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ON A RELATION BETWEEN HIGHER ORDER ASYMPTOTIC RISK SUFFICIENCY AND HIGHER ORDER ASYMPTOTIC SUFFICIENCY IN A LOCAL SENSE

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1. Introduction. In Takeuchi [4] higher order asymptotic risk sufficiency of maximum likelihood estimator has been discussed. In this paper we try to find some relations between asymptotic risk sufficiency with a special loss function and asymptotic sufficiency in a local sense.

Let $\mathcal{P} = \{P_\theta; \theta \in \Theta\}$ be a family of probability distributions on a measurable space $(\mathcal{X}, \mathcal{A}_n)$ with an index set $\Theta$ which is a subset of an Euclidean space with the usual norm $| \cdot |$. For a sub $\sigma$-field $\mathcal{C}$ of $\mathcal{A}_n$, real number $c \geq 0$ and $\theta, \theta' \in \Theta$ let $r^\mathcal{C}_n(c; \theta, \theta') = \inf (1+c)^{-1} \{1 - \mathbb{E}_{P_\theta}(\phi) + c \mathbb{E}_{P_{\theta'}}(\phi); \phi$ are $\mathcal{C}$-measurable statistical test functions on $\mathcal{X}\}$. We note that $r^\mathcal{C}_n(c; \theta, \theta')$ means the Bayes risk of statistical problem of testing a hypothesis 'P$_\theta$ is true' against an alternative 'P$_{\theta'}$ is true' with experiment $(\mathcal{X}, \mathcal{C}, \{P_{\theta'}, P_\theta\})$ relative to a prior probability distribution $(c/(1+c), 1/(1+c))$ on $\theta'$ provided that the loss function is simple.

Let $\{\mathcal{B}_n; n=1, 2, \ldots\}$ be a sequence of sub $\sigma$-fields of $\{\mathcal{A}_n\}$. In this paper we give a sufficient condition about the Bayes risk $r^\mathcal{B}_n$ for $\{\mathcal{B}_n\}$ to be a higher order locally asymptotically sufficient sequence of $\sigma$-fields. More precisely our main result in this paper is the following: Under some conditions if for some positive number $\alpha > 0$ and every compact subset $K$ of $\Theta$, then for every $\beta$ satisfying $0 < \beta < 3^{-1} \alpha$ $\{\mathcal{B}_n\}$ is locally asymptotically sufficient for $\{\mathcal{P}_n\}$ with order $o(n^{-p})$ in the sense that for each $n=1, 2, \ldots$ and each $\theta_0 \in \Theta$ there exists a family $\{Q^\theta_{\theta_0}; \theta \in \Theta\}$ of probability distributions on $(\mathcal{X}, \mathcal{A}_n)$ for which $\mathcal{P}_n$ is sufficient $\sigma$-field and that for every $b > 0$

$$\sup_{\theta : n^{1/2} |\theta - \theta_0| \leq b} ||P_{\theta_0} - Q^\theta_{\theta_0}||_{L_n} = o(n^{-p})$$

uniformly in $\theta_0$ over every compact subsets of $\Theta$. Here $|| \cdot ||_{L_n}$ means the total variation norm over $\mathcal{A}_n$.

We have discussed such a problem in the case $\alpha = \beta = 0$ in Suzuki [3].
non-local situation. In LeCam [1], Chap. 5 he discusses some relations between insufficiency and deficiency in his terminology.

In Section 2 some auxiliary results about the order of asymptotic sufficiency are proved. The main theorem is stated and followed by some discussions about the asymptotic sufficiency in non-local sense in Section 3.

2. Auxiliary results. For each \( n \in \mathbb{N} = \{1, 2, \ldots \} \) let \( \mathcal{P}_n = \{P_{\theta,n}; \theta \in \Theta \} \) be a family of probability distributions on a measurable space \((\mathcal{X}, \mathcal{A}_n)\) with an index set \( \Theta \). For a subset \( U(=\phi) \) of \( \Theta \) we shall denote by \( \mathcal{P}_n^U \) the totality of \( P_{\theta,n} \)'s satisfying \( \theta \in U \). We assume that for each \( n \in \mathbb{N} \) \( \mathcal{P}_n \) is dominated by a \( \sigma \)-finite measure \( \mu_n \) on \((\mathcal{X}, \mathcal{A}_n)\). The probability density function of \( P_{\theta,n} \) relative to \( \mu_n \) will be denoted by \( p_{\theta,n}(x, \theta) \). Without loss of generality we assume in the following that \( \mu_n \) is a probability measure on \((\mathcal{X}, \mathcal{A}_n)\).

For each \( \theta, \theta' \in \Theta \) let \( S_n(\theta) = \{x; p_{\theta,n}(x, \theta) > 0\} \) and let \( h_n(x; \theta, \theta') = p_{\theta,n}(x, \theta) / p_{\theta,n}(x, \theta') \) if \( x \in S_n(\theta) \), \(+\infty\) if \( x \in S_n(\theta') \cap S_n(\theta)^c \), \(-1\) if \( x \in S_n(\theta)^c \cap S_n(\theta') \). We put \( \beta_n(\theta, \theta') = P_{\theta,n} \{S_n(\theta')\} \).

For each \( \theta, \theta' \in \Theta \) and real number \( s \geq 1 \) we define

\[
J_n(s; \theta, \theta') = E_{P_{\theta,n}}[\{h_n(x; \theta, \theta')\}^s].
\]

We note that \( \beta_n(\theta, \theta') = 1 - J_n(1; \theta, \theta') \).

Let \( \{U_n\} \) be a sequence of nonempty subsets of \( \Theta \). For \( \{U_n\} \) we consider the following assumption.

**Assumption 1.** There exist a sequence \( \{\theta_n^*\} \subseteq U_n \) and a positive number \( \gamma \) such that

\[
(a) \quad \lim sup_{b \to \infty} \sup_{\theta \in U_n} J_n(s; \theta, \theta_n^*) < \infty,
\]

\[
(b) \quad \sup_{\theta \in U_n} \beta_n(\theta, \theta_n^*) = o(n^{-\gamma}).
\]

For a sub-\( \sigma \)-field \( \mathcal{C} \) of \( \mathcal{A}_n \) we denote by \( \Phi(\mathcal{C}) \) the family of \( \mathcal{C} \)-measurable statistical test functions on \( \mathcal{X} \). For each \( \theta, \theta' \in \Theta \) and each real number \( c \geq 0 \) we define

\[
r^C(c; \theta, \theta') = \inf (1 + c)^{-1}\{1 - E_{P_{\theta,n}} (\phi) + c E_{P_{\theta,n}} (\phi); \phi \in \Phi(\mathcal{C})\}.
\]

Let \( \{\mathcal{B}_n\} \) be a sequence of sub-\( \sigma \)-fields of \( \{\mathcal{A}_n\} \) (\( \mathcal{B}_n \subseteq \mathcal{A}_n \)). For each \( \theta \in \Theta \) define \( \bar{p}_{\theta,n}(x, \theta) = E_{P_{\theta,n}} [p_{\theta,n}(x, \theta) | \mathcal{B}_n] \) the conditional expectation of \( p_{\theta,n}(x, \theta) \) given \( \mathcal{B}_n \) with respect to \( \mu_n \) and put \( S_n^*(\theta) = \{x; \bar{p}_{\theta,n}(x, \theta) > 0\} \). For \( \theta, \theta' \in \Theta \) define \( g_n(x; \theta, \theta') = \bar{p}_{\theta,n}(x, \theta) / \bar{p}_{\theta,n}(x, \theta') \) if \( x \in S_n^*(\theta') \), \(+\infty\) if \( x \in S_n^*(\theta)^c \cap S_n^*(\theta) \), \(-1\) if \( x \in S_n^*(\theta)^c \cap S_n^*(\theta') \). For \( c > 0 \) and \( \delta > 0 \) let

\[
E_n(c, \theta, \delta) = \{x; g_n(x; \theta, \theta_n^*) < c + \delta \leq h_n(x; \theta, \theta_n^*)\} \quad \text{and} \quad E_n^*(c, \theta, \delta) = \{x; g_n(x; \theta, \theta_n^*) > c - \delta > h_n(x; \theta, \theta_n^*)\}.
\]
**Proposition.** Suppose that for some positive number \( \alpha \) and a sequence \( \{ \theta_n^* \}_{n \in \mathbb{N}} \) \( (\theta_n^* \in U_n) \)

\[
(2.1) \quad \sup_{c > 0} \sup_{\theta \in U_n} \{ r_{\mathcal{B}_n}(c; \theta, \theta_n^*) - r_{\mathcal{A}_n}(c; \theta, \theta_n^*) \} = o(n^{-\alpha}) .
\]

Then we have

\[
(2.2) \quad \sup_{c > 0, \delta > 0} \delta (1 + c)^{-1} \lambda_n(c, \delta) = o(n^{-\alpha}) , \quad \text{and} \quad \sup_{c > 0, \delta > 0} \delta (1 + c)^{-1} \lambda'_n(c, \delta) = o(n^{-\alpha})
\]

where \( \lambda_n(c, \delta) = \sup_{\theta \in U_n} P_{\mathcal{B}_n}(E_n(c, \theta, \delta)) \) and \( \lambda'_n(c, \delta) = \sup_{\theta \in U_n} P_{\mathcal{A}_n}(E_n(c, \theta, \delta)) \).

This proposition can be proved in the same way as the proof of the first and second steps of Theorem 1 in Suzuki [3]. So we shall omit the proof of the proposition.

**Theorem 1.** Suppose that Assumption 1 is satisfied with a sequence \( \{ \theta_n^* \}_{n \in \mathbb{N}} \) and \( \gamma > 0 \), and that \( \{ \mathcal{B}_n \} \) has the property (2.1) with \( \beta > 0 \). Then for every \( \beta \) satisfying \( 0 < \beta < 3^{-1}\alpha \) and \( \beta \leq \gamma \), \( \{ \mathcal{B}_n \} \) is asymptotically sufficient for \( \{ \mathcal{A}_n \} \) with order \( o(n^{-\beta}) \) in the following sense: For each \( n \in \mathbb{N} \) there exists a family \( \{ q_n(x; \theta, \theta_n^*) ; \theta \in \Theta \} \) of probability density functions on \( (\mathcal{X}, \mathcal{A}_n) \) relative to \( \mu_n \) such that

(i) each \( q_n \) can be factorized as follows:

\[
q_n(x; \theta, \theta_n^*) = r_n(x; \theta, \theta_n^*) p_n(x, \theta_n^*)
\]

where \( r_n \) is a \( \mathcal{B}_n \)-measurable function, and

(ii) \( \sup_{\theta \in U_n} \int_{\mathcal{X}} | p_n(x, \theta) - q_n(x; \theta, \theta_n^*) | d\mu_n = o(n^{-\beta}) \).

Proof. We shall divide the proof into several steps.

The first step. Suppose that Assumption 1 is satisfied with a sequence \( \{ \theta_n^* \}_{n \in \mathbb{N}} \) and \( \gamma > 0 \), and that \( \{ \mathcal{B}_n \}_{n \in \mathbb{N}} \) has the property (2.1) with \( \alpha > 0 \). Let \( \beta \) be any number satisfying \( 0 < \beta < 3^{-1}\alpha \) and \( \beta \leq \gamma \). Take \( \varepsilon_1 \) be any number satisfying \( 0 < \varepsilon_1 < 3^{-1}(\alpha - 3\beta) \). Let \( \alpha_n = n^{-\beta}(\log n)^{-1} \), \( m_n = n^{\varepsilon_1} \) and \( i_n = [m_n \alpha_n^{-1}] + 1 \) where \([a]\) means the maximum integer not exceeding \( a \). Put \( (\gamma_n =) \gamma_n(x; \theta, \theta_n^*) = | h_n(x; \theta, \theta_n^*) - g_n(x; \theta, \theta_n^*) | , (\gamma'_n =) \gamma'_n(x; \theta, \theta_n^*) = | h_n(x; \theta, \theta_n^*) - I_{W_n}(x) g_n(x; \theta, \theta_n^*) | \) and

\[
\rho_n(\theta, \theta_n^*) = \int_{\mathcal{X}} \gamma'_n(x; \theta, \theta_n^*) dP_{\mathcal{B}_n}
\]

where \( W_n = W_n(\theta, \theta_n^*) = \{ x ; g_n(x; \theta, \theta_n^*) \leq m_n \} \) and \( I_{W_n} \) means the indicator function of \( W_n \).

We have
\[
\sup_{\theta \in U_n} \rho_n(\theta, \theta^*) \leq \sup_{\theta \in U_n} \int_{W_n^*} \gamma_n dP_{\theta^*} + \sup_{\theta \in U_n} \int_{W_n^*} h_n(x; \theta, \theta^*) dP_{\theta^*} = J_n^* + J_n^{**},
\]
(2.3) and
\[
J_n^* = \sup_{\theta \in U_n} \int_{W_n^*} \gamma_n dP_{\theta^*} \leq \alpha_n + \sup_{\theta \in U_n} \int_{D_n \cap W_n^*} \gamma_n dP_{\theta^*} = \alpha_n + I_n
\]
\[
(D_n = \{x; \gamma_n \leq \alpha_n\}).
\]

Furthermore we have
\[
I_n = \sup_{\theta \in U_n} \int_{D_n \cap W_n^*} \gamma_n dP_{\theta^*} \leq \sup_{\theta \in U_n} \int_{D_n \cap \tilde{W}_n} \gamma_n dP_{\theta^*} + \sup_{\theta \in U_n} \int_{\tilde{W}_n^*} \gamma_n dP_{\theta^*}
\]
where \(W'_n = \{x; h_n(x; \theta, \theta^*) \leq m_n\} \), \(\tilde{W}_n = W_n \cap W_n^*\) and \(W_n^* = W_n \cap (W_n')^c\).

The second step. It holds that
\[
I_n = \sup_{\theta \in U_n} \int_{D_n \cap \tilde{W}_n} \gamma_n dP_{\theta^*} \leq \sum_{i=1}^{2n} \sup_{\theta \in U_n} \int_{B_i} \gamma_n dP_{\theta^*}
\]
(2.4)
\[
= I_{n,1}^* + I_{n,2}^*
\]
where \(B_j = \tilde{W}_n \cap \{x; h_n(x; \theta, \theta^*) \geq 2^{-1}(i+1) \alpha_n, g_n(x; \theta, \theta^*) < 2^{-1} i \alpha_n\} \) and \(C_i = \tilde{W}_n \cap \{x; h_n(x; \theta, \theta^*) < 2^{-1}(i+1) \alpha_n, g_n(x; \theta, \theta^*) \geq 2^{-1}(i+2) \alpha_n\}.\) Using the property (2.2) in Proposition we can evaluate \(I_{n,i}^*(i=1,2)\) as follows. Taking account of \(3\varepsilon_n \leq \alpha - 3\beta\) we have
\[
I_{n,1}^* = \sum_{i=1}^{2n} \sup_{\theta \in U_n} \int_{B_i} \gamma_n dP_{\theta^*}
\]
\[
\leq 2i_n m_n \sup_{1 \leq i \leq 2n, 2} \sup_{\theta \in U_n} P_{\theta^*}(x; h_n(x; \theta, \theta^*) \geq 2^{-1}(i+1) \alpha_n, g_n(x; \theta, \theta^*) < 2^{-1} i \alpha_n)
\]
(2.5)
\[
\leq 4i_n m_n \sup_{1 \leq i \leq 2n} \alpha_n(1+2^{-1} i \alpha_n) n^{-\varepsilon} \eta_n (\eta_n = o(1))
\]
\[
\leq 24i_n m_n n^{-\varepsilon} \eta_n
\]
\[
\leq A_1 n^{-\varepsilon} (\log n)^2 \eta_n (A_1 is a constant)
\]
\[
= o(n^{\varepsilon}).
\]

Similarly we have
(2.6)
\[
I_{n,2}^* = o(n^{\varepsilon}).
\]

Thus from (2.4) and (2.5) we have
(2.7)
\[
I_n^* = o(n^{\varepsilon}).
\]
The third step. Next we evaluate $I'_n$ as follows. For every $s > 1$ we have

$$I'_n = \sup_{\theta \in U_n} \int_{\mathcal{W}_n} \gamma_n dP_{\theta_{n+1}^*} \leq \sup_{\theta \in U_n} \int_{\{h_n > m_n\}} h_n(x; \theta, \theta_{n+1}^*) dP_{\theta_{n+1}^*} \leq (m_n)^{-1-s} \sup_{\theta \in U_n} J_n(s; \theta, \theta_{n+1}^*).$$

Hence we have

$$I'_n \leq A_2(s) (m_n)^{1-s} = A_2(s) n^{\alpha - \alpha_1},$$

where $A_2(s)$ is some constant depending only on $s$. We can choose $s > 1$ large enough so that

$$I'_n \approx o(n^{-\theta}).$$

From (2.7) and (2.8) we have

$$I_n = o(n^{-\theta}).$$

Hence from (2.3) we have

$$J^*_n = o(n^{-\theta}).$$

Put $W'_n = \{x; h_n(x; \theta, \theta_{n+1}^*) < 2^{-1} m_n\}$. Then we have

$$J^**_n = \sup_{\theta \in U_n} \int_{W'_n} h_n(x; \theta, \theta_{n+1}^*) dP_{\theta_{n+1}^*} \leq \sup_{\theta \in U_n} \int_{W'_n} h_n(x; \theta, \theta_{n+1}^*) dP_{\theta_{n+1}^*} + \sup_{\theta \in U_n} \int_{W'_n \setminus (W'_n \cup W_n)} h_n(x; \theta, \theta_{n+1}^*) dP_{\theta_{n+1}^*} \leq 2^{-1} m_n \lambda_n(m_n, m_n/2) + (m_n/2)^{1-s} \sup_{\theta \in U_n} J_n(s; \theta, \theta_{n+1}^*).$$

The first term on the right hand side is of order $o(n^{-\theta})$ by Proposition. The similar consideration as the evaluation of $I'_n$ implies that the second term of (2.9) is also of order $o(n^{-\theta})$ for sufficiently large number $s$. Thus we have

$$J^**_n = o(n^{-\theta}).$$

Hence it follows from (2.3) that

$$\sup_{\theta \in U_n} \rho_n(\theta, \theta_{n+1}^*) = o(n^{-\theta}).$$

The fourth step. Let $a_n(\theta, \theta_{n+1}^*) = [\int_{\mathcal{X}} I_{W_n}(x) g_n(x; \theta, \theta_{n+1}^*) dP_{\theta_{n+1}^*}]^{-1} (\leq \infty)$ and let $r_n(x; \theta, \theta_{n+1}^*) = a_n(\theta, \theta_{n+1}^*) I_{W_n}(x) g_n(x; \theta, \theta_{n+1}^*)$ if $a_n(\theta, \theta_{n+1}^*) < \infty$, $= 1$ otherwise. Define $g_n(x; \theta, \theta_{n+1}^*) = r_n(x; \theta, \theta_{n+1}^*) \rho_n(x; \theta, \theta_{n+1}^*)$ and let $Q_{n+1}^*$ be the probability distribution on $(\mathcal{X}, \mathcal{A}_n)$ with density $g_n(x; \theta, \theta_{n+1}^*)$ relative to $\mu_n$. We note that $\mathcal{B}_n$ is suf-
cient $\sigma$-field for the family $\{Q^\Theta_n; \theta \in \Theta\}$ by the factorization theorem. It follows from (2.10) that there exists $n_0$ such that $a_n(\theta, \theta^*) < \infty$ for every $n \geq n_0$ and every $\theta \in U_n$. Therefore we can assume without loss of generality that $a_n(\theta, \theta^*) < \infty$ for every $\theta \in U_n$ and every $n \geq 1$.

Under this circumstances we have

$$\|P_{\theta, n} - Q^\Theta_n\|_{\mathcal{A}_n} = \int_{\mathcal{X}} |p_n(x, \theta) - q_n(x; \theta, \theta^*)| \, d\mu_n$$

$$\leq \int_{S_n(n^*)} |h_n(x; \theta, \theta^*) - a_n(\theta, \theta^*) I_n(x) g_n(x; \theta, \theta^*)|$$

$$\cdot p_n(x, \theta^*) d\mu_n + \beta_n(\theta, \theta^*)$$

$$= \rho_n(\theta, \theta^*) + |1 - a_n(\theta, \theta^*)| + \beta_n(\theta, \theta^*)$$

$$\leq 2 \rho_n(\theta, \theta^*) + 2 \beta_n(\theta, \theta^*).$$

Here $\|\nu\|_{\mathcal{A}_n}$ means the total variation norm of a signed measure $\nu$ on $\mathcal{L}$, $\mathcal{A}_n$. From Assumption 1, (b) and (2.10) we have

$$\sup_{\theta \in U_n} \|P_{\theta, n} - Q^\Theta_n\|_{\mathcal{A}_n} = o(n^{-\beta}).$$

This completes the proof of the theorem.

3. The order of local asymptotic sufficiency. In this section the index set $\Theta$ is assumed to be a subset of $p$-dimensional Euclidean space $\mathbb{R}^p$. We denote by $| \cdot |$ the usual Euclidean norm in $\mathbb{R}^p$. For $\theta \in \Theta$ and $b > 0$ let $U_n(\theta, b) = \{\theta' \in \Theta; n^{1/2} |\theta' - \theta| \leq b\}.$

Let $\{\mathcal{B}_n\}_{n \in \mathbb{N}}$ be the sequence of sub $\sigma$-fields $\mathcal{B}_n \subset \mathcal{A}_n$ as in the previous section. We consider the following assumption which will be used to prove our main theorem, Theorem 2.

**Assumption 2.** For every compact subset $K$ of $\Theta$ and $b > 0$

(a) $\limsup_{n \to \infty} \sup_{\theta \in U_n(\theta, b)} \int_{\mathcal{X}} j_n(s; \theta, \theta^*) < \infty$ (for $s > 1$), and

(b) $\sup_{\theta^* \in K} \beta_n(\theta, \theta^*) = o(n^{-\gamma}).$

Let $\alpha$ be a given positive number. We state a result about higher order locally asymptotic sufficiency of $\{\mathcal{B}_n\}$ for $\{\mathcal{P}_n\}$.

**Theorem 2.** Suppose that Assumption 2 is satisfied with $\gamma > 0$, and that for every compact subset $K$ of $\Theta$ and every $b > 0$

$$\sup_{c > 0} \sup_{\theta^* \in K} \sup_{\theta \in U_n(\theta^*, b)} \{r_n(c; \theta, \theta^*) - r_n(\mathcal{A}_n(c; \theta, \theta^*))\} = o(n^{-\alpha}).$$

Then for every positive number $\beta$ satisfying $\beta < 3^{-1}\alpha$ and $\beta \leq \gamma$ $\{\mathcal{B}_n\}_{n \in \mathbb{N}}$ is locally asymptotically sufficient for $\{\mathcal{P}_n\}$ with order $o(n^{-\beta})$ in the following sense: For each
n ∈ N and each θ ∈ Θ there exists a family $Q_{	heta} = \{Q_{\theta}^n; \theta \in \Theta\}$ of probability distributions on $(X, A_n)$ such that

(i) $B_n$ is sufficient for $Q_{\theta}^n$, and

(ii) for every compact subset $K$ of $\Theta$ and every $b > 0$

$$\sup_{\theta \in K} \sup_{\theta' \in U_n(\theta, b)} ||P_{\theta, n} - Q_{\theta, n}||_{A_n} = o(n^{-b}).$$

Since the above result follows directly from Theorem 1 we shall omit the proof.

It is open problem whether non-local version of Theorem 2 still holds or not, i.e., whether any conditions such as in Theorem 2 imply the followings or not: There exists a sequence $Q_{\theta} = \{Q_{\theta, n}; \theta \in \Theta\}$ of probability distributions on $(X, A_n)$ such that $B_n$ is sufficient for $Q_{\theta, n}$, and that for every compact subset $K$ of $\Theta$

$$\sup_{\theta \in K} ||P_{\theta, n} - Q_{\theta, n}||_{A_n} = o(n^{-b}).$$

The case of $\alpha = \beta = 0$ has been discussed in Suzuki [3] in such a non-local situation.

It is well known that under some regularity conditions there exist a sequence $\{\theta_n\} \subset \mathbb{N}$ of estimators of $\theta$, a positive number $\gamma$ and a number $\nu \geq 1$ having the following property: For every compact subset $K$ of $\Theta$ there corresponds $a(K)$ such that

$$\sup_{\theta \in K} \{n^{1/2}|\hat{\theta}_n(x) - \theta| \geq a(K) (\log n)^{\nu/2}\} = o(n^{-\gamma})$$

(c.f. Matsuda [2], Chap. 3).

Using such an estimator $\{\hat{\theta}_n\}$ we may be able to construct $\{Q_{\theta, n}; \theta \in \Theta\}$ satisfying the property (3.2), and for which $B_n$ is sufficient.

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References


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