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A PROBABILISTIC PROOF OF THE GAUSS-BONNET-CHERN THEOREM FOR MANIFOLDS WITH BOUNDARY

Dedicated to Professor Nobuyuki Ikeda on his 60-th birthday

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1. Introduction

A probabilistic method to solve a heat equation on differential forms on a closed compact Riemannian manifold is now well-known ([9], [12]). It is based on the notion of *stochastic moving frame* over the manifold. This stochastic moving frame can be realized on a Wiener space by solving a stochastic differential equation over the orthonormal frame bundle and the solution of the initial value problem for the heat equation is given by an integral (expectation) of a certain Wiener functional. By refining the Wiener functional expectation in the framework of the Malliavin calculus to include the notion of generalized Wiener functional expectations ([18], [19]), the heat kernel, i.e, the fundamental solution for the initial value problem, can be expressed as a generalized Wiener functional expectation. As a typical example of applications of this stochastic representation of heat kernels, we can give a simpler proof of the Patodi's cancellation [14] in the proof of the Gauss-Bonnet-Chern theorem ([10], [16], [20]).

In the case of a Riemannian manifold with boundary, Conner [3] investigated the initial value problems for differential forms under absolute or relative boundary conditions and Ray-Singer [15] constructed the fundamental solution by the parametrix method. Further Gilkey [5] proved the Gauss-Bonnet-Chern theorem for a manifold with boundary. We will prove it by a *probabilistic method*. To construct the fundamental solution, we adopt a probabilistic approach to the initial value problem due to Airault [1] and Ikeda-Watanabe [8], [9]. Combining this result with a modified Malliavin calculus, we can still express the fundamental solution for the initial value problem as a generalized Wiener functional expectation. We will then evaluate the short time asymptotic behavior of this generalized expectation probabilistically to compute directly the Gauss-Bonnet-Chern theorem in the case of manifolds with boundary.

In this paper, all the stochastic processes are defined on the time interval

$0 \leq t \leq 1$. This is because of the simplicity and also that it is sufficient for applications discussed here.

2. A modified Malliavin calculus

In applying the Malliavin calculus to solutions of stochastic differential equations (SDE's) with boundary conditions, a difficulty arises that these solutions are usually not smooth in the sense of Malliavin. For a typical example of one-dimensional reflecting Brownian motion $X(t)$ realized on the Wiener space with generic elements w as $X(t)=|w(t)|$ or $X(t)=\max_{0 \leq s \leq t} w(s)-w(t)$, the functional $F(w)=X(1)$ is no longer smooth in the sense of Malliavin. To overcome this difficulty, we regard one component of the Wiener process as fixed and apply the Malliavin calculus for remaining components of the Wiener process. This kind of ideas has been used already in, e.g., Bismut [2] and Ikeda-Kusuoka [7]. The main purpose of this section is to formulate such a modified Malliavin calculus. Before proceeding, however, we review very quickly some of the essential in the Malliavin calculus, c.f., e.g. [18], [19] for details.

Let (W, P) be the r-dimensional Wiener space: $W=W_0^r$ is the Banach space of all continuous paths w: $[0, 1] \rightarrow \boldsymbol{R}^r$ with $w(0)=0$ endowed with the supremum norm and P is the standard r-dimensional Wiener measure. Let $H(\subset W)$ be the Cameron-Martin Hilbert space formed of all $w \in W$ which are absolutely continuous with square-integrable derivatives and endowed with the norm $\|w\|_H= \{\int_0^1 \left|\frac{dw}{dt}\right|^2(t)\, dt\}^{1/2}$.

Let E be a real separable Hilbert space and $L_p(E)$, $1<p<\infty$, be the usual L_p-space of E-valued Wiener functionals over the Wiener space (W, P). We introduce a family $\boldsymbol{D}_p^s(E)$, $1<p<\infty$, $s \in \boldsymbol{R}$, of Sobolev spaces of E-valued Wiener functionals so that $\boldsymbol{D}_p^0(E)=L_p(E)$. Roughly, $\boldsymbol{D}_p^s(E)=(I-L)^{-s/2}(L_p(E))$ and the norm $\|\ \|_{p,s}$ on $\boldsymbol{D}_p^s(E)$ is defined by

$$\|F\|_{p,s} = \|(I-L)^{s/2} F\|_p$$

where L is the Ornstein-Uhlenbeck operator. Set $\boldsymbol{D}^\infty(E)=\bigcap_{s>0} \bigcap_{1<p<\infty} \boldsymbol{D}_p^s(E)$ and $\boldsymbol{D}^{-\infty}(E)=\bigcup_{s>0} \bigcup_{1<p<\infty} \boldsymbol{D}_p^{-s}(E)$. $\boldsymbol{D}^\infty(E)$ is the Fréchet space of (E-valued) test Wiener functionals and $\boldsymbol{D}^{-\infty}(E)$ is its topological dual. We denote these spaces simply by $\boldsymbol{D}_p^s$, $\boldsymbol{D}^\infty$, $\boldsymbol{D}^{-\infty}$ in the case $E=\boldsymbol{R}$. Similarly $L_p(\boldsymbol{R})$ is denoted by L_p. The H-derivative is extended to a closed differential operator D: $\boldsymbol{D}^{-\infty}(E) \rightarrow \boldsymbol{D}^{-\infty}(H \otimes E)$ and its dual is a differential operator D^*: $\boldsymbol{D}^{-\infty}(H \otimes E) \rightarrow \boldsymbol{D}^{-\infty}(E)$. These operators continuously send $\boldsymbol{D}_p^{s+1}(E)$ into $\boldsymbol{D}_p^s(H \otimes E)$ and $\boldsymbol{D}_p^{s+1}(H \otimes E)$ into $\boldsymbol{D}_p^s(E)$, respectively. It holds that $L=-D^*D$.

Let $F \in \boldsymbol{D}^\infty(\boldsymbol{R}^d)$, i.e. $F=(F^1, F^2, \cdots, F^d)$ with $F^i \in \boldsymbol{D}^\infty$. Set $\sigma_F^{ij}=\langle DF^i,$

$DF^j\rangle_H$ ($H\otimes\boldsymbol{R}$ is identified with H) and call $\sigma_F=(\sigma_F^{ij})$ the *Malliavin covariance* of the d-dimensional Wiener functional F. Suppose further that

$$(\det \sigma_F)^{-1}\in L_{\infty-}:=\bigcap_{1<p<\infty} L_p\,.$$

Then, for any Schwartz tempered distribution $T\in\mathcal{S}'(\boldsymbol{R}^d)$ on $\boldsymbol{R}^d$, *the composite* $T\circ F$ (or *the pull-back* $T\circ F$ *of* T *under the d-dimensional Wiener map* F: $W\to\boldsymbol{R}^d$) can be defined as an element in $\tilde{\boldsymbol{D}}^{-\infty}:=\bigcup_{s>0}\bigcap_{1<p<\infty}\boldsymbol{D}_p^{-s}$. Hence for every $G\in\tilde{\boldsymbol{D}}^{\infty}:=\bigcap_{s>0}\bigcup_{1<p<\infty}\boldsymbol{D}_p^{s}$, $G\cdot T\circ F\in\boldsymbol{D}^{-\infty}$ is well-defined and the *generalized expectation*

$$E[G\cdot T\circ F]=\langle G\cdot T\circ F,\mathbf{1}\rangle=\langle T\circ F, G\rangle$$

is well-defined. Here $\langle\ ,\ \rangle$ denotes the natural coupling between $\boldsymbol{D}_p^s$ and its dual $(\boldsymbol{D}_p^s)'=\boldsymbol{D}_q^{-s}$, $\dfrac{1}{p}+\dfrac{1}{q}=1$, and $\mathbf{1}\in\boldsymbol{D}^{\infty}$ is the Wiener functional identically equal to 1.

Now we modify this Malliavin calculus so that Wiener functionals may depend on a parameter $\alpha\in A$ in a finite measure space $(A,\mathcal{A},m)$. For simplicity, we assume that $(A,\mathcal{A},m)$ is a complete probability measure space. As before, let E be a real separable Hilbert space. For every $1<p<\infty$, $1<p'<\infty$ and $s\in\boldsymbol{R}$, let $L_{p'}(\boldsymbol{D}_p^s(E))$ be the $L_{p'}$-space with values in the Banach space $\boldsymbol{D}_p^s(E)$. Hence $L_{p'}(\boldsymbol{D}_p^s(E))$ is a Banach space with the norm $\|\ \|_{p,s\,;\,p'}$ whose element $F=\{F(w,\alpha)\}$ is a map (to be precise, an equivalence class of maps coinciding to each other m-almost everywhere) $\alpha\in A\to F(\cdot,\alpha)\in\boldsymbol{D}_p^s(E)$ which is $\mathcal{A}$-measurable and $\int_A\|F(\cdot,\alpha)\|_{p,s}^{p'}\,m(d\alpha)=:\|F\|_{p,s\,;\,p'}^{p'}<\infty$. Note that the dual of $L_{p'}(\boldsymbol{D}_p^s(E))$ is $L_{q'}(\boldsymbol{D}_q^{-s}(E))$ with $\dfrac{1}{p}+\dfrac{1}{q}=1$, $\dfrac{1}{p'}+\dfrac{1}{q'}=1$. If $F=\{F(w,\alpha)\}\in L_{p'}(\boldsymbol{D}_p^s(E))$ and $G=\{G(w,\alpha)\}\in L_{q'}(\boldsymbol{D}_q^{-s}(E))$, then for almost all $\alpha\in A$ (m), $F(\cdot,\alpha)\in\boldsymbol{D}_p^s(E)$ and $G(\cdot,\alpha)\in\boldsymbol{D}_q^{-s}(E)=(\boldsymbol{D}_p^s(E))'$, and hence the coupling $\langle F(\cdot,\alpha), G(\cdot,\alpha)\rangle$ is defined. The coupling of F and G is defined by

$$\int_A\langle F(\cdot,\alpha), G(\cdot,\alpha)\rangle\, m(d\alpha)$$

which is well-defined because

$$\begin{aligned}
&\int_A|\langle F(\cdot,\alpha), G(\cdot,\alpha)\rangle|\, m(d\alpha)\\
&\leq\int_A\|F(\cdot,\alpha)\|_{p,s}\|G(\cdot,\alpha)\|_{q,-s}\, m(d\alpha)\\
&\leq\Big(\int_A\|F(\cdot,\alpha)\|_{p,s}^{p'}\, m(d\alpha)\Big)^{1/p'}\Big(\int_A\|G(\cdot,\alpha)\|_{q,-s}^{q'}\, m(d\alpha)\Big)^{1/q'}\\
&=\|F\|_{p,s\,;\,q'}\,\|G\|_{q,-s\,;\,p'}\,.
\end{aligned}$$

Define

(2.1) $$L_{\infty-}(\boldsymbol{D}^{\infty}(E)) = \bigcap_{s>0} \bigcap_{1<p<\infty} \bigcap_{1<p'<\infty} L_{p'}(\boldsymbol{D}_p^s(E))$$

(2.2) $$L_{1+}(\tilde{\boldsymbol{D}}^{\infty}(E)) = \bigcap_{s>0} \bigcup_{1<p<\infty} \bigcup_{1<p'<\infty} L_{p'}(\boldsymbol{D}_p^s(E))$$

(2.3) $$L_{\infty-}(\tilde{\boldsymbol{D}}^{-\infty}(E)) = \bigcup_{s>0} \bigcap_{1<p<\infty} \bigcap_{1<p'<\infty} L_{p'}(\boldsymbol{D}_p^{-s}(E))$$

(2.4) $$L_{1+}(\boldsymbol{D}^{-\infty}(E)) = \bigcup_{s>0} \bigcup_{1<p<\infty} \bigcup_{1<p'<\infty} L_{p'}(\boldsymbol{D}_p^{-s}(E)) .$$

$L_{\infty-}(\boldsymbol{D}^{\infty}(E))$ is a Fréchet space and $L_{1+}(\boldsymbol{D}^{-\infty}(E))$ is its topological dual. Note also that there is a natural coupling between $L_{1+}(\tilde{\boldsymbol{D}}^{-\infty}(E))$ and $L_{\infty-}(\tilde{\boldsymbol{D}}^{-\infty}(E))$. It is easy to see that $L_{\infty-}(\boldsymbol{D}^{\infty})$ is an algebra; if $F=\{F(w, \alpha)\}$, $G=\{G(w, \alpha)\} \in L_{\infty-}(\boldsymbol{D}^{\infty})$, then

$$F\cdot G = \{F(w, \alpha)\cdot G(w, \alpha)\} \in L_{\infty-}(\boldsymbol{D}^{\infty}) .$$

Furthermore, for $F\in L_{\infty-}(\boldsymbol{D}^{\infty})$ and $\Phi\in L_{1+}(\boldsymbol{D}^{-\infty})$, $F\cdot\Phi\in L_{1+}(\boldsymbol{D}^{-\infty})$ is defined in usual way by using the natural coupling:

$$\langle G, F\cdot\Phi\rangle = \langle G\cdot F, \Phi\rangle, \quad {}^{\forall}G\in L_{\infty-}(\boldsymbol{D}^{\infty}) .$$

Similarly, for $F\in L_{1+}(\tilde{\boldsymbol{D}}^{\infty})$ and $\Phi\in L_{\infty-}(\tilde{\boldsymbol{D}}^{-\infty})$, $F\cdot\Phi\in L_{1+}(\boldsymbol{D}^{-\infty})$ is well-defined. The map $\boldsymbol{I}$: $\alpha\in A\rightarrow \boldsymbol{1}\in\boldsymbol{D}_{\infty}$ is an element in $L_{\infty-}(\boldsymbol{D}^{\infty})$ and, for every $\Phi=\{\Phi(w, \alpha)\}\in L_{1+}(\boldsymbol{D}^{-\infty})$, the coupling $\langle\boldsymbol{I}, \Phi\rangle$ coincides with

$$\int_A E(\Phi(\cdot, \alpha))\, m(d\alpha)$$

where, for almost every fixed α, $E(\Phi(\cdot, \alpha))$ is the generalized expectation of $\Phi(\cdot, \alpha)\in\boldsymbol{D}^{-\infty}$. Note that $E(\Phi(\cdot, \alpha))$, as a function of α, belongs to $L_{1+}=\bigcup_{1<p<\infty} L_p$.

Let $F=\{F(w, \alpha)\}\in L_{\infty-}(\boldsymbol{D}^{\infty}(\boldsymbol{R}^d))$, i.e. $F=(F^1, F^2, \cdots, F^d)$ with $F^i\in L_{\infty-}(\boldsymbol{D}^{\infty})$, $i=1, 2, \cdots, d$. We assume the following non-degeneracy conditions on the Malliavin covariance of F:

(2.5) For almost all $\alpha\in A\,(m)$, $(\det \sigma_{F(\cdot,\alpha)})^{-1}\in L_{\infty-}$.

(2.6) For every $p>1$, $\|(\det\sigma_{F(\cdot,\alpha)})^{-1}\|_p$, as a function of α, belongs to $L_{\infty-}(A; m) = \bigcap_{1<p<\infty} L_p(A; m)$.

Theorem 2.1. *For every Schwartz distribution $T\in\mathcal{S}'(\boldsymbol{R}^d)$, the composite $T\circ F(\cdot, \alpha)\in\tilde{\boldsymbol{D}}^{-\infty}$ is defined for almost all $\alpha\in A\,(m)$ and*

(2.7) $$T\circ F = \{T\circ F(\cdot, \alpha)\}\in L_{\infty-}(\tilde{\boldsymbol{D}}^{-\infty}) .$$

Proof is almost obvious from that for the corresponding theorem in the Malliavin calculus (cf. Theorem 1.12 of [18]). Hence, for every $G=\{G(w, \alpha)\}$

$\in L_{1+}(\tilde{\boldsymbol{D}}^{\infty})$, $G\cdot T\circ F\in L_{1+}(\boldsymbol{D}^{-\infty})$ is well-defined and hence

$$\int_A E[G(\cdot,\alpha)\cdot T\circ F(\cdot,\alpha)]\, m(d\alpha)$$

is well-defined.

Consider a family $F_\varepsilon=\{F_\varepsilon(w,\alpha)\}\in L_{\infty-}(\boldsymbol{D}^{\infty}(\boldsymbol{R}^d))$ depending on the parameter $\varepsilon\in[0,1]$. We suppose that $\varepsilon\in[0,1]\to F_\varepsilon\in L_{\infty-}(\boldsymbol{D}^{\infty}(\boldsymbol{R}^d))$ is C^1 as a Fréchet space valued function of ε. Suppose that F_ε satisfies the non-degeneracy conditions (2.5) and (2.6) for every $\varepsilon\in[0,1]$ and furthermore that conditions (2.5) and (2.6) are uniform in ε, i.e., for every $p>1$ and $p'>1$,

$$\sup_{\varepsilon\in[0,1]}\|\ \|(\det\sigma_{F_\varepsilon(\cdot,\alpha)})^{-1}\|_p\|_{L_{p'}(A\,;\,m)}<\infty\ . \tag{2.8}$$

Then, by Theorem 2.1, $T\circ F_\varepsilon\in L_{\infty-}(\tilde{\boldsymbol{D}}^{-\infty})$ is defined for $T\in\mathcal{S}'(\boldsymbol{R}^d)$ and $\varepsilon\in[0,1]$.

Theorem 2.2. *For every* $k=0,1,2,\cdots$,

$$T\circ F_\varepsilon=\sum_{\boldsymbol{n},|\boldsymbol{n}|\le k}\frac{1}{\boldsymbol{n}!}\partial^{\boldsymbol{n}}T\circ F_0\cdot(F_\varepsilon-F_0)^{\boldsymbol{n}}+R_\varepsilon^{(k)} \tag{2.9}$$

and $R_\varepsilon^{(k)}=\{R_\varepsilon^{(k)}(w,\alpha)\}\in L_{\infty-}(\tilde{\boldsymbol{D}}^{-\infty})$ *satisfies*

$$R_\varepsilon^{(k)}=O(\varepsilon^{k+1})\quad as\quad \varepsilon\downarrow 0\quad in\quad L_{\infty-}(\tilde{\boldsymbol{D}}^{-\infty}) \tag{2.10}$$

in the sense that $s>0$ *exists such that*

$$\|\ \|R_\varepsilon^{(k)}(\cdot,\alpha)\|_{p,-s}\|_{L_{p'}(A\,;\,m)}=O(\varepsilon^{k+1})\quad as\quad \varepsilon\downarrow 0$$

for every $p>1$ *and* $p'>1$.

In (2.9), we used the multi-index notation: $\boldsymbol{n}=(n_1,n_2,\cdots,n_d)\in\boldsymbol{Z}_+^d$, $\boldsymbol{n}!=n_1!n_2!\cdots n_d!$, $|\boldsymbol{n}|=n_1+n_2+\cdots+n_d$, $a^{\boldsymbol{n}}=a_1^{n_1}a_2^{n_2}\cdots a_d^{n_d}$ for $a=(a_1,a_2,\cdots,a_d)\in\boldsymbol{R}^d$ and $\partial^{\boldsymbol{n}}=\left(\frac{\partial}{\partial x^1}\right)^{n_1}\left(\frac{\partial}{\partial x^2}\right)^{n_2}\cdots\left(\frac{\partial}{\partial x^d}\right)^{n_d}$. Note also that $(F_\varepsilon-F_0)^{\boldsymbol{n}}\in L_{\infty-}(\boldsymbol{D}^{\infty})$ and $\partial^{\boldsymbol{n}}T\circ F_0\cdot(F_\varepsilon-F_0)^{\boldsymbol{n}}$ is in the sense of multiplication of elements in $L_{\infty-}(\boldsymbol{D}_{\infty})$ and $L_{1+}(\boldsymbol{D}^{-\infty})$ explained above. As for this multiplication, it holds generally that if $G\in L_{\infty-}(\boldsymbol{D}^{\infty})$ and $\Phi\in L_{\infty-}(\tilde{\boldsymbol{D}}^{-\infty})$, then $G\cdot\Phi\in L_{\infty-}(\tilde{\boldsymbol{D}}^{-\infty})$. Hence $\partial^{\boldsymbol{n}}T\circ F_0\cdot(F_\varepsilon-F_0)^{\boldsymbol{n}}\in L_{\infty-}(\tilde{\boldsymbol{D}}^{-\infty})$. Proof is easily provided by applying the following formula successively:

$$T\circ F_\varepsilon-T\cdot F_0=\sum_{i=1}^d\int_0^\varepsilon\frac{\partial T}{\partial x^i}\circ F_u\cdot\frac{dF_u^i}{du}\,du\ .$$

This formula can be obtained by approximating the Schwartz distribution by smooth functions.

As an application of the modified Malliavin calculus discussed so far, we give a stochastic representation of heat kernels on a domain with boundary. For simplicity, we consider the case of upper half space $D=\{x\in\boldsymbol{R}^m;\ x^m\geq 0\}$ of $\boldsymbol{R}^m$. Consider the following heat equation

$$\frac{\partial u}{\partial t}=\frac{1}{2}\left(\left(\frac{\partial}{\partial x^m}\right)^2 u+\sum_{i,j=1}^{m-1} a^{ij}(x)\frac{\partial^2 u}{\partial x^i\partial x^j}\right)+\sum_{i=1}^{m} b^i(x)\frac{\partial u}{\partial x^i} \tag{2.11}$$

with the Neumann boundary condition

$$\left.\frac{\partial u}{\partial x^m}\right|_{x^m=0}=0 \tag*{$(2.12)_N$}$$

and with the Dirichlet boundary condition

$$u|_{x^m=0}=0\,. \tag*{$(2.12)_D$}$$

We assume that $a^{ij}(x)$, $b^i(x)$ are C^∞-functions on D with bounded derivatives of all orders and $a(x)=(a^{ij}(x))$ is uniformly elliptic, i.e., a constant $c>0$ exists such that

$$\sum_{i,j=1}^{m-1} a^{ij}(x)\,\xi^i\,\xi^j\geq c|\xi|^2 \qquad \text{for all}\quad \xi=(\xi^1,\xi^2,\cdots,\xi^{m-1})\in\boldsymbol{R}^{m-1}\quad\text{and}\quad x\in D\,.$$

Let $\sigma(x)=(\sigma_k^i(x))$, $i,k=1,2,\cdots,m-1$ be the square root of $a(x)$ and (W,P) be the Wiener space with $r=m-1$. In the following, we write $x=(x^1,x^2,\cdots,x^m)\in\boldsymbol{R}^m$ as $x=(\bar{x},x^m)$ so that $\bar{x}=(x^1,x^2,\cdots,x^{m-1})\in\boldsymbol{R}^{m-1}$. Set $A=\{\varphi\in C([0,1]\to\boldsymbol{R});\ \varphi(0)=0\}$. Let $\varphi\in A$ be fixed and consider the following SDE on $\boldsymbol{R}^{m-1}$ for given $x=(\bar{x},x^m)\in\boldsymbol{R}^m$;

$$\begin{aligned} dX^i(t)&=\sum_{k=1}^{m-1}\sigma_k^i(\bar{X}(t),\,|x^m+\varphi(t)|)\,dw^k(t)+b^i(\bar{X}(t),\,|x^m+\varphi(t)|)\,dt \\ X^i(0)&=x^i\,,\quad i=1,2,\cdots,m-1 \end{aligned} \tag{2.13}$$

where $\bar{X}(t)=(X^1(t),X^2(t),\cdots,X^{m-1}(t))$. The solution of (2.13) is denoted by $\bar{X}(t,x,w;\varphi)$. Let $P^{(1)}$ be the one dimensional standard Wiener measure on A and set

$$\begin{aligned} M(t,x,w;\varphi)=\exp\Big\{&\int_0^t \hat{b}^m(\bar{X}(s,x,w;\varphi),\,x^m+\varphi(s))\,d\varphi(s) \\ &-\frac{1}{2}\int_0^t|\hat{b}^m(\bar{X}(s,x,w;\varphi),\,x^m+\varphi(s))|^2\,ds\Big\} \end{aligned} \tag{2.14}$$

where

$$\hat{b}^m(x)=b^m(\bar{x},\,|x^m|)\,\mathrm{sgn}(x^m)\,,\quad x=(\bar{x},x^m)\in\boldsymbol{R}^m\,.$$

It is well-known that the solutions $u^{\pm}(t,x)$ to the initial value problem (2.11) with $u(0,x)=f(x)$ and the boundary conditions $(2.12)_N$ and $(2.12)_D$, respectively, are given by

$$u^{\pm}(t,x)=E\times E^{(1)}\,[M(t,x,w;\varphi)\,f_{\pm}(\bar{X}(t,x,w;\varphi),\,x^m+\varphi(t))]\,,\quad t>0,\ x\in D \tag{2.15}$$

where

$$f_+(y)=f(\bar{y},\,|y^m|)$$

and

$$f_-(y)=f(\bar{y},\,|y^m|)\,\mathrm{sgn}(y^m)\,,\quad y=(\bar{y},y^m)\in \boldsymbol{R}^m\,.$$

$E\times E^{(1)}$ is the integral (expectation) with respect to the product measure $P(dw)\times P^{(1)}(d\varphi)$ on $W\times A$. Now we introduce a parameter $\varepsilon>0$ and consider the *SDE*

$$(2.13)_\varepsilon\qquad \begin{aligned} dX^i(t) &= \varepsilon\sum_{k=1}^{m-1}\sigma^i_k(\bar{X}(t),\,|x^m+\varepsilon\varphi(t)|)\,dw^k(t)\\ &\quad+\varepsilon^2\,b^i(\bar{X}(t),\,|x^m+\varepsilon\varphi(t)|)\,dt\\ X^i(0) &= x^i\,,\quad i=1,2,\cdots,m-1\,. \end{aligned}$$

Denote the solution by $\bar{X}^\varepsilon(t,x,w;\varphi)$ and define $M^\varepsilon(t,x,w;\varphi)$ by

$$(2.14)_\varepsilon\qquad \begin{aligned} M^\varepsilon(t,x,w;\varphi) &= \exp\{\varepsilon\int_0^t \hat{b}^m(\bar{X}^\varepsilon(s,x,w;\varphi),\,x^m+\varepsilon\varphi(s))\,d\varphi(s)\\ &\quad-\frac{\varepsilon^2}{2}\int_0^t|\hat{b}^m(\bar{X}^\varepsilon(s,x,w;\varphi),\,x^m+\varepsilon\varphi(s))|^2\,ds\}\,. \end{aligned}$$

Then noting the scaling property of the Wiener process, the solutions $u^{\pm}(t,x)$ in (2.15) can be also expressed as

$$(2.15)_\varepsilon\qquad u^{\pm}(\varepsilon^2,x)=E\times E^{(1)}\,[M^\varepsilon(1,x,w;\varphi)\,f_{\pm}(\bar{X}^\varepsilon(1,x,w;\varphi),\,x^m+\varepsilon\varphi(1))]\,,\quad \varepsilon>0,\ x\in D\,.$$

Let $p^+(t,x,y)$ and $p^-(t,x,y)$ be the fundamental solutions (with respect to the Lebesgue measure dy in D) for the heat equation (2.11) with boundary conditions $(2.12)_N$ and $(2.12)_D$, respectively. Let $m(d\varphi)=P^{1;0}_{0;0}(d\varphi):=P^{(1)}(d\varphi\,|\,\varphi(1)=0)$ be the pinned Winer measure (the Brownian bridge) on A. So it is the image measure of $P^{(1)}$ by the map $\varphi\to T\varphi$ on A defined by $(T\varphi)(t)=\varphi(t)-t\varphi(1)$. Then appealing to the modified Malliavin calculus discussed above depending on the parameter $\varphi\in A$, we can give the following stochastic representation for the heat kernels $p^{\pm}(t,x,y)$.

For this, we take generally $x,y\in\boldsymbol{R}^m$ and $\varepsilon>0$ and fix them. We write $x=(\bar{x},x^m)$, $y=(\bar{y},y^m)$ so that $\bar{x},\bar{y}\in\boldsymbol{R}^{m-1}$. Set

$$\hat{X}^\varepsilon(t)=\bar{X}^\varepsilon(t,x,w;T^{x^m,y^m}_\varepsilon(\varphi))$$

and

$$\hat{M}^\varepsilon(t) = M^\varepsilon(t, x, w; T_\varepsilon^{x^m, y^m}(\varphi))$$

where $T_\varepsilon^{a,b}(\varphi) \in A$, $a, b \in \boldsymbol{R}$ is defined by

$$T_\varepsilon^{a,b}(\varphi)(t) = \varphi(t) + \frac{b-a}{\varepsilon} \cdot t, \qquad 0 \le t \le 1, \varphi \in A.$$

In the definition of $\hat{M}^\varepsilon(t)$ by $(2.14)_\varepsilon$, a stochastic integral with respect to the process $\psi(t) = T_\varepsilon^{x^m, y^m}(\varphi)(t)$ is involved and the exact definition will be given in the proof of Lemma 2.2 below.

Lemma 2.1.

$$\hat{X}^\varepsilon(1) \in L_{\infty-}(\boldsymbol{D}^\infty(\boldsymbol{R}^{m-1}))$$

and it satisfies the non-degeneracy condition (2.5) *and* (2.6).

Proof. Fixing $\varphi \in A$, $\hat{X}^\varepsilon(1)$ is a solution of *SDE* $(2.13)_\varepsilon$ in which $\varphi \in A$ is replaced by $T_\varepsilon^{x^m, y^m}(\varphi)(t)$ whose coefficients are therefore time dependent through $T_\varepsilon^{x^m, y^m}(\varphi)(t)$. We can however apply the Malliavin calculus in such a case (cf. Taniguchi [17]) to conclude easily the assertion of the lemma. □

Lemma 2.2.

$$\hat{M}^\varepsilon(1) \in L_{\infty-}(\boldsymbol{D}^\infty)$$

Proof. First we make precise the definition of $\hat{M}^\varepsilon(t)$. $\psi(t) := T_\varepsilon^{x^m, y^m}(\varphi)(t)$ is a stochastic process on the probalility space (A, m) and it is a semimartingale with respect to the natural filtration $\mathcal{A}_t$: Indeed

$$\begin{aligned}\psi(t) &= \varphi(t) + \frac{x^m - y^m}{\varepsilon} \cdot t \\ &= B(t) - \int_0^t \frac{\varphi(s)}{1-s}\, ds + \frac{x^m - y^m}{\varepsilon} \cdot t\end{aligned}$$

where $B(t)$ is an $(\mathcal{A}_t)$-Brownian motion. The stochastic integral

$$\int_0^t \hat{b}^m(\hat{X}^\varepsilon(s), x^m + \varepsilon\psi(s))\, d\psi(s)$$

is, *by definition,* equal to

$$\begin{aligned}&\int_0^t \hat{b}^m(\hat{X}^\varepsilon(s), x^m + \varepsilon\psi(s))\, dB(s) - \int_0^t \hat{b}^m(\hat{X}^\varepsilon(s), x^m + \varepsilon\psi(s)) \frac{\varphi(s)}{1-s}\, ds \\ &\quad + \frac{y^m - x^m}{\varepsilon} \int_0^t \hat{b}^m(\hat{X}^\varepsilon(s), x^m + \varepsilon\psi(s))\, ds.\end{aligned}$$

So $\hat{M}^\varepsilon(t)$ is defined to be

$$\hat{M}^\varepsilon(t) = \exp\{\varepsilon \int_0^t \hat{b}^m(\hat{X}^\varepsilon(s), x^m + \varepsilon\psi(s))\, dB(s)$$

$$-\varepsilon\int_0^t \hat{b}^m(\hat{X}^\varepsilon(s), x^m+\varepsilon\psi(s))\frac{\varphi(s)}{1-s}\,ds$$
$$+(y^m-x^m)\int_0^t \hat{b}^m(\hat{X}^\varepsilon(s), x^m+\varepsilon\psi(s))\,ds\,.$$
$$-\frac{\varepsilon^2}{2}\int_0^t |\hat{b}^m(\hat{X}^\varepsilon(s), x^m+\varepsilon\psi(s))|^2\,ds\}\,.$$

Next, we show that

$$\hat{M}^\varepsilon(1)\in L_{\infty-}(W\times A, P\times m)\,.$$

It is sufficient to show that

$$\exp\{\int_0^1 \frac{|\varphi(s)|}{1-s}\,ds\}\in L_{\infty-}(A, m)\,.$$

But $m=P^{1;0}_{0;0}$ is the Gaussian measure on A and, by Fernique's famous theorem (cf. [11]), it is easy to see that

$$\exp\{\int_0^1\left|\frac{\varphi(s)}{1-s}\right|^\gamma ds\}\in L_{\infty-}(A, m)$$

if $0<\gamma<2$.

Now in order to prove $\hat{M}^\varepsilon(1)\in L_{\infty-}(\boldsymbol{D}^\infty)$, it is sufficient to show that $\int_0^1 \hat{b}^m(\hat{X}^\varepsilon(s), x^m+\varepsilon\psi(s))\,d\psi(s)\in L_{\infty-}(\boldsymbol{D}^\infty)$. It is clear that $\hat{b}^m(\hat{X}^\varepsilon(s), x^m+\varepsilon\psi(s))\in L_{\infty-}(\boldsymbol{D}^\infty)$ and $\sup_{s\in[0,1]}\|\hat{b}^m(\hat{X}^\varepsilon(s), x^m+\varepsilon\psi(s))\|_{p,k\,;\,p'}<\infty$ for every $p>1, p'>1$ and $k>0$. Then the assertion is a consequence of the following:

Lemma 2.3. *Let $B(t, \varphi)$ be an $(\mathcal{A}_t)$-Brownian motion on (A, m) and $\Xi(t; w, \varphi)$ be a function jointly measurable on $[0, 1]\times W\times A$ such that, for every $t\in[0, 1]$, $\Xi(t; w, \varphi)\in L_{\infty-}(\boldsymbol{D}^\infty)$,*

$$\sup_{t\in[0,1]}\|\Xi(t; w, \varphi)\|_{p,k\,;\,p'}<\infty \qquad \text{for every} \quad p>1, p'>1 \quad \text{and} \quad k>0$$

and furthermore, for each t, the map $\varphi\to\Xi(t; \cdot, \varphi)\in\boldsymbol{D}^\infty$ is $\mathcal{A}_t$-measurable. Then the stochastic integral

$$F(w, \varphi)=\int_0^1 \Xi(t; w, \varphi)\,dB(t, \varphi)$$

is well-defined and $F\in L_{\infty-}(\boldsymbol{D}^\infty)$.

Proof. We give a main point of the proof since others are routine. For each $k=1, 2, \cdots$,

$$D^kF(w, \varphi)=\int_0^1 D^k\,\Xi(t; w, \varphi)\,dB(t, \varphi)\in H\otimes\overset{k}{\cdots}\otimes H\,.$$

By Burkholder's inequality for stochastic integrals with values in a Hilbert space $H\otimes\overset{k}{\cdots}\otimes H$, we have for $p\geq 2$ and almost all $w\in W(P)$,

$$\int_A |D^k F(w,\varphi)|_{HS}^p\, m(d\varphi)\leq C_p\int_A \{\int_0^1 |D^k\,\Xi(t;w,\varphi)|_{HS}^2\,dt\}^{p/2}\, m(d\varphi)$$

where C_p is a positive constant. Hence

$$\begin{aligned}||F||_{p,k;p}^p &= \int_A\int_W |D^k F(w,\varphi)|_{HS}^p\, P(dw)\, m(d\varphi)\\ &\leq C_p\int_A\int_W \{\int_0^1 |D^k\,\Xi(t;w,\varphi)|_{HS}^2\,dt\}^{p/2}\, P(dw)\, m(d\varphi)\\ &\leq C_p\int_A\int_W\int_0^1 |D^k\,\Xi(t;w,\varphi)|_{HS}^p\,dt\, P(dw)\, m(d\varphi)\\ &\leq C_p'\int_0^1 ||\Xi(t;w,\varphi)||_{p,k;p}^p\,dt<\infty\,.\end{aligned}$$

This proves $F\in L_{\infty-}(\boldsymbol{D}^\infty)$.

Now we can apply Theorem 2.1 to define the following kernel $\Pi(x,y;\varepsilon)$ for fixed $x\in D$, $y\in\boldsymbol{R}^m$ and $\varepsilon>0$:

$$\begin{aligned}\Pi(x,y;\varepsilon) &= \int_A E[\hat{M}^\varepsilon(1)\cdot\delta_{\bar{y}}(\hat{X}^\varepsilon(1))]\, m(d\varphi)\\ &\Big(=\int_A E[M^\varepsilon(t,x,w;T_\varepsilon^{x^m,y^m}(\varphi))\\ &\qquad\times\delta_{\bar{y}}(\bar{X}^\varepsilon(t,x,w;T_\varepsilon^{x^m,y^m}(\varphi)))]\,P_{0;0}^{1;0}(d\varphi)\Big)\,.\end{aligned}\tag{2.16}$$

From $(2.15)_\varepsilon$, we deduce the following:

Theorem 2.3. *For given $x=(\bar{x},x^m)$, $y=(\bar{y},y^m)\in D$ and $\varepsilon>0$,*

$$\begin{aligned}p^\pm(\varepsilon^2,x,y) &= \frac{1}{\sqrt{2\pi}\,\varepsilon}\exp\left\{-\frac{(x^m-y^m)^2}{2\varepsilon^2}\right\}\Pi(x,y;\varepsilon)\\ &\pm\frac{1}{\sqrt{2\pi}\,\varepsilon}\exp\left\{-\frac{(x^m+y^m)^2}{2\varepsilon^2}\right\}\Pi(x,\check{y};\varepsilon)\end{aligned}\tag{2.17}$$

where $\check{y}=(\bar{y},-y^m)\in\boldsymbol{R}^m$.

As an application of the stochastic representations (2.17), we can reproduce the result of McKean-Singer in §5 of [33] by evaluating probabilistically the right-hand side of (2.17), cf. [20] for details.

3. Heat kernels on differential forms with absolute boundary conditions

Let M be a compact, oriented, smooth Riemannian manifold of dimension m with boundary. We denote by $\Lambda(M)=\sum_{p=0}^m\oplus\Lambda_p(M)$ the space of smooth differential forms. Thus $\Lambda(M)$ is the vector space formed of all C^∞-sections ω: $M\to\Lambda T^*M$ where ΛT^*M is the exterior product bundle of the cotangent bundle T^*M. For $\omega\in\Lambda(M)$, we denote by ω_{norm} the normal component of ω.

DEFINITION 3.1. A differential form $\omega\in\Lambda(M)$ is said to satisfy the *absolute boundary conditions* if

$$\omega_{\mathrm{norm}}=0 \quad \text{and} \quad (d\omega)_{\mathrm{norm}}=0 \quad \text{on } \partial M$$

where d is the exterior derivative.

Consider the following initial and boundary value problem on $\Lambda(M)$:

$$\text{(3.1)} \qquad \begin{cases} \dfrac{\partial u}{\partial t}=\dfrac{1}{2}\Box u \\ \lim\limits_{t\downarrow 0} u(t,\cdot)=f \\ u \text{ satisfies the absolute boundary conditions,} \end{cases}$$

where $\Box=-(d+d^*)(d+d^*)$ is the Laplacian of Hodge-de Rham-Kodaira. The solution $u(t, x)$ of (3.1) is given in the form

$$\text{(3.2)} \qquad u(t,x)=\int_M e(t,x,y)f(y)\,m(dy)\,.$$

Here $e(t,x,y)\in\mathrm{Hom}(\Lambda T_y^*(M), \Lambda T_x^*(M))$ is the fundamental solution for (3.1) and $m(dy)$ is the Riemannian volume. Define

$$\text{(3.3)} \qquad \mathrm{Str}[e(t,x,x)]=\mathrm{tr}[(-1)^F e(t,x,x)]$$

where $(-1)^F\in\mathrm{End}(\Lambda T_x^*(M))$ is determined by

$$(-1)^F\omega=(-1)^p\omega \quad \text{if} \quad \omega\in\Lambda^p T_x^*(M)\,, \quad p=0,1,2,\cdots,m\,.$$

By the eigenfunction expansion combined with the de Rham theorem, we deduce

$$\text{(3.4)} \qquad \int_M \mathrm{Str}[e(t,x,x)]\,m(dx)=\chi(M)$$

$\chi(M)$ being the Euler-Poincaré characteristic of M. We would evaluate the left-hand side of (3.4) to obtain an integral formula of $\chi(M)$.

If $x\in\mathring{M}=M\backslash\partial M$, it was established by Patodi [14] (c.f., [10], [20] for a probabilistic proof) that

$$\text{(3.5)} \qquad \mathrm{Str}[e(t,x,x)]=C(x)+o(1) \quad \text{as} \quad t\downarrow 0\,,$$

$C(x)$ being an explicit polynomial in components of the Riemann curvature tensor known as the *Chern polynomial*, or m-form $C(x)\sqrt{g(x)}\,dx^1\wedge\cdots\wedge dx^m:=e(TM)$ known as the *Euler form*. Hence in order to obtain the asymptotic as $t\downarrow 0$ of the left-hand side of (3.4), it is sufficient to obtain the asymptotic as $t\downarrow 0$ of the integral

(3.6) $$\int_U \mathrm{Str}\,[e(t, x, x)]\, m(dx)$$

where U is any small neighborhood of boundary point $x_0 \in \partial M$. Choose V a coordinate neighborhood of $x_0 \in \partial M$ such that

$$V \cap \overset{\circ}{M} = \{x = (x^1, \cdots, x^m) \in V;\ x^m > 0\}$$

and

$$V \cap \partial M = \{x = (x^1, \cdots, x^m) \in V;\ x^m = 0\}\ .$$

We regard V as a part of $\boldsymbol{R}^m_+ = \{x = (x^1, \cdots, x^m) \in \boldsymbol{R}^m;\ x^m \geq 0\}$ and extend the components $g_{ij}(x)$ of the Riemann metric tensor in V to whole $\boldsymbol{R}^m_+$ so that $g_{ij}(x) = \delta_{ij}$ outside a bounded set in $\boldsymbol{R}^m_+$. If $\tilde{e}(t, x, y)$ is the heat kernel for (3.1) corresponding to this extended Riemannian metric on $\boldsymbol{R}^m_+$, then for any $U \subset\subset V$, it can be deduced by a standard argument ([13]) that

$$\int_U \|e(t, x, x) - \tilde{e}(t, x, x)\| dx = o(e^{-c/t}) \quad \text{as} \quad t \downarrow 0$$

where c is a positive constant and $\|\ \ \|$ is the operator norm in $\mathrm{End}(\Lambda T^*_x(M))$. Hence in order to evaluate (3.6), *we may assume from the beginning that $M = \boldsymbol{R}^m_+$, the components $g_{ij}(x)$ in the global Euclidean coordinate satisfy $g_{ij}(x) = \delta_{ij}$ outside a bounded set.* Furthermore, by choosing a semi-geodesic coordinate in V above, we may without loss of generality assume that

(3.7) $$g_{mm}(x) \equiv 1 \quad \text{and} \quad g_{im}(x) \equiv 0\,, \quad i = 1, \cdots, m-1\ .$$

$\omega \in \Lambda_p(M)$ is given, in the global Euclidean coordinate, as

$$\omega = \sum_{1 \leq i_1 < \cdots < i_p \leq m} \omega_{i_1 i_2 \cdots i_p}(x)\, dx^{i_1} \wedge \cdots \wedge dx^{i_p}\ .$$

Then the tangent component $\omega_{\tan}$ and the normal component ω_{norm} of ω are given by

$$\omega_{\tan} = \sum_{1 \leq i_1 < \cdots < i_p \leq m-1} \omega_{i_1 i_2 \cdots i_p}(x)\, dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

and

$$\omega_{\mathrm{norm}} = \sum_{1 \leq i_1 < \cdots < i_{p-1} \leq m-1} \omega_{i_1 i_2 \cdots i_{p-1} m}(x)\, dx^{i_1} \wedge \cdots \wedge dx^{i_{p-1}} \wedge dx^m\ .$$

Hence ω satisfies the absolute boundary conditions if and only if

(3.8) $$\omega_{i_1 \cdots i_{p-1} m}(x) = 0 \quad \text{on } \partial M \qquad \text{for all} \quad 1 \leq i_1 < \cdots < i_{p-1} \leq m-1$$

and

(3.9) $$\frac{\partial \omega_{i_1 \cdots i_p}}{\partial x^m}(x) = 0 \quad \text{on } \partial M \qquad \text{for all} \quad 1 \leq i_1 < \cdots < i_p \leq m-1\ .$$

We will now represent the heat kernel by a generalized Wiener functional expectation as discussed in the prevoius section and use this representation to compute the asymptotic of (3.6).

Before proceeding, however, we must recall how the initial and boundary value problem (3.1) can be solved probabilistically by a Wiener functional expectation, c.f., [1], [8], [9]. First of all, we review very rapidly how this is done in the case of manifolds without boundary. A basic tool is the *stochastic moving frame* $r(t)=(X(t), e(t))$ *over* M: $X(t)$ is a Brownian motion on M (i.e., the diffusion generated by the $\frac{1}{2}\Delta$, Δ being the Laplace-Beltrami operator on M) and $e(t)$ is the stochastic parallel translation of a frame $e=e(0)$ at x along the Brownian curve $X(t)$. Here, by a frame e at $x\in M$, we mean an orthonormal basis (ONB) in the tangent space $T_x(M)$. This $r(t)$ can be constructed by solving a SDE: For simplicity, we assume $M=\boldsymbol{R}^m$ and the component of the Riemann metric tensor $g_{ij}(x)$ with respect to the global Euclidean coordinate satisfies $g_{ij}(x)=\delta_{ij}$ outside a bounded set in $\boldsymbol{R}^m$. For $x\in\boldsymbol{R}^m$ and an ONB $e=[e_1, \cdots, e_m]$ in $T_x(M)$ given in the globalE uclidean coordinate as $x=(x^1, \cdots, x^m)$ and $e_\alpha=e_\alpha^i\frac{\partial}{\partial x^i}$, $\alpha=1, 2, \cdots, m$, (*we always omit the summation sign for repeated indices*), consider the following SDE for $X(t)=(X^i(t))$ and $e(t)=(e_j^i(t))$ on the Wiener space (W, P) with $r=m$:

$$
(3.10)\qquad
\begin{cases}
dX^i(t) = \sigma_k^i(X(t))\, dw^k(t)-\dfrac{1}{2}\, g^{jk}(X(t))\, \Gamma_{jk}^i(X(t))\, dt \\
de_j^i(t) \;=\; -\Gamma_{lk}^i(X(t))\, e_j^k(t)\circ dX^l(t) \\
X^i(0) \;= x^i \\
e_j^i(0) \;\;= e_j^i\,.
\end{cases}
$$

Here $(\sigma_j^i(x))$ is the square root of $(g^{ij}(x))=(g_{ij}(x))^{-1}$, $\Gamma_{jk}^i(x)$ are the Christoffel symbols in the global Euclidean coordinate and $\circ$ denotes the Stratonovich differential of semimartingales. The unique solution of (3.10) is denoted by

$$r(t) = (X(t), e(t))$$

or, to clarify the dependence on $r:=(x, e)$ and $w\in W$, by

$$r(t, r, w) = (X(t, r, w),\ e(t, r, w))\,.$$

This is a realization of the stochastic moving frame starting at a frame e at x. Note that all pairs $r=(x, e)$ of $x\in M$ and a frame at x constitute a manifold $O(M)$ called the *orthonormal frame bundle over* M which is a principal fibre bundle over M with the structure group $O(m)$.

We consider the initial value problem of the heat equation on differential forms on $M(=\boldsymbol{R}^m)$

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2} \square u \\ u|_{t=0} = f. \end{cases} \tag{3.11}$$

To represent the solution $u(t, \cdot)$ by an expectation of a Wiener functional defined in terms of the stochastic moving frame, we need several notions and notations concernig exterior algebra $\Lambda \boldsymbol{R}^m$ over $\boldsymbol{R}^m$.

Let $\delta^1, \delta^2, \cdots, \delta^m$ be the canonical ONB of $\boldsymbol{R}^m$, i.e., $\delta^i=(0, \cdots, \overset{i-th}{1}, 0, \cdots. 0)$. The exterior algebra or the Grassmann algebra $\Lambda \boldsymbol{R}^m = \sum_{p=0}^m \oplus \Lambda^p \boldsymbol{R}^m$ over $\boldsymbol{R}^m$ is, as usual, the 2^m-dimensional Euclidean space with the canonical ONB $\delta^{i_1} \wedge \cdots \wedge \delta^{i_p}$, $1 \leq i_1 < \cdots < i_p \leq m$, $p=0, 1, \cdots, m$. For $\omega \in \Lambda^p \boldsymbol{R}^m$ and $\lambda \in \Lambda^q \boldsymbol{R}^m$, the exterior product $\omega \wedge \lambda \in \Lambda^{p+q} \boldsymbol{R}^m$ is defined as usual and satisfies $\omega \wedge \lambda = (-1)^{p+q} \lambda \wedge \omega$. Let $\mathrm{End}(\Lambda \boldsymbol{R}^m)$ be the algebra formed of all linear transformations on $\Lambda \boldsymbol{R}^m$. For each $i=1, \cdots, m$, define $a_i^* \in \mathrm{End}(\Lambda \boldsymbol{R}^m)$ by

$$a_i^*(\omega) = \delta^i \wedge \omega\,, \quad \omega \in \Lambda \boldsymbol{R}^m \tag{3.12}$$

and $a_i \in \mathrm{End}(\Lambda \boldsymbol{R}^m)$ by the dual of a_i^*. For $\alpha=(\alpha_{ij}) \in \boldsymbol{R}^m \otimes \boldsymbol{R}^m$, define $D_1(\alpha) \in \mathrm{End}(\Lambda \boldsymbol{R}^m)$ by

$$D_1(\alpha) = \alpha_{ij}\, a_j^*\, a_i \tag{3.13}$$

and, for $\beta=(\beta_{ijkl}) \in \boldsymbol{R}^m \otimes \boldsymbol{R}^m \otimes \boldsymbol{R}^m \otimes \boldsymbol{R}^m$, define $D_2(\beta) \in \mathrm{End}(\Lambda \boldsymbol{R}^m)$ by

$$D_2(\beta) = \beta_{ijkl}\, a_l^*\, a_k\, a_j^*\, a_i\,. \tag{3.14}$$

Given $r=(x, \boldsymbol{e}) \in O(M)$, there is a *canonical isomorphism*

$$\tilde{r}\colon \Lambda^p \boldsymbol{R}^m \to \Lambda^p T_x^*(M)\,, \quad p = 0, \cdots, m$$

and hence an isomorphism

$$\tilde{r}\colon \Lambda \boldsymbol{R}^m \to \Lambda T_x^*(M)$$

defined by

$$\tilde{r}(\delta^{i_1} \wedge \cdots \wedge \delta^{i_p}) = f^{i_1} \wedge \cdots \wedge f^{i_p}$$

where $[f^1, \cdots, f^m]$ is the ONB in $T_x^*(M)$ which is dual to the ONB $\boldsymbol{e}=[e_i, \cdots, e_m]$ in $T_x(M)$. If $(R_{ijkl}(x))$ are components of the Riemann curvature tensor, its scalarization or equivariant representation $J(r)=(J_{\alpha\beta\gamma\delta}(r)) \in \boldsymbol{R}^m \otimes \boldsymbol{R}^m \otimes \boldsymbol{R}^m \otimes \boldsymbol{R}^m$ is defined by

$$\begin{aligned} J_{\alpha\beta\gamma\delta}(r) &= R_{ijkl}(x)\, e_\alpha^i\, e_\beta^j\, e_\gamma^k\, e_\delta^l \\ &\qquad r = (x, \boldsymbol{e} = [e_1, \cdots, e_m])\,, \quad e_\alpha = e_\alpha^i \frac{\partial}{\partial x^i}\,. \end{aligned}$$

Given $r=(x, \boldsymbol{e})$, we construct the stochastic moving frame $r(t, r, w)=(X(t, r, w),$

$e(t, r, w)$) as above and define $M(t, r, w) \in \mathrm{End}(\Lambda \boldsymbol{R}^m)$ by the solution of

$$\text{(3.15)} \qquad \begin{cases} \dfrac{dM(t)}{dt} = M(t)\, D_2\Big(\dfrac{1}{2} J(r(t, r, w))\Big) \\ M(0) = I \text{ (the identity)}. \end{cases}$$

For a given $f \in \Lambda(M)$ $(= C^\infty(\Lambda T^* M))$ whose components (in the global Euclidean coordinate) are all tempered C^∞-functions on M (i.e., C^∞-functions whose derivatives of all orders are of polynomial growth order), define $u(t, r) \in \Lambda T_x^*(M)$ by the following expectation on (W, P):

$$\text{(3.16)} \qquad u(t, r) = \tilde{r} E[M(t, r, w)\, \tilde{r}(t, r, w)^{-1} f(X(t, r, w))]\,.$$

(Remember that $\tilde{r}(t, r, w)$ is the canonical isomorphism $\Lambda \boldsymbol{R}^m \to \Lambda T^*_{X(t,r,w)}(M)$.) We can deduce that $u(t, r)$ depends on x only and is independent of a choice of $\boldsymbol{e}$. This is a consequence of the fact that

$$r(t, r, w)\, g = r(t, rg, g^{-1} w)\,, \quad g \in O(m)$$

where $r \cdot g$ is the action of $O(m)$ on the principal fibre bundle $O(M)$ and the map $w \to g^{-1} w$ on W preserves the measure P. Hence we may write $u(t, r)$ as $u(t, x)$.

Theorem 3.1. *$u(t, x)$ is the unique tempered solution of the initial value problem* (3.11).

For details, c.f., [9]. Since $\boldsymbol{e}$ is irrelevant in the definition of $u(t, x)$, we may always assume that $r = (x, \boldsymbol{e} = [e_1, \cdots, e_m])$ with $e_\alpha = \sigma^i_\alpha(x)\, \partial/\partial x^i$, $\alpha = 1, 2, \cdots, m$. In this case, we denote r by $r(x)$ and also $r(t, r, x) = (X(t, r, w), \boldsymbol{e}(t, r, w))$ and $M(t, r, w)$ by $r(t, x, w) = (X(t, x, w), \boldsymbol{e}(t, x, w))$ and $M(t, x, w)$. Since $M = \boldsymbol{R}^m$, we can identify $T_x^*(M)$ and $\boldsymbol{R}^m$ by identifying $(dx^i)_x$ and δ^i (this identification is, of course, an identification as vector spaces, not as inner product spaces). Under this identification, $f(x) \in \Lambda T_x^*(M) = \Lambda \boldsymbol{R}^m$. And $\tilde{r}(t, x, w)^{-1} \in \mathrm{End}(\Lambda \boldsymbol{R}^m)$ is given by

$$\text{(3.17)} \qquad \tilde{r}(t, x, w)^{-1}\colon \delta^{i_1} \wedge \cdots \wedge \delta^{i_p} \to e^{i_1}(t) \wedge \cdots \wedge e^{i_p}(t)$$

where $e^i(t) = (e^i_1(t), e^i_2(t), \cdots, e^i_m(t)) \in \boldsymbol{R}^m$, $i = 1, 2, \cdots, m$. Set also

$$\text{(3.18)} \qquad \Pi(t)\ (= \Pi(t, x, w)) = \tilde{r}(x)\, \tilde{r}(t, x, w)^{-1}$$

and

$$\text{(3.19)} \qquad \tilde{M}(t)\ (= \tilde{M}(t, x, w)) = \tilde{r}(x)\, M(t, x, w)\, \tilde{r}(x)^{-1}\,.$$

Now $u(t, x) \in \Lambda \boldsymbol{R}^m$ is also expressed in the form

$$\text{(3.20)} \qquad u(t, x) = E[\tilde{M}(t)\, \Pi(t) f(X(t, x, w))]\,.$$

Note that, from (3.15) that $M(t)$ is the unique solution of

$$
(3.21) \qquad \begin{cases} \dfrac{dM(t)}{dt} = M(t)\,\tilde{r}(x)\,D_2\left(\dfrac{1}{2}J(r(t,x,w))\right)\tilde{r}(x)^{-1} \\ \mathrm{M}(0) = I\,. \end{cases}
$$

It is easy to see from (3.17) and (3.10) that $\Pi(t)\in\mathrm{End}(\Lambda\boldsymbol{R}^m)$ is the unique solution of the following integral equation:

$$
(3.22) \qquad \Pi(t) = I+\int_0^t \Pi(s)\circ d\Theta(s)
$$

where $\Theta(t)$ is an $\mathrm{End}(\Lambda\boldsymbol{R}^m)$-valued semimartingale defined by

$$
(3.23) \qquad \Theta(t) = D_1(\theta(t)) = \theta_{ij}(t)\,a_j^*\,a_i
$$

and $\theta\,(t)$ is an $\boldsymbol{R}^m\otimes\boldsymbol{R}^m$-valued semimartingale given by

$$
(3.24) \qquad \theta_{ij}(t) = -\int_0^t \Gamma^i_{lj}(X(s,x,w))\circ dX^l(s,x,w)\,.
$$

If we set $K(t)=\tilde{M}(t)\,\Pi(t)\in\mathrm{End}(\Lambda\boldsymbol{R}^m)$, it is easily seen from (3.21) and (3.22) that $K(t)$ is the unique solution of

$$
(3.25) \qquad \begin{aligned} K(t) = I&+\int_0^t K(s)\,\Pi(s)^{-1}\,\tilde{r}(x)\,D_2\left(\frac{1}{2}J(r(s,x,w))\right)\tilde{r}(x)^{-1}\,\Pi(s)\,ds \\ &+\int_0^t K(s)\circ d\Theta(s)\,. \end{aligned}
$$

Then $u(t,x)$ is expressed simply by

$$
(3.26) \qquad u(t,x) = E[K(t)f(X(t,x,w))]\,.
$$

REMARK 3.1. It is easy to see that if $\alpha=(\alpha_{ij})\in\boldsymbol{R}^m\otimes\boldsymbol{R}^m$,

$$
\tilde{r}(x)\,D_1(\alpha)\,\tilde{r}(x)^{-1} = D_1(\tilde{\alpha})
$$

where

$$
\tilde{\alpha}_{ij} = \alpha_{ab}\,\sigma^i_a(x)\,\tau^b_j(x)
$$

and, if $\beta=(\beta_{ijkl})\in\boldsymbol{R}^m\otimes\boldsymbol{R}^m\otimes\boldsymbol{R}^m\otimes\boldsymbol{R}^m$,

$$
\tilde{r}(x)\,D_2(\beta)\,\tilde{r}(x)^{-1} = D_2(\tilde{\beta})
$$

where

$$
\tilde{\beta}_{ijkl} = \beta_{abcd}\,\sigma^i_a(x)\,\sigma^k_c(x)\,\tau^b_j(x)\,\tau^d_l(x)\,.
$$

Here $(\sigma^i_k(x))$ is, as above, the square root of $(g^{ij}(x))$ and $(\tau^i_k(x))=(\sigma^i_k(x))^{-1}$. Hence

$$
(3.27) \qquad \tilde{r}(x)\,D_2\left(\frac{1}{2}J(r(s,x,w))\right)\tilde{r}(x)^{-1} = D_2\left(\frac{1}{2}\mathcal{J}(s,x,w)\right)
$$

where $\mathcal{J}(s, x, w)=(\mathcal{J}_{ijkl}(s, x, w))\in \boldsymbol{R}^m\otimes\boldsymbol{R}^m\otimes\boldsymbol{R}^m\otimes\boldsymbol{R}^m$ is given by

$$\mathcal{J}_{ijkl}(s, x, w) = J_{abcd}(r(s, x, w))\, \sigma_a^i(x)\, \sigma_c^k(x)\, \tau_j^b(x)\, \tau_l^d(x)\,. \tag{3.28}$$

Then (3.25) is also written in the form

$$K(t) = I+\int_0^t K(s)\,\Pi(s)^{-1} D_2\left(\frac{1}{2}\mathcal{J}(s, x, w)\right)\Pi(s)\, ds + \int_0^t K(s)\circ d\Theta(s)\,. \tag{3.25$'$}$$

Now we come back to the case of $M=\boldsymbol{R}_+^m$ and the initial and boundary value problem (3.1). This time, the moving frame $r(t)=(X(t), \boldsymbol{e}(t))$ is defined in a similar way where $X(t)$ is the *reflecting barrier Brownian motion* on M and $\boldsymbol{e}(t)$ is the stochastic parallel translation of $\boldsymbol{e}(0)$ along the path $X(t)$. Keeping in mind that we want to apply the modified Malliavin calculus of the previous section, we realize $r(t)$ in the following way. First we set

$$\begin{aligned} b^i(y) &= -\frac{1}{2}\, g^{jk}(y)\, \Gamma_{jk}^i(y) \\ &= \frac{1}{2\sqrt{\det g(y)}}\, \frac{\partial(g^{ij}(y)\sqrt{\det g(y)})}{\partial y_j} \end{aligned} \tag{3.29}$$

so that, by noting (3.7), we have

$$\frac{1}{2}\Delta = \frac{1}{2}\left(\sum_{i,j=1}^{m-1} g^{ij}(y)\frac{\partial^2}{\partial y^i\,\partial y^j}+\frac{\partial^2}{(\partial y^m)^2}\right)+\sum_{i=1}^m b^i(y)\frac{\partial}{\partial y^i}\,. \tag{3.30}$$

Note that by (3.7),

$$b^m(y) = \frac{1}{4}\,\frac{1}{\det g(y)}\,\frac{\partial(\det g(y))}{\partial y^m}\,. \tag{3.31}$$

Let (W, P) be the Wiener space with $r=m-1$ and set $A=\{\varphi\in C([0, 1]\to\boldsymbol{R});\ \varphi(0)=0\}$. If $P^{(1)}$ is the one-dimensional Wiener measure on A, then $(W\times A, P\times P^{(1)})$ is nothing but the m-dimensional Wiener space. Given $x\in M=\boldsymbol{R}_+^m$ written as $x=(\bar{x}, x^m)$ so that $\bar{x}\in\boldsymbol{R}^{m-1}$, we consider the following *SDE* on $\boldsymbol{R}^{m-1}$ for each $\varphi\in A$:

$$\begin{cases} dX^i(t) = \displaystyle\sum_{k=1}^{m-1}\sigma_k^i(\bar{X}(t), |x^m+\varphi(t)|)\, dw^k(t)+b^i(\bar{X}(t), |x^m+\varphi(t)|)\, dt \\ \qquad i = 1, \cdots, m-1\,, \\ \bar{X}(0) = \bar{x} \end{cases} \tag{3.32}$$

where $\bar{X}(t)=(X^1(t), \cdots, X^{m-1}(t))$. The solution of (3.32) is denoted by $\bar{X}(t, x, w; \varphi)$. Let $\tilde{P}$ be a probability on $W\times A$ which is absolutely continuous with respect to $P\times P^{(1)}$ with the density given by

$$\frac{d\tilde{P}}{d(P\times P^{(1)})}=m(1,x,w;\varphi):=\exp\{\int_0^1\hat{b}^m(\bar{X}(s,x,w;\varphi),x^m+\varphi(s))\,d\varphi(s)\tag{3.33}$$
$$-\frac{1}{2}\int_0^1|\hat{b}^m(\bar{X}(s,x,w;\varphi),x^m+\varphi(s))|^2\,ds\}$$

where $\hat{b}^m(y)=b^m(\bar{y},|y^m|)\operatorname{sgn}(y^m)$, $y=(\bar{y},y^m)\in\boldsymbol{R}^m$. Then, by the Girsanov theorem, we deduce that the $X(t)=X(t,x,w;\varphi)$:

$$X(t)=(\bar{X}(t,x,w;\varphi),X^m(t)=|x^m+\varphi(t)|)\in M\tag{3.34}$$

is a realization of the reflecting Brownian motion on M under the probabilty $\tilde{P}$. Hence, if $\boldsymbol{e}(t)=\boldsymbol{e}(t,x,w;\varphi)=(e^i_j(t))$ is the solution of SDE:

$$\begin{cases} de^i_j(t)=-\Gamma^i_{lk}(X(t))\,e^k_j(t)\circ dX^l(t)\,, \\ e^i_j(0)=\sigma^i_j(x)\,, \end{cases}\tag{3.35}$$

then, under the probability $\tilde{P}$, $r(t)=r(t,x,w;\varphi)=(X(t),\boldsymbol{e}(t))$ is a realization of the stochastic moving frame.

Next, we have to define $K(t)\in\operatorname{End}(\Lambda\boldsymbol{R}^m)$. Let $P,Q\in\operatorname{End}(\Lambda\boldsymbol{R}^m)$ be defined by

$$P=a^*_m\,a_m \quad\text{and}\quad Q=a_m\,a^*_m\,.\tag{3.36}$$

It is easy to see generally that $\{a_i,a_j\}=\{a^*_i,a^*_j\}=0$ and $\{a^*_i,a_j\}=\delta_{ij}\,I$ where $\{A,B\}=AB+BA$ is the anticommutator. Hence we have

$$P^2=P,\ Q^2=Q,\ P=I-Q \quad\text{and}\quad PQ=0\,.\tag{3.37}$$

Also it is clear that, under the above identification of $\Lambda\boldsymbol{R}^m$ and $\Lambda T^*_x(M)$ as vector spaces,

$$P[\omega]=\omega_{\text{norm}},\ Q[\omega]=\omega_{\tan},\quad\text{for}\quad \omega\in\Lambda(M)\,.\tag{3.38}$$

For $X(t)=(X^i(t))$ and $\boldsymbol{e}(t)=(e^i_j(t))$ given by (3.34) and (3.35), we define $\Pi(t)\in\operatorname{End}(\Lambda\boldsymbol{R}^m)$ by (3.18) where $r(x)=(x,\boldsymbol{e}(x)=(\sigma^i_j(x)))$. Similarly $\Theta(t)\in\operatorname{End}(\Lambda\boldsymbol{R}^m)$ and $\theta(t)\in\boldsymbol{R}^m\otimes\boldsymbol{R}^m$ are defined and $\Pi(t)$ satisfies the same integral equation (3.22). Following [1] and [9], we introduce the following equation for $K(t)\in\operatorname{End}(\Lambda\boldsymbol{R}^m)$ which is a modified form of (3.25)$'$:

$$\begin{cases} dK(t)=K(t)\,\Pi(t)^{-1}\,D_2\Big(\frac{1}{2}\,\mathcal{J}(t,x,w;\varphi))\Big)\,\Pi(t)\,dt \\ \qquad\qquad +I_{\{X^m(t)>0\}}\,K(t)\circ d\Theta(t)\,, \\ K(t)\,P=0 \quad\text{when}\quad X^m(t)=0\,, \\ K(0)=\begin{cases} I=P+Q & \text{if}\quad x^m=X^m(0)>0\,, \\ Q & \text{if}\quad x^m=X^m(0)=0\,. \end{cases} \end{cases}\tag{3.39}$$

where $\mathcal{J}(t,x,w;\varphi)=(\mathcal{J}_{ijkl}(t,x,w;\varphi))\in\boldsymbol{R}^m\otimes\boldsymbol{R}^m\otimes\boldsymbol{R}^m\otimes\boldsymbol{R}^m$ is given by

(3.40) $\quad \mathcal{J}_{ijkl}(t, x, w; \varphi) = J_{abcd}(r(t, x, w; \varphi))\, \sigma^i_a(x)\, \sigma^k_c(x)\, \tau^b_j(x)\, \tau^d_l(x)$.

An exact meaning of (3.39) is given by the following integral equation:

$$(3.39)' \quad \begin{aligned} K(t)P &= I_{\{\sigma>t\}}\Big[P+\int_0^t K(s)\,\Pi(s)^{-1} D_2\Big(\frac{1}{2}\mathcal{J}(s, x, w; \varphi)\Big)\,\Pi(s)\, dsP \\ &\qquad +\int_0^t K(s)\circ d\tilde{\Theta}(s)\, P\Big] \\ &\quad +I_{\{\sigma\le t\}}\Big[\int_{\tau(t)}^t K(s)\,\Pi(s)^{-1} D_2\Big(\frac{1}{2}\mathcal{J}(s, x, w; \varphi)\Big)\,\Pi(s)\, dsP \\ &\qquad +\int_{\tau(t)}^t K(s)\circ d\tilde{\Theta}(s)\, P\Big] \\ K(t)\,Q &= Q+\int_0^t K(s)\,\Pi(s)^{-1} D_2\Big(\frac{1}{2}\mathcal{J}(s, x, w; \varphi)\Big)\,\Pi(s)\, dsQ \\ &\quad +\int_0^t K(s)\circ d\tilde{\Theta}(s)\, Q \end{aligned}$$

where

$$\tilde{\Theta}(t) = \int_0^t I_{\{X^m(s)>0\}}\, d\Theta(s)$$

and

(3.41) $\quad \sigma = \inf\{t;\, X^m(t) = 0\}, \quad \tau(t) = \sup\{s<t;\, X^m(s) = 0\}$

with the conventions that $\inf \phi=\infty$ and $\sup \phi=0$. The solution $K(t)=K(t, x, w; \varphi)$ of the equation (3.39)′ exists and is unique as was shown in [8] or [9].

Theorem 3.2. *For $f\in\Lambda(M)$ with tempered C^∞-components, define $u(t, x)\in \Lambda T^*_x(M)=\Lambda \boldsymbol{R}^m$ (by the above identification of these vector spaces) by*

(3.42) $\quad u(t, x) = E^{P\times P^{(1)}}\,[m(1, x, w; \varphi)\, K(t, x, w; \varphi) f(X(t, x, w; \varphi))]$.

Then any solution of the initial and boundary value problem (3.1) *with tempered C^∞-components must coincide with $u(t, x)$.*

For the proof, c.f., [8], [9].

Now we can represent the heat kernel $e(t, x, y)\in \mathrm{Hom}(\Lambda T^*_y(M), \Lambda T^*_x(M))$ by a generalized Wiener functional expectation in the framework of the modified Malliavin calculus, For this, it is more convenient to introduce a parameter $\varepsilon>0$ as in the previous section. So let $x=(\bar{x}, x^m)\in M=\boldsymbol{R}^m_+$ and $\varepsilon>0$ be given and fixed. We modify the equation (3.32) as

$$(3.32)_\varepsilon \quad \begin{cases} dX^i(t) = \varepsilon \sum_{k=1}^{m-1} \sigma^i_k(\bar{X}(t),\, |x^m+\varepsilon\varphi(t)|)\, dw^k(t) \\ \qquad\qquad +\varepsilon^2\, b^i(\bar{X}(t),\, |x^m+\varepsilon\varphi(t)|)\, dt\,, \quad i = 1, \cdots, m-1\,, \\ \bar{X}(0) = \bar{x} \end{cases}$$

and denote the solution by $\bar{X}^\varepsilon(t)=\bar{X}^\varepsilon(t,x,w;\varphi)=(X^{\varepsilon,1}(t,x,w;\varphi),\cdots,X^{\varepsilon,m-1}(t,x,w;\varphi))$. Set $X^\varepsilon(t)=X^\varepsilon(t,x,w;\varphi)=(\bar{X}^\varepsilon(t,x,w;\varphi),X^{\varepsilon,m}(t)=|x^m+\varepsilon\varphi(t)|)$. Set

$$m^\varepsilon(1,x,w;\varphi):=\exp\{\varepsilon\int_0^1 \hat{b}^m(\bar{X}^\varepsilon(s,x,w;\varphi),x^m+\varepsilon\varphi(s))\,d\varphi(s) -\frac{\varepsilon^2}{2}\int_0^1 |\hat{b}^m(\bar{X}^\varepsilon(s,x,w;\varphi),x^m+\varepsilon\varphi(s))|^2\,ds\} \tag{3.33$_\varepsilon$}$$

where $\hat{b}^m(y)=b^m(\bar{y},|y^m|)\operatorname{sgn}(y^m)$ for $y=(\bar{y},y^m)\in\boldsymbol{R}^m$. Define $\boldsymbol{e}^\varepsilon(t)=\boldsymbol{e}^\varepsilon(t,x,w;\varphi)$, $\Pi^\varepsilon(t)=\Pi^\varepsilon(t,x,w;\varphi)$ and $\Theta^\varepsilon(t)=\Theta^\varepsilon(t,x,w;\varphi)$ in the same way as above from $X^\varepsilon(t)=X^\varepsilon(t,x,w;\varphi)$ instead of $X(t)$. Finally define $K^\varepsilon(t)=K^\varepsilon(t,x,w;\varphi)$ to be the unique solution of

$$\begin{aligned} K(t)\,P &= I_{\{\sigma^\varepsilon>t\}}\Big[P+\varepsilon^2\int_0^t K(s)\,\Pi^\varepsilon(s)^{-1}D_2\Big(\frac{1}{2}\mathcal{J}^\varepsilon(s,x,w;\varphi)\Big) \\ &\qquad\times\Pi^\varepsilon(s)\,dsP+\int_0^t K(s)\circ d\tilde{\Theta}^\varepsilon(s)\,P\Big] \\ &\quad+I_{\{\sigma^\varepsilon\le t\}}\Big[\varepsilon^2\int_{\tau^\varepsilon(t)}^t K(s)\,\Pi^\varepsilon(s)^{-1}D_2\Big(\frac{1}{2}\mathcal{J}^\varepsilon(s,x,w;\varphi)\Big) \\ &\qquad\times\Pi^\varepsilon(s)\,dsP+\int_{\tau^\varepsilon(t)}^t K(s)\circ d\tilde{\Theta}^\varepsilon(s)\,P\Big] \\ K(t)\,Q &= Q+\varepsilon^2\int_0^t K(s)\,\Pi^\varepsilon(s)^{-1}D_2\Big(\frac{1}{2}\mathcal{J}^\varepsilon(s,x,w;\varphi)\Big)\Pi^\varepsilon(s)\,dsQ \\ &\quad+\int_0^t I_{\{X^{\varepsilon,m}(s)>0\}}K(s)\circ d\tilde{\Theta}^\varepsilon(s)\,Q \end{aligned} \tag{3.39$'_\varepsilon$}$$

where

$$\tilde{\Theta}^\varepsilon(t)=\int_0^t I_{\{X^{\varepsilon,m}(s)>0\}}\,d\Theta^\varepsilon(s)\,,$$

$r^\varepsilon(t)=r^\varepsilon(t,x,w;\varphi)=(X^\varepsilon(t,x,w;\varphi),\boldsymbol{e}^\varepsilon(t,x,w;\varphi))$ and $\mathcal{J}^\varepsilon(t,x,w;\varphi)=(\mathcal{J}^\varepsilon_{ijkl}(t,x,w;\varphi))$ with

$$\mathcal{J}^\varepsilon_{ijkl}(t,x,w;\varphi)=J_{abcd}(r^\varepsilon(t,x,w;\varphi))\,\sigma^i_a(x)\,\sigma^k_c(x)\,\tau^b_j(x)\,\tau^d_l(x)\,.$$

Furthermore, $\sigma^\varepsilon=\inf\{t;\,X^{\varepsilon,m}(t)=0\}$ and $\tau^\varepsilon(t)=\sup\{s<t;\,X^{\varepsilon,m}(s)=0\}$. Noting the scaling property of the Wiener measure, we deduce easily that the expectation $u(t,x)$ of (3.42) can also be expressed as

$$u(\varepsilon^2,x)=E^{P\times P^{(1)}}[m^\varepsilon(1,x,w;\varphi)\,K^\varepsilon(1,x,w;\varphi)f(X^\varepsilon(1,x,w;\varphi))]\,. \tag{3.43}$$

For each $y=(\bar{y},y^m)\in\boldsymbol{R}^m$ and $\varphi\in A$, define $T^{x^m,y^m}_\varepsilon(\varphi)\in A$ as in the previous section by

$$T^{x^m,y^m}_\varepsilon(\varphi)\,(t)=\varphi(t)+\frac{y^m-x^m}{\varepsilon}\,t\,,\quad 0\le t\le 1\,.$$

Set

$$\hat{X}^{\varepsilon}(t) = \bar{X}^{\varepsilon}(t, x, w; T_{\varepsilon}^{x^m, y^m}(\varphi)) \in \boldsymbol{R}^{m-1}$$
$$\hat{m}^{\varepsilon}(1) = m^{\varepsilon}(1, x, w; T_{\varepsilon}^{x^m, y^m}(\varphi)) \in (0, \infty)$$
$$\hat{e}^{\varepsilon}(t) = \boldsymbol{e}^{\varepsilon}(t, x, w; T_{\varepsilon}^{x^m, y^m}(\varphi)) \in GL(m; \boldsymbol{R})$$
$$\hat{\Pi}^{\varepsilon}(t) = \Pi^{\varepsilon}(t, x, w; T_{\varepsilon}^{x^m, y^m}(\varphi)) \in \mathrm{End}(\Lambda \boldsymbol{R}^m)$$

and

$$\hat{K}^{\varepsilon}(t) = K^{\varepsilon}(t, x, w; T_{\varepsilon}^{x^m, y^m}(\varphi)) \in \mathrm{End}(\Lambda \boldsymbol{R}^m)\,.$$

Let $m(d\varphi) = P_{0;0}^{1;0}(d\varphi)$ be the pinned Wiener measure on A. As in the previous section

$$\hat{X}^{\varepsilon}(1) \in L_{\infty-}(\boldsymbol{D}^{\infty}(\boldsymbol{R}^{m-1}))\,,$$
$$\hat{m}^{\varepsilon}(1) \in L_{\infty-}(\boldsymbol{D}^{\infty})\,,$$
$$\hat{\Pi}^{\varepsilon}(1), \hat{K}^{\varepsilon}(1) \in L_{\infty-}(\boldsymbol{D}^{\infty}(\mathrm{End}(\Lambda \boldsymbol{R}^m)))\,.$$

Therefore, we can apply Theorem 2.1 to define the following kernel $\mathcal{E}(x, y; \varepsilon) \in \mathrm{End}(\Lambda \boldsymbol{R}^m)$ for fixed $x = (\bar{x}, x^m) \in M = \boldsymbol{R}_+^m$, $y = (\bar{y}, y^m) \in \boldsymbol{R}^m$ and $\varepsilon > 0$:

$$\begin{aligned} \mathcal{E}(x, y; \varepsilon) &= \int_A E[\hat{m}^{\varepsilon}(1)\, \hat{K}^{\varepsilon}(1)\, \delta_{\bar{y}}(\hat{X}^{\varepsilon}(1))]\, m(d\varphi) \\ &= \int_A E[\hat{m}^{\varepsilon}(1, x, w; T_{\varepsilon}^{x^m, y^m}(\varphi))\, \hat{K}^{\varepsilon}(1, x, w; T_{\varepsilon}^{x^m, y^m}(\varphi)) \\ &\qquad \times \delta_{\bar{y}}(\bar{X}^{\varepsilon}(1, x, w; T_{\varepsilon}^{x^m, y^m}(\varphi)))]\, P_{0;0}^{1;0}(d\varphi)\,. \end{aligned} \tag{3.44}$$

From (3.43), we can finally deduce the following:

Theorem 3.3. *For given $x = (\bar{x}, x^m)$, $y = (\bar{y}, y^m) \in M = \boldsymbol{R}_+^m$ and $\varepsilon > 0$, the fundamental solution $e(t, x, y)$ to the initial and boundary value problem* (3.1) *with respect to the Riemannian volume is expressed as*

$$\begin{aligned} e(\varepsilon^2, x, y) = \frac{1}{\sqrt{\det g(y)}} \Big[&\frac{1}{\sqrt{2\pi}\varepsilon} \exp\left\{-\frac{(x^m - y^m)^2}{2\varepsilon^2}\right\} \mathcal{E}(x, y; \varepsilon) \\ &+ \frac{1}{\sqrt{2\pi}\varepsilon} \exp\left\{-\frac{(x^m + y^m)^2}{2\varepsilon^2}\right\} \mathcal{E}(x, \check{y}; \varepsilon) \Big] \end{aligned} \tag{3.45}$$

where $\check{y} = (\bar{y}, -y^m) \in \boldsymbol{R}^m$.

4. Gauss-Bonnet-Chern theorem for manifolds with boundary

Our main object in this section is to evaluate the integral (3.6) by using the expression of $e(t, x, x)$ given by Theorem 3.3. We can thereby obtain an integral formula for the Euler-Poincaré characteristic of the manifold M as explained in the previous section. This formula is known as the Gauss-Bonnet-Chern theorem for manifolds with boundary.

First, we introduce the following notations in order to apply Theorem 3.3 for the heat kernel $e(t, x, x)$ on the *diagonal.* For given $x=(\bar{x}, x^m)\in \boldsymbol{R}^m_+$, $0<\varepsilon\leq 1$ and $\varphi\in A$, define maps T_1, T_2 on A by

$$T_1(\varphi)=\varphi \tag{4.1}$$

and

$$T_2(\varphi)(t)=\varphi(t)-\frac{2x^m}{\varepsilon}t, \qquad 0\leq t\leq 1. \tag{4.2}$$

Set, for each $i=1, 2$,

$$\bar{X}^\varepsilon_{(i)}(t)=\bar{X}^\varepsilon(t, x, w; T_i(\varphi))\in \boldsymbol{R}^{m-1}, \tag{4.3}$$

$$m^\varepsilon_{(i)}(1)=m^\varepsilon(1, x, w; T_i(\varphi))\in(0, \infty), \tag{4.4}$$

$$\boldsymbol{e}^\varepsilon_{(i)}(t)=\boldsymbol{e}^\varepsilon(t, x, w; T_i(\varphi))\in GL(m, \boldsymbol{R}), \tag{4.5}$$

$$\Pi^\varepsilon_{(i)}(t)=\Pi^\varepsilon(t, x, w; T_i(\varphi))\in \mathrm{End}(\Lambda \boldsymbol{R}^m), \tag{4.6}$$

$$K^\varepsilon_{(i)}(t)=K^\varepsilon(t, x, w; T_i(\varphi))\in \mathrm{End}(\Lambda \boldsymbol{R}^m). \tag{4.7}$$

Then by Theorem 3.3, we have

$$\begin{aligned} e(\varepsilon^2, x, x) &= \frac{1}{\sqrt{\det g(x)}}\frac{1}{\sqrt{2\pi}\varepsilon}\int_A E[m^\varepsilon_{(1)}(1)K^\varepsilon_{(1)}(1)\delta_{\bar{x}}(\bar{X}^\varepsilon_{(1)}(1))]P^{1,0}_{0,0}(d\varphi) \\ &\quad +\frac{1}{\sqrt{\det g(x)}}\frac{1}{\sqrt{2\pi}\varepsilon}\exp\left\{-\frac{2(x^m)^2}{\varepsilon^2}\right\} \\ &\quad \times\int_A E[m^\varepsilon_{(2)}(1)K^\varepsilon_{(2)}(1)\delta_{\bar{x}}(\bar{X}^\varepsilon_{(2)}(1))]P^{1,0}_{0,0}(d\varphi). \end{aligned} \tag{4.8}$$

By applying the time dependent Malliavin calculus ([17]) and estimates given in § 2 (Lemma 2.1 and 2.2), we see that $\bar{X}^\varepsilon_{(i)}(1)\in L_{\infty-}(\boldsymbol{D}^\infty(\boldsymbol{R}^{m-1}))$, $m^\varepsilon_{(i)}(1)\in L_{\infty-}(\boldsymbol{D}^\infty)$, $K^\varepsilon_{(i)}(1)\in L_{\infty-}(\boldsymbol{D}^\infty(\mathrm{End}(\Lambda\boldsymbol{R}^m)))$. Furthermore, we can see easily that, for every $k=1, 2, \cdots$ and $p, p'\in(1, \infty)$, (denoting by $K_1, K_2, \cdots$, positive constants depending on p, p' and k but not on $x\in\boldsymbol{R}^m_+$ and $0<\varepsilon\leq 1$)

$$\begin{aligned} &\|\bar{X}^\varepsilon_{(i)}(1)\|_{p,k;p'}\leq K_1 \\ &\|m^\varepsilon_{(i)}(1)\|_{p,k;p'}\leq K_2 \\ &\|m^\varepsilon_{(2)}(1)\|_{p,k;p'}\leq K_3\exp\{K_4x^m\} \\ &\|\Pi^\varepsilon_{(1)}(1)\|_{p,k;p'}\leq K_5 \\ &\|\Pi^\varepsilon_{(2)}(1)\|_{p,k;p'}\leq K_6\exp\{K_7x^m\} \\ &\|K^\varepsilon_{(1)}(1)\|_{p,k;p'}\leq K_8 \\ &\|K^\varepsilon_{(2)}(1)\|_{p,k;p'}\leq K_9\exp\{K_{10}x^m\}. \end{aligned}$$

It holds that

$$(4.9)\qquad \bar{X}^{\varepsilon}_{(i)}(1) = \bar{x}+\varepsilon\int_0^1 \sigma(\bar{x},\ |x^m+\varepsilon T_i(\varphi)(t)|)dw(t)+O(\varepsilon^2)$$
$$\text{as}\quad \varepsilon\downarrow 0\ \text{ in }\ L_{\infty-}(\boldsymbol{D}^\infty(\boldsymbol{R}^{m-1}))$$

and $O(\varepsilon^2)$ is uniform in $x\in\boldsymbol{R}^m_+$. By setting

$$F_i = \frac{\bar{X}^{\varepsilon}_{(i)}(1)-\bar{x}}{\varepsilon}\in L_{\infty-}(\boldsymbol{D}^\infty(\boldsymbol{R}^{m-1}))\,,$$

we see easily that for every $p,\ p'\in(1,\ \infty)$

$$(4.10)\qquad \sup_{x,\varepsilon}\|(\det\sigma_{F_i})^{-1}\|_{p,p'}<\infty$$

where $\|\ \ \|_{p,p'}$ is $L_{p'}(A,\ P^{1;0}_{0;0};\ L_p(W,\ P))$-norm. Hence, by Theorem 2.2 and its obvious modification, we can conclude that

$$(4.11)\qquad \begin{aligned} \delta_{\bar{x}}(\bar{X}^{\varepsilon}_{(i)}(1)) &= \delta_{\bar{x}}(\bar{x}+\varepsilon F_i)\\ &= \varepsilon^{-m+1}\delta_0(F_i)\\ &= \varepsilon^{-m+1}[\delta_0(\int_0^1 \sigma(\bar{x},\ |x^m+\varepsilon T_i(\varphi)(t)|)dw(t))+O(\varepsilon)]\quad \text{in}\quad L_{\infty-}(\tilde{\boldsymbol{D}}^{-\infty})\,,\\ &\qquad i=1,\ 2\,. \end{aligned}$$

Next, we have

$$(4.12)\qquad m^{\varepsilon}_{(1)}(1) = 1+O(\varepsilon)\qquad \text{in}\quad L_{\infty-}(\boldsymbol{D}^\infty)$$

uniformly in x and

$$(4.13)\qquad m^{\varepsilon}_{(2)}(1) = 1+O(\varepsilon)e^{Kx^m/\varepsilon}\qquad \text{in}\quad L_{\infty-}(\boldsymbol{D}^\infty)$$

in the following sense: Generally, for $\Phi(\varepsilon,\ x)=\{\Phi(\varepsilon,\ x,\ w;\ \varphi)\}\in L_{\infty-}(\boldsymbol{D}^\infty)$, we write

$$\Phi(\varepsilon,\ x) = O(\varepsilon^l)e^{Kx^m/\varepsilon}\qquad \text{in}\quad L_{\infty-}(\boldsymbol{D}^\infty)\ \text{as}\ \varepsilon\downarrow 0$$

if, for every $p,\ p'\in(1,\ \infty)$ and $k>0$, there exist $K_i=K_i\ (p,\ k;\ p')>0,\ i=1,\ 2,$ such that

$$\|\Phi(\varepsilon,\ x)\|_{p,k\,;\,p'}\le K_1\varepsilon^l e^{K_2x^m/\varepsilon}\,.$$

Next, we have to estimate $K^{\varepsilon}_{(i)}(1)\in\mathrm{End}\,(\Lambda\boldsymbol{R}^m)$. The cancellation of the supertrace is based on the following *Berezin formula*. As before, $\mathrm{Str}\,[A]$ for $A\in\mathrm{End}\,(\Lambda\boldsymbol{R}^m)$ is defined to be

$$\mathrm{Str}\,[A] = \mathrm{tr}\,[(-1)^F A]$$

where $(-1)^F \in \mathrm{End}(\Lambda \boldsymbol{R}^m)$ is determined by

$$(-1)^F \omega = (-1)^p \omega \qquad \text{if} \quad \omega \in \mathrm{End}(\Lambda^p \boldsymbol{R}^m)\,,\ p = 0, \cdots, m\,.$$

Lemma 4.3. (Berezin formula, cf. [4])

Every $A \in \mathrm{End}(\Lambda \boldsymbol{R}^m)$ *is uniquely expressed as*

$$A = \sum_{K,L} \alpha_{\{K,\hat{L}\}}(A) a_K^* a_L \qquad (\alpha_{\{K,\hat{L}\}}(A) \in \boldsymbol{R})\,. \tag{4.14}$$

Here, K, L range over all subsets of $\{1, \cdots, m\}$ and

$$a_K^* a_L = a_{k_1}^* \cdots a_{k_p}^* a_{l_1} \cdots a_{l_q}$$

if $K = \{k_1 < \cdots < k_p\}$ and $L = \{l_1 < \cdots < l_q\}$ (Of course, $a_\phi^* = a_\phi = I$). Moreover, $\hat{L}$ means $\hat{L} = \{\hat{l}_1, \hat{l}_2, \cdots, \hat{l}_q\}$ and the symbol $\wedge$ is to emphasize that it is related to not a_l^* but a_l. Then $\mathrm{Str}[A]$ *is given by*

$$\mathrm{Str}[A] = (-1)^{m(m+1)/2} \alpha_{\{1,\cdots,m,\hat{1}\cdots,\hat{m}\}}[A]\,. \tag{4.15}$$

Noting that $Q = a_m a_m^*$ (cf. (3.36)) and the commutation relations of a_i, a_j^*, we have

Corollary. *For every* $A \in \mathrm{End}(\Lambda \boldsymbol{R}^m)$,

$$\mathrm{Str}[AQ] = (-1)^{m(m-1)/2} \alpha_{\{1,\cdots,m-1,\hat{1},\cdots,\widehat{m-1}\}}[A]\,. \tag{4.16}$$

Define $\mathrm{End}(\Lambda \boldsymbol{R}^m)$-valued semimartingales $\Xi_{(i)}^\varepsilon(t)$, $i = 1, 2$, by

$$\begin{aligned}\Xi_{(i)}^\varepsilon(t) = {} & \varepsilon^2 \int_0^t \Pi_{(i)}^\varepsilon(s)^{-1} D_2\Big(\frac{1}{2}\mathcal{J}_{(i)}^\varepsilon(s)\Big)\Pi_{(i)}^\varepsilon(s)ds \\ & + \int_0^t I_{\{x^m + \varepsilon T_i(\varphi)(s) > 0\}}\, d\Theta_{(i)}^\varepsilon(s)\,,\end{aligned} \tag{4.17}$$

where $\Pi_{(i)}^\varepsilon(t) = \Pi^\varepsilon(t, x, w; T_i(\varphi))$, $\Theta_{(i)}^\varepsilon(t) = \Theta^\varepsilon(t, x, w; T_i(\varphi))$ and $\mathcal{J}_{(i)}^\varepsilon(t) = \mathcal{J}^\varepsilon(t, x, w; T_i(\varphi))$, $i = 1, 2$. Then, noting $P + Q = I$, the equation $(3.39)'_\varepsilon$ can be written in the form

$$\begin{aligned}K_{(i)}^\varepsilon(t) &= I_{\{\sigma_{(i)}^\varepsilon > t\}}[I + \int_0^t K_{(i)}^\varepsilon(s) \circ d\Xi_{(i)}^\varepsilon(s)]P \\ &\quad + I_{\{\sigma_{(i)}^\varepsilon \le t\}} \int_{\tau_{(i)}^\varepsilon(t)}^t K_{(i)}^\varepsilon(s) \circ d\Xi_{(i)}^\varepsilon(s) P \\ &\quad + [I + \int_0^t K_{(i)}^\varepsilon(s) \circ d\Xi_{(i)}^\varepsilon(s)]Q \\ &= I_{\{\sigma_{(i)}^\varepsilon > t\}}[I + \int_0^t K_{(i)}^\varepsilon(s) \circ d\Xi_{(i)}^\varepsilon(s)] \\ &\quad + I_{\{\sigma_{(i)}^\varepsilon \le t\}}\Big[\int_{\tau_{(i)}^\varepsilon(t)}^t K_{(i)}^\varepsilon(s) \circ d\Xi_{(i)}^\varepsilon(s) P\end{aligned} \tag{4.18}$$

$$+\{I+\int_0^t K^{\varepsilon}_{(i)}(s)\circ d\Xi^{\varepsilon}_{(i)}(s)\}Q] .$$

Here, $\sigma^{\varepsilon}_{(i)}=\inf\{t: x^m+\varepsilon T_i(\varphi)(t)=0\}$ and $\tau^{\varepsilon}_{(i)}(t)$ is defined similarly. We substitute this expression of $K^{\varepsilon}_{(i)}(t)$ into $K^{\varepsilon}_{(i)}(s)$ appearing in the right-hand side of (4.18). We iterate this substitution m-times. Then, noting the Berezin formulas (4.15) and (4.16), the fact that $\Xi^{\varepsilon}_{(1)}(t)=O(\varepsilon)$ in $L_{\infty-}(\boldsymbol{D}^{\infty}(\Lambda\boldsymbol{R}^m))$ and also that $PQ=QP=0$ and $P^2=Q^2=I$, we can easily deduce that

$$\begin{aligned}K^{\varepsilon}_{(1)}(1) = {} & I_{\{\sigma^{\varepsilon}_{(1)}>1\}}[O(\varepsilon^m)+R^{\varepsilon}_1] \\ & +I_{\{\sigma^{\varepsilon}_{(1)}\leq 1\}}[\sum_{l=0}^{m-1}\int_0^1\circ\Xi^{\varepsilon}_{(1)}d(s_1)\int_0^{s_1}\circ d\Xi^{\varepsilon}_{(1)}(s_2)\cdots \\ & \times\int_0^{s_{l-1}}\circ d\Xi^{\varepsilon}_{(1)}(s_l)Q+O(\varepsilon^m)+R^{\varepsilon}_2] \\ & \qquad \text{in} \quad L_{\infty-}(\boldsymbol{D}^{\infty}(\mathrm{End}(\Lambda\boldsymbol{R}^m))) ,\end{aligned} \tag{4.19}$$

where $\mathrm{Str}(R^{\varepsilon}_1)=\mathrm{Str}(R^{\varepsilon}_2)=0$ and $O(\varepsilon^m)$ are uniform in x. Similarly we can deduce that

$$\begin{aligned}K^{\varepsilon}_{(2)}(1) = {} & I_{\{\sigma^{\varepsilon}_{(2)}>1\}}[O(\varepsilon^m)\exp\{Kx^m/\varepsilon\}+R^{\varepsilon}_3] \\ & +I_{\{\sigma^{\varepsilon}_{(2)}\leq 1\}}[\sum_{l=0}^{m-1}\int_0^1\circ d\Xi^{\varepsilon}_{(2)}(s_1)\int_0^{s_1}\circ d\Xi^{\varepsilon}_{(2)}(s_2)\cdots \\ & \times\int_0^{s_{l-1}}\circ d\Xi^{\varepsilon}_{(2)}(s_l)Q+O(\varepsilon^m)\exp\{Kx^m/\varepsilon\}+R^{\varepsilon}_4] \\ & \qquad \text{in} \quad L_{\infty-}(\boldsymbol{D}^{\infty}(\mathrm{End}(\Lambda\boldsymbol{R}^m))) ,\end{aligned} \tag{4.20}$$

where $\mathrm{Str}(R^{\varepsilon}_3)=\mathrm{Str}(R^{\varepsilon}_4)=0$ and $O(\varepsilon^m)$ are uniform in x. Set

$$\begin{aligned}\Xi_i(=\Xi_i(x,\varepsilon,w;\varphi)) = {} & \sum_{l=0}^{m-1}\int_0^1\circ d\Xi^{\varepsilon}_{(i)}(s_1)\int_0^{s_1}\circ d\Xi^{\varepsilon}_{(i)}(s_2)\cdots \\ & \times\int_0^{s_{l-1}}\circ d\Xi^{\varepsilon}_{(i)}(s_l)Q\in L_{\infty-}(\boldsymbol{D}^{\infty}(\mathrm{End}(\Lambda\boldsymbol{R}^m))) ,\end{aligned}$$

the term correponding to $l=0$ being I and $s_0=1$. Then, by (4.8),

$$\begin{aligned}& \mathrm{Str}[e(\varepsilon^2,x,x)] \\ & =\frac{1}{\sqrt{\det g(x)}}\frac{1}{\sqrt{2\pi\varepsilon}} \\ & \quad\times\int_A E[m^{\varepsilon}_{(1)}(1)\,\mathrm{Str}[K^{\varepsilon}_{(1)}(1)]\delta_{\bar{x}}(\bar{X}^{\varepsilon}_{(1)}(1))]P^{1;0}_{0;0}(d\varphi) \\ & \quad+\frac{1}{\sqrt{\det g(x)}}\frac{1}{\sqrt{2\pi\varepsilon}}\exp\left\{-\frac{2(x^m)^2}{\varepsilon^2}\right\} \\ & \quad\times\int_A E[m^{\varepsilon}_{(2)}(1)\,\mathrm{Str}[K^{\varepsilon}_{(2)}(1)]\delta_{\bar{x}}(\bar{X}^{\varepsilon}_{(2)}(1))]P^{1;0}_{0;0}(d\varphi)\end{aligned}$$

and by the above estimates (4.11), (4.12), (4.13), (4.19) and (4.20), we have

$$
\begin{aligned}
&\operatorname{Str}[e(\varepsilon^2, x, x)] \\
&= \frac{1}{\sqrt{2\pi\cdot\det g(x)}}\varepsilon^{-m}\int_A I_{\{\sigma^\varepsilon_{(1)}\le 1\}} \\
&\quad \times E[\operatorname{Str}[\Xi_1]\delta_0(\int_0^1 \sigma(\bar{x}, |x^m+\varepsilon\varphi(t)|)dw(t))]P^{1;0}_{0;0}(d\varphi) \\
(4.21)\quad &\quad +\frac{1}{\sqrt{2\pi\cdot\det g(x)}}\varepsilon^{-m}\exp\left\{-\frac{2(x^m)^2}{\varepsilon^2}\right\} \\
&\quad \times\int_A E[\operatorname{Str}[\Xi_2]\delta_0(\int_0^1 \sigma(\bar{x}, |x^m+\varepsilon\varphi^m(t)-2x^m t|)dw(t))]P^{1;0}_{0;0}(d\varphi) \\
&\quad +O(1)+\exp\left\{-\frac{2(x^m)^2}{\varepsilon^2}+\frac{Kx^m}{\varepsilon}\right\}\cdot O(1) \qquad \text{as} \quad \varepsilon\downarrow 0,
\end{aligned}
$$

for some $K>0$ which is independent of x and ε and $O(1)$ is uniform in x. By the reflection principle applied to the pinned process $\{\varphi(t), P^{1;0}_{0;0}\}$, we see easily that

$$
\begin{aligned}
&\int_A I_{\{\sigma^\varepsilon_{(1)}\le 1\}} E[\operatorname{Str}[\Xi_1]\delta_0(\int_0^1 \sigma(\bar{x}, |x^m+\varepsilon\varphi(t)|)dw(t))]P^{1;0}_{0;0}(d\varphi) \\
&= \exp\left\{-\frac{2(x^m)^2}{\varepsilon^2}\right\}\int_A E[\operatorname{Str}[\Xi_2]\delta_0(\int_0^1 \sigma(\bar{x}, |x^m(1-2t)+\varepsilon\varphi(t)|)dw(t))]P^{1;0}_{0;0}(d\varphi).
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&\operatorname{Str}[e(\varepsilon^2, x, x)] \\
&= \frac{2}{\sqrt{2\pi\cdot\det g(x)}}\varepsilon^{-m}\exp\left\{-\frac{2(x^m)^2}{\varepsilon^2}\right\} \\
(4.22)\quad &\quad \times\int_A E[\operatorname{Str}[\Xi_2]\delta_0(\int_0^1 \sigma(\bar{x}, |x^m(1-2t)+\varepsilon\varphi(t)|)dw(t))]P^{1;0}_{0;0}(d\varphi) \\
&\quad +O(1)+\exp\left\{-\frac{2(x^m)^2}{\varepsilon^2}+\frac{Kx^m}{\varepsilon}\right\}\cdot O(1) \qquad \text{as} \quad \varepsilon\downarrow 0.
\end{aligned}
$$

As is explained in § 3, we need to estimate the integral (3.6) in which a neighborhood U is of the form

$$U=\bar{U}\times\{0<x^m<\gamma\}, \qquad \bar{U}\subset \boldsymbol{R}^{m-1}.$$

Then

$$
\begin{aligned}
&\int_U \operatorname{Str}[e(\varepsilon^2, x, x)]m(dx) \\
(4.23)\quad &= \int_{\bar{U}}\sqrt{\det g(\bar{x}, 0)}\,d\bar{x}\left\{\int_0^\gamma \operatorname{Str}[e(\varepsilon^2, (\bar{x}, x^m), (\bar{x}, x^m))]\frac{\sqrt{\det g(\bar{x}, x^m)}}{\sqrt{\det g(\bar{x}, 0)}}dx^m\right\}.
\end{aligned}
$$

We set

$$(4.24)\qquad I(\varepsilon, \bar{x}; \gamma) = \int_0^\gamma \mathrm{Str}\,[e(\varepsilon^2, (\bar{x}, x^m), (\bar{x}, x^m))] \frac{\sqrt{\det g(\bar{x}, x^m)}}{\sqrt{\det g(\bar{x}, 0)}} dx^m .$$

Also we introduce the following notation:

$$A(\varepsilon, \gamma) = o_{\gamma\downarrow 0}(\varepsilon^k) \qquad \text{as} \quad \varepsilon \downarrow 0$$

if

$$\lim_{\gamma\downarrow 0} \left\{ \varlimsup_{\varepsilon\downarrow 0} \left[\frac{A(\varepsilon, \gamma)}{\varepsilon^k} \right] \right\} = 0 .$$

For example, $O(\varepsilon)+O(\gamma)=o_{\gamma\downarrow 0}(1)$ and $\exp\left\{-2\left(\frac{\gamma}{\varepsilon}\right)\right\}=o_{\gamma\downarrow 0}(\varepsilon^k)$ for every $k>0$. Our aim now is to obtain the following estimate:

$$(4.25)\qquad I(\varepsilon, \bar{x}; \gamma) = I(\bar{x}) + o_{\gamma\downarrow 0}(1) \qquad \text{as} \quad \varepsilon \downarrow 0 \text{ uniformly in } \bar{x} \in \boldsymbol{R}^{m-1}$$

and obtain an explicit form of $I(\bar{x})$. So we fix $\bar{x} \in \boldsymbol{R}^{m-1}$. Without loss of generality, we may assume that our local coordinate has been so chosen that $g_{ij}(\bar{x}, 0)=\delta_{ij}$, $\Gamma^k_{i,j}(\bar{x}, 0)=0$ for every $1\le i, j, k \le m-1$. Then, setting $x^m=\varepsilon\xi$, we see easily that

$$(4.26)\qquad \begin{aligned} \Xi^{\varepsilon}_{(2)}(t) = {} & \varepsilon^2 \left[\frac{1}{2} R_{ijkl}(\bar{x}, 0) a_i^* a_j a_k^* a_l \right] t - \varepsilon [\Gamma^i_{mj}(\bar{x}, 0) a_j^* a_i] \\ & \times \int_0^t I_{\{|\xi(1-2s)+\varphi(s)|>0\}} d\,|\xi(1-2s)+\varphi(s)| + C_{ij}(t) a_j^* a_i \end{aligned}$$

with $C_{ij}(t)=O(\varepsilon^2)\exp(K\xi)$ in $L_{\infty-}(\boldsymbol{D}^\infty)$. On the probability space $(A, P^{1;0}_{0;0}(d\varphi))$, the semimartingale $|\varphi(t)+\xi(1-2t)|$ has the following decomposition:

$$|\varphi(t)+\xi(1-2t)| - \xi = \int_0^t \mathrm{sgn}(\varphi(s)+\xi(1-2s)) d\varphi(s) + L(t, \xi)$$

where

$$(4.27)\qquad L(t, \xi) = \lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon} \int_0^t I_{(-\varepsilon,\varepsilon)}(\varphi(s)+\xi(1-2s)) ds .$$

Then

$$\begin{aligned} & \int_0^t I_{\{|\xi(1-2s)+\varphi(s)|>0\}} d\,|\xi(1-2s)+\varphi(s)| \\ & = \int_0^t \mathrm{sgn}\,(\varphi(s)+\xi(1-2s)) d\varphi(s) \\ & = |\varphi(t)+\xi(1-2t)| - \xi - L(t, \xi) . \end{aligned}$$

In particular

$$(4.28)\qquad \int_0^1 I_{\{|\xi(1-2s)+\varphi(s)|>0\}} d\,|\xi(1-2s)+\varphi(s)| = -L(1, \xi) .$$

Now we can deduce from (4.16), (4.17) and (4.28) that

$$
\begin{aligned}
\operatorname{Str}[\Xi_2] &= (-1)^{m(m-1)/2}\alpha_{\{1,\cdots,m-1,\hat{1},\cdots,\widehat{m-1}\}}\Big[\sum_{p=1}^{m-1}\frac{1}{p!}\Big\{\varepsilon^2\frac{1}{2}R_{ijkl}(\bar{x},0) \\
&\qquad \times a_i^* a_j a_k^* a_l + \varepsilon\Gamma_{mj}^i(\bar{x},0)a_j^* a_i L(1,\xi)\Big\}^p\Big] + O(\varepsilon^m)\exp\{K\xi\} \\
&= (-1)^{m(m-1)/2}\varepsilon^{m-1}\alpha_{\{1,\cdots,m-1,\hat{1},\cdots,\widehat{m-1}\}}\Big[\sum_{\substack{0\le\nu,\mu\le m-1\\ 2\nu+\mu=m-1}} \\
&\qquad \times\frac{1}{\nu!}\Big(\frac{1}{2}R_{ijkl}(\bar{x},0)a_i^* a_j a_k^* a_l\Big)^\nu \\
&\qquad \times\frac{1}{\mu!}(\Gamma_{mj}^i(\bar{x},0)a_j^* a_i L(1,\xi))^\mu\Big] + O(\varepsilon^m)\exp\{K\xi\} \\
&\qquad\qquad \text{in} \quad L_{\infty-}(\boldsymbol{D}^\infty).
\end{aligned} \tag{4.29}
$$

Also, setting $x^m=\varepsilon\xi$ again,

$$
\begin{aligned}
&\delta_0\Big(\int_0^1 \sigma(\bar{x}, |x^m(1-2t)+\varepsilon\varphi(t)|)dw(t)\Big) \\
&\quad = \delta_0(\sigma(\bar{x},0)w(1)) + O(\varepsilon)(1+\xi) \\
&\quad = \sqrt{\det g(\bar{x},0)}\,\delta_0(w(1)) + O(\varepsilon)(1+\xi) \qquad \text{in} \quad L_{\infty-}(\tilde{\boldsymbol{D}}^{-\infty}).
\end{aligned} \tag{4.30}
$$

Hence, by (4.22) and by changing the variable $x^m(=\varepsilon\xi)$ into ξ in (4.24), we have

$$
\begin{aligned}
I(\varepsilon,\bar{x},\gamma) &= (-1)^{m(m-1)/2}\sum_{\substack{0\le\nu,\mu\le m-1\\ 2\nu+\mu=m-1}}\frac{1}{\nu!}\frac{1}{\mu!} \\
&\qquad \times\alpha_{\{1,\cdots,m-1,\hat{1},\cdots,\widehat{m-1}\}}\Big[\Big(\frac{1}{2}R_{ijkl}(\bar{x},0)a_i^* a_j a_k^* a_l\Big)^\nu \\
&\qquad \times(\Gamma_{mj}^i(\bar{x},0)a_j^* a_i)^\mu\Big]E[\delta_0(w(1))]\frac{2}{\sqrt{2\pi}} \\
&\qquad \times\int_0^{\gamma/\varepsilon}\exp\{-2\xi^2\}E_{0;0}^{1;0}[L(1,\xi)^\mu]d\xi + O(\varepsilon) + O(\gamma) \\
&= (-1)^{m(m-1)/2}\frac{2}{(\sqrt{2\pi})^{m-1}}\sum_{\substack{0\le\nu,\mu=m-1\\ 2\nu+\mu=m-1}}\frac{1}{\nu!}\frac{1}{\mu!} \\
&\qquad \times\alpha_{\{1,\cdots,m-1,\hat{1},\cdots,\widehat{m-1}\}}\Big[\Big(\frac{1}{2}R_{ijkl}(\bar{x},0)a_i^* a_j a_k^* a_l\Big)^\nu \\
&\qquad \times(\Gamma_{mj}^i(\bar{x},0)a_j^* a_i)^\mu\Big]\frac{1}{\sqrt{2\pi}} \\
&\qquad \times\int_0^\infty \exp\{-2\xi^2\}E_{0;0}^{1;0}[L(1,\xi)^\mu]d\xi + o_{\gamma\downarrow 0}(1) \\
&\qquad\qquad \text{as} \quad \varepsilon\downarrow 0.
\end{aligned} \tag{4.31}
$$

(Here, $E^{1;0}_{0;0}[\Phi]=\int_A \Phi(\varphi)P^{1;0}_{0;0}(d\varphi)$ in notation.)

Lemma 4.2.

$$\int_0^\infty \frac{1}{\sqrt{2\pi}}\exp\{-2\xi^2\}E^{1;0}_{0;0}[L(1,\xi)^\mu]d\xi \tag{4.32}$$
$$= \frac{\mu!}{2^{\mu/2+2}\Gamma\left(\frac{\mu}{2}+1\right)}, \qquad \mu=0,1,2,\cdots.$$

Proof. Let $P^{1;b}_{0;a}$ be the probability law of pinned Brownian motion such that $x(0)=a$ and $x(1)=b$. If $l(t)=\lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_0^t I_{(-\varepsilon,\varepsilon)}(x(s))ds$, then clearly

$$E^{1;0}_{0;0}[L(1,\xi)^\mu]=E^{1;-\xi}_{0;\xi}[l(1)^\mu]$$

where $E^{1;b}_{0;a}$ denotes the expectation with respect to $P^{1;b}_{0;a}$. It is easy to see that

$$\frac{1}{\sqrt{2\pi}}\exp\{-2\xi^2\}E^{1;-\xi}_{0;\xi}[l(1)^\mu]$$
$$=\mu!\int\cdots\int_{0\le t_1\le t_2\le\cdots\le t_\mu\le 1} dt_1dt_2\cdots dt_\mu\frac{1}{\sqrt{t_1}}\exp\left(-\frac{\xi^2}{2t_1}\right)\frac{1}{\sqrt{2\pi(t_2-t_1)}}$$
$$\times\cdots\frac{1}{\sqrt{2\pi(t_\mu-t_{\mu-1})}}\frac{1}{\sqrt{2\pi(1-t_\mu)}}\exp\left(-\frac{\xi^2}{2(1-t_\mu)}\right).$$

Hence

$$\int_0^\infty\frac{1}{\sqrt{2\pi}}\exp\{-2\xi^2\}E^{1;-\xi}_{0;\xi}[l(1)^\mu]d\xi$$
$$=\frac{\mu!}{2\sqrt{2\pi}^\mu}\int\cdots\int_{0\le t_1\le t_2\le\cdots\le t_\mu\le 1}\frac{1}{\sqrt{t_\mu-t_{\mu-1}}\sqrt{t_{\mu-1}-t_{\mu-2}}\cdots\sqrt{t_2-t_1}\sqrt{1-t_\mu+t_1}} \tag{4.33}$$
$$\times dt_1\cdots dt_\mu$$

because

$$\int_0^\infty\frac{1}{\sqrt{2\pi t_1\cdot 2\pi(1-t_\mu)}}\exp\left(-\frac{\xi^2}{2t_1}-\frac{\xi^2}{2(1-t_\mu)}\right)d\xi=\frac{1}{2\sqrt{2\pi}}\frac{1}{\sqrt{1-t_\mu+t_1}}.$$

The integral (4.33) can be computed successively to obtain

$$(4.33)=\frac{\mu!}{2(\sqrt{2\pi})^\mu}B\left(\frac12,\frac12\right)B\left(\frac12,\frac22\right)\cdots B\left(\frac12,\frac{\mu-2}{2}\right)B\left(\frac32,\frac{\mu-1}{2}\right)$$
$$=\frac{\mu!}{2^{\mu/2+2}\Gamma\left(\frac{\mu}{2}+1\right)}.$$

By the Bianchi identity for the curvature tensor $\{R_{ijkl}\}$, it is easy to deduce that

$$\begin{aligned}(4.34)\qquad &\frac{1}{2}R_{ijkl}(\bar{x},0)a_i^*a_ja_k^*a_l\\ &=-\frac{1}{4}R_{ijkl}(\bar{x},0)a_i^*a_j^*a_ka_l+\text{lower terms in } a_i^* \text{ and } a_j\,.\end{aligned}$$

Also

$$(4.35)\qquad \Gamma_{mj}^i(\bar{x},0)a_j^*a_i=-\Gamma_{ij}^m(\bar{x},0)a_i^*a_j\,.$$

Now (4.31) implies our desired estimate (4.25) and, combining this with (4.32), (4.34) and (4.35), we see easily that $I(\bar{x})$ in (4.25) is given as follows:

$$\begin{aligned}(4.36)\qquad I(\bar{x})&=\frac{2}{(\sqrt{2\pi})^{m-1}}(-1)^{m(m-1)/2}\sum_{\substack{0\leq\nu,\mu\leq m-1\\ 2\nu+\mu=m-1}}\frac{1}{\nu!}\frac{1}{\mu!}\frac{\mu!}{2^{\mu/2+2}\Gamma\left(\frac{\mu}{2}+2\right)}\\ &\times\alpha_{\{1,\cdots,m-1,\hat{1},\cdots,\widehat{m-1}\}}\left[\left(-\frac{1}{4}R_{ijkl}(\bar{x},0)a_i^*a_j^*a_ka_l\right)^{\nu}\right.\\ &\left.\times(-\Gamma_{ij}^m(\bar{x},0)a_i^*a_j)^{\mu}\right]\\ &=\frac{2}{(\sqrt{2\pi})^{m-1}}\frac{(-1)^{m(m-1)/2}}{2^{1/2(m+3)}}\sum_{\substack{0\leq\nu,\mu\leq m-1\\ 2\nu+\mu=m-1}}(-1)^{\nu+\mu}\frac{1}{\nu!}\frac{1}{\Gamma\left(\frac{\mu}{2}+2\right)}\\ &\times\alpha_{\{1,\cdots,m-1,\hat{1},\cdots,\widehat{m-1}\}}\left[\left(\frac{1}{2}R_{ijkl}(\bar{x},0)a_i^*a_j^*a_ka_l\right)^{\nu}\right.\\ &\left.\times(\Gamma_{ij}^m(\bar{x},0)a_i^*a_j)^{\mu}\right]\end{aligned}$$

The estimate (4.25) is clearly uniform in $\bar{x}\in\boldsymbol{R}^{m-1}$.

We will proceed to calculate $I(\bar{x})$, $\bar{x}=(x^1,\cdots,x^{m-1})\in\boldsymbol{R}^{m-1}$. For this, we consider the following cases separately.

Case I. *m* is odd. We write $m-1=2n$. Then μ in (4.36) is even and we set $\mu=2p$. Now, by (4.36),

$$\begin{aligned}I(\bar{x})&=\frac{1}{2}\cdot\frac{(-1)^n}{\pi^n2^{2n}}\sum_{\substack{0\leq\nu,p\leq n\\ \nu+p=n}}\frac{1}{\nu!}\frac{1}{p!}\\ &\times\alpha_{\{1,\cdots,m-1,\hat{1},\cdots,\widehat{m-1}\}}\left[\left(-\frac{1}{2}R_{ijkl}(\bar{x},0)a_i^*a_j^*a_ka_l\right)^{\nu}\right.\\ (4.37)\qquad&\left.\times(\Gamma_{ij}^m(\bar{x},0)a_i^*a_j\Gamma_{kl}^m(\bar{x},0)a_k^*a_l)^{p}\right]\\ &=\frac{1}{2}\cdot\frac{1}{\pi^n2^{2n}}\alpha_{\{1,\cdots,m-1,\hat{1},\cdots,\widehat{m-1}\}}\left[\frac{1}{n!}\left(\frac{1}{2}\hat{R}_{ijkl}(\bar{x},0)a_i^*a_j^*a_ka_l\right)^{n}\right]\end{aligned}$$

where

$$\hat{R}_{ijkl}(\bar{x}, 0) = R_{ijkl}(\bar{x}, 0) + 2\Gamma^m_{ij}(\bar{x}, 0)\Gamma^m_{kl}(\bar{x}, 0) . \tag{4.38}$$

$\{\hat{R}_{ijkl}(\bar{x}, 0)\}_{0\le i,j,k,l\le m-1}$ can be identified with the curvature tensor of submanifold ∂M. Introducing the curvature form $\hat{\Omega}_{ij}$ for ∂M by

$$\hat{\Omega}_{ij} = \frac{1}{2}\sum_{k,l=1}^{m-1} \hat{R}_{ijkl}(\bar{x}, 0)dx^k \wedge dx^l , \qquad i, j = 1, 2, \cdots, m-1 , \tag{4.39}$$

we see easily that (4.37) is equivalent to the following:

$$\begin{aligned} &I(\bar{x})dx^1 \wedge \cdots \wedge dx^{m-1} \\ &= \frac{1}{2}\cdot\frac{1}{\pi^n 2^{2n} n!}\sum_{\sigma\in\mathfrak{S}(2n)} \operatorname{sgn}(\sigma)\hat{\Omega}_{\sigma(1)\sigma(2)} \wedge \hat{\Omega}_{\sigma(3)\sigma(4)} \wedge \cdots \wedge \hat{\Omega}_{\sigma(2n-1)\sigma(2n)} \end{aligned} \tag{4.40}$$

where $\mathfrak{S}(2n)$ is the symmetric group of order $2n$ (i.e., the group of permutations over $\{1, 2, \cdots, 2n\}$). By the definition of the Euler form for closed Riemannian manifold of even dimension (cf. [6]), we can finally identify $I(\bar{x})$ as

$$I(\bar{x})dx^1 \wedge \cdots \wedge dx^{m-1} = \frac{1}{2} e(T\partial M) \tag{4.41}$$

where $e(T\partial M)$ is the Euler form of ∂M.

Case II. *m* is even. We write $m=2n$. In (4.36),

$$\mu = m-1-2\nu = 2n-2\nu-1$$

and

$$\Gamma\left(\frac{\mu}{2}+1\right) = \Gamma\left(n-\nu+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{n-\nu}} 1\cdot 3 \cdots (2n-2\nu-1) .$$

Introducing $\Omega_{ij} \in \Lambda_2(M)|_{\partial M}$ and $\omega_{im} \in \Lambda(M)|_{\partial M}$ by

$$\Omega_{ij} = \frac{1}{2}\sum_{k,l=1}^{m-1} R_{ijkl}(\bar{x}, 0)dx^k \wedge dx^l, \qquad i, j = 1, 2, \cdots, m-1 , \tag{4.42}$$

and

$$\omega_{im} = \sum_{j=1}^{m-1} \Gamma^m_{ij}(\bar{x}, 0)dx^j, \qquad i = 1, 2, \cdots, m-1 , \tag{4.43}$$

we see easily that (4.36) is equivalent to the following:

$$I(\bar{x})dx^1 \wedge \cdots \wedge dx^{m-1} = \sum_{\nu=0}^{n-1} Q_{\nu,m}(TM, T\partial M) \tag{4.44}$$

where

$$\begin{aligned} Q_{\nu,m}(TM,\ T\partial M) = & \frac{1}{\pi^n \nu! 2^{n+\nu} 1\cdot 3 \cdots (2n-2\nu-1)} \\ & \times \sum_{\sigma\in\mathfrak{S}(2n-1)} \operatorname{sgn}(\sigma)\Omega_{\sigma(1)\sigma(2)}\wedge\cdots\wedge\Omega_{\sigma(2\nu-1)\sigma(2\nu)} \\ & \wedge\omega_{\sigma(2\nu+1)m}\wedge\cdots\wedge\omega_{\sigma(2n-1)m}\,. \end{aligned} \tag{4.45}$$

Now we return to our starting situation of compact Riemannian manifold M with boundary. We have obtained above that, for any coordinate neighborhood U of a boundary point in the form $U=\{x=(\bar{x},\ x^m);\ \bar{x}=(x^1,\cdots,x^{m-1})\in\bar{U},\ 0\leq x^m<\gamma\}$,

$$\int_U e(\varepsilon^2,\ x,\ x)m(dx) = \int_{\bar{U}} e(T\partial M)+o_{\gamma\downarrow 0}(1) \qquad \text{as} \quad \varepsilon\downarrow 0$$

if m is odd, and

$$\int_U e(\varepsilon^2,\ x,\ x)m(dx) = \int_{\bar{U}} \sum_{\nu=0}^{n-1} Q_{\nu,m}(T\partial M)+o_{\gamma\downarrow 0}(1) \qquad \text{as} \quad \varepsilon\downarrow 0$$

if m is even. If $U\subset\subset M$, then we know that

$$\int_U e(\varepsilon^2,\ x,\ x)m(dx) = o(1) \qquad \text{as} \quad \varepsilon\downarrow 0$$

if m is odd, and

$$\int_U e(\varepsilon^2,\ x,\ x)m(dx) = \int_U e(TM)+o(1) \qquad \text{as} \quad \varepsilon\downarrow 0$$

if m is even. Therefore, we can finally deduce the following (Gilkey [5], [6]):

Theorem 4.1 (Gauss-Bonnet-Chern theorem for manifolds with boundary).

$$\chi(M) = \frac{1}{2}\int_{\partial M} e(T\partial M) \qquad \textit{if } m \textit{ is odd,}$$

and

$$\chi(M) = \int_M e(TM)+\int_{\partial M}\sum_{\nu=0}^{n-1} Q_{\nu,m}(TM,\ T\partial M) \qquad \textit{if } m=2n \textit{ is even}\,.$$

REMARK 4.1. When $\partial M\neq\phi$, the Euler-Poincaré characteristic $\chi(M,\ \partial M)$ with respect to the relative homology $H_*(M,\ \partial M)$ is also well-known. For this $\chi(M,\ \partial M)$, the following formula is known (Gilkey [6] p. 246):

$$\chi(M,\ \partial M) = \int_M \operatorname{Str}[e_{\mathrm{rel}}(t,\ x,\ x)]m(dx)$$

where $e_{\mathrm{rel}}(t,\ x,\ x)$ is the fundamental solution for the following heat equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\Box u \\ \lim\limits_{t \downarrow 0} u(t, \cdot) = f \\ u \quad \text{satisfies the relative boundary conditions}\,. \end{cases}$$

Here $\omega \in \Lambda(M)$ is said to satisfy the *relative boundary conditions* (Conner [3], Ray-Singer [15]) if

$$\omega_{\tan} = 0 \quad \text{and} \quad (d^*\omega)_{\tan} = 0 \qquad \text{on} \quad \partial M\,.$$

For this fundamental solution, it holds that

$$e_{\mathrm{rel}}(t, x, y) = *^{-1} e(t, x, y) *$$

where $*$ is the Hodge $*$-operator. By using this fact, we can easily see that

$$\chi(M, \partial M) = (-1)^m \chi(M)\,.$$

Thus the Gauss-Bonnet-Chern theorem for $\chi(M, \partial M)$ is easily deduced from Theorem 4.1.

REMARK 4.2. In this paper, we follow the definition of the curvature tensor $\{R_{ijkl}\}$ to [9]: Note that the curvature tensor in [4], [6], [14] is of opposite sign and therefore certain powers of -1 appear in the definition of $e(T\partial M)$, $e(TM)$ and $Q_{\nu,m}(TM, T\partial M)$.

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