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A RELATIONSHIP BETWEEN BROWNIAN MOTIONS WITH OPPOSITE DRIFTS VIA CERTAIN ENLARGEMENTS OF THE BROWNIAN FILTRATION

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1. Introduction

Let $B = \{B_t, t \geqq 0\}$ be a one-dimensional standard Brownian motion starting from 0. To $B^{(\mu)} = \{B_t^{(\mu)} = B_t + \mu t, t \geqq 0\}$, a Brownian motion with constant drift μ, we associate the exponential additive functional

$$A_t^{(\mu)} = \int_0^t \exp\bigl(2B_s^{(\mu)}\bigr)ds, \qquad t \geqq 0,$$

which is the quadratic variation process of the corresponding geometric Brownian motion $e^{(\mu)} = \{e_t^{(\mu)} = \exp\bigl(B_t^{(\mu)}\bigr), t \geqq 0\}$.

These Wiener functionals play important roles in many fields; mathematical finance (see, e.g., Yor [33], [34], Geman-Yor [10], [11], Leblanc [17]), diffusion processes in random environment (Comtet-Monthus [5], Comtet-Monthus-Yor [6], Kawazu-Tanaka [16]), probabilistic studies related to Laplacians on hyperbolic spaces (Gruet [12], Ikeda-Matsumoto [13]) and so on. The reader will find more related topics and references in [37].

However, even for fixed t, the joint law of $\bigl(A_t^{(\mu)}, B_t^{(\mu)}\bigr)$ or, equivalently, that of $\bigl(A_t = A_t^{(0)}, B_t\bigr)$ due to the Cameron-Martin relationship between $B^{(\mu)}$ and B is fairly complicated, although it is quite tractable (see [33]). As an example of the description of this law, we present the following conditional Laplace transform ([2], [13], [18]):

$$\begin{aligned} E\left[\exp\left(-\frac{u^2 A_t}{2}\right) \,\Big|\, B_t = x\right] \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right) \\ = \int_{|x|}^{\infty} \frac{z}{\sqrt{2\pi t^3}} \exp\left(-\frac{z^2}{2t}\right) J_0(u\phi(x,z))dz, \end{aligned}$$

where $u \geqq 0$, J_0 is the Bessel function of the first kind of order 0 and $\phi(x,z) = \sqrt{2}e^{x/2}(\cosh z - \cosh x)^{1/2}$, $z \geqq |x|$.

A quite different description of the law of $A_t^{(\mu)}$ is as follows (Geman-Yor [10], Yor [34]). Let T_λ be an exponential random variable with parameter λ, independent of B. Moreover let $Z_{1,a}$, Z_b be a Beta and a Gamma random variables with parameters

$(1, a)$, b, respectively, and U be a uniform random variable on $[0, 1]$. We assume that $Z_{1,a}$ and Z_b are independent and that Z_b is also independent of U. Then it holds that

$$A_{T_\lambda}^{(\mu)} \overset{\text{(law)}}{=} \frac{Z_{1,a}}{2Z_b} \overset{\text{(law)}}{=} \frac{1-U^{1/a}}{2Z_b}, \tag{1.1}$$

where $a = (\kappa+\mu)/2, b = (\kappa-\mu)/2$ and $\kappa = \sqrt{2\lambda+\mu^2}$.

In any case, despite this complexity, a number of identities about A_t or $A_t^{(\mu)}$ for different μ's are known. In particular, let us recall the simple Bougerol's identity ([4]):

$$\sinh(B_t) \overset{\text{(law)}}{=} \gamma_{A_t} \tag{1.2}$$

for any fixed t, where $\{\gamma_s, s \geqq 0\}$ is another Brownian motion independent of B. This formula makes it easy to calculate the Mellin transform of the probability law of A_t. See Alili-Dufresne-Yor [1] for a simple proof of (1.2).

The main result of the present paper goes into another direction, in that it exhibits a relation — in terms of exponential functionals — between $B^{(-\mu)}$ and $B^{(\mu)}$, which we present in the following way, strongly inspired by Dufresne [9].

Theorem 1.1. *Let $\mu > 0$. Then one has the identity in law*

$$\left\{\frac{1}{A_t^{(-\mu)}}, t>0\right\} \overset{\text{(law)}}{=} \left\{\frac{1}{A_t^{(\mu)}} + \frac{1}{\tilde{A}_\infty^{(-\mu)}}, t>0\right\}, \tag{1.3}$$

where $\tilde{A}_\infty^{(-\mu)}$ is a copy of $A_\infty^{(-\mu)}$, independent of $B^{(\mu)}$.

In fact, Theorem 1.1 is a reinforcement of Dufresne's result [9]. He showed that the identity (1.3) in law holds for any fixed time t from the knowledge of the laws of $A_{T_\lambda}^{(\pm\mu)}$ as given in (1.1) above and some algebraic identities in law between Beta and Gamma random variables ([8]).

As a check on Theorem 1.1, let us discuss how we became convinced that (1.3) may hold at the process level. Indeed, assuming Theorem 1.1, we find that

$$\left(\frac{1}{A_t^{(-\mu)}} - \frac{1}{A_\infty^{(-\mu)}}, \frac{1}{A_\infty^{(-\mu)}}\right) \overset{\text{(law)}}{=} \left(\frac{1}{A_t^{(\mu)}}, \frac{1}{\tilde{A}_\infty^{(-\mu)}}\right) \tag{1.4}$$

and, consequently,

$$\frac{A_\infty^{(-\mu)} - A_t^{(-\mu)}}{A_t^{(-\mu)}} \overset{\text{(law)}}{=} \frac{\tilde{A}_\infty^{(-\mu)}}{A_t^{(\mu)}} \tag{1.5}$$

holds for any fixed t. But now, (1.5) is easily checked by using the Markov property of $B^{(-\mu)}$ and the time reversal for $\{B_u^{(-\mu)}, 0 \leqq u \leqq t\}$ at time t. Note, on the other hand, that the identity in law (1.4) for fixed t does not seem obvious a priori. In the end, the conjunction of the facts that (1.5) holds and that the identity in law holds for fixed t in (1.3) according to Dufresne [9] made us think that the full identity in law between processes in (1.3) might hold.

We also take this opportunity to note that the identity (1.3) gives a very simple check on the identities (101) and (102) in Comtet-Monthus-Yor [6], which are nothing else but

$$E\left[\frac{1}{A_t^{(-\mu)}}\right] = E\left[\frac{1}{A_t^{(\mu)}}\right] + E\left[\frac{1}{A_\infty^{(-\mu)}}\right].$$

We now give a detailed plan of the rest of this paper. In Section 2, we present a number of variants and consequences of Theorem 1.1. Among them, the most important result is Theorem 2.2, which expresses $B^{(-\mu)}$ in terms of $B^{(\mu)}$ and an independent Gamma variable.

In Section 3, we give two proofs of Theorem 2.2: the first one relies upon the enlargement of the filtration of $B^{(-\mu)}$ with the variable $A_\infty^{(-\mu)}$, whereas the second one uses Lamperti's representation

$$\exp\left(B_t^{(-\mu)}\right) = R^{(-\mu)}_{A_t^{(-\mu)}}, \qquad t \geqq 0,$$

where $R^{(-\mu)} = \{R_u^{(-\mu)}, u \geqq 0\}$ denotes a Bessel process with index $-\mu$ and we condition $R^{(-\mu)}$ upon its lifetime

$$T_0^{(-\mu)} \equiv \inf\{u; R_u^{(-\mu)} = 0\} \equiv A_\infty^{(-\mu)}.$$

In Section 4, to show the versatility of our approach, we enlarge the filtration of a $2D$-Brownian motion, that of $\left(B^{(-\mu)}, \gamma^{(\nu)}\right)$, with the variable

$$X^{(\mu,\nu)} \equiv \int_0^\infty \exp\left(B_s^{(-\mu)}\right) d\gamma_s^{(\nu)}$$

by using recent results about the law of $X^{(\mu,\nu)}$ obtained by Paulsen [24] and rediscovered by Yor [36]; see also [3]. Although, in this $2D$ case, the results are not so striking as that expressed by Theorem 1.1, they are still easy enough to present and we compare them with the $1D$ story.

As a conclusion to this Introduction, we would like to indicate, apart from its own interest which we have discussed, that Theorem 1.1 is the essential starting step in our proofs of the following extension of Pitman's $2M - X$ theorem ([26]): for any $c > 0$, the stochastic process $L^{(\mu),c} = \{L_t^{(\mu),c}, t \geqq 0\}$ defined by

$$L_t^{(\mu),c} = c \log\left(\int_0^t \exp\left(\frac{2}{c} B_s^{(\mu)}\right) ds\right) - B_t^{(\mu)}$$

is a diffusion process and has the same distribution as $L^{(-\mu),c}$. Moreover we give two quite different proofs of this result in [20] and [21] which, we think, would be easier to read once Theorem 1.1 has been presented separately, hence the present paper. See also [19], where the above result has been announced together with related topics. Finally, a multi-dimensional extension of Theorem 1.1 or rather Proposition 3.1 below is being used in [23] in connection with some queueing problems.

2. Brownian Motions with Opposite Drifts

Let $B^{(\mu)} = \{B_t^{(\mu)}, t \geqq 0\}$ be a Brownian motion with drift $\mu \in \mathbf{R}$ starting from 0 as in the Introduction. Then, thanks to the Cameron-Martin theorem, if $W^{(\mu)}$ denotes the probability law of $B^{(\mu)}$ on the canonical path space $\Omega = C([0,\infty);\mathbf{R})$, then for any $\mu, \nu \in \mathbf{R}$ the laws $W^{(\mu)}, W^{(\nu)}$ are related by

$$W^{(\mu)}|_{\mathscr{B}_t} = \exp\left((\mu-\nu)X_t - \frac{\mu^2-\nu^2}{2}t\right) W^{(\nu)}|_{\mathscr{B}_t},$$

where $X_t(w) = w(t), t \geqq 0$, is the coordinate process and $\mathscr{B}_t = \sigma\{X_s; s \leqq t\}$ is its natural filtration. In particular, $h(x) = \exp(2\mu x)$ is a harmonic function for $B^{(-\mu)}$ and $B^{(\mu)}$ is the Doob h-transform of $B^{(-\mu)}$.

We give another relationship between $B^{(\mu)}$ and $B^{(-\mu)}$.

Theorem 2.1. *Letting $\mu > 0$ and γ_μ be a Gamma random variable with parameter μ, independent of $B^{(\mu)}$, we set*

$$A_t^{(\mu)} = \int_0^t \exp\big(2B_s^{(\mu)}\big)ds \qquad \text{and} \qquad \hat{B}_t^{(\mu)} = B_t^{(\mu)} - \log\left(1 + 2\gamma_\mu A_t^{(\mu)}\right).$$

Then the following identity in law holds:

$$\left\{B_t^{(-\mu)}, t \geqq 0\right\} \stackrel{\text{(law)}}{=} \left\{\hat{B}_t^{(\mu)}, t \geqq 0\right\}. \tag{2.1}$$

The following remark explains how the Gamma variable γ_μ comes into the picture. For this purpose, note that

$$\int_0^\infty \exp\big(2\hat{B}_s^{(\mu)}\big)ds = \int_0^\infty \frac{\exp\big(2B_s^{(\mu)}\big)}{\big(1+2\gamma_\mu A_s^{(\mu)}\big)^2}ds = \frac{1}{2\gamma_\mu}.$$

Then, assuming that (2.1) holds, we recover the known result

$$A_\infty^{(-\mu)} \equiv \int_0^\infty \exp\big(2B_s^{(-\mu)}\big)ds \stackrel{\text{(law)}}{=} \frac{1}{2\gamma_\mu} \tag{2.2}$$

(cf. Dufresne [9], Yor [35]).

This remark can be developed so as to provide a method for a proof of Theorem 2.1, which we now present in the following equivalent form.

Theorem 2.2. *Let* $\mu > 0$. *Then*
(i) $A_\infty^{(-\mu)} \stackrel{\text{(law)}}{=} (2\gamma_\mu)^{-1}$.

(ii) *Given* $A_\infty^{(-\mu)} = 1/2c$, *the process* $\{B_t^{(-\mu)}, t \geqq 0\}$ *is distributed as* $\{B_t^{(\mu)} - \log\bigl(1 + 2cA_t^{(\mu)}\bigr), t \geqq 0\}$.

We postpone the proof of Theorem 2.2 until the next section and we give some consequences of it in the rest of this section. In particular, the following is an immediate consequence of (2.1) and (2.2) and, considering the quadratic variation process of $e^{(-\mu)}$ and its counterpart in (2.3) below, we obtain Theorem 1.1.

Theorem 2.3. *Let* $\mu > 0$ *and* $\tilde{A}_\infty^{(-\mu)}$ *be a copy of* $A_\infty^{(-\mu)}$, *independent of* $B^{(\mu)}$. *Then one has*

$$\left\{ \left(\frac{A_\infty^{(-\mu)} e_t^{(-\mu)}}{A_\infty^{(-\mu)} - A_t^{(-\mu)}}, e_t^{(-\mu)} \right), t \geqq 0 \right\} \stackrel{\text{(law)}}{=} \left\{ \left(e_t^{(\mu)}, \frac{\tilde{A}_\infty^{(-\mu)} e_t^{(\mu)}}{\tilde{A}_\infty^{(-\mu)} + A_t^{(\mu)}} \right), t \geqq 0 \right\}. \tag{2.3}$$

We next look at Theorem 2.1 from the point of view of stochastic calculus: Theorem 2.1 gives a non-canonical semimartingale decomposition of $B^{(-\mu)}$, or, equivalently, it tells us that the canonical decomposition of the stochastic process

$$\hat{B}_t^{(\mu)} = B_t^{(\mu)} - \log\bigl(1 + 2\gamma_\mu A_t^{(\mu)}\bigr), \qquad t \geqq 0,$$

in its own filtration $\hat{\mathscr{B}}_t^{(\mu)} = \sigma\{\hat{B}_s^{(\mu)}; s \leqq t\}$ is $\beta_t - \mu t$, where $\{\beta_t, t \geqq 0\}$ is a $\bigl(\hat{\mathscr{B}}_t^{(\mu)}\bigr)$-Brownian motion. Thus we have the following two expressions for $\hat{B}_t^{(\mu)}$:

$$\begin{aligned} \hat{B}_t^{(\mu)} &= B_t + \mu t - \int_0^t \frac{2\gamma_\mu \exp\bigl(2B_s^{(\mu)}\bigr)}{1 + 2\gamma_\mu A_s^{(\mu)}} ds \\ &= \beta_t - \mu t. \end{aligned}$$

Therefore we obtain

$$E\left[\frac{\gamma_\mu \exp\bigl(2B_s^{(\mu)}\bigr)}{1 + 2\gamma_\mu A_s^{(\mu)}} \;\Big|\; \hat{\mathscr{B}}_s^{(\mu)} \right] = \mu \tag{2.4}$$

for every $s \geqq 0$.

The following confirms and amplifies this identity:

Proposition 2.4. *For every $s > 0$, the following identity in law holds*:

$$\begin{aligned}(2.5)\qquad &\left(\gamma_\mu \exp\left(-B_s^{(-\mu)}\right), \{B_u^{(-\mu)}, u \leqq s\}\right)\\ &\stackrel{\text{(law)}}{=} \left(\gamma_\mu \exp\left(B_s^{(\mu)}\right), \{B_u^{(\mu)} - \log\left(1 + 2\gamma_\mu A_u^{(\mu)}\right), u \leqq s\}\right),\end{aligned}$$

where, on both hand sides, the Gamma random variable γ_μ with parameter μ is assumed to be independent of $\{B_u^{(\pm\mu)}, u \leqq s\}$.

Consequently, for any fixed $s > 0$, a simple algebraic manipulation in (2.5) shows that $\gamma_\mu \exp\left(2B_s^{(\mu)}\right)/\left(1 + 2\gamma_\mu A_s^{(\mu)}\right)$ is independent of $\hat{\mathscr{B}}_s^{(\mu)}$ and is distributed as γ_μ, which confirms (2.4). It may also be useful for further studies about exponential Brownian functionals to record the following two-dimensional consequence of (2.5):

$$(2.6)\qquad \left(\frac{\gamma_\mu \exp\left(2B_s^{(\mu)}\right)}{1 + 2\gamma_\mu A_s^{(\mu)}}, \frac{\exp\left(B_s^{(\mu)}\right)}{1 + 2\gamma_\mu A_s^{(\mu)}}\right) \stackrel{\text{(law)}}{=} \left(\gamma_\mu, \exp\left(B_s^{(-\mu)}\right)\right).$$

The identity between the first members of each side of (2.6) may be understood as a consequence of the following:

$$A_\infty^{(-\mu)} = A_s^{(-\mu)} + \exp\left(2B_s^{(-\mu)}\right)\tilde{A}_\infty^{(-\mu)} \stackrel{\text{(law)}}{=} \frac{1}{2\gamma_\mu},$$

whereas the identity between the second members of each side of (2.6) is the application of (2.1) to one-dimensional marginals.

3. Two Proofs of Theorem 2.2

In this section we give two different proofs of Theorem 2.2. As was mentioned in the previous section, (i) is known. Thus, it remains to prove (ii). The first proof is based on the theory of the enlargements of filtrations (cf. Jeulin [14] and Yor [31], Chapter 12) and the other one is based on Lamperti's relation and some stability properties of the laws of Bessel processes under time reversal and time inversion. (See, e.g., [32], where some of these arguments have been already used).

First Proof of Theorem 2.2. Let $\mathscr{B}_t^{(\mu)} = \sigma\{B_s^{(\mu)}; s \leqq t\}$, which in fact does not depend on μ, and $\hat{\mathscr{B}}_t^{(-\mu)} = \mathscr{B}_t^{(-\mu)} \vee \sigma\{A_\infty^{(-\mu)}\}$. Then, applying the main result for (semi)martingale decompositions in the set-up of the initial enlargement of filtrations (cf. Yor [31]), we can show that there exists a $\left(\hat{\mathscr{B}}_t^{(-\mu)}\right)$-Brownian motion $\{B_t^*\}_{t\geqq 0}$ independent of $A_\infty^{(-\mu)}$ such that

$$(3.1)\qquad B_t^{(-\mu)} = B_t^* + \mu t - \int_0^t \frac{\exp\left(2B_s^{(-\mu)}\right)}{A_\infty^{(-\mu)} - A_s^{(-\mu)}} ds.$$

Therefore, under the condition $A_\infty^{(-\mu)} = 1/2c$, $\{B_t^{(-\mu)}, t \geqq 0\}$ is the solution of

$$z_t = B_t^* + \mu t - \int_0^t \frac{\exp(2z_s)}{1/2c - \int_0^s \exp(2z_u)du} ds.$$

As will be shown in the Appendix, this equation, considered as an ordinary equation with the initial data $\{B_t^* + \mu t, t \geqq 0\}$, has a unique solution

$$z_t = B_t^{*(\mu)} - \log\left(1 + 2cA_t^{*(\mu)}\right),$$

where $B_t^{*(\mu)} = B_t^* + \mu t$ and

$$A_t^{*(\mu)} = \int_0^t \exp\left(2B_s^{*(\mu)}\right) ds. \qquad \square$$

To summarize this first proof and for the ease of future references, we give the following.

Proposition 3.1. *There exists a Brownian motion* $B_t^{*(\mu)} \equiv B_t^* + \mu t, t \geqq 0$, *with drift* $\mu > 0$, *with respect to the filtration* $\left(\hat{\mathscr{B}}_t^{(-\mu)}, t \geqq 0\right)$ *such that*

$$B_t^{(-\mu)} = B_t^{*(\mu)} - \log\left(1 + \frac{A_t^{*(\mu)}}{A_\infty^{(-\mu)}}\right) = B_t^{*(\mu)} + \log\left(1 - \frac{A_t^{(-\mu)}}{A_\infty^{(-\mu)}}\right). \tag{3.2}$$

In particular, $B^{*(\mu)}$ *is independent of* $A_\infty^{(-\mu)}$ *and it holds that*

$$\left(1 - \frac{A_t^{(-\mu)}}{A_\infty^{(-\mu)}}\right)\left(1 + \frac{A_t^{*(\mu)}}{A_\infty^{(-\mu)}}\right) = 1. \tag{3.3}$$

Note that (3.3) and some trivial algebra yield identity (1.3), where, instead of an identity in law, we have an almost sure equality, $A_t^{(\mu)}$ and $\tilde{A}_\infty^{(-\mu)}$ being changed into $A_t^{*(\mu)}$ and $A_\infty^{(-\mu)}$, respectively.

Second proof of Theorem 2.2. By Lamperti's relation, for any $\nu \in \mathbf{R}$ there exists a Bessel process $\{R_t^{(\nu)}, t \geqq 0\}$ with index ν starting from 1 such that

$$e_t^{(\nu)} \equiv \exp\left(B_t^{(\nu)}\right) = R_{A_t^{(\nu)}}^{(\nu)}, \quad \nu \in \mathbf{R}.$$

We now use this relation for $\nu = \pm\mu$. Setting

$$\tilde{T}_0 = \tilde{A}_\infty^{(-\mu)}, \quad H_t = \frac{\tilde{T}_0}{\tilde{T}_0 + A_t^{(\mu)}} \quad \text{and} \quad K_t = \frac{\tilde{T}_0 A_t^{(\mu)}}{\tilde{T}_0 + A_t^{(\mu)}},$$

where, as usual with our tilde notation, $\tilde{T}_0$ is assumed to be independent of $\{B_t^{(\mu)}, t \geqq 0\}$; we have

$$\frac{\tilde{A}_\infty^{(-\mu)} e_t^{(\mu)}}{\tilde{A}_\infty^{(-\mu)} + A_t^{(\mu)}} = H_t R^{(\mu)}_{A_t^{(\mu)}}.$$

Therefore, if we show

$$\left\{ R^{(-\mu)}_{A_t^{(-\mu)}}, t \geqq 0 \right\} \stackrel{\text{(law)}}{=} \left\{ H_t R^{(\mu)}_{A_t^{(\mu)}}, t \geqq 0 \right\}$$

or, equivalently,

$$\left\{ R_u^{(-\mu)}, u \leqq T_0 \right\} \stackrel{\text{(law)}}{=} \left\{ H_{k_u} R^{(\mu)}_{A_{k_u}^{(\mu)}}, u \leqq \tilde{T}_0 \right\} \tag{3.4}$$

for the inverse function k_u of K_t, we obtain the assertion of Theorem 2.2. Furthermore, noting that

$$A_{k_u}^{(\mu)} = \frac{\tilde{T}_0 u}{\tilde{T}_0 - u} \quad \text{and} \quad H_{k_u} = 1 - \frac{u}{\tilde{T}_0},$$

we can easily show (3.4) from the following lemma.

Lemma 3.1. *Let $\{R_t^{(\pm\mu)}, t \geqq 0\}$ be Bessel processes respectively with indices μ and $-\mu$ both starting from 1 and define T_0 by*

$$T_0 = \inf \left\{ u; R_u^{(-\mu)} = 0 \right\}.$$

Then the following identity in law holds:

$$\left\{ R_u^{(-\mu)}, u \leqq T_0 \right\} \stackrel{\text{(law)}}{=} \left\{ \left(1 - \frac{u}{\tilde{T}_0} \right) R^{(\mu)}_{\tilde{T}_0 u/(\tilde{T}_0 - u)}, u \leqq \tilde{T}_0 \right\}, \tag{3.5}$$

where, on the right hand side, $\tilde{T}_0$ is assumed to be independent of $R^{(\mu)}$.

Proof. From D.Williams' result ([27], [28], [29]) on the time reversal of Bessel processes, and conditioning with respect to a last passage time, the law of $\{R_u^{(-\mu)}, u \leqq T_0\}$, conditioned on $\{T_0 = t\}$, is that of the bridge of $\{R_u^{(\mu)}, 0 \leqq u \leqq t\}$, given $R_t^{(\mu)} = 0$. But, from Theorem 5.8, p.324 of [27], this bridge can be represented as

$$\left(1 - \frac{u}{t} \right) R^{(\mu)}_{tu/(t-u)}, \quad u < t,$$

which finishes the proof of (3.5). □

In order to connect even better our two proofs and the existing literature on Bessel bridges, let us recall that $\{R_u^{(\mu)}, u \leqq t\}$, conditioned on $R_t^{(\mu)} = 0$, is the solution of

$$r(u) = 1 + \beta_u + \left(\frac{1}{2} + \mu\right) \int_0^u \frac{ds}{r(s)} - \int_0^u \frac{r(s)}{t - s} ds, \qquad 0 \leqq u \leqq t, \tag{3.6}$$

(see the equation (12.8) in [31], p.36).

Thus the arguments just developed in the proof of Lemma 3.1 show that the decomposition of $\{R_u^{(-\mu)}, u \leqq T_0\}$ in its own filtration, originally enlarged with T_0, is

$$R_u^{(-\mu)} = 1 + \beta_u + \left(\frac{1}{2} + \mu\right) \int_0^u \frac{ds}{R_s^{(-\mu)}} - \int_0^u \frac{R_s^{(-\mu)}}{T_0 - s} ds, \tag{3.7}$$

where $\{\beta_u, u \geqq 0\}$ is a Brownian motion in this enlarged filtration. On the other hand, if we apply Itô's formula to $e_t^{(-\mu)} = \exp\bigl(B_t^{(-\mu)}\bigr)$, starting from the equation (3.1), we obtain

$$e_t^{(-\mu)} = 1 + \int_0^t e_s^{(-\mu)} d\tilde{B}_s + \left(\frac{1}{2} + \mu\right) \int_0^t e_s^{(-\mu)} ds - \int_0^t \frac{\bigl(e_s^{(-\mu)}\bigr)^3}{A_\infty^{(-\mu)} - A_s^{(-\mu)}} ds. \tag{3.8}$$

Now, Lamperti's relation, $e_t^{(-\mu)} = R^{(-\mu)}_{A_t^{(-\mu)}}$, yields $A_\infty^{(-\mu)} = T_0(R^{(-\mu)})$ and the equation (3.8) becomes, after time-changing it with the inverse of $\{A_t^{(-\mu)}, t \geqq 0\}$, precisely the equation (3.7).

4. Enlarging the 2D Brownian Filtration with a Subordinated Perpetuity

Let $\gamma^{(\nu)} = \{\gamma_t^{(\nu)}, t \geqq 0\}$ be another Brownian motion with drift ν which is independent of the original Brownian motion $B^{(\mu)}$. We consider the stochastic process $X^{(\mu,\nu)} = \{X_t^{(\mu,\nu)}, t \geqq 0\}$ defined by

$$X_t^{(\mu,\nu)} = \int_0^t \exp\left(B_s^{(\mu)}\right) d\gamma_s^{(\nu)}, \qquad t \geqq 0.$$

The purpose of this section is, assuming that $\mu > 0$ and enlarging the original filtration $\mathscr{F}_t \equiv \sigma\{B_s, \gamma_s; s \leqq t\}$ with $X_\infty^{(-\mu,\nu)} = \lim_{t\to\infty} X_t^{(-\mu,\nu)}$, to obtain a canonical decomposition of the pair $\{B_t, \gamma_t\}$ in this enlarged filtration, which we denote by $\hat{\mathscr{F}}_t, t \geqq 0$.

Our hope in developing this identity was to obtain an identity in law between some functionals of $\left(B^{(\mu)}, \gamma^{(\nu)}\right)$ and $\left(B^{(-\mu)}, \gamma^{(\nu)}\right)$, which might lead to further extensions of Pitman's theorem just as (1.3) led to [19]–[21]; this, together with our original interest in the enlargement of filtrations, motivated our derivation of Theorem 4.2 below. Unfortunately, we have neither discovered identities in law similar to (1.3) nor further extensions of Pitman's theorem involving γ and exponential functionals of $B^{(\pm\mu)}$, which does not mean that such extensions do not exist!

The above motivations being explained, we still find it of interest to develop the enlargement formulae with respect to $\{\hat{\mathscr{F}}_t\}$. First we determine the law of the so-called subordinated perpetuity $X_\infty^{(-\mu,\nu)}$: it has recently been shown (cf. [24], [22], [36], [3]) that it obeys the generic type IV Pearson distribution ([15], [25], [30]). Precisely, the following is known.

Theorem 4.1. ([24], [3], [36]) *For every $\mu > 0$ and $\nu \in \mathbf{R}$, the probability law of $X_\infty^{(-\mu,\nu)}$ admits the density*

$$f_{\mu,\nu}(x) = \frac{C_{\mu,\nu}}{(1+x^2)^{1/2+\mu}} \exp(2\nu \arctan(x)), \tag{4.1}$$

where the normalizing constant $C_{\mu,\nu}$ is given by

$$C_{\mu,\nu} = \frac{|\Gamma(1/2+\mu+\sqrt{-1}\nu)|^2}{\pi 2^{1-2\mu}\Gamma(2\mu)}.$$

REMARK 4.1. We note that the family $\{C_{\mu,\nu}\}_\mu$ enjoys the recurrence formula

$$C_{\mu-1,\nu} = \frac{(\mu+1/2)^2+\nu^2}{\mu(\mu+1/2)} C_{\mu,\nu}.$$

The following functions will play important roles in the sequel:

$$\varphi(z) = \frac{d}{dz}(\log f_{\mu,\nu}(z)) = \frac{2\nu-(2\mu+1)z}{1+z^2},$$
$$\psi(z) = z\varphi(z)+1 = -2\mu + \frac{2\nu z+1+2\mu}{1+z^2}.$$

The main result of this section is the following semimartingale decomposition of the pair $\{B_t, \gamma_t\}$ in $\{\hat{\mathscr{F}}_t\}$.

Theorem 4.2. *Let $\mu > 0$ and $\nu \in \mathbf{R}$. Then there exists a two-dimensional $(\hat{\mathscr{F}}_t)$-Brownian motion $\{(\hat{B}_t, \hat{\gamma}_t), t \geqq 0\}$ such that*

$$B_t = \hat{B}_t - \int_0^t \psi\left(Y_s^{(\mu,\nu)}\right) ds \qquad \textit{and} \qquad \gamma_t = \hat{\gamma}_t - \int_0^t \varphi\left(Y_s^{(\mu,\nu)}\right) ds, \tag{4.2}$$

where

$$Y_s^{(\mu,\nu)} = \left(X_\infty^{(-\mu,\nu)} - X_s^{(-\mu,\nu)}\right) \exp\left(-B_s^{(-\mu)}\right).$$

Proof. We first remark that the presentation of the initial enlargement formula given in Yor [31], Chapter 12, pp.33–34, is applicable with only one change, made necessary by the fact that our filtration $\{\mathscr{F}_t\}$ is generated by a two-dimensional Brownian motion instead of a one-dimensional one. We may summarize as follows.

We denote by $\phi_x(t) = \phi(t,x)$ the density of the conditional distribution of $X_\infty^{(-\mu,\nu)}$ given $\mathscr{F}_t$ and write

$$\phi_x(t) \; = \; \phi_x(0)\exp\left[\int_0^t (\rho_1(s,x)dB_s + \rho_2(s,x)d\gamma_s) - \frac{1}{2}\int_0^t \left(\rho_1(s,x)^2 + \rho_2(s,x)^2\right)ds\right].$$

Then, for a generic $(\mathscr{F}_t)$-martingale $M = \{M_t, t \geqq 0\}$ given by

$$M_t \; = \; \int_0^t (m_1(s)dB_s + m_2(s)d\gamma_s),$$

there exists a $\left(\hat{\mathscr{F}}_t\right)$-martingale $\hat{M} = \{\hat{M}_t, t \geqq 0\}$ such that

$$M_t \; = \; \hat{M}_t + \int_0^t \left\{m_1(s)\rho_1\left(s, X_\infty^{(-\mu,\nu)}\right) + m_2(s)\rho_2\left(s, X_\infty^{(-\mu,\nu)}\right)\right\} ds.$$

Thus, in order to prove Theorem 4.2, it only remains to find ρ_1 and ρ_2.

We first note

$$X_\infty^{(-\mu,\nu)} \; = \; X_t^{(-\mu,\nu)} + \int_t^\infty e_s^{(-\mu)} d\gamma_s^{(\nu)} \overset{\text{(law)}}{=} X_t^{(-\mu,\nu)} + e_t^{(-\mu)}\tilde{X}_\infty^{(-\mu,\nu)},$$

where $\tilde{X}_\infty^{(-\mu,\nu)}$ is a copy of $X_\infty^{(\mu,\nu)}$ independent of $\mathscr{F}_t$. Then, by virtue of Theorem 4.1, we get, for every Borel function $g : \mathbf{R} \to \mathbf{R}_+$,

$$\begin{aligned}
\int_{\mathbf{R}} g(x)\phi_x(t)dx &= \int_{\mathbf{R}} g(x)P\left(X_t^{(-\mu,\nu)} + e_t^{(-\mu)}\tilde{X}_\infty^{(-\mu,\nu)} \in dx\right) \\
&= \int_{\mathbf{R}} g\left(X_t^{(-\mu,\nu)} + e_t^{(-\mu)}y\right) f_{\mu,\nu}(y)\,dy \\
&= \int_{\mathbf{R}} g(x) f_{\mu,\nu}\left(\frac{x - X_t^{(-\mu,\nu)}}{e_t^{(-\mu)}}\right)\frac{1}{e_t^{(-\mu)}}dx.
\end{aligned}$$

Therefore, the process $\{\phi_x(t), t \geqq 0\}$ of conditional probability densities of $X_\infty^{(-\mu,\nu)}$ given $\{\mathscr{F}_t\}$ is found to be

$$\phi_x(t) \; = \; \frac{1}{e_t^{(-\mu)}} f_{\mu,\nu}\left(\frac{x - X_t^{(-\mu,\nu)}}{e_t^{(-\mu)}}\right).$$

Note, in particular, that this provides a family of $(\mathscr{F}_t)$-martingales.

Then, writing $\phi_x(t)$ in the exponential form by using Itô's formula, we obtain

$$\rho_1(s,x) = -\psi\left(\frac{x - X_s^{(-\mu,\nu)}}{e_s^{(-\mu)}}\right) \qquad \text{and} \qquad \rho_2(s,x) = -\varphi\left(\frac{x - X_s^{(-\mu,\nu)}}{e_s^{(-\mu)}}\right),$$

which completes the proof of Theorem 4.2. $\square$

Finally let us discuss the enlargement formula (4.2) by comparing it with the enlargement formula for the filtration of $\{B_t, t \geqq 0\}$ enlarged with

$$A_\infty^{(-\mu)} = \int_0^\infty \exp\left(2B_s^{(-\mu)}\right) ds,$$

which is presented in Section 3 and asserts that, setting $\mathscr{F}_t^* = \mathscr{F}_t \vee \sigma\{A_\infty^{(-\mu)}\}$, there exists a $(\mathscr{F}_t^*)$-Brownian motion $\{B_t^*, t \geqq 0\}$ such that

$$B_t^{(-\mu)} = B_t^{*(\mu)} - \int_0^t \frac{\exp\left(2B_s^{(-\mu)}\right)}{A_\infty^{(-\mu)} - A_s^{(-\mu)}} ds$$

or

$$B_t = B_t^* + 2\mu t - \int_0^t \frac{\exp\left(2B_s^{(-\mu)}\right)}{A_\infty^{(-\mu)} - A_s^{(-\mu)}} ds, \tag{4.3}$$

where $B_t^{*(\mu)} = B_t^* + \mu t$.

Next let us consider the case $\nu = 0$ in Theorem 4.2, so that the function ψ becomes

$$\psi(z) = -2\mu + \frac{1+2\mu}{1+z^2}.$$

Let us further remark that the formula (4.3) is also an enlargement formula in $\mathscr{G}_t^* = \mathscr{F}_t \vee \sigma\{A_\infty^{(-\mu)}\} \vee \sigma\{\hat{\gamma}_u, u \geqq 0\}$, where $\{\hat{\gamma}_u, u \geqq 0\}$ denotes the Brownian motion associated with the martingale given by

$$X_t^{(-\mu)} \equiv X_t^{(-\mu,0)} \equiv \int_0^t \exp\left(B_s^{(-\mu)}\right) d\gamma_s = \hat{\gamma}_{A_t^{(-\mu)}}.$$

Since $\hat{\mathscr{F}}_t = \mathscr{F}_t \vee \sigma\{X_\infty^{(-\mu)}\} \subset \mathscr{G}_t^*$ for any $t > 0$, in order that the formulae (4.2) and (4.3) be coherent, the following conditional expectation relation must hold:

$$E\left[\frac{\left(e_s^{(-\mu)}\right)^2}{A_\infty^{(-\mu)} - A_s^{(-\mu)}} \,\middle|\, \hat{\mathscr{F}}_s\right] = \frac{1+2\mu}{1+\left(\left(X_\infty^{(-\mu)} - X_s^{(-\mu)}\right)/e_s^{(-\mu)}\right)^2}$$

or, equivalently,

$$E\left[\frac{1}{A_\infty^{(-\mu)} - A_s^{(-\mu)}} \,\middle|\, \hat{\mathscr{F}}_s\right] = \frac{1+2\mu}{\left(e_s^{(-\mu)}\right)^2 + \left(X_\infty^{(-\mu)} - X_s^{(-\mu)}\right)^2}. \tag{4.4}$$

But, this relationship also follows from the known fact (see, e.g., Dufresne [7], Yor [35]): conditionally on $\mathscr{F}_s$, the two-dimensional random variable $\left(X_\infty^{(-\mu)} - X_s^{(-\mu)}, A_\infty^{(-\mu)} - A_s^{(-\mu)}\right)$ is distributed as $\left(e_s^{(-\mu)}\sqrt{H}N, \left(e_s^{(-\mu)}\right)^2 H\right)$, where, for a $Gamma(\mu)$ variable γ_μ, $H = 1/2\gamma_\mu$ and N is a standard normal variable independent of H.

Indeed, this fact being recalled, we may write (4.4) in the equivalent form

$$E\left[H^{-1} \mid N\sqrt{H} = x\right] = \frac{1+2\mu}{1+x^2}, \tag{4.5}$$

which is deduced from the following elementary lemma.

Lemma 4.1. *Let γ_α be a Gamma random variable with parameter α and N be a standard normal variable, independent of γ_α.*

(i) *For any Borel function $f : \mathbf{R} \to \mathbf{R}_+$, one has*

$$E\left[f\left(\frac{N}{\sqrt{2\gamma_\alpha}}\right)2\gamma_\alpha\right] = 2\alpha E\left[f\left(\frac{N}{\sqrt{2\gamma_{\alpha+1}}}\right)\right].$$

(ii) *The probability density of the* (*Student*) *variable $N/\sqrt{2\gamma_\alpha}$ is given by*

$$f_{\alpha,0}(x) = \frac{C_{\alpha,0}}{(1+x^2)^{\alpha+1/2}} = \frac{\Gamma(\alpha+1/2)}{\sqrt{\pi}\Gamma(\alpha)}\frac{1}{(1+x^2)^{\alpha+1/2}}.$$

(iii) *One has*

$$E\left[2\gamma_\alpha \,\middle|\, \frac{N}{\sqrt{2\gamma_\alpha}} = x\right] = 2\alpha\frac{f_{(\alpha+1),0}(x)}{f_{\alpha,0}(x)} = \frac{1+2\alpha}{1+x^2}.$$

Appendix : On a simple ordinary differential equation

In this appendix we show that, given continuous functions φ and f, the ordinary differential equation

$$z_t = \varphi(t) + \int_0^t \exp(\alpha z_s) f\left(\int_0^s \exp(\alpha z_u)du\right) ds \tag{4.6}$$

for $z = \{z_t\}_{t \geqq 0}$ has a unique solution; furthermore, it admits an explicit representation in terms of φ and f.

For this purpose, we consider the primitive F of f :

$$F(x) = \int_0^x f(y)dy.$$

Then we have

$$z_t = \varphi(t) + F\left(\int_0^t \exp(\alpha z_s)ds\right)$$

and

$$\exp\left[-\alpha F\left(\int_0^t \exp(\alpha z_s)ds\right)\right]\exp(\alpha z_t) = \exp(\alpha\varphi(t)).$$

Hence, setting

$$G(x) = \int_0^x \exp(-\alpha F(\xi))d\xi,$$

we obtain

$$G\left(\int_0^t \exp(\alpha z_s)ds\right) = \int_0^t \exp(\alpha\varphi(s))ds.$$

Therefore we can write the solution z of (4.6) as

$$z_t = \varphi(t) + (F\circ G^{-1})\left(\int_0^t \exp(\alpha\varphi(s))\,ds\right).$$

In our example, which appeared in the first proof of Theorem 2.2, we have $\alpha = 2$, $f(x) = (x - 1/2c)^{-1}$. Therefore we obtain

$$F(x) = \log(1-2cx), \qquad G(x) = \frac{1}{2c}\left(\frac{1}{1-2cx} - 1\right)$$

and, finally,

$$\left(F\circ G^{-1}\right)(x) = -\log(1+2cx).$$

Note added in Proof. D. Dufresne: The integral of geometric Brownian motion, Adv. Appl. Prob., **33** (2001), 223–241, presents an impressive survey of results on exponential Brownian functionals. In particular, his Theorem 3.1 and Corollary 3.3 are the fixed-time analogues of our Theorem 1.1.

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