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A CHARACTERIZATION OF THE SIMPLE GROUP J_4

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In [4] Janko describes the properties of a simple group of order $2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$ denoted by J_4 . It has exactly one conjugacy class of elements of order 3 and if π is one of them, then the centralizer of π in J_4 is isomorphic to the 6-fold cover of the Mathieu group M_{22} . We show in this paper that these properties characterize the group J_4 , we prove namely the

Theorem A. *Let G be a finite group containing an element π of order 3 such that $C_G(\pi)$ is isomorphic to the 6-fold cover M_{22} . If G is not 3-normal then G is isomorphic to J_4 .*

In the first section we shall list some properties of the 6-fold cover of M_{22} , which will be needed in the proof. The second section is then devoted to the proof of Theorem A. In the last section we remark that the following holds:

Theorem B *There exists no simple group G which is not 3-normal and contains an element π such that $C_G(\pi)$ is isomorphic to the triple cover of M_{22} .*

The Frattini subgroup of a group X is denoted by $D(X)$. The other notation is hopefully standard.

In the whole paper with the exception of the last section G denotes a simple group satisfying the assumptions of Theorem A and π is an element of G of order 3 such that $C_G(\pi)$ is isomorphic to the 6-fold cover of M_{22} .

1. Some known results and structure of $N_G(\langle \pi \rangle)$

We first list some well known results which will be used in the proof of our theorems.

Lemma 1.1 (Gaschütz). *Let A be an abelian normal subgroup of the group X contained in the subgroup B of X with $(|X:B|, |A|)=1$. Then if A has a complement in B , A has a complement in X .*

Proof. See [1].

Lemma 1.2 (Thompson). *If the group X admits a fixed-point-free automorphism of prime order then X is nilpotent.*

Proof. See [2; 10.2.1].

Lemma 1.3 (Thompson). *Let T_0 be a maximal subgroup of an S_2 -subgroup of the group X . If X does not have a subgroup with index two then all involutions of X are conjugate to elements of T_0 in X .*

Proof. See [10. Lemma 5.38].

Lemma 1.4 (Burnside). *Let P be an S_p -subgroup of the group X and assume that $N_X(P) = C_X(P)$. Then X has a normal p -complement.*

Proof. See [2; 7.4.3].

Lemma 1.5. *Let P be a p -group and let Q be a noncyclic abelian q -group of automorphisms of P , q a prime distinct from p . Then $P = \langle C_P(x) \mid 1 \neq x \in Q \rangle$.*

Proof. See [2; 5.3.16].

Lemma 1.6. *Any involution t of the group X which does not lie in the maximal normal 2-subgroup of X inverts a nontrivial element of X of odd order.*

Proof. Let t be an involution of X with $t \notin O_2(X)$. Then there exists a conjugate t_1 of t in X such that the dihedral group $\langle t, t_1 \rangle$ is not a 2-group by [2; 3.8.2]. Since the index of the cyclic subgroup $\langle t_1 t \rangle$ has index two in $\langle t, t_1 \rangle$ we see that $O(\langle t_1 t \rangle)$ is nontrivial and is inverted by t since t inverts $t_1 t$.

The following three lemmas are taken from [4; (2.1), (2.3), (2.4)].

Lemma 1.7. *Let $X \cong M_{22}$ and let T be an S_2 -subgroup of X . Then T possesses precisely two distinct elementary abelian subgroups E_1 and E_2 of order 16 and they are both normal in T . We have $N_X(E_1)$ is a splitting extension of E_1 by A_6 , $N_X(E_2)$ is a splitting extension of E_2 by S_5 and $N_X(E_i)$ acts transitively on E_i^* , $i = 1, 2$. The group X has the order $2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$ and exactly one conjugacy class of involutions with the representative $e \in E_1$ and we have $C_X(e) = C(e) \cap N_X(E_1)$. An S_3 -subgroup P of X is elementary abelian of order 9 and we have $C_X(P) = P$ and $N_X(P) = PQ$ where Q is quaternion and acts regularly on P . The group X has exactly one conjugacy class of elements of order 3 and if σ is one of them, then $C_X(\sigma) \cong \langle \sigma \rangle \times A_4$ and $N_X(\langle \sigma \rangle) = \langle \sigma \rangle B$ where $B \cong S_4$.*

Lemma 1.8. *Let $X \cong \text{Aut}(M_{22})$ so that $X' \cong M_{22}$ and $|X : X'| = 2$. The group X possesses exactly two conjugacy classes of involutions which are contained in $X - X'$ with the representatives t_1 and t_2 . If E_1 and E_2 are the only elementary abelian subgroups of rank 4 of an S_2 -subgroup of X' as described in (1.7) then t_1 and t_2 can be chosen to lie in $C_X(E_2) = A = \langle E_2, t_2 \rangle$ which is elementary abelian of*

order 32. Then $N_X(A) = AB$ where $B \subseteq X'$, $B \cong S_5$ and B operates transitively on E_2^* and operates on $A - E_2$ in two orbits of sizes 10 and 6 represented respectively by t_1 and t_2 . We have $N_X(E_1)$ is a splitting and faithful extension of E_1 by S_6 .

Lemma 1.9. *Every maximal subgroup of the simple group M_{24} is isomorphic to one of the following groups:*

$PSL(2, 23)$, M_{23} , $\text{Aut}(M_{22})$, $\text{Aut}(M_{12})$, $PSL(2, 7)$,

The holomorph of an elementary abelian group of order 16,

An extension of M_{21} by S_3 ,

A splitting and faithful extension of an elementary abelian group of order 64 by a subgroup Y where $|0_3(Y)| = 3$, $Y/0_3(Y) \cong S_6$, $|Y:Y'| = 2$, $Y'' = Y'$ and $C_Y(0_3(Y)) = Y'$,

A splitting and faithful extension of an elementary abelian group of order 64 by $S_3 \times PSL(2, 7)$.

In the next lemma we list some properties of $N_G(\langle \pi \rangle)$ which can be easily deduced from (1.7) and (1.8) and are essentially proved in [4]. Throughout the paper we shall fix the notation which will be introduced in the following lemma.

Lemma 1.10. *The following hold in G :*

(i) Let $H = N_G(\langle \pi \rangle)$. Then $|H:H'| = 2$, $H'' = H' = C_H(\pi)$, $Z(H')$ is cyclic of order 6, $H'/Z(H') \cong M_{22}$ and $H/Z(H') \cong \text{Aut}(M_{22})$. Let us denote the involution in $Z(H')$ by z .

(ii) Let T be an S_2 -subgroup of H and let $T_0 = T \cap H'$. Then T_0 contains exactly two elementary abelian subgroups E_1 and E_2 of rank 5. These are normal in T and we have $C_{H'}(E_i) = E_i \langle \pi \rangle$, $i = 1, 2$ and

$N_{H'}(E_1) = E_1 B_1$ where B_1 is isomorphic to the triple cover of A_6 ,

$C_H(E_1) = E_1 \langle \pi \rangle$, $N_H(E_1)/C_H(E_1) \cong S_6$,

$N_{H'}(E_2) = E_2 \langle \pi \rangle \times B_2$ where $B_2 \cong S_5$ and acts transitively on $(E_2/\langle z \rangle)^*$,

$N_H(E_2) = \langle \pi \rangle E_2^* B_2$ where E_2^* is an abelian group of order 64, B_2 normalizes E_2^* , $\langle z \rangle \geq D(E_2^*)$, E_2^* is elementary abelian if and only if there exist involutions in $H - H'$ and if so then E_2^* is the only elementary abelian subgroup of T of order 64. Furthermore $E_2^* \cap H' = E_2$ and $E_2^* \leq T$.

(iii) Let P be an S_3 -subgroup of B_1 . Then P is an S_3 -subgroup of G . $P = \langle \pi, \sigma, \tau \rangle$ is extraspecial of order 27 and exponent 3, where the generators of P are chosen in such a way that $C_{E_1}(\tau) = \langle z \rangle$ and $C_{E_1}(\sigma) = E_0$ is of order 8. We have $N_{H'}(P) = \langle z \rangle \times PQ$ where Q is quaternion and acts regularly on $P/\langle \pi \rangle$. There exists exactly one conjugacy class of elements of order 3 in $H - \langle \pi \rangle$ and exactly one conjugacy class of subgroups of order 9 in H' represented by $M = \langle \pi, \sigma \rangle$. We have $C_G(M) = M \times E_0$ and $N_{H'}(M)/C_{H'}(M) \cong S_3$.

(iv) We have $C_{H'}(E_0) = E_1 M$.

(v) $E_1 \cap E_2$ is of order 8 and we have $C_{H'}(E_1 \cap E_2) = \langle \pi \rangle \times E_1 E_2$ and

$N_{H'}(E_1 \cap E_2) = C_{H'}(E_1 \cap E_2)U$ where $U \cong S_3$ and $O_3(U)$ acts regularly on $E_1 E_2 / \langle z \rangle$. Furthermore $E_1 \cap E_2 \subseteq T' = \langle T' \cap E_1, T' \cap E_2 \rangle$ and $T' \cap E_i$, $i=1, 2$, are the only elementary abelian subgroups of T' of rank four. So $E_1 \cap E_2$ is normal in $N_G(T)$. We have $C(E_1) \cap E_2^* = E_1 \cap E_2$.

Proof. $C_G(\pi)$ is isomorphic to the 6-fold cover M_{22} , i.e. $C_G(\pi)' = C_G(\pi)$, $Z(C_G(\pi))$ is cyclic of order 6 and $C_G(\pi)/Z(C_G(\pi)) \cong M_{22}$.

Let P be an S_3 -subgroup of $C_G(\pi)$. Then $\langle \pi \rangle \subseteq Z(P)$ and P does not split over by $\langle \pi \rangle$ (1.1). So $D(P) = \langle \pi \rangle$ by (1.7) and hence P is an S_3 -subgroup of G . Let R be an S_2 -subgroup of $N(P) \cap C_G(\pi)$. Since R operates transitively on $(P/D(P))^*$ by (1.7) we see that P is extraspecial of order 27 and exponent 3. Furthermore we have $R/\langle z \rangle \cong Q_8$ where z is the involution in $Z(C_G(\pi))$. So R must split over $\langle z \rangle$ and we get $N(P) \cap C_G(\pi) = \langle z \rangle \times PQ$ where $Q \cong Q_8$ and acts regularly on $P/D(P)$. In particular there exists exactly one conjugacy class of elements of order 3 in $C_G(\pi) - \langle \pi \rangle$ and hence exactly one conjugacy class of subgroups of order 9.

Since G is not 3-normal, π must be conjugate to an element in $P - \langle \pi \rangle$ and hence to π^{-1} . So $|N_G(\langle \pi \rangle): C_G(\pi)| = 2$. Let $H = N_G(\langle \pi \rangle)$. Then $H' = C_G(\pi)$ and $H/Z(H')$ is isomorphic to $\text{Aut}(M_{22})$ or $Z_2 \times M_{22}$ since $|\text{Aut}(M_{22}): M_{22}| = 2$ by (1.8). But the second case is not possible since otherwise there would exist a 2-element in $H - H'$ which operates trivially on $P/D(P)$ and inverts $D(P)$ and this is absurd. So $H/Z(H') \cong \text{Aut}(M_{22})$.

Let T be an S_2 -subgroup of H . Then all assertions of (ii) are proved in [4; Proposition 1 and 3] and we shall use them in the following.

Since an S_3 -subgroup of $B_1 \subseteq H$ is also an S_3 -subgroup of H we can assume that $P \subseteq B_1$. By the action of the non-cyclic abelian 3-group $P/D(P)$ on the 2-group E_1 we see that there is an element σ in P with $E_0 = E_1 \cap C(\sigma)$ is elementary abelian of order 8 and an element τ in P with $\langle z \rangle = E_1 \cap C(\tau)$. Let $M = \langle \pi, \sigma \rangle$. Then $C_G(M) = E_0 \times M$ and $N_{H'}(M)/C_{H'}(M) \cong S_3$ by (1.7). This completes the proof of the first three assertions of the lemma.

For the proof of (iv) let $\bar{H}' = H'/Z(H')$, which is isomorphic to M_{22} . By (1.7) we have $C_{\bar{H}}(\bar{\sigma}) = \langle \bar{\sigma} \rangle x \bar{E}_0 \langle \bar{\tau} \rangle$ where $\bar{E}_0 \langle \bar{\tau} \rangle$ is isomorphic to A_4 and $N_{\bar{H}'}(\langle \bar{\sigma} \rangle) / \langle \bar{\sigma} \rangle$ is isomorphic to S_4 . This gives that $C_{\bar{H}}(\bar{E}_0) \cap N_{\bar{H}'}(\langle \bar{\sigma} \rangle) = \bar{E}_0 \langle \bar{\sigma} \rangle$. By Burnside's transfer theorem we get $C_{\bar{H}'}(\bar{E}_0) = O_3(C_{\bar{H}'}(\bar{E}_0)) \langle \bar{\sigma} \rangle$. By the structure of M_{22} , $\bar{K} = O_3(C_{\bar{H}}(\bar{E}_0))$ is a 2-group containing \bar{E}_1 .

Suppose that $\bar{K} \neq \bar{E}$. Then the non-trivial group \bar{K}/\bar{E}_1 is normalized by $\bar{P} = \langle \bar{\sigma}, \bar{\tau} \rangle$. Since P is not cyclic there is by (1.5) a non-trivial element x in \bar{P} such that $C_{\bar{K}/\bar{E}_1}(x) = C_{\bar{K}}(x)\bar{E}_1/\bar{E}_1 \neq 1$. As $\bar{\sigma}$ operates regularly on \bar{K}/\bar{E}_1 and normalizes $C_{\bar{K}/\bar{E}_1}(x)$ we get that $\bar{K}/\bar{E}_1 = C_{\bar{K}/\bar{E}_1}(x)$ is elementary abelian of order four since $|\bar{K}/\bar{E}_1| \leq 8$. By the structure of the centralizer of an element of order three in M_{22} we get that $C_{\bar{K}}(x)$ is four group and that $\bar{K} = \bar{E}_1 C_{\bar{K}}(x)$.

$\bar{S} = C(C_{\bar{K}}(x)) \cap \bar{E}_1$ is non-trivial and is normalized by x which operates

regularly on it. This yields that $|\bar{S}| \geq 4$. So $\bar{D} = C_{\bar{K}}(x)\bar{S}$ is elementary abelian of rank at least four. Since an S_2 -subgroup of M_{22} contains exactly two elementary abelian subgroups of rank four by (1.7) we see that \bar{D} is conjugate in H' to \bar{E}_2 . But \bar{D} is normalized by \bar{P} whereas an S_3 -subgroup of $N_{\bar{H}}(\bar{E}_2)$ is of order 3. This contradiction shows that $\bar{K} = \bar{E}_1$ and hence $C_{H'}(E_0) = E_1M$.

For the proof of (v) observe that E_1E_2/E_1 is a non-trivial elementary abelian 2-group of $N_{H'}(E_1)/E_1 = \bar{B}_1$ which is isomorphic to the triple cover of A_6 . Since an S_2 -subgroup of \bar{B}_1 is dihedral of order 8 there exists a four group \bar{V} of \bar{B}_1 containing \bar{E}_2 . By the structure of \bar{B}_1 and by (1.1) we get that $N(\bar{V}) \cap \bar{B}_1 = \langle \pi \rangle x \bar{V} \bar{U}$ where $\bar{U} \cong S_3$ and operates faithfully on \bar{V} . Let U_0 be an S_3 -subgroup of the inverse image of \bar{U} . By (1.6) we can assume that U_0 is inverted by an involution x in $T_0 - E_1E_2$ such that $\langle U_0, x \rangle$ maps into \bar{U} and $\langle U_0, x \rangle \cong S_3$. U_0 normalizes the inverse image V of \bar{V} . Since $E_1E_2 \subseteq V$ and E_1 and E_2 are the only elementary abelian 2-groups of T_0 hence of V of rank 5 we see that U_0 normalizes both E_1 and E_2 and hence E_1E_2 . This implies that $\bar{V} = \bar{E}_2$ and $E_1 \cap E_2$ is of order 8. Furthermore U_0 maps onto an S_3 -subgroup of $N_{H'}(E_2)/E_2$. Since B_2 operates transitively on $(E_2/\langle z \rangle)^*$ by (ii) we obtain that U_0 operates regularly on $E_1/\langle z \rangle$. Since T_0 does not split over $\langle z \rangle$ we see that $\langle z \rangle$ is properly contained in $(E_1E_2)' = [E_1, E_2] \subseteq E_1 \subset E_2$ by (ii). Since U_0 operates regularly on $E_1 \cap E_2/\langle z \rangle$ we get that $(E_1E_2)' = E_1 \cap E_2$ and hence $D(E_1E_2) = E_1 \cap E_2$. Since U_0 acts regularly on $E_1 \cap E_2/\langle z \rangle$ we see that $E_1 \cap E_2$ is not centralized by x and hence $C_{H'}(E_1 \cap E_2) = \langle \pi \rangle \times E_1E_2$. By (iv) we get that $E_1 \cap E_2$ is not normalized by an S_3 -subgroup of H' , because otherwise it would be centralized by a subgroup of order 9 and would be conjugate to E_0 by (iii). This implies that U_0 operates regularly also on $E_1/\langle z \rangle$, because otherwise an S_3 -subgroup of B_1 containing $\langle \pi, U_0 \rangle$ would normalize $\langle [E_1, U_0], z \rangle = E_1 \cap E_2$. So U_0 acts regularly on $E_1E_2/\langle z \rangle$ and we have $C(E_1) \cap E_2 = E_1 \cap E_2$.

So we have seen that the elementary abelian group $E_1E_2/E_1 \cap E_2$ of rank 4 is normalized by $\langle U_0, x \rangle \cong S_3$ such that U_0 operates regularly on it. This shows that $T'_0/E_1 \cap E_2 = C(x(E_1 \cap E_2) \cap (E_1E_2/E_1 \cap E_2))$ and hence that $T'_0 = \langle T'_0 \cap E_1, T'_0 \cap E_2 \rangle$ where $T'_0 \cap E_i$, $i=1, 2$, is of order 16. Since $T/E_1 \cong Z_2 \times D_8$ we see that there exists an element t in $(T - T_0) \cap E_2^*$ such that $T/E_1 = \langle tE_1 \rangle x(T_0/E_1)$. Thus $[t, T_0] \subseteq E_1 \cap E_2^* = E_1 \cap E_2$. This implies that $T' = T'_0$. Since T_0 contains exactly two elementary abelian subgroups of order 32, T' is not abelian. This yields that $T' \cap E_i$, $i=1, 2$, are the only elementary abelian subgroups of T' of order 16. Since T' is normal in $N_G(T)$ we get that $E_1 \cap E_2 = (T' \cap E_1) \cap (T' \cap E_2)$ is normal in $N_G(T)$.

This completes the proof of the lemma.

2. Proof of Theorem A

In this section we prove Theorem A in a sequence of lemmas. We shall

use the notation introduced in (1.10).

Lemma 2.1. *We have $N_G(M)/C_G(M) \cong GL(2, 3)$ and $N_G(M)$ is contained in $N_G(E_0)$.*

Proof. Since G is not 3-normal and there exists precisely one conjugacy class of elements of order 3 in $H - \langle \pi \rangle$ represented by σ we have that $\pi \sim \sigma$ in G . So there exists an element g in G such that $\sigma^g = \pi$ and $C_P(\sigma)^g = M^g \subseteq P$. Since there exists in H' exactly one conjugacy class of subgroups of order 9 we can assume that $M^g = M$. So $\pi \sim \sigma$ in $N_G(M)$.

Since $M^\#$ is the union of $N_{H'}(M)$ -orbits of sizes 1, 1 and 6 represented by π, π^{-1}, σ respectively we get that $|N_G(M)/C_G(M)| = |GL(2, 3)|$ and hence $N_G(M)/C_G(M) \cong GL(2, 3)$.

Since $E_0 = 0_2(C_G(M))$ by (1.10.iii) we see that $E_0 \triangleleft N_G(M)$.

Lemma 2.2. *We have $C_G(E_1) = 0_2(C_G(E_1))\langle \pi \rangle$ where $0_2(C_G(E_1))$ is either equal to E_1 or is an elementary abelian group of order 2^{11} .*

Proof. By (1.10.ii) we have $C_H(E_1) = E_1\langle \pi \rangle$. Burnside's transfer theorem yields then that $C_G(E_1) = 0_3(C_G(E_1))\langle \pi \rangle$ since $\langle \pi \rangle$ is an S_3 -subgroup of $C_G(E_1)$ by (1.10.iii).

Let $K = 0_3(C_G(E_1))$. Since $C_H(E_1) = E_1\langle \pi \rangle$ we see that π operates regularly on K/E_1 . Thus K/E_1 is nilpotent by (1.2). As $E_1 \subseteq Z(K)$ we get that K is nilpotent. Furthermore K is normalized by P and hence we have $K = \langle C_K(x) \mid 1 \neq x \in M \rangle$, by (1.5).

We have $E_0 = C(x) \cap E_1 \subseteq Z(C_K(x))$ for any $x \in M - \langle \pi \rangle$. Since $N_G(M)$ is contained in $N_G(E_0)$ by (2.1) we see that $C(E_0) \cap C_G(x) = 0_2(C(E_0) \cap C_G(x))M$ for any $1 \neq x \in M$, where the maximal normal 2-subgroup of $C(E_0) \cap C_G(x)$ is elementary abelian of order 32 by (1.10.iv). So $C_K(x)$ is an elementary abelian 2-group of order at most 32. On the other hand π operates regularly on $C_K(x)E_0$ for $x \in M - \langle \pi \rangle$. This implies that we have either $C_K(x) = 0_2(C(E_0) \cap C_G(x))$ or $C_K(x) = E_0$ for $x \in M - \langle \pi \rangle$. Since all elements of the set $\{C_K(x) \mid x \in M - \langle \pi \rangle\}$ are conjugate to each other via τ we have either

$$C_K(x) = E_0 \quad \text{for all } x \in M - \langle \pi \rangle, \text{ i.e. } K = E_1,$$

or

$$\begin{aligned} C_K(x) &= 0_2(C(E_0) \cap C_G(x)) \quad \text{for all } 1 \neq x \in M, \text{ i.e.} \\ K &= \langle 0_2(C(E_0) \cap C_G(x)) \mid 1 \neq x \in M \rangle \end{aligned}$$

where $0_2(C(E_0) \cap C_G(x))$ is elementary abelian of order 32 for all $1 \neq x \in M$. We can assume that we are in the second case.

Let S be an S_2 -subgroup of $N_G(M)$. Then S acts transitively on $M^\#$ and normalizes E_0 . So S acts transitively on the set $\{0_2(C(E_0) \cap C_G(x)) \mid 1 \neq x \in M\}$

and hence normalizes K . Since $E_1 = 0_2(C(E_0) \cap C_G(\pi)) \subseteq Z(K)$ we get that $K \subseteq Z(K)$ and hence that K is elementary abelian of order 2^{11} since

$$\bar{K} = K/E_1 = C_{\bar{K}}(\sigma) \times C_{\bar{K}}(\sigma\pi) \times C_{\bar{K}}(\sigma\pi^{-1})$$

is of order 2^6 .

Lemma 2.3. *If $0_2(C_G(E_1)) = E_1$, then T is an S_2 -subgroup of G .*

Proof. By (2.2) and the assumption of this lemma we have $C_G(E_1) = E_1 \times \langle \pi \rangle$. Then $N_G(E_1)$ normalizes $\langle \pi \rangle$ and hence we get $N_G(E_1) = N_H(E_1)$. Thus T is an S_2 -subgroup of $N_G(E_1)$.

Suppose that T is not an S_2 -subgroup of G . Then there exists a 2-group $T\langle x \rangle$ in G with $|T\langle x \rangle| = 2$. If $E_1^\# \subseteq T_0$ we get $E_1^\# = E_1$ by (1.10.ii). This contradicts the fact that T is an S_2 -subgroup of $N_G(E_1)$. So $E_1^\# \not\subseteq T_0$ and thus $T - T_0$ contains involutions. Then $E_2^\#$ is the only elementary abelian subgroup of T of order 64 by (1.10.ii) and hence x normalizes $E_2^\#$.

Since x normalizes $T' \cap E_2^\# = T' \cap E_2$ and since T' contains exactly two elementary abelian subgroups of rank four, namely $T' \cap E_i$, $i=1, 2$, we see that x also normalizes $T' \cap E_1$. Since $E_1 \cap E_2 \triangleleft N_G(T)$ by (1.10.v) we get that $E_1 E_2^\# = C_T(E_1 \cap E_2) \triangleleft N_G(T)$ and hence $X = E_1 E_2^\# \cap C(T' \cap E_1) = E_1 C_{E_2^\#}(T' \cap E_1)$ is normalized by x . (1.10.v) gives then that $Z(X) = T' \cap E_1$ and that E_1 and $(T' \cap E_1) \times (E_2^\# \cap C(T' \cap E_1))$ are the only elementary abelian subgroups of X of rank five. Since $E_2^\# \cap C(T' \cap E_1)$ is normalized by x we get that $E_1^\# = E_1$ which is a contradiction. Thus T is a Sylow 2-subgroup of G .

Lemma 2.4. *If T is an S_2 -subgroup of G then the centralizer of the involution z in G is H .*

Proof. Let $C = C_G(z)$ and denote the homomorphic image of any subset X of C in $C/\langle z \rangle$ by \bar{X} . Obviously H is contained in C .

Then T is an S_2 -subgroup of C by our assumption and \bar{T} is isomorphic to an S_2 -subgroup of $\text{Aut}(M_{22})$. Since $H \subseteq C$ all involutions in \bar{T}_0 are conjugate to $\bar{e} \in Z(\bar{T})$ in \bar{C} and all involutions in $\bar{T} - \bar{T}_0$ are conjugate to involutions in $\bar{E}_2^\# - \bar{E}_2$ in \bar{C} where $\bar{E}_2^\# = C_{\bar{T}}(\bar{E}_2)$ is the only elementary abelian subgroup of \bar{T} of order 32 by (1.7) and (1.8). Furthermore we have $N_{\bar{H}}(\bar{E}_2^\#) = \bar{E}_2^\# \bar{B}_2$ where $\bar{B}_2 \cong S_5$ and $(\bar{E}_2^\#)^*$ splits into \bar{B}_2 -orbits of sizes 15, 6 and 10 represented respectively by \bar{e} , \bar{f}_1 and \bar{f}_2 where \bar{f}_1 and \bar{f}_2 are in $\bar{E}_2^\# - \bar{E}_2$ by (1.8).

If \bar{C} has no subgroups of index two then \bar{f}_i , $i=1, 2$, must be conjugate to an element of \bar{T}_0 hence to \bar{e} in \bar{C} by (1.3), Thompson's transfer lemma. But this conjugation must take place in $N_{\bar{C}}(\bar{E}_2^\#)$ since $\bar{E}_2^\#$ is the only elementary abelian subgroup of \bar{T} of rank 5. So we get by the above paragraph that all involutions of $\bar{E}_2^\#$ are conjugate to each other in $N_{\bar{C}}(\bar{E}_2^\#)$. In particular 31 divides the order of the group $N_{\bar{C}}(\bar{E}_2^\#)/C_{\bar{C}}(\bar{E}_2^\#)$.

Let $\tilde{N} = N_{\mathcal{C}}(\bar{E}_2^*)/C_{\mathcal{C}}(\bar{E}_2^*)$. Then \tilde{N} is isomorphic to a subgroup of $GL(5, 2)$, has dihedral S_2 -subgroups of order 8 and contains a subgroup \tilde{B}_2 which is isomorphic to S_5 . So $\tilde{N}/0(\tilde{N})$ is either isomorphic to A_7 or to a subgroup of $PTL(2, q)$ containing $PSL(2, q)$ where q is an odd prime power by [3].

Assume first that $31 \mid |0(\tilde{N})|$. Let \tilde{S} be an S_{31} -subgroup of $0(\tilde{N})$. Since 31^2 does not divide the order of $GL(5, 2)$ and since \tilde{N} is isomorphic to a subgroup of $GL(5, 2)$ we see that \tilde{S} is cyclic of order 31. By Frattini's argument we get that $N_{\tilde{N}}(\tilde{S})$ covers $\tilde{N}/0(\tilde{N})$ and hence that $N_{\tilde{N}}(\tilde{S})/N_{0(\tilde{N})}(\tilde{S})$ contains a subgroup isomorphic to S_5 by the above paragraph. Since $\text{Aut}(\tilde{S})$ is cyclic we conclude that \tilde{S} is centralized by an element \tilde{a} of order 5. But $C(\tilde{a}) \cap \bar{E}_2^*$ is nontrivial and is normalized by \tilde{S} . But this is not possible since \tilde{S} operates regularly on \bar{E}_2^* . Thus $31 \nmid |0(\tilde{N})|$ and hence $31 \mid |N/0(N)|$.

So $\tilde{N}/0(\tilde{N})$ is isomorphic to a subgroup of $PTL(2, q)$ containing $PSL(2, q)$. Since \tilde{N} is isomorphic to a subgroup of $GL(5, 2)$ and $|GL(5, 2)| = 2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$ this is possible only if $q = 31$. But $|PSL(2, 31)| = 2^5 \cdot 3 \cdot 5 \cdot 31$ whereas $|\tilde{N}|_2 = 8$. This contradiction shows that \bar{C} contains a subgroup \bar{C}_0 with index 2.

We have $1 \neq \bar{C}_0 \cap \bar{T}$. Thus $Z(\bar{T}) = \langle \bar{e} \rangle$ is contained in \bar{C}_0 . Since all involutions of \bar{T}_0 are conjugate to \bar{e} in $\bar{H}' \subseteq \bar{C}$ and since \bar{T}_0 is generated by its involutions we get $\bar{T}_0 \subseteq \bar{C}_0$. In particular \bar{T}_0 is an S_2 -subgroup of \bar{C}_0 and $\bar{H}' \subseteq \bar{C}_0$.

Let now \bar{Y} be a minimal normal subgroup of \bar{C}_0 . Since $0(\bar{Y})$ is characteristic in \bar{Y} we get either $0(\bar{Y}) = \bar{Y}$ or $0(\bar{Y}) = 1$.

Suppose that $0(\bar{Y}) = 1$. Then $\bar{T}_0 \cap \bar{Y}$ is nontrivial and hence $\bar{T}_0 \leq \bar{Y}$ as above. Thus $\bar{H}' \subseteq \bar{Y}$. So \bar{Y} is a direct product of isomorphic, non-abelian simple groups. Since \bar{P} is an S_3 -subgroup of \bar{Y} and $Z(\bar{P})$ is cyclic we see that \bar{Y} is simple. \bar{T}_0 is an S_2 -subgroup of the simple group \bar{Y} and is isomorphic to an S_2 -subgroup of M_{22} . So we get by [6; Corollary 1.3] that \bar{Y} is isomorphic to one of the following groups: M_{22} , M_{23} , McL , $PSL(4, q)$, $q \equiv 3 \pmod{8}$, $PSU(4, 1)$, $q \equiv 5 \pmod{8}$. An S_3 -subgroup of McL is of order 3^6 , M_{22} and M_{23} have abelian S_3 -subgroups, and $PSL(4, q)$ and $PSU(4, q)$ have S_3 -subgroups which are not isomorphic to P by [6; Lemma 2.1 and 2.2]. This contradiction shows that $0(\bar{Y}) = \bar{Y}$.

If $\bar{Y} \cap \bar{H}' = 1$ then $\bar{\pi}$ acts regularly on \bar{Y} and hence \bar{Y} is nilpotent by (12). We have $\bar{Y} = \langle C_{\bar{Y}}(\bar{x}) \mid 1 \neq \bar{x} \in \bar{M} \rangle$ by (1.5). Since $C_{\bar{Y}}(\bar{x})$ is isomorphic to a subgroup of \bar{H}' for any $\bar{x} \in \bar{M}$ we get that $\pi(\bar{Y}) \subseteq \{5, 7, 11\}$ and $C_{\bar{Y}}(\bar{x})$ is cyclic of prime order or 1 by (1.7). Since $\bar{\pi}$ acts regularly on $C_{\bar{Y}}(\bar{x})$ we get that $C_{\bar{Y}}(\bar{x})$ is of order 7 for $\bar{x} \in \bar{M} - \langle \bar{\pi} \rangle$. Since \bar{P} operates nontrivially on \bar{M} and normalizes $Z(\bar{Y}) \neq 1$ we get by (1.5) $Z(\bar{Y}) = \langle C_{Z(\bar{Y})}(\bar{x}) \mid 1 \neq \bar{x} \in \bar{M} \rangle = \bar{Y}$. Thus \bar{Y} is elementary abelian of order 7^3 . Since $|GL(3, 7)| = 2^6 \cdot 3^4 \cdot 7^3 \cdot 19$ we get that \bar{H}' cannot operate faithfully on \bar{Y} , i.e. \bar{H}' centralizes \bar{Y} . But this is not possible. Thus $\bar{Y} \cap \bar{H}' \neq 1$. Since \bar{Y} is of odd order and $\bar{Y} \cap \bar{H}'$ is normal in \bar{H}' we get

that $\bar{Y} \cap \bar{H}' = \langle \bar{\pi} \rangle$ and hence by (1.4) $\bar{Y} = 0_{3'}(\bar{Y}) \langle \bar{\pi} \rangle$ since $\bar{H}' = N(\langle \bar{\pi} \rangle) \cap \bar{C}_0$. Since \bar{Y} is a minimal normal subgroup of \bar{C}_0 we obtain $0_3(\bar{Y}) = 1$ and hence $\langle \bar{\pi} \rangle < \bar{C}_0$. This yields that $\bar{C}_0 = \bar{H}'$ and thus $C_G(z) = H$.

Lemma 2.5. $0_2(C_G(E_1))$ is elementary abelian of order 2^{11} .

Proof. Assume that $0_2(C_G(E_1))$ is not of order 2^{11} . Then we get by (2.2) that $0_2(C_G(E_1)) = E_1$ and hence by (2.3) and (2.4) that T is an S_2 -subgroup of G and $C_G(z) = H$.

Let $F = C_G(E_0)$ and $\bar{F} = F/E_0$. M is an S_3 -subgroup of F by (1.10.iii). We show first that $0_{3'}(F) = E_0$.

Let $K = 0_{3'}(F)$. Then K is a characteristic subgroup of F and hence normal in $N_G(E_0)$. Furthermore we have by (1.5) that $K = \langle C_K(x) \mid 1 \neq x \in M \rangle$. Since $N_G(M) \subseteq N_G(E_0)$ by (2.1) and $N_G(M)$ operates transitively on $M^\#$ we see that $N_G(M)$ operates transitively on the set $\{C_K(x) \mid 1 \neq x \in M\}$. Since $E_0 \subseteq Z(C_K(x))$ for any $x \in M$ we get by (1.10.iv) as in the proof of (2.2) that K/E_0 is an elementary abelian group of order 2^8 if $K \neq E_0$. But this is not possible since T is an S_2 -subgroup of G . So $K = E_0$ and hence $0_{3'}(F) = 1$.

We have $N_{\bar{F}}(\bar{M}) = \bar{M}\bar{Q}$ where \bar{Q} is a 2-group which acts regularly on \bar{M} by (2.1). Since $N_{\bar{F}}(\bar{M})$ is normalized by an element a of order 3 contained in $N_G(E_0)/E_0$ we can assume by Frattini's argument that a normalizes \bar{Q} . By the structure of $\text{Aut}(M) \cong GL(2, 3)$ we see that \bar{Q} is not of order 4 because otherwise a would centralize \bar{Q} . So \bar{Q} is either isomorphic to the quaternion group or cyclic of order two. In the second case we get $\bar{F} = 0_{3'}(\bar{F})N_{\bar{F}}(\bar{M})$ by [9, II]. Since $0_{3'}(\bar{F}) = 1$ this implies that $\bar{F} = N_{\bar{F}}(\bar{M})$ which is not possible since $E_1 \subseteq \bar{F}$. So we have $N_{\bar{F}}(\bar{M}) = \bar{M}\bar{Q}$ where \bar{Q} is quaternion and acts regularly on \bar{M} .

Let \bar{Y} be a minimal normal subgroup of \bar{F} . Since $0_{3'}(\bar{F}) = 1$ we have $\bar{M} \cap \bar{Y} \neq 1$. Since \bar{Q} operates transitively on $M^\#$ we obtain $\bar{M} \subseteq \bar{Y}$. As \bar{M} not normal in \bar{F} , \bar{Y} is not solvable. Furthermore \bar{Y} is the unique minimal normal subgroup of \bar{F} . Thus \bar{Y} is normal in $N_G(E_0)/E_0$. So there exists an element a of order 3 in $N_G(E_0)/E_0$ which normalizes $N_{\bar{F}}(\bar{M})$. The argument we used above to show that an S_2 -subgroup of $N_{\bar{F}}(\bar{M})$ is quaternion applies also to this situation and we get that $N_{\bar{Y}}(\bar{M}) = N_{\bar{F}}(\bar{M})$. Since \bar{Q} is quaternion we see that \bar{Y} must be simple. By Frattini's argument we get furthermore that $\bar{Y} = \bar{F}$.

We have $C_{\bar{F}}(\bar{\pi}) = \bar{E}_1\bar{M} \cong Z_3xA_4$ and all elements of $\bar{M}^\#$ are conjugate to $\bar{\pi}$ in \bar{F} . So [7] gives that \bar{F} is isomorphic to one of the following groups: $PSL(3, 7)$, $PSU(3, 5^2)$, M_{22} , M_{23} , HS , $PSL(5, 2)$, $PSp(4, 4)$, M_{24} , R , J_2 . The last three of these groups have S_3 -subgroups of order 27 but \bar{F} has an S_3 -subgroup of order 9. $PSL(5, 2)$, $PSp(4, 4)$, M_{22} , M_{23} , HS have 2-subgroups of order $\geq 2^7$. But T is an S_2 -subgroup of G and is of order 2^9 . We have $19 \mid |PSL(3, 7)|$ and $5^3 \mid |PSU(3, 5^2)|$ but $F \subseteq C_G(z) = H$ and $|H| = 2^9 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$. This is a

contradiction.

This contradiction completes the proof of the lemma.

Lemma 2.6. *G is isomorphic to J_4 .*

Proof. By (2.2) and (2.5) we have $C_G(E_1) = K\langle\pi\rangle$ where K is elementary abelian of order 2^{11} and is normal in $C_G(E_1)$. Let $N = N_G(K)$. Then we have

- (i) $C_G(K) = K$ since $C_G(K) \subseteq C_G(E_1) = K\langle\pi\rangle$.
- (ii) $N_G(E_1) \subseteq N$ where $N_{H'}(E_1)/E_1$ is isomorphic to the triple cover of A_6 and $N_H(E_1)/E_1\langle\pi\rangle \cong S_6$ by (1.10.ii).
- (iii) $C_N(M) = Mx E_0$ and $N_N(M) = N_G(M)$ as we have seen in the proof of (2.2).

We first show that $0_{3'}(N) = K$. Since $0_{3'}(N)$ is normalized by M we get by (1.5) that $0_{3'}(N) = \langle C(x) \cap 0_{3'}(N) \mid 1 \neq x \in M \rangle$. We have $C_K(\pi) = E_1$ and $C(\pi) \cap 0_{3'}(N)/C_K(\pi) \subseteq 0_{3'}(N_{H'}(E_1)/E_1) = 1$. Thus $C(\pi) \cap 0_{3'}(N) = C_K(\pi)$. By (iii) this yields that $C_K(x) = C(x) \cap 0_{3'}(N)$ for all $1 \neq x \in M$ and hence $0_{3'}(N) = K$.

Let $\bar{N} = N/K$ and let \bar{Y} be a minimal normal subgroup of \bar{N} . Since $0_{3'}(\bar{N}) = 1$ we see that $\bar{P} \cap \bar{Y} \neq 1$. Thus $Z(\bar{P})$ is contained in \bar{Y} which implies that $\bar{M} \subseteq \bar{Y}$ by (iii). By (ii) we see that \bar{Y} is not solvable. Since $C_{\bar{N}}(\bar{\pi})$ is isomorphic to the triple cover of A_6 by (ii) and $\langle\pi\rangle \trianglelefteq C_{\bar{Y}}(\bar{\pi}) \trianglelefteq C_{\bar{N}}(\bar{\pi})$ we get that $C_{\bar{Y}}(\bar{\pi}) = C_{\bar{N}}(\bar{\pi})$. In particular \bar{Y} is simple since \bar{P} is an S_3 -subgroup of \bar{Y} and $Z(\bar{P})$ is cyclic. Since $C_{\bar{N}}(\bar{\pi}) \subseteq \bar{Y}$ we get $N(\bar{M}) \cap C_{\bar{N}}(\bar{\pi}) \subseteq N_{\bar{Y}}(\bar{M})$ where $N(\bar{M}) \cap C_{\bar{N}}(\bar{\pi})/\bar{M}$ is isomorphic to S_3 by (1.10.iii). Since $N_{\bar{N}}(\bar{M})/\bar{M}$ is isomorphic to $GL(2, 3)$ by (iii) and (2.1) and since $N_{\bar{Y}}(\bar{M}) \trianglelefteq N_{\bar{N}}(\bar{M})$ we get by the structure of $GL(2, 3)$ that $N_{\bar{Y}}(\bar{M}) \trianglelefteq N_{\bar{N}}(\bar{M})$. So we have seen that \bar{Y} is a simple group containing an element $\bar{\pi}$ of order 3 such that $C_{\bar{Y}}(\bar{\pi})/\langle\bar{\pi}\rangle$ is isomorphic to $A_6 \cong PSL(2, 9)$ and an elementary abelian subgroup \bar{M} of order 9 all identity elements of which are conjugate to $\bar{\pi}$ in \bar{Y} . So [7] gives that \bar{Y} is isomorphic to M_{24} or R or J_2 . But J_2 is 3-normal by [5] and R cannot operate faithfully on an elementary abelian 2-group of order 2^{11} since $29 \mid |R|$ and $29 \nmid (2^k - 1)$ for $1 \leq k \leq 11$. So $\bar{Y} \cong M_{24}$. On the other hand \bar{P} is an S_3 -subgroup the normal subgroup \bar{Y} of \bar{N} and hence $N_{\bar{N}}(\bar{P})$ covers \bar{N}/\bar{Y} . Since $N_{\bar{N}}(\bar{P}) \subseteq N_{\bar{N}}(Z(\bar{P}))$ and $N_{\bar{N}}(\bar{P})/\bar{P}$ is a 2-group we get that \bar{N}/\bar{Y} is a 2-group. Since $\text{Aut}(M_{24}) = M_{24}$ we obtain then that $\bar{N} = \bar{Y}$, for otherwise every element in $\bar{N} - \bar{Y}$ would induce a nontrivial outer automorphism of M_{24} by the structure of $N_{\bar{N}}(\bar{P})$.

Now we can apply [8; Theorem A] and obtain that K splits into two N -classes of involutions the sizes of which are either 759 and 1288 or 1771 and 276. Since $z \in K$ is centralized by an S_3 -subgroup of N the number of conjugates of z in N is either $1288 = 2^3 \cdot 7 \cdot 23$ or $1771 = 7 \cdot 11 \cdot 23$. In the first case we have $|C_N(z)/K| = 2^4 \cdot 3^3 \cdot 5 \cdot 11$. By (1.9) we get then that $C_N(z)/K \cong \text{Aut}(M_{12})$. We have $(C_N(z)/K)' \cong M_{12}$ and $N_{H'}(E_1)K/K$ is contained in $(C_N(z)/K)'$. This implies that M_{12} contains an element of order 3 which centralizes a dihedral

group of order 8. But M_{12} has exactly two classes of involutions the centralizers of which in M_{12} are isomorphic to a faithful extension of $Q_8^*Q_8$ by S_3 or to $Z_2 \times S_5$. So there exists no dihedral subgroup of M_{12} of order 8 which is centralized by an element of order 3. This contradiction shows that K splits into two N -orbits of sizes 1771 and 276.

So z lies in the center of an S_2 -subgroup of N . We shall show that $0(C_G(z))=W$ is trivial. Since $H \subseteq C_G(z)$ and $W \cap H \subseteq 0(H)=\langle \pi \rangle$ we have either $W \cap H=1$ or $W \cap H=\langle \pi \rangle$. In the second case we get by (1.4) that $W=0_3(W)\langle \pi \rangle$ and hence $C_G(z)=WH$ by the Frattini's argument. But this is not possible since $2^{21} \mid |C_G(z)|$. So $W \cap H=1$. Then W is nilpotent by (1.2) and we have $W=\langle C_W(x) \mid 1 \neq x \in M \rangle$ by (1.5). Since G has exactly one conjugacy class of elements of order 3, $C_W(x)$ is conjugate to a subgroup of H . Since π operates regularly on $C_W(x)$ for any $x \in M^*$ we get that $C_W(x)$ is cyclic of order 7 or 1. Since P normalizes W and acts nontrivially on $M-\langle \pi \rangle$ we get that $Z(W)=W$ is elementary abelian of order 7^3 or 1. In any case H' centralizes W . This implies that $W=1$.

So we can apply [8; Theorem B] and see that either $|G|=|M(24)'|$ or $G \cong J_4$. But the first case is not possible since $3^{16} \mid |M(24)'|$. So G is isomorphic to J_4 . This completes the proof of the lemma and the proof of Theorem A.

3. Proof of Theorem B

A slight modification of the proof of Theorem A gives Theorem B. We shall only indicate where differences are to be made.

Let G be a simple group which is not 3-normal and contains an element π such that $C_G(\pi)$ is isomorphic to the triple cover of M_{22} . Then Lemma (1.10) is valid for G where H is to be replaced by $H/\langle z \rangle$. We shall use the same notation as in the second section which was introduced in (1.10) with their corresponding new meanings. Then we have

Lemma 3.1. *We have $N_G(M)/C_G(M) \cong GL(2, 3)$ and $N_G(M)$ is contained in $N_G/(E_0)$ where $E_0=0_2(C_G(M))$ is a four group.*

Proof. The same as in (2.1).

Lemma 3.2. *We have $C_G(E_1)=0_2(C_G(E_1))\langle \pi \rangle$ where either $0_2(C_G(E_1))=E_1$ or $0_2(C_G(E_1))$ is elementary abelian of order 2^{10} .*

Proof. The same as in (2.2).

Lemma 3.3. *If $0_2(C_G(E_1))=E_1$ then T is an S_2 -subgroup of G .*

Proof. The same as in (2.3).

Lemma 3.4. *T is not an S_2 -subgroup of G and hence $O_2(C_G(E_1))$ is of order 2^{10} .*

Proof. The argument we have used in (2.4) to show that $C_G(z)$ contains a subgroup C_0 with index two applies also to this case and yields that G has a subgroup with index two. But this is a contradiction since G is simple.

Conclusion 3.5. *G does not exist.*

Proof. Otherwise we get as in (2.6) that $N_G(K)/K$ is isomorphic to J_2 or M_{24} or R , where $K=O_2(C_G(E_1))$ is elementary abelian of order 2^{10} . But M_{24} and R cannot operate faithfully on a 2-group of order 2^{10} . Since J_2 is 3-normal by [5] we obtain a contradiction since we can see that $N_G(K)$ is not 3-normal as in (2.6).

This completes the proof of Theorem B.

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