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ON A RELATION BETWEEN HIGHER ORDER ASYMPTOTIC RISK SUFFICIENCY AND HIGHER ORDER ASYMPTOTIC SUFFICIENCY IN A LOCAL SENSE

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1. Introduction. In Takeuchi [4] higher order asymptotic risk sufficiency of maximum likelihood estimator has been discussed. In this paper we try to find some relations between asymptotic risk sufficiency with a special loss function and asymptotic sufficiency in a local sense.

Let $\mathcal{P}_n = \{P_{\theta,n}; \theta \in \Theta\}$ be a family of probability distributions on a measurable space $(\mathcal{X}, \mathcal{A}_n)$ with an index set $\Theta$ which is a subset of an Euclidean space with the usual norm $| \cdot |$. For a sub $\sigma$-field $\mathcal{C}$ of $\mathcal{A}_n$, real number $c \geq 0$ and $\theta, \theta' \in \Theta$ let $r_n^\mathcal{C}(c; \theta, \theta') = \inf \{ (1+c)^{-1} \{ 1 - E_{P_{\theta,n}}(\phi) + c E_{P_{\theta',n}}(\phi) \}; \phi \text{ are } \mathcal{C}\text{-measurable statistical test functions on } \mathcal{X} \}$. We note that $r_n^\mathcal{C}(c; \theta, \theta')$ means the Bayes risk of statistical problem of testing a hypothesis 'P$_{\theta',n}$ is true' against an alternative 'P$_{\theta,n}$ is true' with experiment $(\mathcal{X}, \mathcal{C}, \{P_{\theta,n}, P_{\theta',n}\})$ relative to a prior probability distribution $(c/(1+c), 1/(1+c))$ on $\{\theta', \theta\}$ provided that the loss function is simple.

Let $\{\mathcal{B}_n; n=1, 2, \ldots\}$ be a sequence of sub $\sigma$-fields of $\{\mathcal{A}_n\} (\mathcal{B}_n \subset \mathcal{A}_n)$. In this paper we give a sufficient condition about the Bayes risk $r_n^{\mathcal{B}_n}$ for $\{\mathcal{B}_n\}$ to be higher order locally asymptotically sufficient sequence of $\sigma$-fields. More precisely our main result in this paper is the following: Under some conditions if for some positive number $\alpha \sup \sup \sup \{ r_n^\mathcal{B}_n(c; \theta, \theta') - r_n^\mathcal{A}_n(c; \theta, \theta') \} = o(n^{-\alpha})$ for every $b>0$ and every compact subset $K$ of $\Theta$, then for every $\beta$ satisfying $0 < \beta < 3^{-1} \alpha$ the family $\{P_{\theta_n}; \theta \in \Theta\}$ is locally asymptotically sufficient for $\{\mathcal{A}_n\}$ with order $o(n^{-\beta})$ in the sense that for each $n=1, 2, \ldots$ and each $\theta_0 \in \Theta$ there exists a family $\{Q_{\theta_n}; \theta \in \Theta\}$ of probability distributions on $(\mathcal{X}, \mathcal{A}_n)$ for which $\mathcal{P}_n$ is sufficient $\sigma$-field and that for every $b>0$

$$\sup_{\theta : n^{1/2} |\theta-\theta_0| \leq b} ||P_{\theta,n} - Q_{\theta_n}||_{\mathcal{A}_n} = o(n^{-\beta})$$

uniformly in $\theta_0$ over every compact subsets of $\Theta$. Here $|| \cdot ||_{\mathcal{A}_n}$ means the total variation norm over $\mathcal{A}_n$.

We have discussed such a problem in the case $\alpha = \beta = 0$ in Suzuki [3] under
non-local situation. In LeCam [1], Chap. 5 he discusses some relations between insufficiency and deficiency in his terminology.

In Section 2 some auxiliary results about the order of asymptotic sufficiency are proved. The main theorem is stated and followed by some discussions about the asymptotic sufficiency in non-local sense in Section 3.

2. Auxiliary results. For each $n \in \mathbb{N} = \{1, 2, \ldots\}$ let $\mathcal{P}_n = \{P_{\theta, n}; \theta \in \Theta\}$ be a family of probability distributions on a measurable space $(\mathcal{X}, \mathcal{A}_n)$ with an index set $\Theta$. For a subset $U(\pm \phi)$ of $\Theta$ we shall denote by $\mathcal{P}_{U}^\phi$ the totality of $P_{\theta, n}$'s satisfying $\theta \in U$. We assume that for each $n \in \mathbb{N}$ $\mathcal{P}_n$ is dominated by a $\sigma$-finite measure $\mu_n$ on $(\mathcal{X}, \mathcal{A}_n)$. The probability density function of $P_{\theta, n}$ relative to $\mu_n$ will be denoted by $p_{n}(x, \theta)$. Without loss of generality we assume in the following that $\mu_n$ is a probability measure on $(\mathcal{X}, \mathcal{A}_n)$. For each $\theta, \theta' \in \Theta$ let $S_n(\theta) = \{x; p_{n}(x, \theta) > 0\}$ and let $h_n(x; \theta, \theta') = p_{n}(x, \theta)p_{n}(x, \theta')$ if $x \in S_n(\theta)$, $= + \infty$ if $x \in S_n(\theta) \cap S_n(\theta')$, $= 1$ if $x \in S_n(\theta) \cap S_n(\theta')$. We put $\beta_n(\theta, \theta') = P_{\theta, n}\{S_n(\theta')\}$. For each $\theta, \theta' \in \Theta$ and real number $s \geq 1$ we define

$$J_n(s; \theta, \theta') = E_{P_{\theta', n}}[\{h_n(x; \theta, \theta')\}^s].$$

We note that $\beta_n(\theta, \theta') = 1 - J_n(1; \theta, \theta')$.

Let $\{U_{n}\}$ be a sequence of nonempty subsets of $\Theta$. For $\{U_{n}\}$ we consider the following assumption.

ASSUMPTION 1. There exist a sequence $\{\theta^*_n\}_{n=1}^{\infty}$ and a positive number $\gamma$ such that

(a) For every $s \geq 1$

$$\lim sup_{b \to \infty} sup_{\theta \in U_{n}} J_n(s; \theta, \theta^*_n) < \infty,$$

(b) $\sup_{\theta \in U_{n}} \beta_n(\theta, \theta^*_n) = o(n^{-\gamma})$.

For a sub $\sigma$-field $\mathcal{C}$ of $\mathcal{A}_n$ we denote by $\Phi(\mathcal{C})$ the family of $\mathcal{C}$-measurable statistical test functions on $\mathcal{X}$. For each $\theta, \theta' \in \Theta$ and each real number $c \geq 0$ we define

$$r_{\mathcal{C}}(c; \theta, \theta') = \inf (1+c)^{-1}\{1-E_{P_{\theta, n}}(\phi) + cE_{P_{\theta', n}}(\phi); \phi \in \Phi(\mathcal{C})\}.$$

Let $\{\mathcal{B}_n\}$ be a sequence of sub $\sigma$-fields of $\{\mathcal{A}_n\}$ $(\mathcal{B}_n \subset \mathcal{A}_n)$. For each $\theta \in \Theta$ define $p_{n}(x, \theta) = E_{P_{\theta, n}}[p_n(x, \theta) | \mathcal{B}_n]$ the conditional expectation of $p_n(x, \theta)$ given $\mathcal{B}_n$ with respect to $\mu_n$ and put $S_{n}(\theta) = \{x; p_{n}(x, \theta) > 0\}$. For $\theta, \theta' \in \Theta$ define $g_n(x; \theta, \theta') = P_{n}(x, \theta)p_{n}(x, \theta')$ if $x \in S_{n}(\theta)$, $= + \infty$ if $x \in S_{n}(\theta) \cap S_{n}(\theta')$, $= 1$ if $x \in S_{n}(\theta) \cap S_{n}(\theta')$. For $c > 0$ and $\delta > 0$ let $E_{n}(c, \theta, \delta) = \{x; g_{n}(x; \theta, \theta^*_n) > c - \delta > h_{n}(x; \theta, \theta^*_n)\}$ and $E_{n}(c, \theta, \delta) = \{x; g_{n}(x; \theta, \theta^*_n) > c - \delta > h_{n}(x; \theta, \theta^*_n)\}$.
Proposition. Suppose that for some positive number \( \alpha \) and a sequence \( \{\theta_n^*\}_{n \in \mathbb{N}} \) \((\theta_n^* \in U_n)\)

\begin{equation}
(2.1) \quad \sup_{c>0} \sup_{\theta \in U_n} \{r_{\mathcal{B}_n}(c; \theta, \theta_n^*) - r_n^*(c; \theta, \theta_n^*)\} = o(n^{-\alpha}) .
\end{equation}

Then we have

\begin{equation}
(2.2) \quad \sup_{c>0, \delta > 0} \delta (1+\varepsilon)^{-1} \lambda_n(c, \delta) = o(n^{-\alpha}) , \quad \text{and}
\end{equation}

\begin{equation}
(2.3) \quad \sup_{c>0, \delta > 0} \delta (1+\varepsilon)^{-1} \lambda_n(c, \delta) = o(n^{-\alpha})
\end{equation}

where \( \lambda_n(c, \delta) = \sup_{\theta \in U_n} P_n^*(E_n(c, \theta, \delta)) \) and \( \lambda_n(c, \delta) = \sup_{\theta \in U_n} P_n^*(E_n(c, \theta, \delta)) \).

This proposition can be proved in the same way as the proof of the first and second steps of Theorem 1 in Suzuki [3]. So we shall omit the proof of the proposition.

Theorem 1. Suppose that Assumption 1 is satisfied with a sequence \( \{\theta_n^*\}_{n \in \mathbb{N}} \) and \( \gamma > 0 \), and that \( \{\mathcal{B}_n\} \) has the property (2.1) with \( \beta > 0 \). Then for every \( \beta \) satisfying \( 0 < \beta < 3^{-1}\alpha \) and \( \beta \geq \gamma \), \( \{\mathcal{B}_n\} \) is asymptotically sufficient for \( \{\mathcal{B}_n\} \) with order \( o(n^{-\beta}) \) in the following sense: For each \( n \in \mathbb{N} \) there exists a family \( \{q_n(x; \theta, \theta_n^*) ; \theta \in \Theta\} \) of probability density functions on \((\mathcal{X}, \mathcal{A}_n)\) relative to \( \mu_n \) such that

(i) each \( q_n \) can be factorized as follows:

\[ q_n(x; \theta, \theta_n^*) = r_n(x; \theta, \theta_n^*) p_n(x, \theta_n^*) \]

where \( r_n \) is a \( \mathcal{B}_n \)-measurable function, and

(ii) \( \sup_{\theta \in U_n} \int_{\mathcal{X}} | p_n(x, \theta) - q_n(x; \theta, \theta_n^*) | d\mu_n = o(n^{-\beta}) . \)

Proof. We shall divide the proof into several steps.

The first step. Suppose that Assumption 1 is satisfied with a sequence \( \{\theta_n^*\}_{n \in \mathbb{N}} \) and \( \gamma > 0 \), and that \( \{\mathcal{B}_n\}_{n \in \mathbb{N}} \) has the property (2.1) with \( \alpha > 0 \). Let \( \beta \) be any number satisfying \( 0 < \beta < 3^{-1}\alpha \) and \( \beta \geq \gamma \). Take \( \varepsilon \) be any number satisfying \( 0 < \varepsilon < 3^{-1}(\alpha - 3\beta) \). Let \( \alpha_n = n^{-\beta} \log n \), \( m_n = n^\alpha \), and \( \lambda_n = [m_n \alpha_n] + 1 \) where \([a]\) means the maximum integer not exceeding \( a \). Put \((\gamma_n, \gamma_n') = (\gamma_n, \gamma_n') \) and

\[ \rho_n(\theta, \theta_n^*) = \int_{\mathcal{X}} \gamma_n'(x; \theta, \theta_n^*) dP_n^*(x, \theta_n^*) \]

where \( W_n = W_n(\theta, \theta_n^*) = \{x; g_n(x; \theta, \theta_n^*) \leq m_n\} \) and \( I_{W_n} \) means the indicator function of \( W_n \).

We have
\[ \sup_{\theta \in U_n} \rho_n(\theta, \theta^*) \leq \sup_{\theta \in U_n} \int_{W_n} \gamma_n dP_{\theta,n} + \sup_{\theta \in U_n} \int_{W_n} h_n(x; \theta, \theta^*) dP_{\theta,n} = J_1^* + J_2^* , \]

(2.3) and

\[ J_1^* = \sup_{\theta \in U_n} \int_{W_n} \gamma_n dP_{\theta,n} \leq \alpha_n + \sup_{\theta \in U_n} \int_{D_n \cap W_n} \gamma_n dP_{\theta,n} = \alpha_n + I_n \]

\[ (D_n = \{ x ; \gamma_n \leq \alpha_n \}) . \]

Furthermore we have

\[ I_n = \sup_{\theta \in U_n} \int_{D_n \cap W_n} \gamma_n dP_{\theta,n} \leq \sup_{\theta \in U_n} \int_{D_n \cap \tilde{W}_n} \gamma_n dP_{\theta,n} + \sup_{\theta \in U_n} \int_{\tilde{W}_n} \gamma_n dP_{\theta,n} \]

where \( W_n' = \{ x ; h_n(x; \theta, \theta^*) \leq m_2 \} \), \( \tilde{W}_n = W_n \cap W_n' \) and \( W_n^* = W_n \cap (W_n')^c \). The second step. It holds that

\[ I_1^* = \sup_{\theta \in U_n} \int_{D_n \cap \tilde{W}_n} \gamma_n dP_{\theta,n} \leq \sum_{i=1}^{2n-2} \sup_{\theta \in U_n} \int_{B_i} \gamma_n dP_{\theta,n} \]

(2.4)

\[ + \sum_{i=3}^{2n-3} \sup_{\theta \in U_n} \int_{C_i} \gamma_n dP_{\theta,n} \]

\[ = I_{1,1}^* + I_{1,2}^* \]

where \( B_i = \tilde{W}_n \cap \{ x ; h_n(x; \theta, \theta^*) \geq 2^{-1}(i+1) \alpha_n, g_n(x; \theta, \theta^*) < 2^{-1} i \alpha_n \} \) and \( C_i = \tilde{W}_n \cap \{ x ; h_n(x; \theta, \theta^*) < 2^{-1}(i+1) \alpha_n, g_n(x; \theta, \theta^*) \geq 2^{-1}(i+2) \alpha_n \} \). Using the property (2.2) in Proposition we can evaluate \( I_{1,i}^* (i=1, 2) \) as follows. Taking account of \( 3\varepsilon_1 < \alpha - 3\beta \) we have

\[ I_{1,1}^* = \sum_{i=1}^{2n-2} \sup_{\theta \in U_n} \int_{B_i} \gamma_n dP_{\theta,n} \]

\[ \leq 2i_n^* m_n \left[ \sup_{1 \leq i \leq 2n-2} \sup_{\theta \in U_n} P_{\theta,n}^* (x ; h_n(x; \theta, \theta^*) \geq 2^{-1}(i+1) \alpha_n, \right. \]

\[ g_n(x; \theta, \theta^*) < 2^{-1} i \alpha_n \} \]

(2.5)

\[ \leq 2i_n^* m_n \left[ \sup_{1 \leq i \leq 2n-2} \lambda_n(2^{-1} i \alpha_n, 2^{-1} \alpha_n) \right] \]

\[ \leq 4i_n^* m_n \left[ \sup_{1 \leq i \leq 2n-2} \alpha_n^{-1} (1 + 2^{-1} i \alpha_n) n^{-n} \eta'_n \right] (\eta'_n = o(1)) \]

\[ \leq 4i_n^* m_n n^{-n} \eta'_n \]

\[ \leq A_1 n^{-1(3-3\beta)} (\log n)^{\delta} \eta'_n \quad (A_1 \text{ is a constant}) \]

\[ = o(n^{-\delta}) . \]

Similarly we have

(2.6) \[ I_{1,2}^* = o(n^{-\delta}) . \]

Thus from (2.4) and (2.5) we have

(2.7) \[ I_1^* = o(n^{-\delta}) . \]
The third step. Next we evaluate $I'_n$ as follows. For every $s > 1$ we have

$$I'_n = \sup_{\theta \in U_n} \int_{\mathcal{W}_n} \gamma_n dP_{\theta_{n,n}} \leq \sup_{\theta \in U_n} \int_{\{h_n > m_n\}} h_n(x; \theta, \theta_n^s) dP_{\theta_{n,n}} \leq (m_n)^{-1-s} \sup_{\theta \in U_n} J_n(s; \theta, \theta_n^s).$$

Hence we have

$$I'_n \leq A_2(s)(m_n)^{-1-s} = A_2(s) n^{(1-s)/t},$$

where $A_2(s)$ is some constant depending only on $s$. We can choose $s > 1$ large enough so that

$$I'_n = o(n^{-p}).$$

From (2.7) and (2.8) we have

$$I_n = o(n^{-p}).$$

Hence from (2.3) we have

$$J^*_n = o(n^{-p}).$$

Put $W'_n = \{x; h_n(x; \theta, \theta_n^s) < 2^{-1} m_n\}$. Then we have

$$J^{**}_n = \sup_{\theta \in U_n} \int_{W'_n} h_n(x; \theta, \theta_n^s) dP_{\theta_{n,n}} \leq \sup_{\theta \in U_n} \int_{W'_n} h_n(x; \theta, \theta_n^s) dP_{\theta_{n,n}} + \sup_{\theta \in U_n} \int_{W'_n} h_n(x; \theta, \theta_n^s) dP_{\theta_{n,n}} \leq 2^{-1} m_n \lambda_n(m_n, m_n/2) + (m_n/2)^{1-s} \sup_{\theta \in U_n} J_n(s; \theta, \theta_n^s).$$

The first term on the right hand side is of order $o(n^{-p})$ by Proposition. The similar consideration as the evaluation of $I'_n$ implies that the second term of (2.9) is also of order $o(n^{-p})$ for sufficiently large number $s$. Thus we have

$$J^{**}_n = o(n^{-p}).$$

Hence it follows from (2.3) that

$$\sup_{\theta \in U_n} \rho_n(\theta, \theta_n^s) = o(n^{-p}).$$

The fourth step. Let $a_n(\theta, \theta_n^s) = \int_{\mathcal{X}} I_{W_n}(x) g_n(x; \theta, \theta_n^s) dP_{\theta_{n,n}} = 1(\leq \infty)$ and let $r_n(x; \theta, \theta_n^s) = a_n(\theta, \theta_n^s) I_{W_n}(x) g_n(x; \theta, \theta_n^s)$ if $a_n(\theta, \theta_n^s) < \infty$, $= 1$ otherwise. Define $g_n(x; \theta, \theta_n^s) = r_n(x; \theta, \theta_n^s) P_n(x, \theta_n^s)$ and let $Q_n$ be the probability distribution on $(\mathcal{X}, \mathcal{A}_n)$ with density $g_n(x; \theta, \theta_n^s)$ relative to $\mu_n$. We note that $B_n$ is suf-
cient \( \sigma \)-field for the family \( \{ Q_{\theta_n}^\theta \; ; \; \theta \in \Theta \} \) by the factorization theorem. It follows from (2.10) that there exists \( n_0 \) such that \( a_n(\theta, \theta^*) < \infty \) for every \( n \geq n_0 \) and every \( \theta \in U_n \). Therefore we can assume without loss of generality that \( a_n(\theta, \theta^*) < \infty \) for every \( \theta \in U_n \) and every \( n \geq 1 \).

Under this circumstances we have

\[
\| P_{\theta, n} - Q_{\theta_n}^\theta \|_{L^*_a} = \int \left| p_n(x, \theta) - q_n(x; \theta, \theta^*) \right| d\mu \\
\leq \int \left| h_n(x; \theta, \theta^*) - a_n(\theta, \theta^*) I_{w_n}(x) g_n(x; \theta, \theta^*) \right| \\
\cdot p_n(x, \theta^*) d\mu_n + \beta_n(\theta, \theta^*) \\
= \rho_n(\theta, \theta^*) + |1 - a_n(\theta, \theta^*)^{-1}| + \beta_n(\theta, \theta^*) \\
\leq 2 \rho_n(\theta, \theta^*) + 2 \beta_n(\theta, \theta^*)
\]

Here \( \| \nu \|_{L^*_a} \) means the total variation norm of a signed measure \( \nu \) on \( (\mathcal{X}, \mathcal{A}_a) \).

From Assumption 1, (b) and (2.10) we have

\[
\sup_{\theta \in U_n} \| P_{\theta, n} - Q_{\theta_n}^\theta \|_{L^*_a} = o(n^{-\beta})
\]

This completes the proof of the theorem.

3. The order of local asymptotic sufficiency. In this section the index set \( \Theta \) is assumed to be a subset of \( p \)-dimensional Euclidean space \( \mathbb{R}^p \). We denote by \( | \cdot | \) the usual Euclidean norm in \( \mathbb{R}^p \). For \( \theta \in \Theta \) and \( b > 0 \) let \( U_n(\theta, b) = \{ \theta' \in \Theta \; ; \; n^{-b} |\theta' - \theta| \leq b \} \).

Let \( \{ \mathcal{B}_n \} \in \mathbb{N} \) be the sequence of sub \( \sigma \)-fields \( \mathcal{B}_n \subset \mathcal{A}_a \) as in the previous section. We consider the following assumption which will be used to prove our main theorem, Theorem 2.

**Assumption 2.** For every compact subset \( \mathcal{K} \) of \( \Theta \) and \( b > 0 \)

(a) \( \lim_{n \to \infty} \sup_{\theta^* \in \mathcal{K}} \sup_{\theta \in U_n(\theta^*, b)} \int_{\mathcal{S}}(s; \theta, \theta^*) < \infty \) (\( \forall s > 1 \)), and

(b) \( \sup_{\theta^* \in \mathcal{K}} \sup_{\theta \in U_n(\theta^*, b)} \beta_n(\theta, \theta^*) = o(n^{-\gamma}) \).

Let \( \alpha \) be a given positive number. We state a result about higher order locally asymptotic sufficiency of \( \{ \mathcal{B}_n \} \) for \( \{ \mathcal{D}_n \} \).

**Theorem 2.** Suppose that Assumption 2 is satisfied with \( \gamma > 0 \), and that for every compact subset \( \mathcal{K} \) of \( \Theta \) and every \( b > 0 \)

(3.1) \( \sup_{c > 0} \sup_{\theta^* \in \mathcal{K}} \sup_{\theta \in U_n(\theta^*, b)} \{ r_{\mathcal{D}_n}(c; \theta, \theta^*) - r_{\mathcal{A}_n}(c; \theta, \theta^*) \} = o(n^{-\alpha}) \).

Then for every positive number \( \beta \) satisfying \( \beta < 3^{-1} \alpha \) and \( \beta \leq \gamma \) \( \{ \mathcal{B}_n \} \in \mathbb{N} \) is locally asymptotically sufficient for \( \{ \mathcal{D}_n \} \) with order \( o(n^{-\beta}) \) in the following sense: For each
For each \( n \in \mathbb{N} \) and each \( \theta_0 \in \Theta \) there exists a family \( Q_{\theta_0}^n = \{Q_{\theta_0,n}; \theta \in \Theta\} \) of probability distributions on \((\mathcal{X}, \mathcal{A}_n)\) such that

(i) \( \mathcal{B}_n \) is sufficient for \( Q_{\theta_0}^n \), and

(ii) for every compact subset \( K \) of \( \Theta \) and every \( b > 0 \)

\[
\sup_{\theta \in K} \sup_{\theta \in \mathcal{U}_n(\theta, b)} \|P_{\theta,n} - Q_{\theta_0,n}\|_{\mathcal{L}_n} = o(n^{-p}).
\]

Since the above result follows directly from Theorem 1 we shall omit the proof.

It is open problem whether non-local version of Theorem 2 still holds or not, i.e., whether any conditions such as in Theorem 2 imply the followings or not: There exists a sequence \( Q_n = \{Q_{\theta,n}; \theta \in \Theta\} \) of probability distributions on \((\mathcal{X}, \mathcal{A}_n)\) such that \( \mathcal{B}_n \) is sufficient for \( Q_n \), and that for every compact subset \( K \) of \( \Theta \)

\[
\sup_{\theta \in K} ||P_{\theta,n} - Q_{\theta,n}||_{\mathcal{L}_n} = o(n^{-p}).
\]

The case of \( \alpha = \beta = 0 \) has been discussed in Suzuki [3] in such a non-local situation.

It is well known that under some regularity conditions there exist a sequence \( \{\hat{\theta}_n\} \) of estimators of \( \theta \), a positive number \( \gamma \) and a number \( v \geq 1 \) having the following property: For every compact subset \( K \) of \( \Theta \) there corresponds \( a(K) \) such that

\[
\sup_{\theta \in K} P_{\theta,n} \{n^{1/2} |\hat{\theta}_n(x) - \theta| \geq a(K) (\log n)^{v/2} \} = o(n^{-\gamma})
\]

(c.f. Matsuda [2], Chap. 3).

Using such an estimator \( \{\hat{\theta}_n\} \) we may be able to construct \( \{Q_{\theta,n}; \theta \in \Theta\} \) satisfying the property (3.2), and for which \( \mathcal{B}_n \) is sufficient.

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References


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