<table>
<thead>
<tr>
<th>Title</th>
<th>On the BP,*-Hopf invariant</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Hirashima, Yasumasa</td>
</tr>
<tr>
<td>Citation</td>
<td>Osaka Journal of Mathematics. 12(1) P.187-P.196</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1975</td>
</tr>
<tr>
<td>Text Version</td>
<td>publisher</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.18910/5816">https://doi.org/10.18910/5816</a></td>
</tr>
<tr>
<td>DOI</td>
<td>10.18910/5816</td>
</tr>
<tr>
<td>rights</td>
<td></td>
</tr>
</tbody>
</table>
In this paper we will consider the $BP_*$-Hopf invariant, $\pi_*(S^0) \to \text{Ext}_{BP_*BP}(BP_*, BP_*)$, i.e. the Hopf invariant defined by making use of the homology theory of the Brown-Peterson spectrum $BP$. The $BP_*$-Hopf invariant is essentially "the functional coaction character". Similarly we will define the $BP_*$-e invariant ("the functional Chern-Dold character") and show that the $BP_*$-Hopf invariant coincides with the $BP_*$-e invariant by the $BP$-analogue of Buhstabner-Panov's theorem ([6], [7]). As applications we give a proof of the non-existence of elements of Hopf invariant 1, and detect $\alpha$-series.

We will use freely notations of Adams [2], [3], [4]. For example, $S, H, HZ_p$ and $HZ_{(p)}$, denote the sphere spectrum, the Eilenberg-MacLane spectrum, $Z_p$ coefficient Eilenberg-MacLane spectrum and $Z_{(p)}$ coefficient Eilenberg-MacLane spectrum respectively, where $Z_{(p)}$ is the ring of integers localized at the fixed prime $p$.

We list some well known facts:

$$\pi_*(BP) = BP_*(S^0) = BP_* = Z[v_1, v_2, \ldots], \text{deg} v_k = |v_k| = 2(p^k-1).$$

$$H_*(BP) = HZ_{(p)}(BP) = Z_{(p)}[n_1, n_2, \ldots], \text{deg} n_k = |n_k| = 2(p^k-1).$$

The Hurewicz map

$$h^H = (i_H \wedge 1_{BP})_* : \pi_*(BP) \to H_*(BP)$$

is decided by the formula [5]

$$h^H(v_k) = bn_k - \sum_{0 < c < k} h^H(v_{k-c})^c n_c.$$

$$BP_*(BP) = BP_*[t_1, t_2, \ldots], \text{deg} t_k = |t_k| = 2(p^k-1).$$

The Thom map $BP \to HZ$ induces

$$BP_*(BP) \to HZ_{(p)}(BP) = H_*(BP), \quad \mu(t_k) = n_k, \quad \mu(v_k, 1) = 0$$

$(k > 0)$ and ([10])
where $c$ is the conjugation map of the Hopf algebra $(HZ)_*(HZ_p)$ and $\xi_k (k=1, 2, \cdots)$ are Milnor's basis of a polynomial subalgebra $Z_p [\xi_1, \xi_2, \cdots] \subset (HZ)_*(HZ_p)$. $BP*(BP) \cong BP_* \otimes Z^E \{ r_E \}$, where $E$ runs through sequences of non-negative integers $E=(e_1, e_2, \cdots)$ in which all but finite number of terms are zero and $\deg r_E = |E| = |r_E| = 2(\sum_{k \geq 1} e_k (p^k - 1))$.

### 1. BP-analogue of Panov's theorem

To compute $\text{Ext}^{1, *}_{BP*}(BP_*, BP_*)$ we define some subquotient group of $H_*(BP)$ and compute this group and next relate this with $\text{Ext}^1_{BP*}(BP_*, BP_*)$.

We may regard $\pi_*(BP)$ as a submodule of $H_*(BP)$ by the Hurewicz map $h_H$. Cohomology operations $r_E$ act on $H_*(BP)$ so that we define $N = \cap_{E \in 0} r_1^{-1}(\text{Im} h^H)$ and $N/\text{Im} h^H$.

We fix a prime $p$ and discuss the Brown-Peterson spectrum associated with this prime, then for $n=2k(p-1)$ $(N/\text{Im} h^H)_n=0$ as $H_*(BP)=0$, thus it remains to decide the groups $(N/\text{Im} h^H)_{2k(p-1)}$.

**Theorem 1.1.** For odd prime $p$ $(N/\text{Im} h^H)_{2k(p-1)}=Z_{p\gamma(k)+1}$ with generator $v_1^{k+1}p^{\gamma(k)+1}$ where $v_1(k)$ denotes the exponent of highest power of $p$ dividing $k$. For $p=2$ $(N/\text{Im} h^H)_{2k}=Z_2 (k: \text{odd})$, $Z_4 (k=2)$ and $Z_{2^{k+2}} (k>2, \text{even})$ with generators $v_1^{k+2}, v_1^{-4}$ and $v_1^{k+2}+v_1^{k-3}v_1/2$ respectively.

Similar theorem for $MU$ spectrum was first computed by Panov [7], and Landweber [6] gave a shortened proof of which $BP$-analogue we follow faithfully.

Exponent sequences $E=(e_1, e_2, \cdots)$, $F=(f_1, f_2, \cdots)$ are ordered as follows: $E>F$ if

1. $|E| > |F|$, or
2. $|E| = |F|$, and $n(E) = \sum_{k \geq 1} e_k < n(F)$, or
3. $E=F$, $n(E)=n(F)$ and there exist a $k$ such that $e_k > f_k$, $e_i = f_i$ ($i > k$).

We have that if $E>E'$ and $F>F'$ then $E+F>E'+F'$, where the sum is componentwise. We say that an element $a$ of $N$ has type $E$ if $r_E(a) \not\in (p) = p \cdot \text{Im} h^H$ and $r_F(a) \not\in (p)$ for any $F>E$, especially $a$ has type 0 if $a$ has type $(0, 0, \cdots)$. If $a$ has type $E$, such a $E$ is denoted by $t(a)$.

**Lemma 1.2.**

1. $v_{k+1}$ has type $p\triangle_k$ ($k \geq 1$) and $v_1$ has type 0 (i.e. $t(v_{k+1})=p\triangle_k$, $t(v_1)=0$).
Using the formula ([10])

\[ r_{E}(n_k) = \begin{cases} n_i, & E = p^i \Delta_j (i+j = k); \\ 0, & \text{otherwise,} \end{cases} \]

and \( v^E \) means \( v_1^e v_2^e \cdots \).

The lemma can be proved by a routine induction on \( k \), so we omit it.

By the above lemma we get \( t(v^E) \neq t(v^F) \) for \( E \neq F \), \( |E| = |F| \).

Theorem 1.1 is divided into three lemmas as Landweber did in \( MU \) case.

**Lemma 1.3.** \( (N/\text{Im } h^H)_{2k(p-1)} \) is cyclic (i.e., has one generator).

**Lemma 1.4.**

1. \( v_1^k | p^e v_1^{(k+1)} \in N_{2k(p-1)} \), and
2. if \( p \) is odd, or \( p=2 \) and \( k \) is odd, or \( p=2 \) and \( k=2 \), then \( v_1^k | p^e v_1^{(k+1)} \)

represents the generator of \( (N/\text{Im } h^H)_{2k(p-1)} \).

**Lemma 1.5.** If \( p=2 \) and \( k>2 \), then \( v_1^k | 2 v_1^{(k+2)} + v_1^{k-3} v_2 / 2 \) represents the generator of \( (N/\text{Im } h^H)_{2k} \).

Proof of Lemma 1.3. Let \( a \in N_{2k(p-1)} \) represent an element of order \( p \) in \( (N/\text{Im } h^H)_{2k(p-1)} \), then \( pa \in \text{Im } h^H \). Write \( pa = \lambda v_1^k + \lambda_1 v_1^{E_1} + \lambda_2 v_1^{E_2} + \cdots + \lambda_\ell v_1^{E_\ell} \),
with \( \lambda, \lambda_j \in \mathbb{Z}/(p), |E_j| = 2k(p-1) \) and \( t(v_1^{E_1}) < t(v_1^{E_2}) < \cdots < t(v_1^{E_\ell}) \). Apply \( r_{t(v_1^{E_j})} \) to the element \( pa \). We get \( \lambda_i \equiv 0 \mod (p) \) since \( \lambda_j v_1^{E_j} = v_1^{E_j} \equiv 0 \mod (p) \). Next apply \( r_{t(v_1^{E_\ell})} \). By the same argument we have \( \lambda_{i-1} \equiv 0 \mod (p) \). Continue these argument, then we get \( \lambda_1 \equiv \lambda_2 \equiv \cdots \equiv \lambda_i \equiv 0 \mod (p) \). So we conclude

\[ pa = \lambda v_1^k \mod (p) \]

and hence

\[ a = \lambda v_1^k | p \] in \( (N/\text{Im } h^H)_{2k(p-1)} \).

This implies Lemma 1.3.

Proof of Lemma 1.4. (1) We get by induction

\[ r_{E}(v_1^k) = \begin{cases} \binom{k}{e} p^e v_1^{k-e}, & E = e \Delta_i; \\ 0, & \text{otherwise,} \end{cases} \]
Using the formula ([6], [7])
\[ \nu_{p}\begin{pmatrix}k \\ e \end{pmatrix} = \nu_{p}(k) - \nu_{p}(e) \] for \( e \leq p^{\nu_{p}(k)} \),
we have
\[ \nu_{p}\begin{pmatrix}k \\ e \end{pmatrix} p^{e} \geq \nu_{p}(k) + 1 . \]
The equality holds for \( e = 1 \). Hence
\[ v_{1}^{k}/p^{\nu_{p}(k)+1} \subseteq N_{2k(p-1)} \quad \text{and} \quad v_{1}^{k}/p^{\nu_{p}(k)+2} \subseteq N_{2k(p-1)}. \]

(2) For odd prime \( p \), or \( p = 2 \) and \( k \): odd, or \( p = 2 \) and \( k = 2 \), \( v_{1}^{k}/p^{\nu_{p}(k)+1} \) has type \( \Delta_{l}(<t(v^{k}), |E| = 2k(p-1), E \equiv k\Delta_{l}) \) by the above argument.

If \[ a = \lambda \cdot v_{1}^{k}/p^{\nu_{p}(k)+2} + \sum_{E \equiv k\Delta_{l}} \lambda_{E} \cdot v^{E} \in N_{2k(p-1)}, \lambda, p\lambda_{E} \in Z_{(p)}, \]
then \( pa \) has type 0 so that \( p \mid \lambda, p \mid \lambda_{E} \) by the same type-argument as the proof of Lemma 1.3. This shows that there is no element \( a \) such that \( v_{1}^{k}/p^{\nu_{p}(k)+1} \subseteq pa \) in \( \langle N/Imh^{H} \rangle_{2k(p-1)} \). This implies Lemma 1.4.

Proof of Lemma 1.5. In case \( p = 2 \) and \( k > 2 \),
\[ \nu_{2}\begin{pmatrix}k \\ e \end{pmatrix} = \nu_{2}(k)+1 \quad (e = 1, 2) \]
and
\[ \nu_{2}\begin{pmatrix}k \\ e \end{pmatrix} p^{e} > \nu_{2}(k)+1 \quad (e > 2) . \]
These imply that \( v_{1}^{k}/2^{\nu_{2}(k)+1} \) has type \( 2\Delta_{l} \). After routine computations we obtain
\[ r_{\Delta_{l}}(v_{1}^{k}/2^{\nu_{2}(k)+1}) = v_{1}^{k-1} \equiv r_{\Delta_{l}}(v_{1}^{k-3}v_{2}) \mod (2) \]
\[ r_{2\Delta_{l}}(v_{1}^{k}/2^{\nu_{2}(k)+1}) = v_{1}^{k-2} \equiv r_{2\Delta_{l}}(v_{1}^{k-3}v_{2}) \mod (2) . \]
So we conclude that \( v_{1}^{k}/2^{\nu_{2}(k)+1} + v_{1}^{k-3}v_{2} \) has type 0, and thus \( v_{1}^{k}/2^{\nu_{2}(k)+2} + v_{1}^{k-3}v_{2}/2 \subseteq N_{2k} \). Put \( P = v_{1}^{k}/2^{\nu_{2}(k)+2} + v_{1}^{k-3}v_{2}/2 \). We decide the type of \( P \) in several steps.
\[ t(P) = \Delta_{2} \quad \text{or} \quad t(P) > \Delta_{2} \quad \text{by} \quad r_{2\Delta_{l}}(P) = v_{1}^{k-3} . \]
The Cartan formula implies \( r_{E}(P) \equiv 0 \mod (2) \) for \( E > \Delta_{2} \) and \( E \equiv i\Delta_{2} \), so that \( t(P) = \Delta_{2} \) or \( i\Delta_{1} (i \geq 4) \). For \( i \geq 4 \)
\[ r_{i\Delta_{1}}(P) \equiv \left( \begin{pmatrix} k \\ i \end{pmatrix} \right) 2^{i}/2^{\nu_{2}(k)+2} v_{1}^{i-k} \equiv 0 \mod (2) , \]
thus
\[ \left( \begin{pmatrix} k \\ i \end{pmatrix} \right) 2^{i}/2^{\nu_{2}(k)+2} \equiv 0 \mod (2) \quad \text{if} \quad \nu_{2}(k) = 1 , \]
and if \( \nu_{2}(k) \geq 2 \).
This implies that \( t(P) = 4\Delta \) if \( \nu_2(k) = 1 \) and \( t(P) = 4\Delta \) if \( \nu_2(k) \geq 2 \).

There is no element \( a \in \text{Im} h^H \) such that \( r_E(P) \equiv r_E(a) \mod (2) \) for any \( E \). If not, \( a \) is represented by a linear combination of \( v_1^k, v_1^{t-3}v_2 \) and \( v_1^{t-2}v_2^2 \), then \( r_E(a) \equiv 0 \mod (2) \) which contradicts the assumption. This implies Lemma 1.5 and also completes the proof of Theorem 1.1.

Next we lift the group \( \langle N \mid \text{Im} h^H \rangle \) to a subquotient group of \( BP_*(BP) \) by Thom map \( BP_*(BP) \rightarrow H_*(BP) \). We denote by \( (r_E)_{*} \) the right action of \( r_E \) on \( BP_*(BP) \) which is compatible under Thom map \( \mu \) with the action on \( H_*(BP) \).

We consider the groups

\[
N^{BP} = \bigcap_{n \in \mathbb{Z}} (r_E)_{*}^{-1}(\text{Im} h^{BP}) \quad \text{and} \quad N^{BP}/\text{Im} h^{BP} + BP_{*}, 1,
\]

on which Thom map induces the group homomorphism

\[
N^{BP}/\text{Im} h^{BP} + BP_{*}, 1 \xrightarrow{\hat{\mu}} N/\text{Im} h^H.
\]

**Theorem 1.6.** \( \hat{\mu} \) is isomorphic.

**Proof.** In \( BP_*(BP) \otimes \mathbb{Q} = BP_*(\mathbb{Q}[n_1, n_2, \ldots]) \)

\[
\bigcap_{n \in \mathbb{Z}} (r_E)_{*}^{-1}(0) = BP_*(\mathbb{Q})
\]

so that in \( BP_*(BP) \)

\[
\bigcap_{n \in \mathbb{Z}} (r_E)_{*}^{-1}(0) = BP_{*}, 1 \quad (= BP_{*} \otimes 1).
\]

We get easily

\[
\text{Ker} \mu \cap N^{BP} \subset \bigcap_{n \in \mathbb{Z}} (r_E)_{*}^{-1}(0) = BP_{*}, 1
\]

and

\[
\text{Ker} \hat{\mu} = \{(\text{Im} h^{BP} + \text{Ker} \mu) \cap N^{BP} + \text{Im} h^{BP} + BP_{*}, 1\}/\text{Im} h^{BP} + BP_{*}, 1
\]

\[
= 0
\]

so that \( \hat{\mu} \) is monomorphic.

For any prime \( p \), \( h^{BP}(v_1) = v_1 = v_1, 1 + pt_1 \), and thus

\[
v_1^k - v_1, 1 = \sum_{0 \leq t \leq k} \binom{k}{t} p^t v_1^{k-t} t_1^e.
\]

We get

\[
(v_1^k - v_1, 1)/(p^{y(k)+1}) \in BP_{2k(p-1)}(BP)
\]
and

\[(v_i^k - v_i^k \cdot 1)/\rho^{(k)+1} \in N^BP_{2k+1} \, .\]

In case of \(p=2\) and \(k \geq 2\),

\[h_{BP}(v_2) = v_2 = v_2 \cdot 1 - 3v_i^3t_i - 5v_i^4t_i^2 + 2t_i - 4t_i^3, \quad (3).\]

We get

\[\begin{align*}
(v_i^k - v_i^k \cdot 1)/2^k & = (1/2)v_i^{k-1}t_i + (1/2)v_i^{k-2}t_i^2 + A, \\
(v_i^{k-2}v_2 - (v_i^{k-3}v_2)\cdot 1)/2 & = (-1/2)v_i^{k-1}t_i + (-1/2)v_i^{k-2}t_i^2 + B,
\end{align*}\]

where \(A, B \in BP_*(BP)\), and thus

\[\begin{align*}
(v_i^k - v_i^k \cdot 1)/2^k & = (v_i^{k-2}v_2 - (v_i^{k-3}v_2)\cdot 1)/2 \in BP_{2k}(BP).
\end{align*}\]

We have easily

\[\frac{(v_i^k - v_i^k \cdot 1)/2^k}{(v_i^{k-2}v_2 - (v_i^{k-3}v_2)\cdot 1)/2} \in N^{BP}_{2k} \, .\]

These conclude that \(\hat{\rho}\) is epimorphic and complete the proof of Theorem 1.6.

The conjugation map \(c\) of the Hopf algebra \(BP_*(BP)\) induces the isomorphism

\[\hat{c}: N^BP/\text{Im } h^BP + BP_* 1 \rightarrow \bigcap_{x \neq 0} r_E'(BP_* 1)/\text{Im } h^BP + BP_* 1,\]

but \(\hat{c}\) preserves the generators given in Theorem 1.6 up to sign, so that we obtain

\[\textbf{Corollary 1.7.}\]

\[N^BP/\text{Im } h^BP + BP_* 1 = \bigcap_{x \neq 0} r_E'(BP_* 1)/\text{Im } h^BP + BP_* 1.\]

We next show that

\[\text{Ext}_*(BP_* BP, BP_*) = \bigcap_{x \neq 0} r_E'(BP_* 1)/\text{Im } h^BP + BP_* 1 \approx N/\text{Im } h^H.\]

Let \(S \rightarrow BP \rightarrow I\) be the cofibration obtained from the unit \(S \rightarrow BP, I^{(k)} = I \wedge I \wedge \cdots \wedge I\) (k-factors) and \(d_k\) be the composition \(BP \wedge I^{(k)} \rightarrow BP \wedge I^{(k+1)} \rightarrow BP \wedge I^{(k+1)}\) (or equivalently \(BP \wedge I^{(k)} \rightarrow BP \wedge BP \wedge I^{(k)} \rightarrow BP \wedge I^{(k+1)}\)). Then we obtain the geometric resolution of Adams \([4]\]

\[BP \rightarrow BP \wedge I \rightarrow BP \wedge I^{(2)} \rightarrow BP \wedge I^{(3)} \rightarrow \cdots ,\]

which defines a chain complex of a spectrum \(X\)

\[BP_*(X) \rightarrow (BP \wedge I)_*(X) \rightarrow (BP \wedge I^{(2)})_*(X) \rightarrow \cdots .\]
and

\[ \text{Ext}^*_{BP^*}(BP^*, BP^*(X)) = \text{Ker} (d_k)*/\text{Im} (d_{k-1})*. \]

For \( X = S^p \), \( (d_j)_* = p^*h^BP \) and \( (d_j)_* = (p_\otimes 1)_\Psi_I \) where \( BP^*(BP) \overset{p^*}{\to} BP^*(I) = BP^*(BP)/BP^*_1 \) is the canonical projection, \( BP^*(I) \overset{\Psi_I}{\longrightarrow} BP^*(BP) \otimes_{BP^*} BP^*(I) \) is the coaction map of \( I \) for which

\[
\Psi_I(x) = \sum_{E} t^E \otimes r_E(x).
\]

**Remark.** This coaction map is twisted by the conjugation map \( c \) of \( BP^*(BP) \) from the one defined by Adams [2].

\[
\text{Ker} (d_1)_* = \{ x \in BP^*(BP)/BP^*_1 | r_E(x) = 0, \ E \neq 0 \}
\]

\[ = \bigcap_{E \neq 0} r_E^I (BP^*_1)/BP^*_1, \]

\[ \text{Im} (d_0)_* = \text{Im} h^BP/\text{Im} h^BP \cap BP^*_1 = \text{Im} h^BP + BP^*_1/\text{BP}^*_1. \]

Hence we obtain

**Theorem 1.8.**

\[ \text{Ext}^*_{BP^*}(BP^*, BP^*) = \bigcap r_E^I(BP^*_1)/\text{Im} h^BP + BP^*_1. \]

**Corollary 1.9.**

\[ \text{Ext}^*_{BP^*}(BP^*, BP^*) \overset{\hat{\mu}}{\cong} \frac{N}{\text{Im} h^H}. \]

2. The \( BP^* \)-Hopf invariant

Since \( BP_*(BP) \) is flat over \( BP_*, BP_*(BP) \) comodules and \( BP_*(BP) \) comodule homomorphisms form a relative abelian category so that similar construction of Adams [1] is valid for \( BP_* \) homology theory. We review the construction of the \( BP_* \)-Hopf invariant quickly; for a morphism \( f: X \to Y \) of CW-spectra in homotopy category such that \( f_* = 0 \), we have a short exact sequence

\[ E(f): 0 \to BP_*(Y) \to BP_*(C_f) \to BP_*(SX) \to 0, \]

which is regarded as an element of \( \text{Ext}^*_{BP^*}(BP_*(X), BP_*(Y)) \). This is the \( BP_* \)-Hopf invariant of \( f \).

For \( X = S^q \) \((q = 2(p - 1))\), \( Y = S^p \) the \( BP_* \)-Hopf invariant is defined on the whole group \( \pi_{(p-1)}(S^p) \). For a short exact sequence \( E(f) \) we apply the Adams
resolution $BP \rightarrow BP \wedge I \rightarrow BP \wedge I^{(2)} \rightarrow \cdots$ then we obtain a short exact sequence of chain complexes

$$
\begin{array}{cccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & BP_*^*(S^0) & (BP \wedge I)_*^*(S^0) & (BP \wedge I^{(2)})_*^*(S^0) \rightarrow \cdots \\
\downarrow & i_* & \downarrow & \downarrow \\
0 & BP_*^*(C_{f}) & (BP \wedge I)_*^*(C_{f}) & (BP \wedge I^{(2)})_*^*(C_{f}) \rightarrow \cdots \\
\downarrow & j_* & \downarrow & \downarrow \\
0 & BP_*^*(S^{kq}) & (BP \wedge I)_*^*(S^{kq}) & (BP \wedge I^{(2)})_*^*(S^{kq}) \rightarrow \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0
\end{array}
$$

Let $\sigma_{kq} \in BP_*(S^{kq})$ be a generator and $\delta(\sigma_{kq}) = [i_*^{-1}(d_0)_*^{-1}j_*^{-1}(\sigma_{kq})] \in \text{Ext}_{BP_*(BP_*,BP_*)}^1$ then the element $\delta(\sigma_{kq}) = E(f)_*^*(\sigma_{kq})$ is just the element $E(f)$ by well known technique of homological algebra. This construction is considered as follows; let $\sigma_0 = i_*(\sigma_0)$, $\mu_{kq}$ are generators of $BP_*(C_{f})$ of dimension 0, dimension $kq$ respectively so that $j_*^*(\mu_{kq}) = \sigma_{kq}$ and let $\eta_R: BP = S \wedge BP \rightarrow BP \wedge BP$ be the Boardman map and put $\eta_R^*(\mu_{kq}) = A_f \sigma_0 + \mu_{kq} (A_f \in BP_*(BP))$. Then $A_f$ represents $E(f)$. Replacing $BP \rightarrow BP \wedge BP$ by $BP \rightarrow H \wedge BP$, we have the $BP_*-\varepsilon$ invariant (or the functional Chern-Dold character) and is equivalent to the $BP_*$-Hopf invariant by Corollary 1.9.

### 3. Applications

For an element $f \in \pi_{kq-1}(S^0) (q=2(p-1))$ we get a short exact sequence $0 \rightarrow (HZ_{p})_*^*(S^0) \rightarrow (HZ_{p})_*^*(C_{f}) \rightarrow (HZ_{p})_*^*(S^{kq}) \rightarrow 0$ and can choose generators $\sigma_0'$ and $\mu_{kq}$ of $(HZ_{p})_*^*(C_{f})$ such that $j_*^*(\mu_{kq}) = \sigma_{kq}$ and $j_*^*(\mu_{kq}) = \sigma_{kq}$ where $\sigma_{kq}$ is a canonical generator of $(HZ_{p})_*^*(S^n)$. Let $\Psi: (HZ_{p})_*^*(C_{f}) \rightarrow A_* \otimes (HZ_{p})_*^*(C_{f})$ be the coaction, then the definition of the Hopf invariant in the sense of Steenrod is described as follows; $f(\in \pi_{kq-1}(S^0))$ is said to have mod $p$ Hopf invariant 1 if $\langle P^k, H_f \rangle = 0$, where $P^k$ is the Steenrod reduced power (interpreted as $Sq^{2k}$ if $p=2$) and $\Psi(\mu_{kq}) = H_f \sigma_0' + \mu'_{kq} (H_f \in A_*)$.

**Theorem 3.1** (Adams, Liulevicius, Shimada-Yamanoshita.) If $f$ has mod $p$ Hopf invariant 1 then

1. $k=1, 2$ or 4 for $p=2$;
2. $k=1$ for odd prime $p$.

**Proof.** Consider the following diagram
Let $V(0) = S^0 \cup e^1$ then there exists a map $\phi_k : S^0 \to S^0 V(0) \to V(0)$ such that $\phi_k(\sigma_{0 \theta}) = v_1^k \gamma_0$ where $\sigma_{0 \theta} \in BP_{0 \theta}(S^{0 \theta})$ and $\gamma_0 \in BP_0(V(0))$ are generators (8). $\alpha$-series elements $\alpha_k (k=1,2,\cdots)$ of $\pi_{k \theta-1}(S^n)$ are defined by $\alpha_k = j \phi_k$ where $j: V(0) \to S^1$ is the canonical projection. We detect these elements by means of the $BP_\ast$-Hopf invariant. We have the following diagram of cofibrations;

\[
\begin{array}{ccc}
S^0 & \rightarrow & S^0 \\
\phi_k & \downarrow & \downarrow \\
S^{0 \theta} & \rightarrow & V(0) \\
\alpha_k & \downarrow & \downarrow \\
S^{1 \theta} & \rightarrow & S^1 \\
\end{array}
\]

Considering the above diagram following results are obtained:

- $BP_\ast(S^n)$ \{ generator; $\sigma_n$
- relations; none,
- $BP_\ast(V(0)) = BP_\ast(p)$ \{ generator; $\gamma_0$
- relations; $p \gamma_0 = 0$
- $BP_\ast(C_{\alpha_k})$ \{ generators; $a_{\ast}(\gamma_0), \lambda_{k \theta+1}$
- relations; $(p, v_i^k) \cdot a_{\ast}(\gamma_0) = 0$
- formula; $b_{\ast}(\lambda_{k \theta+1}) = p \sigma_{k \theta+1}$

Since $A_f$ is a multiple of $v_1^k/p \gamma_{k \theta}^{1+1} = p^{k-\gamma_{k \theta}^{1+1}}$ or $v_1^k/2^{2^{k+2}} + 2^{k-2^{k+2}} n_1^k \mod 2 \cdot H_{k \theta}(BP)$ by Theorem 1.1, $H_f$ is a multiple of $p^{k-\gamma_{k \theta}^{1+1}} \xi^k_1$ or $2^{k-2^{k+2}} \xi^{2^k}_1$. In case of an odd prime $p$ $H_f = 0$ for $k > 1$, in case of $p = 2$ and odd number $k$ $H_f = 0$ for $k > 1$, in case of $p = 2$ and even $k$ $H_f = 0$ for $k > 4$. This completes the proof of Theorem 3.1.
\( BP_\ast(C_{a_k}) \)

- generators: \( c_\ast(\sigma_1), \mu_{q+1} \)
- relations: none
- formula: \( d_\ast(\mu_{q+1}) = \sigma_{q+1} \)

and the formula

\[ h_\ast(\lambda_{q+1}) = p\mu_{q+1} - v_1^h c_\ast(\sigma_1) \cdot \]

The coefficient \( v_1^h \) of \( c_\ast(\sigma_1) \) is decided up to a multiple of a unit of \( Z(p) \). The image of these generators of Thom homomorphism are denoted by \( \sigma_n', \gamma_n', \lambda_n' \) and \( \mu_n' \) respectively.

**Theorem 3.2.**

\[ e(\alpha_k) = v_1^h/p \text{ in } N/\text{Im } h^H \cong \text{Ext}^1_{BP_\ast(C_{a_k})}(BP_\ast, BP_\ast). \]

**Proof.** By applying the Chern-Dold character \( BP_\ast(C_{a_k}) \rightarrow (H \wedge BP)_\ast(C_{a_k}) \) to \( \mu_{q+1} \), we get \( B_\ast(\mu_{q+1}) = A_{a_k}c_\ast(\sigma_1) + \mu_{q+1} \). \( A_{a_k} \) represents \( BP_\ast-e \) invariant of \( \alpha_k \) in \( N/\text{Im } h^H \). The computation

\[
p\mu_{q+1} = h_\ast B_\ast(\lambda_{q+1}) = B_\ast(p\mu_{q+1} - v_1^h c_\ast(\sigma_1))
\]

\[ = p\mu_{q+1} + (pA_{a_k} - v_1^h)c_\ast(\sigma_1) \]

implies \( pA_{a_k} = v_1^h \) and this completes the proof.

**Osaka City University**

---

**References**


