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# ON A RELATION BETWEEN HIGHER ORDER ASYMPTOTIC RISK SUFFICIENCY AND HIGHER ORDER ASYMPTOTIC SUFFICIENCY IN A LOCAL SENSE

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**1. Introduction.** In Takeuchi [4] higher order asymptotic risk sufficiency of maximum likelihood estimator has been discussed. In this paper we try to find some relations between asymptotic risk sufficiency with a special loss function and asymptotic sufficiency in a local sense.

Let $\mathcal{P}_n=\{P_{\theta,n}; \theta\in\Theta\}$ be a family of probability distributions on a measurable space $(\mathcal{X}, \mathcal{A}_n)$ with an index set $\Theta$ which is a subset of an Euclidean space with the usual norm $|\cdot|$. For a sub $\sigma$-field $\mathcal{C}$ of $\mathcal{A}_n$, real number $c\geqq 0$ and $\theta, \theta'\in\Theta$ let $r_n^{\mathcal{C}}(c: \theta, \theta')=\inf\ (1+c)^{-1}\{1-E_{P_{\theta,n}}(\phi)+cE_{P_{\theta',n}}(\phi);\ \phi$ are $\mathcal{C}$-measurable statistical test functions on $\mathcal{X}\}$. We note that $r_n^{\mathcal{C}}(c: \theta, \theta')$ means the Bayes risk of statistical problem of testing a hypothesis '$P_{\theta',n}$ is true' against an alternative '$P_{\theta,n}$ is true' with experiment $(\mathcal{X}, \mathcal{C}, \{P_{\theta',n}, P_{\theta,n}\})$ relative to a prior probability distribution $(c/(1+c), 1/(1+c))$ on $\{\theta', \theta\}$ provided that the loss function is simple.

Let $\{\mathcal{B}_n; n=1, 2, \cdots\}$ be a sequence of sub $\sigma$-fields of $\{\mathcal{A}_n\}$ $(\mathcal{B}_n\subset\mathcal{A}_n)$. In this paper we give a sufficient condition about the Bayes risk $r_n^{\mathcal{B}_n}$ for $\{\mathcal{B}_n\}$ to be higher order locally asymptotically sufficient sequence of $\sigma$-fields. More precisely our main result in this paper is the following: Under some conditions if for some positive number $\alpha$ $\sup_{c>0}\sup_{\theta^*\in K}\sup_{\theta:\, n^{1/2}|\theta-\theta^*|\leqq b}\{r_n^{\mathcal{B}_n}(c: \theta, \theta^*)-r_n^{\mathcal{A}_n}(c: \theta, \theta^*)\}=o(n^{-\alpha})$ for every $b>0$ and every compact subset $K$ of $\Theta$, then for every $\beta$ satisfying $0<\beta<3^{-1}\alpha$ $\{\mathcal{B}_n\}$ is locally asymptotically sufficient for $\{\mathcal{P}_n\}$ with order $o(n^{-\beta})$ in the sense that for each $n=1, 2, \cdots$ and each $\theta_0\in\Theta$ there exists a family $\{Q_{\theta,n}^{\theta_0}; \theta\in\Theta\}$ of probability distributions on $(\mathcal{X}, \mathcal{A}_n)$ for which $\mathcal{P}_n$ is sufficient $\sigma$-field and that for every $b>0$

$$\sup_{\theta:\, n^{1/2}|\theta-\theta_0|\leqq b}||P_{\theta,n}-Q_{\theta,n}^{\theta_0}||_{\mathcal{A}_n}=o(n^{-\beta})$$

uniformly in $\theta_0$ over every compact subsets of $\Theta$. Here $||\cdot||_{\mathcal{A}_n}$ means the total variation norm over $\mathcal{A}_n$.

We have discussed such a problem in the case $\alpha=\beta=0$ in Suzuki [3] under non-local situation. In LeCam [1], Chap. 5 he discusses some relations between insufficiency and deficiency in his terminology.

In Section 2 some auxiliary results about the order of asymptotic sufficiency are proved. The main theorem is stated and followed by some discussions about the asymptotic sufficiency in non-local sense in Section 3.

2. **Auxiliary results.** For each $n\in N=\{1, 2, \cdots\}$ let $\mathcal{P}_n=\{P_{\theta,n};\ \theta\in\Theta\}$ be a family of probability distributions on a measurable space $(\mathcal{X}, \mathcal{A}_n)$ with an index set $\Theta$. For a subset $U(\neq\phi)$ of $\Theta$ we shall denote by $\mathcal{P}_n^U$ the totality of $P_{\theta,n}$'s satisfying $\theta\in U$. We assume that for each $n\in N$ $\mathcal{P}_n$ is dominated by a $\sigma$-finite measure $\mu_n$ on $(\mathcal{X}, \mathcal{A}_n)$. The probability density function of $P_{\theta,n}$ relative to $\mu_n$ will be denoted by $p_n(x, \theta)$. Without loss of generality we assume in the following that $\mu_n$ is a probability measure on $(\mathcal{X}, \mathcal{A}_n)$. For each $\theta, \theta'\in\Theta$ let $S_n(\theta)=\{x;\ p_n(x, \theta)>0\}$ and let $h_n(x; \theta, \theta')=p_n(x, \theta)/p_n(x, \theta')$ if $x\in S_n(\theta')$, $=+\infty$ if $x\in S_n(\theta)\cap S_n(\theta')^c$, $=1$ if $x\in S_n(\theta)^c\cap S_n(\theta')^c$. We put $\beta_n(\theta, \theta')=P_{\theta,n}\{S_n(\theta')^c\}$. For each $\theta, \theta'\in\Theta$ and real number $s\geqq 1$ we define

$$J_n(s\colon \theta, \theta') = E_{P_{\theta',n}}[\{h_n(x; \theta, \theta')\}^s] .$$

We note that $\beta_n(\theta, \theta') = 1-J_n(1\colon \theta, \theta')$.

Let $\{U_n\}$ be a sequence of nonempty subsets of $\Theta$. For $\{U_n\}$ we consider the following assumption.

ASSUMPTION 1. There exist a sequence $\{\theta_n^*\}_{n\in N}(\theta_n^*\in U_n)$ and a positive number $\gamma$ such that

(a) For every $s\geqq 1$

$$\lim_{b\to\infty}\ \sup\ \sup_{\theta\in U_n} J_n(s\colon \theta, \theta_n^*)<\infty ,$$

(b) $\sup_{\theta\in U_n} \beta_n(\theta, \theta_n^*) = o(n^{-\gamma})$ .

For a sub $\sigma$-field $\mathcal{C}$ of $\mathcal{A}_n$ we denote by $\Phi(\mathcal{C})$ the family of $\mathcal{C}$-measurable statistical test functions on $\mathcal{X}$. For each $\theta, \theta'\in\Theta$ and each real number $c\geqq 0$ we define

$$r_n^{\mathcal{C}}(c\colon \theta, \theta') = \inf\ (1+c)^{-1}\{1-E_{P_{\theta,n}}(\phi)+cE_{P_{\theta',n}}(\phi);\ \phi\in\Phi(\mathcal{C})\} .$$

Let $\{\mathcal{B}_n\}$ be a sequence of sub $\sigma$-fields of $\{\mathcal{A}_n\}$ $(\mathcal{B}_n\subset\mathcal{A}_n)$. For each $\theta\in\Theta$ define $\bar{p}_n(x, \theta)=E_{\mu_n}[p_n(x, \theta)\,|\,\mathcal{B}_n]$ the conditional expectation of $p_n(x, \theta)$ given $\mathcal{B}_n$ with respect to $\mu_n$ and put $S_n'(\theta)=\{x;\ \bar{p}_n(x, \theta)>0\}$. For $\theta, \theta'\in\Theta$ define $g_n(x; \theta, \theta')=\bar{p}_n(x, \theta)/\bar{p}_n(x, \theta')$ if $x\in S_n'(\theta')$, $=+\infty$ if $x\in S_n'(\theta')^c\cap S_n'(\theta)$, $=1$ if $x\in S_n'(\theta')^c\cap S_n'(\theta)$. For $c>0$ and $\delta>0$ let $E_n(c, \theta, \delta)=\{x;\ g_n(x; \theta, \theta_n^*)<c<c+\delta\leqq h_n(x; \theta, \theta_n^*)\}$ and $E_n'(c, \theta, \delta)=\{x;\ g_n(x; \theta, \theta_n^*)>c>c-\delta>h_n(x; \theta, \theta_n^*)\}$.

**Proposition.** *Suppose that for some positive number $\alpha$ and a sequence $\{\theta_n^*\}_{n\in N}$ $(\theta_n^*\in U_n)$*

$$\sup_{c>0}\sup_{\theta\in U_n}\{r_n^{\mathcal{B}_n}(c:\theta,\theta_n^*)-r_n^{\mathcal{A}_n}(c:\theta,\theta_n^*)\}=o(n^{-\alpha})\,. \tag{2.1}$$

*Then we have*

$$\sup_{c>0,\delta>0}\delta(1+c)^{-1}\lambda_n(c,\delta)=o(n^{-\alpha})\,,\quad \textit{and} \tag{2.2}$$
$$\sup_{c>0,\delta>0}\delta(1+c)^{-1}\lambda_n'(c,\delta)=o(n^{-\alpha})$$

*where* $\lambda_n(c,\delta)=\sup_{\theta\in U_n}P_{\theta_n^*,n}(E_n(c,\theta,\delta))$ *and* $\lambda_n'(c,\delta)=\sup_{\theta\in U_n}P_{\theta_n^*,n}(E_n'(c,\theta,\delta))$.

This proposition can be proved in the same way as the proof of the first and second steps of Theorem 1 in Suzuki [3]. So we shall omit the proof of the proposition.

**Theorem 1.** *Suppose that Assumption 1 is satisfied with a sequence $\{\theta_n^*\}_{n\in N}$ and $\gamma>0$, and that $\{\mathcal{B}_n\}$ has the property (2.1) with $\beta>0$. Then for every $\beta$ satisfying $0<\beta<3^{-1}\alpha$ and $\beta\leqq\gamma$, $\{\mathcal{B}_n\}$ is asymptotically sufficient for $\{\mathcal{P}_n^{U_n}\}$ with order $o(n^{-\beta})$ in the following sense: For each $n\in N$ there exists a family $\{q_n(x;\theta,\theta_n^*);\theta\in\Theta\}$ of probability density functions on $(\mathcal{X},\mathcal{A}_n)$ relative to $\mu_n$ such that*

(i) *each $q_n$ can be factorized as follows:*

$$q_n(x;\theta,\theta_n^*)=r_n(x;\theta,\theta_n^*)\,p_n(x,\theta_n^*)$$

*where $r_n$ is a $\mathcal{B}_n$-measurable function, and*

(ii) $\displaystyle\sup_{\theta\in U_n}\int_{\mathcal{X}}|p_n(x,\theta)-q_n(x;\theta,\theta_n^*)|\,d\mu_n=o(n^{-\beta})\,.$

Proof. We shall divide the proof into several steps.

The first step. Suppose that Assumption 1 is satisfied with a sequence $\{\theta_n^*\}_{n\in N}$ and $\gamma>0$, and that $\{\mathcal{B}_n\}_{n\in N}$ has the property (2.1) with $\alpha>0$. Let $\beta$ be any number satisfying $0<\beta<3^{-1}\alpha$ and $\beta\leqq\gamma$. Take $\varepsilon_1$ be any number satisfying $0<\varepsilon_1<3^{-1}\cdot(\alpha-3\beta)$. Let $\alpha_n=n^{-\beta}(\log n)^{-1}$, $m_n=n^{\varepsilon_1}$ and $i_n=[m_n\alpha_n^{-1}]+1$ where $[a]$ means the maximum integer not exceeding $a$. Put $(\gamma_n=)\,\gamma_n(x;\theta,\theta_n^*)=|h_n(x;\theta,\theta_n^*)-g_n(x;\theta,\theta_n^*)|$, $(\gamma_n'=)\,\gamma_n'(x;\theta,\theta_n^*)=|h_n(x;\theta,\theta_n^*)-I_{W_n}(x)g_n(x;\theta,\theta_n^*)|$ and

$$\rho_n(\theta,\theta_n^*)=\int_{\mathcal{X}}\gamma_n'(x;\theta,\theta_n^*)\,dP_{\theta_n^*,n}$$

where $W_n=W_n(\theta,\theta_n^*)=\{x;g_n(x;\theta,\theta_n^*)\leqq m_n\}$ and $I_{W_n}$ means the indicator function of $W_n$.

We have

$$\sup_{\theta \in U_n} \rho_n(\theta, \theta_n^*) \leqq \sup_{\theta \in U_n} \int_{W_n} \gamma_n \, dP_{\theta_n^*, n} + \sup_{\theta \in U_n} \int_{W_n^c} h_n(x; \theta, \theta_n^*) \, dP_{\theta_n^*, n} = J_n^* + J_n^{**},$$

(2.3) and $J_n^* = \sup_{\theta \in U_n} \int_{W_n} \gamma_n \, dP_{\theta_n^*, n} \leqq \alpha_n + \sup_{\theta \in U_n} \int_{D_n \cap W_n} \gamma_n \, dP_{\theta_n^*, n} = \alpha_n + I_n$

$(D_n = \{x; \gamma_n \geqq \alpha_n\})$.

Furthermore we have

$$I_n = \sup_{\theta \in U_n} \int_{D_n \cap W_n} \gamma_n \, dP_{\theta_n^*, n} \leqq \sup_{\theta \in U_n} \int_{D_n \cap \widetilde{W}_n} \gamma_n \, dP_{\theta_n^*, n} + \sup_{\theta \in U_n} \int_{W_n^*} \gamma_n \, dP_{\theta_n^*, n}$$

where $W_n' = \{x; h_n(x; \theta, \theta_n^*) \leqq m_n\}$, $\widetilde{W}_n = W_n \cap W_n'$ and $W_n^* = W_n \cap (W_n')^c$.

The second step. It holds that

$$\begin{aligned} I_n' = \sup_{\theta \in U_n} \int_{D_n \cap \widetilde{W}_n} \gamma_n \, dP_{\theta_n^*, n} &\leqq \sum_{i=1}^{2i_n-2} \sup_{\theta \in U_n} \int_{B_i} \gamma_n \, dP_{\theta_n^*, n} \\ &+ \sum_{i=0}^{2i_n-3} \sup_{\theta \in U_n} \int_{C_i} \gamma_n \, dP_{\theta_n^*, n} \qquad (2.4) \\ &= I_{n,1}' + I_{n,2}' \end{aligned}$$

where $B_j = \widetilde{W}_n \cap \{x; h_n(x; \theta, \theta_n^*) \geqq 2^{-1}(i+1)\alpha_n, g_n(x; \theta, \theta_n^*) < 2^{-1} i \alpha_n\}$ and $C_i = \widetilde{W}_n \cap \{x; h_n(x; \theta, \theta_n^*) < 2^{-1}(i+1)\alpha_n, g_n(x; \theta, \theta_n^*) \geqq 2^{-1}(i+2)\alpha_n\}$. Using the property (2.2) in Proposition we can evaluate $I_{n,i}'(i=1, 2)$ as follows. Taking account of $3\varepsilon_1 < \alpha - 3\beta$ we have

$$\begin{aligned} I_{n,1}' &= \sum_{i=1}^{2i_n-2} \sup_{\theta \in U_n} \int_{B_i} \gamma_n \, dP_{\theta_n^*, n} \\ &\leqq 2i_n m_n [\sup_{1 \leqq i \leqq 2i_i-2} \sup_{\theta \in U_n} P_{\theta_n^*, n}\{x; h_n(x; \theta, \theta_n^*) \geqq 2^{-1}(i+1)\alpha_n, \\ &\quad g_n(x; \theta, \theta_n^*) < 2^{-1} i \alpha_n\}] \\ &\leqq 2i_n m_n [\sup_{1 \leqq i \leqq 2i_n-2} \lambda_n(2^{-1} i \alpha_n, 2^{-1}\alpha_n)] \qquad (2.5) \\ &\leqq 4i_n m_n [\sup_{1 \leqq i \leqq 2i_n-2} \alpha_n^{-1}(1+2^{-1} i \alpha_n) n^{-\alpha} \eta_n'] \ (\eta_n' = o(1)) \\ &\leqq 4i_n^2 m_n n^{-\alpha} \eta_n' \\ &\leqq A_1 \cdot n^{-(\alpha-2\beta-3\varepsilon_1)} (\log n)^2 \eta_n' \quad (A_1 \text{ is a constant}) \\ &= o(n^{-\beta}). \end{aligned}$$

Similarly we have

$$I_{n,2}' = o(n^{-\beta}). \tag{2.6}$$

Thus from (2.4) and (2.5) we have

$$I_n' = o(n^{-\beta}). \tag{2.7}$$

The third step. Next we evaluate $I_n'$ as follows. For every $s>1$ we have

$$I_n'' = \sup_{\theta \in U_n} \int_{W_n^*} \gamma_n \, dP_{\theta_q^*,n} \leqq \sup_{\theta \in U_n} \int_{\{h_n > m_n\}} h_n(x;\theta,\theta_n^*)\, dP_{\theta_n^*,n}$$
$$\leqq (m_n)^{1-s} \sup_{\theta \in U_n} J_n(s:\theta,\theta_n^*)\,.$$

Hence we have

$$I_n'' \leqq A_2(s)\,(m_n)^{1-s} = A_2(s)\, n^{(1-s)\varepsilon_1}$$

where $A_2(s)$ is some constant depending only on $s$. We can choose $s>1$ large enough so that

$$I_n'' = o(n^{-\beta})\,. \tag{2.8}$$

From (2.7) and (2.8) we have

$$I_n = o(n^{-\beta})\,.$$

Hence from (2.3) we have

$$J_n^* = o(n^{-\beta})\,.$$

Put $W_n'' = \{x; h_n(x;\theta,\theta_n^*) < 2^{-1} m_n\}$. Then we have

$$\begin{aligned} J_n^{**} &= \sup_{\theta \in U_n} \int_{W_n^c} h_n(x;\theta,\theta_n^*)\, dP_{\theta_n^*,n} \\ &\leqq \sup_{\theta \in U_n} \int_{W_n^c \cap W_n''} h_n(x;\theta,\theta_n^*)\, dP_{\theta_n,n} + \sup_{\theta \in U_n} \int_{W_n^c \cap (W_n'')^c} h_n(x;\theta,\theta_n^*)\, dP_{\theta_n^*,n} \\ &\leqq 2^{-1} m_n \lambda_n'(m_n, m_n/2) + (m_n/2)^{1-s} \sup_{\theta \in U_n} J_n(s:\theta,\theta_n^*)\,. \end{aligned} \tag{2.9}$$

The first term on the right hand side is of order $o(n^{-\beta})$ by Proposition. The similar consideration as the evaluation of $I_n''$ implies that the second term of (2.9) is also of order $o(n^{-\beta})$ for sufficiently large number $s$. Thus we have

$$J_n^{**} = o(n^{-\beta})\,.$$

Hence it follows from (2.3) that

$$\sup_{\theta \in U_n} \rho_n(\theta,\theta_n^*) = o(n^{-\beta})\,. \tag{2.10}$$

The fourth step. Let $a_n(\theta,\theta_n^*) = [\int_{\mathcal{X}} I_{W_n}(x)\, g_n(x;\theta,\theta_n^*)\, dP_{\theta_n^*,n}]^{-1} (\leqq \infty)$ and let $r_n(x;\theta,\theta_n^*) = a_n(\theta,\theta_n^*)\, I_{W_n}(x)\, g_n(x;\theta,\theta_n^*)$ if $a_n(\theta,\theta_n^*) < \infty$, $=1$ otherwise. Define $q_n(x;\theta,\theta_n^*) = r_n(x;\theta,\theta_n^*)\, p_n(x,\theta_n^*)$ and let $Q_{\theta,n}^{\theta_n^*}$ be the probability distribution on $(\mathcal{X}, \mathcal{A}_n)$ with density $q_n(x;\theta,\theta_n^*)$ relative to $\mu_n$. We note that $\mathcal{B}_n$ is sufficient $\sigma$-field for the family $\{Q^{\theta_n^*}_{\theta,n};\, \theta\in\Theta\}$ by the factorization theorem. It follows from (2.10) that there exists $n_0$ such that $a_n(\theta, \theta_n^*)<\infty$ for every $n\geqq n_0$ and every $\theta\in U_n$. Therefore we can assume without loss of generality that $a_n(\theta, \theta_n^*)<\infty$ for every $\theta\in U_n$ and every $n\geqq 1$.

Under this circumstances we have

$$\begin{aligned}
\|P_{\theta,n}-Q^{\theta_n^*}_{\theta,n}\|_{\mathcal{A}_n} &= \int_{\mathcal{X}} |p_n(x, \theta)-q_n(x;\theta, \theta_n^*)|\,d\mu_n \\
&\leqq \int_{S_n(\theta_n^*)} |h_n(x;\theta, \theta_n^*)-a_n(\theta, \theta_n^*)\, I_{W_n}(x)\, g_n(x;\theta, \theta_n^*)| \\
&\qquad \cdot p_n(x, \theta_n^*)\, d\mu_n+\beta_n(\theta, \theta_n^*) \\
&= \rho_n(\theta, \theta_n^*)+|1-a_n(\theta, \theta_n^*)^{-1}|+\beta_n(\theta, \theta_n^*) \\
&\leqq 2\,\rho_n(\theta, \theta_n^*)+2\,\beta_n(\theta, \theta_n^*)\,.
\end{aligned}$$

Here $\|\nu\|_{\mathcal{A}_n}$ means the total variation norm of a signed measur $\nu$ on $(\mathcal{X}, \mathcal{A}_n)$. From Assumption 1, (b) and (2.10) we have

$$\sup_{\theta\in U_n} \|P_{\theta,n}-Q^{\theta_n^*}_{\theta,n}\|_{\mathcal{A}_n} = o(n^{-\beta})\,.$$

This completes the proof of the theorem.

**3. The order of local asymptotic sufficiency.** In this section the index set $\Theta$ is assumed to be a subset of $p$-dimensional Euclidean space $R^p$. We denote by $|\cdot|$ the usual Euclidean norm in $R^p$. For $\theta\in\Theta$ and $b>0$ let $U_n(\theta, b) = \{\theta'\in\Theta;\, n^{1/2}|\theta'-\theta|\leqq b\}$.

Let $\{\mathcal{B}_n\}_{n\in N}$ be the sequence of sub $\sigma$-fields $\mathcal{B}_n\subset\mathcal{A}_n$ as in the previous section. We consider the following assumption which will be used to prove our main theorem, Theorem 2.

ASSUMPTION 2. For every compact subset $K$ of $\Theta$ and $b>0$

(a) $\displaystyle\limsup_{n\to\infty}\ \sup_{\theta^*\in K}\ \sup_{\theta\in U_n(\theta^*,b)} J_n(s:\theta, \theta^*)<\infty \quad (\forall\, s>1)$, and

(b) $\displaystyle\sup_{\theta^*\in K}\ \sup_{\theta\in U_n(\theta^*,b)} \beta_n(\theta, \theta^*) = o(n^{-\gamma})$.

Let $\alpha$ be a given positive number. We state a result about higher order locally asymptotic sufficiency of $\{\mathcal{B}_n\}$ for $\{\mathcal{P}_n\}$.

**Theorem 2.** *Suppose that Assumption 2 is satisfied with $\gamma>0$, and that for every compact subset $K$ of $\Theta$ and every $b>0$*

$$\sup_{c>0}\ \sup_{\theta^*\in K}\ \sup_{\theta\in U_n(\theta^*,b)} \{r_n^{\mathcal{B}_n}(c:\theta, \theta^*)-r_n^{\mathcal{A}_n}(c:\theta, \theta^*)\} = o(n^{-\alpha})\,. \tag{3.1}$$

*Then for every positive number $\beta$ satisfying $\beta<3^{-1}\alpha$ and $\beta\leqq\gamma$ $\{\mathcal{B}_n\}_{n\in N}$ is locally asymptotically sufficient for $\{\mathcal{P}_n\}$ with order $o(n^{-\beta})$ in the following sense: For each*

*$n\in N$ and each $\theta_0\in\Theta$ there exists a family $Q_n^{\theta_0}=\{Q_{\theta,n}^{\theta_0};\ \theta\in\Theta\}$ of probability distributions on $(\mathcal{X}, \mathcal{A}_n)$ such that*

(i) *$\mathcal{B}_n$ is sufficient for $Q_n^{\theta_0}$, and*

(ii) *for every compact subset $K$ of $\Theta$ and every $b>0$*

$$\sup_{\theta\in K}\ \sup_{\theta\in U_n(\theta,b)} \|P_{\theta,n}-Q_{\theta,n}^{\theta_0}\|_{\mathcal{A}_n}=o(n^{-\beta}).$$

Since the above result follows directly from Theorem 1 we shall omit the proof.

It is open problem whether non-local version of Theorem 2 still holds or not, i.e., whether any conditions such as in Theorem 2 imply the followings or not: There exists a sequence $Q_n=\{Q_{\theta,n};\ \theta\in\Theta\}$ of probability distributions on $(\mathcal{X}, \mathcal{A}_n)$ such that $\mathcal{B}_n$ is sufficient for $Q_n$, and that for every compact subset $K$ of $\Theta$

$$\sup_{\theta\in K}\|P_{\theta,n}-Q_{\theta,n}\|_{\mathcal{A}_n}=o(n^{-\beta}). \tag{3.2}$$

The case of $\alpha=\beta=0$ has been discussed in Suzuki [3] in such a non-local situation.

It is well known that under some regularity conditions there exist a sequence $\{\hat{\theta}_n\}_{n\in N}$ of estimators of $\theta$, a positive number $\gamma$ and a number $v\geqq 1$ having the following property: For every compact subset $K$ of $\Theta$ there corresponds $a(K)$ such that

$$\sup_{\theta\in K} P_{\theta,n}\{n^{1/2}|\hat{\theta}_n(x)-\theta|\geqq a(K)\,(\log n)^{v/2}\}=o(n^{-\gamma})$$

(c.f. Matsuda [2], Chap. 3).

Using such an estimator $\{\hat{\theta}_n\}$ we may be able to construct $\{Q_{\theta,n};\ \theta\in\Theta\}$ satisfying the property (3.2), and for which $\mathcal{B}_n$ is sufficient.

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