta,\infty}$ is characterized by

$$(X,\mathscr{D}(K))_{\theta,\infty}=\begin{cases} \mathscr{C}^{2\theta}([0,1]), \text{ if } 0<\theta<1/2, \\ \{f\in\mathscr{C}^{1,2\theta-1}([0,1]),\ \alpha_0f(0)-\beta_0f'(0)=\alpha_1f(1)+\beta_1f'(1)=0\}, \\ \text{if } 1/2<\theta<1 \end{cases}$$

([2, p. 205]), where the space $\mathscr{C}^\sigma([0,1])=\mathscr{C}^\sigma([0,1];\boldsymbol{C})$, $0<\sigma<1$.

This being stated, let $\rho>0$ fixed and consider the operator K previously introduced, with $a(\cdot)=1$, $b(\cdot)=c(\cdot)=0=\omega_0$.

Let us introduce another operator H by

$$\mathscr{D}(H)=\mathscr{D}(K),\ (Hu)(x)=a(x)u''(x),\ u\in\mathscr{D}(H),\ 0\le x\le 1,$$

where $a\in\mathscr{C}((0,1);\boldsymbol{R})$, $\inf_x a(x)>0$.

Our general statements allow to handle in the space $\mathscr{C}([0,1])$ the mixed problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \rho\dfrac{\partial^3 u}{\partial x^2 \partial t} + a(x)\dfrac{\partial^4 u}{\partial x^4} = f(t,x),\ 0<t\leq T,\ 0\leq x\leq 1, \\ u(0,x)=u_0(x),\ \dfrac{\partial u}{\partial t}(0,x)=u_1(x),\ 0\leq x\leq 1, \\ \alpha_0 u(t,0)-\beta_0\dfrac{\partial u}{\partial x}(t,0)=\alpha_1 u(t,1)+\beta_1\dfrac{\partial u}{\partial x}(t,1)=0,\ 0<t\leq T, \\ \alpha_0\dfrac{\partial^2 u}{\partial x^2}(t,0)-\beta_0\dfrac{\partial^3 u}{\partial x^3}(t,0)=\alpha_1\dfrac{\partial^2 u}{\partial x^2}(t,1)+\beta_1\dfrac{\partial^3 u}{\partial x^3}(t,1)=0,\ 0<t\leq T, \end{cases}$$

provided that ρ is sufficiently large

Since the interpolation spaces $(X,\mathscr{D}(K))_{\theta,\infty}$ are well-characteriazed, it should be a simple matter to translate the conditions given by Theorems 1–7 into precise regularity properties and compatibility relations to ask for the initial and non homogeneous data.

Example 7. To begin with, we recall that if $-A$ generates an analytic semigroup in X, with non dense domain $\mathscr{D}(A)$, then the operator $-A_0$ defined in $X_0=\mathscr{D}(A)^a$ (=closure of $\mathscr{D}(A)$ in X) by

$$\mathscr{D}(A_0)=\{u\in\mathscr{D}(A);\ Au\in X_0\},\ A_0u=Au,\ u\in\mathscr{D}(A_0),$$

does generate a bounded analytic semigroup in X_0 that is strongly continuous in $t\geq 0$. Further, by [19, Proposition 1.6],

$$(X,\mathscr{D}(A))_{\theta,\infty}=(X_0,\mathscr{D}(A_0))_{\theta,\infty},\ 0<\theta<1.$$

In particular, if Ω is a bounded domain in R^q, $q\geq 1$, with a smooth boundary $\partial\Omega$,

$$X=\mathscr{C}(\bar{\Omega}),\ \mathscr{C}_0(\bar{\Omega})=\{u\in X;\ u|_{\partial\Omega}=0\},$$

and A is given by

$$\mathscr{D}(A)=\{u\in\mathscr{C}_0(\bar{\Omega});\ u(\bar{\Omega})\},\ Au=-\Delta u,\ u\in\mathscr{D}(A),$$

where Δ is understood in the sense of distributions on Ω, then

$$\mathscr{D}(A)^a=\mathscr{C}_0(\bar{\Omega}),\ \mathscr{D}(A^0)=\{u\in\mathscr{C}_0(\bar{\Omega});\ \Delta u\in\mathscr{C}_0(\bar{\Omega})\},\ A_0u=-\Delta u,\ u\in\mathscr{D}(A_0).$$

It is also well known that A_0 satisfies the spectral assumptions of [7], so that $-A_0{}^k$, $k\in \boldsymbol{N}$, do generate analytic semigroups in X_0.

Now, the space $(X,\mathscr{D}(A))_{\theta,\infty}=(X_0,\mathscr{D}(A_0))_{\theta,\infty}$, $0<\theta<1$, $\theta\neq 1/2$, has been characterized by means of Hölder-continuous functions on $\bar{\Omega}$ and, on the other hand, by [23, p. 93],

$$(X_0,\mathscr{D}(A_0^m))_{\theta,\infty}=\{u\in\mathscr{D}(A_0^s);\ A_0^s u\in(\mathscr{C}(\bar{\Omega}),\mathscr{D}(A))_{\theta m-s,\infty}\},$$

where $s=[\theta m]$ is the integer part of θm and $m\in\boldsymbol{N}$.

Hence we could consider some mixed problems connected with the equation

$$\frac{\partial^n u}{\partial t^n}+c_1(-\Delta)^{q_1}\frac{\partial^{n-1}u}{\partial t^{n-1}}+\cdots+c_n(-\Delta)^{q_1+\cdots+q_n}u=f(t,x),\ 0\leq t\leq T,\ x\in\Omega,$$

with $c_1,\cdots,c_n>0$, $q_1>q_2>\cdots>q_n\geq 1$, $q_j\in\boldsymbol{N}$, taking $X=\mathscr{C}_0(\bar{\Omega})$, $B_1=c_1(-\Delta)^{q_1}$, according to the definitions above mentioned, as well as other equations obtained by the preceding one with the help of suitable perturbations that do not change the type of equation.

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