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ON THE BP_*-HOPF INVARIANT

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In this paper we will consider the BP_*-Hopf invariant, $\pi_*(S^0)\to \mathrm{Ext}^{1,*}_{BP_*(BP)}(BP_*, BP_*)$, i.e. the Hopf invariant defined by making use of the homology theory of the Brown-Peterson spectrum BP. The BP_*-Hopf invariant is essentially "the functional coaction character". Similarly we will define the BP_*-e invariant ("the functional Chern-Dold character") and show that the BP_*-Hopf invariant coincides with the BP_*-e invariant by the BP-analogue of Buhstaber-Panov's theorem ([6], [7]). As applications we give a proof of the non-existence of elements of Hopf invariant 1, and detect α-series.

We will use freely notations of Adams [2], [3], [4]. For example, S, H, HZ_p and $HZ_{(p)}$ denote the sphere spectrum, the Eilenberg-MacLane spectrum, Z_p coefficient Eilenberg-MacLane spectrum and $Z_{(p)}$ coeffitient Eilenberg-MacLane spectrum respectively, where $Z_{(p)}$ is the ring of integers localized at the fixed prime p.

We list some well known facts:

$$\pi_*(BP)=BP_*(S^0)=BP_*=Z_{(p)}[v_1, v_2, \cdots]\,,\quad \deg v_k=|v_k|=2(p^k-1)\,.$$
$$H_*(BP)=HZ_{(p)*}(BP)=Z_{(p)}[n_1, n_2, \cdots]\,,\quad \deg n_k=|n_k|=2(p^k-1\,).$$

The Hurewicz map

$$h^H=(i^H\wedge 1_{BP})_*:\pi_*(BP)\to H_*(BP)$$

is decided by the formula [5]

$$h^H(v_k)=pn_k-\sum_{0<s<k} h^H(v_{k-s})^{p^s}n_s\,.$$
$$BP_*(BP)=BP_*[t_1, t_2, \cdots]\,,\quad \deg t_k=|t_k|=2(p^k-1)\,.$$

The Thom map $BP\xrightarrow{\mu} HZ$ induces

$$BP_*(BP)\xrightarrow{\mu} HZ_{(p)*}(BP)=H_*(BP)\,,\quad \mu(t_k)=n_k\,;\quad \mu(v_k\cdot 1)=0$$

($k>0$) and ([10])

$$HZ_{(p)*}(BP) \overset{\mu_*}{\to} (HZ_p)_*(HZ_p)\,, \quad \mu_*(n_k) = c(\xi_k)\,,$$

where c is the conjugation map of the Hopf algebra $(HZ_p)_*(HZ_p)$ and $\xi_k\,(k=1, 2, \cdots)$ are Milnor's basis of a polynomial subalgebra $Z_p[\xi_1, \xi_2, \cdots]\subset (HZ_p)_*(HZ_p)$. $BP^*(BP)=BP_*\hat{\otimes}Z_{(p)}\{r_E\}$, where E runs through sequences of non-negative integers $E=(e_1, e_2, \cdots)$ in which all but finite number of terms are zero and $\deg r_E=|E|=|r_E|=2(\sum_{k\geq 1} e_k(p^k-1))$.

1. *BP*-analogue of Panov's theorem

To compute $\mathrm{Ext}^{1,*}_{BP_*(BP)}(BP_*, BP_*)$ we define some subquotient group of $H_*(BP)$ and compute this group and next relate this with $\mathrm{Ext}^{1,*}_{BP_*(BP)}(BP_*, BP_*)$.

We may regard $\pi_*(BP)$ as a submodule of $H_*(BP)$ by the Hurewicz map h^H. Cohomology operations r_E act on $H_*(BP)$ so that we define

$$N = \bigcap_{E\neq 0} r_E^{-1}(\mathrm{Im}\, h^H) \quad \text{and} \quad N/\mathrm{Im}\, h^H\,.$$

We fix a prime p and discuss the Brown-Peterson spectrum associated with this prime, then for $n\neq 2k(p-1)$ $(N/\mathrm{Im}\, h^H)_n=0$ as $H_n(BP)=0$, thus it remains to decide the groups $(N/\mathrm{Im}\, h^H)_{2k(p-1)}$.

Theorem 1.1. *For odd prime p $(N/\mathrm{Im}\, h^H)_{2k(p-1)}=Z_{p^{\nu_p(k)+1}}$ with generator $v_1{}^k/p^{\nu_p(k)+1}$ where $\nu_p(k)$ denotes the exponent of highest power of p dividing k. For $p=2$ $(N/\mathrm{Im}\, h^H)_{2k}=Z_2$ (k: odd), Z_4 $(k=2)$ and $Z_{2^{\nu_2(k)+2}}$ $(k>2$, even$)$ with generators $v_1{}^k/2$, $v_1{}^2/4$ and $v_1{}^k/2^{\nu_2(k)+2}+v_1{}^{k-3}v_2/2$ respectively.*

Similar theorem for MU spectrum was first computed by Panov [7], and Landweber [6] gave a shortened proof of which BP-analogue we follow faithfully.

Exponent sequences $E=(e_1, e_2, \cdots)$, $F=(f_1, f_2, \cdots)$ are ordered as follows: $E>F$ if

(1) $|E|>|F|$, or

(2) $|E|=|F|$, and $n(E) = \sum_{k\geq 1} e_k<n(F)$, or

(3) $E=F$, $n(E)=n(F)$ and there exist a k such that $e_k>f_k$, $e_i=f_i$ $(i>k)$.

We have that if $E>E'$ and $F>F'$ then $E+F>E'+F'$, where the sum is componentwise. We say that an element a of N has type E if $r_E(a)\notin(p)=p\cdot\mathrm{Im}\, h^H$ and $r_F(a)\in(p)$ for any $F>E$, especially a has type 0 if a has type $(0, 0, \cdots)$. If a has type E, such a E is denoted by $t(a)$.

Lemma 1.2.

(1) *v_{k+1} has type $p\Delta_k$ $(k\geq 1)$ and v_1 has type* 0 (i.e. $t(v_{k+1})=p\Delta_k$, $(t(v_1)=0)$.

(2) $r_{\Delta_{k+1}}(v_{k+1})=p$.

(3) $t(v^E)=(pe_1, pe_2, \cdots)$ *where* $E=(e_1, e_2, \cdots)$

and v^E *means* $v_1^{e_1}v_2^{e_2}\cdots$.

Using the formula ([10])

$$r_E(n_k) = \begin{cases} n_i\,, & E = p^i\Delta_j\ (i+j=k); \\ 0\,, & \text{otherwise,} \end{cases}$$

and

$$v_k = pn_k \sum_{0<s<k} v_{k-s}^{p^s} n_s\,,$$

the lemma can be proved by a routine induction on k, so we omit it.

By the above lemma we get $t(v^E)\neq t(v^F)$ for $E\neq F$, $|E|=|F|$.

Theorem 1.1 is divided into three lemmas as Landweber did in MU case.

Lemma 1.3. $(N/\mathrm{Im}\,h^H)_{2k(p-1)}$ *is cyclic (i.e., has one generator).*

Lemma 1.4.

(1) $v_1^k/p^{\nu_p(k)+1}\in N_{2k(p-1)}$, *and*

(2) *if* p *is odd, or* $p=2$ *and* k *is odd, or* $p=2$ *and* $k=2$, *then* $v_1^k/p^{\nu_p(k)+1}$ *represents the generator of* $(N/\mathrm{Im}\,h^H)_{2k(p-1)}$.

Lemma 1.5. *If* $p=2$ *and* $k>2$, *then* $v_1^k/2^{\nu_2(k)+2}+v_1^{k-3}v_2/2$ *represents the generator of* $(N/\mathrm{Im}\,h^H)_{2k}$.

Proof of Lemma 1.3. Let $a\in N_{2k(p-1)}$ represent an element of order p in $(N/\mathrm{Im}\,h^H)_{2k(p-1)}$, then $pa\in\mathrm{Im}\,h^H$. Write $pa=\lambda v_1^k+\lambda_1 v^{E_1}+\lambda_2 v^{E_2}+\cdots+\lambda_i v^{E_i}$ with $\lambda, \lambda_j\in Z_{(p)}$, $|E_j|=2k(p-1)$ and $t(v^{E_1})<t(v^{E_2})<\cdots<t(v^{E_i})$. Apply $r_{t(v^{E_i})}$ to the element pa. We get $\lambda_i\equiv 0 \bmod (p)$ since $\lambda_i r_{t(v^{E_i})}(v^{E_i})\equiv 0 \bmod (p)$. Next apply $r_{t(v^{E_{i-1}})}$. By the same argument we have $\lambda_{i-1}\equiv 0 \bmod (p)$. Continue these argument, then we get $\lambda_1\equiv\lambda_2\equiv\cdots\equiv\lambda_i\equiv 0 \bmod (p)$. So we conclude

$$pa \equiv \lambda v_1^k \bmod (p)$$

and hence

$$a = \lambda\cdot(v_1^k/p) \quad\text{in}\quad (N/\mathrm{Im}\,h^H)_{2k(p-1)}\,.$$

This implies Lemma 1.3.

Proof of Lemma 1.4. (1) We get by induction

$$r_E(v_1^k) = \begin{cases} \binom{k}{e} p^e v_1^{k-e}, & E = e\Delta_1; \\ 0, & \text{otherwise,} \end{cases}$$

Using the formula ([6], [7])

$$\nu_p\left(\binom{k}{e}\right) = \nu_p(k)-\nu_p(e) \quad \text{for} \quad e \leq p^{\nu_p(k)},$$

we have

$$\nu_p\left(\binom{k}{e}p^e\right) \geq \nu_p(k)+1 .$$

The equality holds for $e=1$. Hence

$$v_1{}^k/p^{\nu_p(k)+1} \in N_{2k(p-1)} \quad \text{and} \quad v_1{}^k/p^{\nu_p(k)+2} \notin N_{2k(p-1)}.$$

(2) For odd prime p, or $p=2$ and k: odd, or $p=2$ and $k=2$, $v_1{}^k/p^{\nu_p(k)+1}$ has type $\Delta_1(<t(v^E)$, $|E|=2k(p-1)$, $E\neq k\Delta_1)$ by the above argument.

If $$a = \lambda\cdot v_1{}^k/p^{\nu_p(k)+2} + \sum_{\substack{E=2k(p-1)\\ E\neq k\Delta_1}} \lambda_E \cdot v^E \in N_{2k(p-1)},\ \lambda,\ p\lambda_E \in Z_{(p)} ,$$

then pa has type 0 so that $p\,|\,\lambda$, $p\,|\,\lambda_E$ by the same type-argument as the proof of Lemma 1.3. This shows that there is no element a such that $v_1{}^k/p^{\nu_p(k)+1}=pa$ in $(N/\operatorname{Im} h^H)_{2k(p-1)}$. This implies Lemma 1.4.

Proof of Lemma 1.5. In case $p=2$ and $k>2$,

$$\nu_2\left(\binom{k}{e}2^e\right) = \nu_2(k)+1 \quad (e = 1, 2)$$

and

$$\nu_2\left(\binom{k}{e}2^e\right) > \nu_2(k)+1 \quad (e>2) .$$

These imply that $v_1{}^k/2^{\nu_2(k)+1}$ has type $2\Delta_1$. After routine computations we obtain

$$r_{\Delta_1}(v_1{}^k/2^{\nu_2(k)+1}) = v_1^{k-1} \equiv r_{\Delta_1}(v_1^{k-3}v_2) \bmod (2)$$
$$r_{2\Delta_1}(v_1{}^k/2^{\nu_2(k)+1}) = v_1^{k-2} \equiv r_{2\Delta_1}(v^{k-3}v_2) \bmod (2) .$$

So we conclude that $v_1{}^k/2^{\nu_2(k)+1}+v_1^{k-3}v_2$ has type 0, and thus $v_1{}^k/2^{\nu_2(k)+2}+v_1^{k-3}v_2/2 \in N_{2k}$. Put $P=v_1{}^k/2^{\nu_2(k)+2}+v_1^{k-3}v_2/2$. We decide the type of P in several steps. $t(P)=\Delta_2$ or $t(P)>\Delta_2$ by $r_{\Delta_2}(P)=v_1^{k-3}$. The Cartan formula implies $r_E(P)\equiv 0 \bmod (2)$ for $E>\Delta_2$ and$E\neq i\Delta_1$, so that $t(P)=\Delta_2$ or $i\Delta_1$ $(i\geqq 4)$. For $i\geqq 4$

$$r_{i\Delta_1}(P) \equiv \left(\binom{k}{i}2^i/2^{\nu_2(k)+2}\right)v_1^{k-i} \bmod (2) ,$$

thus

$$\binom{k}{i}2^i/2^{\nu_2(k)+2} \equiv 0 \bmod (2) \quad \text{if} \quad \nu_2(k) = 1 ,$$

and if $\nu_2(k)\geq 2$

$$\binom{k}{i}2^i/2^{\nu_2(k)+2} \equiv \begin{cases} 2^{i-2-\nu_2(i)}, & i \leq \nu_2(k)+2; \\ 0 \bmod (2), & i > \nu_2(k)+2. \end{cases}$$

This implies that $t(P)=\Delta_2$ if $\nu_2(k)=1$ and $t(P)=4\Delta_1$ if $\nu_2(k)\geq 2$.

There is no element $a \in \operatorname{Im} h^H$ such that $r_E(P) \equiv r_E(a) \bmod (2)$ for any E. If not, a is represented by a linear combination of $v_1{}^k$, $v_1^{k-3}v_2$ and $v_1^{k-6}v_2{}^2$, then $r_{\Delta_2}(a) \equiv 0 \bmod (2)$ which contradicts the assumption. This implies Lemma 1.5 and also completes the proof of Theorem 1.1.

Next we lift the group $(N/\operatorname{Im} h^H)$ to a subquotient group of $BP_*(BP)$ by Thom map $BP_*(BP) \xrightarrow{\mu} H_*(BP)$. We denote by $(r_E)_*$ the right action of r_E on $BP_*(BP)$ which is compatible under Thom map μ with the action on $H_*(BP)$. We consider the groups

$$N^{BP} = \bigcap_{E \neq 0} (r_E)_*^{-1}(\operatorname{Im} h^{BP}) \text{ and } N^{BP}/\operatorname{Im} h^{BP} + BP_* \cdot 1,$$

on which Thom map induces the group homomorphism

$$N^{BP}/\operatorname{Im} h^{BP} + BP_* \cdot 1 \xrightarrow{\hat{\mu}} N/\operatorname{Im} h^H.$$

Theorem 1.6. *$\hat{\mu}$ is isomorphic.*

Proof. In $BP_*(BP) \otimes Q = BP_* \otimes Q[n_1, n_2, \cdots]$

$$\bigcap_{E \neq 0} (r_E)_*^{-1}(0) = BP_* \otimes Q$$

so that in $BP_*(BP)$

$$\bigcap_{E \neq 0} (r_E)_*^{-1}(0) = BP_* \cdot 1 \; (= BP_* \otimes 1).$$

We get easily

$$\operatorname{Ker} \mu \cap N^{BP} \subset \bigcap_{E \neq 0} (r_E)_*^{-1}(0) = BP_* \cdot 1$$

and

$$\begin{aligned} \operatorname{Ker} \hat{\mu} &= \{(\operatorname{Im} h^{BP} + \operatorname{Ker} \mu) \cap N^{BP} + \operatorname{Im} h^{BP} + BP_* \cdot 1\} / \operatorname{Im} h^{BP} + BP_* \cdot 1 \\ &= 0 \end{aligned}$$

so that $\hat{\mu}$ is monomorphic.

For any prime p, $h^{BP}(v_1) = v_1 = v_1 \cdot 1 + pt_1$, and thus

$$v_1{}^k - v_1{}^k \cdot 1 = \sum_{1 \leq e \leq k} \binom{k}{e} p^e v_1{}^{k-e} t_1{}^e.$$

We get

$$(v_1{}^k - v_1{}^k \cdot 1)/p^{\nu_p(k)+1} \in BP_{2k(p-1)}(BP)$$

and

$$(v_1{}^k - v_1{}^k\cdot 1)/p^{\nu_p(k)+1} \in N^{BP}_{2k(p-1)}\,.$$

In case of $p=2$ and $k\geq 2$,

$$h^{BP}(v_2) = v_2 = v_2\cdot 1 - 3v_1{}^2t_1 - 5v_1t_1{}^2 + 2t_2 - 4t_1{}^3\,,\quad ([3])\,.$$

We get

$$(v_1{}^k - v_1{}^k\cdot 1)/2^{\nu_2(k)+2} = (1/2)v_1^{k-1}t_1 + (1/2)v_1^{k-2}t_1{}^2 + A\,,$$
$$(v_1^{k-3}v_2 - (v_1^{k-3}v_2)\cdot 1)/2 = (-1/2)v_1^{k-1}t_1 + (-1/2)v_1^{k-2}t_1{}^2 + B\,,$$

where A, $B\in BP_*(BP)$, and thus

$$(v_1{}^k - v_1{}^k\cdot 1)/2^{\nu_2(k)+2} + (v_1^{k-3}v_2 - (v_1^{k-3}v_2\cdot)1)/2 \in BP_{2k}(BP)\,.$$

We have easily

$$(v_1{}^k - v_1{}^k\cdot 1)/2^{\nu_2(k)+2} + (v_1^{k-3}v_2 - (v_1^{k-3}v_2)\cdot 1)/2 \in N^{BP}_{2k}\,.$$

These conclude that $\hat{\mu}$ is epimorphic and complete the proof of Theorem 1.6.

The conjugation map c of the Hopf algebra $BP_*(BP)$ induces the isomorphism

$$\hat{c}\colon\ N^{BP}/\mathrm{Im}\,h^{BP} + BP_*\cdot 1 \to \bigcap_{E\neq 0} r_E^{-1}(BP_*\cdot 1)/\mathrm{Im}\,h^{BP} + BP_*\cdot 1\,,$$

but $\hat{c}$ preserves the generators given in Theorem 1.6 up to sign, so that we obtain

Corollary 1.7.

$$N^{BP}/\mathrm{Im}\,h^{BP} + BP_*\cdot 1 = \bigcap_{E\neq 0} r_E^{-1}(BP_*\cdot 1)/\mathrm{Im}\,h^{BP} + BP_*\cdot 1\,.$$

We next show that

$$\begin{aligned}\mathrm{Ext}^{1,*}_{BP_*(BP)}(BP_*, BP_*) &= \bigcap_{E\neq 0} r_E^{-1}(BP_*\cdot 1)/\mathrm{Im}\,h^{BP} + BP_*\cdot 1 \\ &\cong N/\mathrm{Im}\,h^H\,.\end{aligned}$$

Let $S\xrightarrow{i} BP \xrightarrow{p} I$ be the cofibration obtained from the unit $S\xrightarrow{i} BP$, $I^{(k)} = I\wedge I\wedge\cdots\wedge I$ (k-factors) and d_k be the composition $BP\wedge I^{(k)} \xrightarrow{p\wedge 1} I^{(k+1)} \xrightarrow{B} BP\wedge I^{(k+1)}$ (or equivalently $BP\wedge I^{(k)} \xrightarrow{B} BP\wedge BP\wedge{}^{(k)} \xrightarrow{1\wedge p\wedge 1} BP\wedge I^{(k+1)}$). Then we obtain the geometric resolution of Adams [4]

$$BP \xrightarrow{d_0} BP\wedge I \xrightarrow{d_1} BP\wedge I^{(2)} \xrightarrow{d_2} BP\wedge I^{(3)} \xrightarrow{d_3} \cdots\,,$$

which defines a chain complex of a spectrum X

$$BP_*(X) \xrightarrow{(d_0)_*} (BP\wedge I)_*(X) \xrightarrow{(d_1)_*} (BP\wedge I^{(2)})_*(X) \to \cdots,$$

and

$$\mathrm{Ext}^{k,*}_{BP_*(BP)}(BP_*, BP_*(X)) = \mathrm{Ker}\,(d_k)_*/\mathrm{Im}\,(d_{k-1})_* .$$

For $X=S^0$, $(d_0)_*=p_*h^{BP}$ and $(d_1)_*=(p_*\otimes 1)\Psi_I$ where $BP_*(BP)\xrightarrow{p_*}BP_*(I)=BP_*(BP)/BP_*\cdot 1$ is the canonical projection, $BP_*(I)\xrightarrow{\Psi_I}BP_*(BP)\otimes_{BP_*}BP_*(I)$ is the coaction map of I for which

$$\Psi_I(x) = \sum_E t^E\otimes r_E(x) .$$

REMARK. This coaction map is twisted by the conjugation map c of $BP_*(BP)$ from the one denfied by Adams [2].

$$\begin{aligned}\mathrm{Ker}\,(d_1)_* &= \{x\in BP_*(BP)/BP_*\cdot 1 \mid r_E(x)=0\,,\ E\neq 0\}\\ &= \bigcap_{E\neq 0} r_E^{-1}(BP_*\cdot 1)/BP_*\cdot 1\,,\end{aligned}$$

$$\mathrm{Im}\,(d_0)_* = \mathrm{Im}\,h^{BP}/\mathrm{Im}\,h^{BP}\cap BP_*\cdot 1 = \mathrm{Im}\,h^{BP}+BP_*\cdot 1/BP_*\cdot 1 .$$

Hence we obtain

Theorem 1.8.

$$\mathrm{Ext}^{1,*}_{BP_*(BP)}(BP_*, BP_*) = \bigcap_{E\neq 0} r_E^{-1}(BP_*\cdot 1)/\mathrm{Im}\,h^{BP}+BP_*\cdot 1 .$$

Corollary 1.9.

$$\mathrm{Ext}^{1,*}_{BP_*(BP)}(BP_*, BP_*) \overset{\hat{\mu}}{\cong} N/\mathrm{Im}\,h^H .$$

2. The BP_*-Hopf invariant

Since $BP_*(BP)$ is flat over BP_*, $BP_*(BP)$ comodules and $BP_*(BP)$ comodule homomorphisms form a relative abelian category so that similar construction of Adams [1] is valid for BP_* homology theory. We review the construction of the BP_*-Hopf invariant quickly; for a morphism $f\colon X\to Y$ of CW-spectra in homotopy category such that $f_*=0$, we have a short exact sequence

$$E(f)\colon\ 0\to BP_*(Y)\to BP_*(C_f)\to BP_*(SX)\to 0\,,$$

which is regarded as an element of $\mathrm{Ext}^{1,*}_{BP_*(BP)}(BP_*(X), BP_*(Y))$. This is the BP_*-Hopf invariant of f.

For $X=S^{kq-1}(q=2(p-1))$, $Y=S^0$ the BP_*-Hopf invariant is defined on the whole group $\pi_{kq-1}(S^0)$. For a short exact sequence $E(f)$ we apply the Adams resolution $BP \to BP\wedge I \to BP\wedge I^{(2)} \to \cdots$ then we obtain a short exact sequence of chain complexes

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & BP_*(S^0) & \to & (BP\wedge I)_*(S^0) & \to & (BP\wedge I^{(2)})_*(S^0) \to \cdots \\
 & & \downarrow & & \downarrow i_* & & \downarrow \\
0 & \to & BP_*(C_f) & \to & (BP\wedge I)_*(C_f) & \to & (BP\wedge I^{(2)})_*(C_f) \to \cdots \\
 & & \downarrow j_* & & \downarrow & & \downarrow \\
0 & \to & BP_*(S^{kq}) & \to & (BP\wedge I)_*(S^{kq}) & \to & (BP\wedge I^{(2)})_*(S^{kq}) \to \cdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
\end{array}$$

Let $\sigma_{kq} \in BP_{kq}(S^{kq})$ be a generator and $\delta(\sigma_{kq}) = [i_*^{-1}(d_0)_* j_*^{-1}(\sigma_{kq})] \in \mathrm{Ext}^{1,kq}_{BP_*(BP)}(BP_*, BP_*)$ then the element $\delta(\sigma_{kq})=E(f)_*(\sigma_{kq})$ is just the element $E(f)$ by well known technique of homological algebra. This construction is considered as follows; let $\sigma_0=i_*(\sigma_0)$, μ_{kq} are generators of $BP_*(C_f)$ of dimension 0, dimension kq respectively so that $j_*(\mu_{kq})=\sigma_{kq}$, and let η_R: $BP\simeq S\wedge BP \to BP\wedge BP$ be the Boardman map and put $\eta_{R^*}(\mu_{kq})=A_f\sigma_0+\mu_{kq}(A_f\in BP_*(BP))$. Then A_f represents $E(f)$. Replacing $BP\to BP\wedge BP$ by $BP\to H\wedge BP$, we have the BP_*-e invariant (or the functional Chern-Dold character) and is equivalent to the BP_*-Hopf invariant by Corollary 1.9.

3. Applications

For an element $f\in\pi_{kq-1}(S^0)$ $(q=2(p-1))$ we get a short exact sequence $0\to(HZ_p)_*(S^0)\to(HZ_p)_*(C_f)\to(HZ_p)_*(S^{kq})\to 0$ and can choose generators σ_0' and μ'_{kq} of $(HZ_p)_*(C_f)$such that $\sigma_0'=i_*(\sigma_0')$ and $j_*(\mu'_{kq})=\sigma'_{kq}$ where σ_n' is a canonical generator of $(HZ_p)_n(S^n)$. Let Ψ: $(HZ_p)_*(C_f)\to A_*\otimes(HZ_p)_*(C_f)$ be the coaction, then the definition of the Hopf invariant in the sence of Steenrod is described as follows; $f(\in\pi_{kq-1}(S^0))$ is said to have mod p Hopf invariant 1 if $\langle P^k, H_f\rangle\neq 0$, where P^k is the Steenrod reduced power (interpreted as Sq^{2k} if $p=2$) and $\Psi(\mu'_{kq})=H_f\sigma_0'+\mu'_{kq}$ $(H_f\in A_*)$.

Theorem 3.1 (Adams, Liulevicius, Shimada-Yamanoshita.) *If f has mod p Hopf invariant* 1 *then*

(2) $k=1, 2$ *or* 4 *for* $p=2$;

(2) $k=1$ *for odd prime* p.

Proof. Consider the following diagram

$$\begin{tikzcd}
BP_*(C_f) \arrow[rr, "\rho_p\mu"] \arrow[d, "B_*"] & & (HZ_p)_*(C_f) \arrow[dl, "\eta_{R^*}"'] \arrow[d, "\Psi"] \\
H_*(BP)\otimes H_*(C_f) \arrow[r, "\mu_*\otimes 1"] & A_*\otimes(HZ_p)_*(C_f) \arrow[r, "c\otimes 1"] & A_*\otimes(HZ_p)_*(C_f)
\end{tikzcd}$$

then

$$\begin{aligned}\Psi(\mu'_{kq}) &= \Psi\mu(\mu_{kq}) = (c\otimes 1)(\mu_*\otimes 1)(A_f\sigma_0'+\mu'_{kq}) \\ &= H_f\sigma_0'+\mu'_{kq}\,.\end{aligned}$$

Since A_f is a multiple of $v_1^k/p^{\nu_p(k)+1}=p^{k-\nu_p(k)-1}n_1^k$ or $v_1^k/2^{\nu_2(k)+2}+v_1^{k-3}v_2\times 2^{k-\nu_2(k)-2}n_1^k \bmod 2\cdot H_{kq}(BP)$ by Theorem 1.1, H_f is a multiple of $p^{k-\nu_p(k)-1}\xi_1^k$ or $2^{k-\nu_2(k)-2}\xi_1^{2k}$. In case of an odd prime p $H_f=0$ for $k>1$, in case of $p=2$ and odd number k $H_f=0$ for $k>1$, in case of $p=2$ and even k $H_f=0$ for $k>4$. This completes the proof of Theorem 3.1.

Let $V(0)=S^0\cup_p e^1$ then there exists a map ϕ_k: $S^{kq}\to S^{kq}V(0)\to V(0)$ such that $\phi_{k*}(\sigma_{kq})=v_1^k\cdot\gamma_0$ where $\sigma_{kq}\in BP_{kq}(S^{kq})$ and $\gamma_0\in BP_0(V(0))$ are generators ([8]). α-series elements α_k $(k=1, 2, \cdots)$ of $\pi_{kq-1}(S^0)$ are defined by $\alpha_k=j\phi_k$ where j: $V(0)\to S^1$ is the canonical projection. We detect these elements by means of the BP_*-Hopf invariant. We have the following diagram of cofibrations;

$$\begin{tikzcd}
 & S^0 \arrow[r, equal] \arrow[d, "i"] & S^0 \arrow[d, dashed] & & \\
S^{kq} \arrow[r, "\phi_k"] \arrow[d, equal] & V(0) \arrow[r, "a"] \arrow[d, "j"] & C_{\phi_k} \arrow[r, "b"] \arrow[d] & S^{kq+1} \arrow[r] \arrow[d, equal] & SV(0) \arrow[d] \\
S^{kq} \arrow[r, "\alpha_k"] & S^1 \arrow[r, "c"] \arrow[d] & C_{\alpha_k} \arrow[r, "d"] \arrow[d] & S^{kq+1} \arrow[r] & S^2 \\
 & S^1 \arrow[r, equal] & S^1 & &
\end{tikzcd}$$

Considering the above diagram following results are obtained;

$$BP_*(S^n)\ \begin{cases}\text{generator; } \sigma_n \\ \text{relations; none,}\end{cases}$$

$$BP_*(V(0)) = BP_*/(p)\ \begin{cases}\text{generator; } \gamma_0 \\ \text{relations; } p\gamma_0 = 0\,,\end{cases}$$

$$BP_*(C_{\phi_k})\ \begin{cases}\text{generators; } a_*(\gamma_0),\ \lambda_{kq+1} \\ \text{relations; } (p,\ v_1^k)\cdot a_*(\gamma_0) = 0 \\ \text{formula; } b_*(\lambda_{kq+1}) = p\sigma_{kq+1}\,,\end{cases}$$

$$BP_*(C_{\alpha_k})\begin{cases}\text{generators; } c_*(\sigma_1),\ \mu_{kq+1}\\ \text{relations;} \quad \text{none}\\ \text{formula;} \quad d_*(\mu_{kq+1})=\sigma_{kq+1}\,,\end{cases}$$

and the formula

$$h_*(\lambda_{kq+1})=p\mu_{kq+1}-v_1{}^kc_*(\sigma_1)\,.$$

The coefficient $v_1{}^k$ of $c_*(\sigma_1)$ is decided up to a multiple of a unit of $Z_{(p)}$. The image of these generators of Thom homomorphism are denoted by σ_n', γ_n', λ_n' and μ_n' respectively.

Theorem 3.2.

$$e(\alpha_k)=v_1{}^k/p \quad in \quad N/\mathrm{Im}\,h^H\cong \mathrm{Ext}^{1,kq}_{BP_*(BP)}(BP_*,BP_*)\,.$$

Proof. By applying the Chern-Dold character $BP_*(C_{\alpha_k})\overset{B_*}{\to}(H\wedge BP)_*(C_{\alpha_k})$ to μ_{kq+1} we get $B_*(\mu_{kq+1})=A_{\alpha_k}c_*(\sigma_1')+\mu'_{kq+1}$. A_{α_k} represents BP_*-e invariant of α_k in $N/\mathrm{Im}\,h^H$. The computation

$$\begin{aligned}p\mu'_{kq+1}&=h_*B_*(\lambda_{kq+1})=B_*(p\mu_{kq+1}-v_1{}^kc_*(\sigma_1))\\&=p\mu'_{kq+1}+(pA_{\alpha_k}-v_1{}^k)c_*(\sigma_1')\end{aligned}$$

implies $pA_{\alpha_k}=v_1{}^k$ and this completes the proof.

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