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ON SUBSECTIONS OF BLOCKS AND BRAUER PAIRS

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Introduction

In Broué [6] and Okuyama [13] relations between blocks of a finite group and those of its subgroups are studied. In Section 1 of the present paper, we show the following theorem. Let H be a subgroup of a group G. Let B and b be blocks of G and H, respectively. Put $\sigma(B,b) = |G:H|\zeta_b(1)/\zeta(1)$ for an irreducible character ζ in B, where ζ_b is the b-component of ζ_H.

Theorem 1.7. With the notation above, assume that $d(B) \geq d(b)$. Let χ be the character of G afforded by a virtually irreducible RG-module in B (in the sense of Knörr [10]). Then

$$\frac{|G:H|\chi_b(a)}{\chi(1)} \equiv \sigma(B,b)\omega_b(a) \mod \pi \qquad \text{for all } a \in ZRH.$$

Further, the following are equivalent.

(i) $\sigma(B,b) \not\equiv 0 \mod p$.

(ii) B and b are linked, and B and b have a common defect group.

(For notation and the definition of "linked", see below.) This theorem reformulates some of Broué's results in [6]. Furthermore, some of Brauer's results in [2] are immediate consequences of this theorem.

In Section 2, we study the invariant $\sigma(B,b)$ in the case when $b^G = B$ and $d(b) = d(B)$.

In Section 3 we consider the canonical characters of Brauer pairs. If P is a p-subgroup of G and b_P is a block of $PC_G(P)$ with defect group P, we call (P, b_P) a Brauer pair in G. Let θ_P be the canonical character of b_P, i.e. θ_P is a unique irreducible character in b_P which is trivial on P. Let (P, b_P) and (Q, b_Q) be Brauer pairs such that $P \rhd Q$. Under this condition, a necessary and sufficient condition for $(b_P)^{PC_G(Q)} = (b_Q)^{PC_G(Q)}$ (i.e. b_P and b_Q are linked) involving the multiplicity $(\theta_Q, \theta_P)_{C_G(P)}$ is known (Brauer [1, (6D)]). We shall improve that condition slightly and show that if b_P and b_Q are linked, then $(\theta_Q, \theta_P)_{C_G(P)} \equiv \pm 1 \pmod p$ (Theorem 3.5).

Notation and terminology

Let us introduce some notation. Let G be a finite group and p a prime. Let (K, R, k) be a p-modular system. We assume that K contains a primitive $|G|$-th root of unity. Let πR be the maximal ideal of R and let ν be the valuation of K normalized so that $\nu(p) = 1$. For an irreducible character χ of G, let ω_χ be the central character of KG corresponding to χ. Let B be a block of G with defect $d(B)$. Put $\omega_B = \omega_\chi$ for an irreducible character χ in B. Let e_B be the block idempotent of RG corresponding to B. Let $G_{p'}$ be the set of p'-elements of G. For an R-linear combination θ of irreducible characters or irreducible Brauer characters in B, we define, $ht(\theta)$, the height of θ, by $ht(\theta) = \nu(\theta(1)) - \nu|G| + d(B)$, and put $\theta^* = \sum \theta(x^{-1})x$, where x runs through $G_{p'}$. For a block b of a subgroup H of G and an R-linear combination θ of irreducible characters of G, let θ_b be the b-component of θ_H. An R-linear combination of irreducible characters in B is called an R-generalized character in B.

Let ZRG be the center of RG. Put

$$Z_0(B) = \{a \in ZRG_{p'}e_B;\quad \omega_B(a) \not\equiv 0 \mod \pi\},$$

where $ZRG_{p'}$ is the R-submodule of ZRG spanned by p-regular conjugacy class sums. Let $s_H : RG \to RH$ be the R-linear map defined by $s_H(x) = x$ if $x \in H$ and $s_H(x) = 0$ if $x \in G - H$. As in [12], we say that B and b are linked if $s_H(Z_0(B))e_b \subseteq Z_0(b)$. Let Tr_H^G be the relative trace map, when RG is considered as a G-algebra in the usual way.

Let (P, b_P) be a Brauer pair in G. We call (P, b_P) a B-Brauer pair if $(b_P)^G = B$. For the canonical character θ_P of b_P, we put $n(P, b_P) = |PC_G(P)|/|P|\theta_P(1)$. Since θ_P may be regarded as an irreducible character of defect 0 of $PC_G(P)/P$, $n(P, b_P)$ is an integer prime to p.

For the definition of virtually irreducible RG-modules (lattices) and basic properties of them, see Knörr [10].

1. Blocks of subgroups and some results on subsections

Throughout this section, we use the following notation: Let G be a finite group and H a subgroup of G. Let B and b be blocks of G and H, respectively.

Lemma 1.1. *Let θ be an R-generalized character in B. Let ξ be an irreducible character of height 0 in b. Then, for $a \in ZRH$, $\theta_b(a)/\xi(1)$ lies in R and*

$$\frac{\theta_b(a)}{\xi(1)} \equiv \frac{\theta_b(1)}{\xi(1)}\omega_\xi(a) \mod \pi.$$

Proof. Put $\theta_b = \sum r_\eta \eta$, where $r_\eta \in R$ and η ranges over the irreducible characters in b. Then

$$\begin{aligned}\frac{\theta_b(a)}{\xi(1)} &= \sum r_\eta \frac{\eta(a)}{\eta(1)}\frac{\eta(1)}{\xi(1)} \\ &\equiv \sum r_\eta \omega_\xi(a)\frac{\eta(1)}{\xi(1)} \quad \bmod \pi \\ &\equiv \frac{\theta_b(1)}{\xi(1)}\omega_\xi(a) \quad \bmod \pi,\end{aligned}$$

as required. □

Lemma 1.2. *Let θ be an R-generalized character in B. Let ζ be an irreducible character of height 0 in B. Then*

$$\omega_\zeta(\theta^*) \equiv \frac{\theta(1)}{\zeta(1)}\omega_\zeta(\zeta^*) \quad \bmod \pi.$$

Proof. A direct computation shows that $\omega_\zeta(\theta^*) = \theta(\zeta^*)/\zeta(1)$. So the result follows from Lemma 1.1 (with $H = G$, $b = B$, $\xi = \zeta$ and $a = \zeta^*$). □

The following proposition is proved in Osima [14, Lemma 1]; the proof below is just a slight modification of Osima's.

Proposition 1.3. *Let χ be an irreducible character of height 0 in B. Then*

$$\frac{|G|\chi(1)}{\mathrm{rank}_R RGe_B} \equiv \omega_\chi(\chi^*) \not\equiv 0 \quad \bmod p.$$

Proof. Put $\theta = \sum |G|^{-1} n(\phi)\phi$, where ϕ ranges over the irreducible Brauer characters in B and $n(\phi)$ is the degree of the projective cover of a module affording ϕ. Then θ is the restriction on $G_{p'}$ of an R-generalized character in B. So, by Lemma 1.2,

$$\omega_\chi(\theta^*) \equiv \frac{\theta(1)}{\chi(1)}\omega_\chi(\chi^*) \quad \bmod \pi,$$

Also $\nu(\theta(1)/\chi(1)) \geq 0$. Further, we have $e_B = \theta^*$. Thus $\omega_\chi(\theta^*) = 1$. Hence $\nu(\theta(1)/\chi(1)) = 0$ and $\chi(1)/\theta(1) \equiv \omega_\chi(\chi^*) \bmod \pi$. Since $\theta(1) = \mathrm{rank}_R RGe_B/|G|$ and $\omega_\chi(\chi^*)$ is a rational integer, the result follows. □

REMARK 1.4. Theorem 1.3 of [12] follows from Lemma 1.2 and Proposition 1.3.

The proof of the following lemma is a simple application of known facts, cf. Broué [6, (P1) in Section 1.1].

Lemma 1.5. *Let χ be the character afforded by a virtually irreducible RG-module in B and let ζ be an irreducible character in B. Let $a \in ZRH$. Then*
(i) $|G : H|\chi_b(a)/\chi(1)$ *lies in R.*
(ii) $|G : H|\chi_b(a)/\chi(1) \equiv |G : H|\zeta_b(a)/\zeta(1) \mod \pi$.

Proof. Clearly we may assume $a \in ZRHe_b$. Define $\omega : RG \to K$ by $\omega(x) = \chi(x)/\chi(1)$, $x \in RG$. Then, for $x \in ZRG$,
(1) $\omega(x) \in R$, and
(2) $\omega(x) \equiv \omega_\zeta(x) \mod \pi$,
cf. 1.7 Remark of [10]. Then for $a \in ZRHe_b$,
(3) $\omega(\mathrm{Tr}_H^G(a)) = |G : H|\chi_b(a)/\chi(1)$.
So (i) follows from (1). Since a formula similar to (3) holds for ω_ζ, (ii) follows from (2). □

Lemma 1.6. *Assume that $d(b) = d(B)$. Then the following are equivalent.*
(i) *B and b are linked.*
(ii) *For every R-generalized character θ in B with $ht(\theta) = 0$, θ_b is of height 0.*
(iii) *For some R-generalized character θ in B, θ_b is of height 0.*
(iv) *For some irreducible character ζ in B with $ht(\zeta) = 0$, ζ_b is of height 0.*
(v) *For some $a \in Z_0(B)$, $s_H(a)e_b \in Z_0(b)$.*

Further, if these conditions are satisfied, then B and b have a common defect group.

Proof. (i) ⇔ (ii) : See [12, Corollary 1.5].
(ii) ⇒ (iii) : Trivial.
(iii) ⇒ (iv) : It follows that there is an irreducible character ζ such that $ht(\zeta_b) = 0$. Then, since $|G : H|\zeta_b(1)/\zeta(1)$ lies in R, we see $ht(\zeta) = 0$.
(iv) ⇒ (ii) : Put $\theta = \sum r_\chi \chi$, where χ ranges over the irreducible characters in B and $r_\chi \in R$. Then, applying Lemma 1.5 with $a = e_b$, we obtain

$$\begin{aligned}\frac{\theta_b(1)|G|}{\zeta(1)|H|} &= \sum r_\chi \frac{\chi_b(1)|G|}{\chi(1)|H|}\frac{\chi(1)}{\zeta(1)} \\ &\equiv \frac{\zeta_b(1)|G|}{\zeta(1)|H|}\frac{\theta(1)}{\zeta(1)} \not\equiv 0 \mod \pi.\end{aligned}$$

So θ_b is of height 0.
(v) ⇒ (iii) : There is an R-generalized character θ such that $\theta^* = a$ by [12, Corollary 1.4]. Then $(\theta_b)^* = s_H(a)e_b \in Z_0(b)$. So $ht(\theta_b) = 0$ by [12, Theorem 1.3].
(i) ⇒ (v) : Trivial.

If B and b are linked, then the standard argument using defect classes of blocks shows that a defect group of b is contained in a defect group of B. So the last assertion follows. This completes the proof. □

For the character χ of G afforded by a virtually irreducible RG-module in B, put $\sigma(B, b) = |G : H|\chi_b(1)/\chi(1)$. By Lemma 1.5, $\sigma(B, b)$ lies in R, and $\sigma(B, b)$ modulo p is determined uniquely by B and b only (and does not depend on the choice of χ).

The following theorem may be considered as a reformulation of some of Broué's results in [6], see Remark 1.8 below. See also Okuyama [13, Corollary 1].

Theorem 1.7. *Assume that $d(B) \geq d(b)$. Let χ be the character of G afforded by a virtually irreducible RG-module in B. Then*

$$\frac{|G : H|\chi_b(a)}{\chi(1)} \equiv \sigma(B, b)\omega_b(a) \mod \pi \qquad \text{for all } a \in ZRH.$$

Further, the following are equivalent.

(i) $\sigma(B, b) \not\equiv 0 \mod p$.

(ii) *B and b are linked, and B and b have a common defect group.*

Proof. To prove the first assertion, we may assume $a \in ZRHe_b$. Let ζ be an irreducible character of height 0 in B. Then, by Lemma 1.5, we get

$$\frac{|G : H|\chi_b(a)}{\chi(1)} \equiv \frac{|G : H|\zeta_b(a)}{\zeta(1)} \mod \pi.$$

Let ξ be an irreducible character of height 0 in b. We can write $|G : H|\zeta_b(a)/\zeta(1) = (|G : H|\xi(1)/\zeta(1))(\zeta_b(a)/\xi(1))$. Then $|G : H|\xi(1)/\zeta(1)$ lies in R, since $d(B) \geq d(b)$. Further

$$\frac{\zeta_b(a)}{\xi(1)} \equiv \frac{\zeta_b(1)}{\xi(1)}\omega_\xi(a) \mod \pi$$

by Lemma 1.1. So we get

$$\begin{aligned}\frac{|G : H|\chi_b(a)}{\chi(1)} &\equiv \frac{|G : H|\xi(1)}{\zeta(1)}\frac{\zeta_b(1)}{\xi(1)}\omega_\xi(a) \mod \pi \\ &\equiv \frac{|G : H|\zeta_b(1)}{\zeta(1)}\omega_\xi(a) \mod \pi.\end{aligned}$$

Since $|G : H|\zeta_b(1)/\zeta(1) \equiv \sigma(B, b) \bmod \pi$, the first assertion is proved. Further, since

$$\nu\left(\frac{|G : H|\zeta_b(1)}{\zeta(1)}\right) = d(B) - d(b) + ht(\zeta_b),$$

we see that $\sigma(B, b) \not\equiv 0 \bmod \pi$ if and only if $d(B) = d(b)$ and $ht(\zeta_b) = 0$. Since $ht(\zeta) = 0$, the last condition is equivalent to (ii) by Lemma 1.6. This completes the proof. □

REMARK 1.8. (i) Proposition 2.1.1 (b) of Broué [6] states if $d(B) \geq d(b)$, then

$$\mathrm{Tr}_H^G(J(ZRH)e_b)e_B \subseteq J(ZRG)e_B. \tag{1.1}$$

The formula in the above theorem, namely,

$$\frac{|G:H|\chi_b(a)}{\chi(1)} \equiv \sigma(B,b)\omega_b(a) \mod \pi \qquad \text{for all } a \in ZRH, \tag{1.2}$$

may be considered as a restatement of (1.1). In fact, if $a = e_b$, then (1.2) is true by definition. On the other hand, if $a \in J(ZRH)e_b$, then (1.2) yields $|G : H|\chi_b(a)/\chi(1) \equiv 0 \mod \pi$, which is (1.1).

(ii) By Proposition 2.2.2 (a) of Broué [6], if $d(B) \geq d(b)$, we obtain

$$\sigma(B,b) \equiv \lambda(B,b)\omega_b(s_H(e_B)) \mod \pi, \tag{1.3}$$

where $\lambda(B,b) = |G : H|^2 \mathrm{rank}_R RHe_b/\mathrm{rank}_R RGe_B$. From this, we can obtain the equivalence (i) $\Leftrightarrow$ (ii) in Theorem 1.7. In fact,

$$\begin{aligned}\text{(i)} &\iff d(B) = d(b) \quad \text{and} \quad \omega_b(s_H(e_B)) \not\equiv 0 \mod \pi \ \text{(by (1.3))}\\ &\iff \text{(ii) (by Lemma 1.6 (v)).}\end{aligned}$$

For the value of $\sigma(B,b)$, see Section 2 below.

The following corollary (and Lemma 1.5) extends Brauer [2, (3E)(i), (iii), (3F), (4C)] and Okuyama [13, Theorem 1].

Corollary 1.9. *Let χ be the character of G afforded by a virtually irreducible RG-module in B. Then, for $x \in H$,*

$$\nu(\chi_b(x)) \geq \nu|C_H(x)| - d(B) + ht(\chi).$$

If $d(B) \geq d(b)$, then the equality holds if and only if B and b are linked, B and b have a common defect group and $\omega_b(\hat{K}_x) \not\equiv 0 \mod \pi$, where $\hat{K}_x$ denotes the class sum of the conjugacy class of H containing x.

Proof. Apply Lemma 1.5 (i) and Theorem 1.7 with $a = \hat{K}_x$. □

The following is a special case of Corollary 1.9. For different proofs, see Broué [5, Proposition 3.4.1], Watanabe [16, Lemma] (see also Corollary 2.6 below).

Corollary 1.10 (Brauer [2, (4C)]). *Let u be a p-element of G. Let B be a block of G and let b be a block of $C_G(u)$ such that $b^G = B$ and that $d(b) = d(B)$; that is, (u,b) is a major subsection associated with B. Let χ be an irreducible character in*

B. Then

$$\nu(\chi_b(u)) = \nu|C_G(u)| - d(B) + ht(\chi).$$

In particular, $\chi_b(u) \neq 0$.

Proof. We apply Corollary 1.9 for $C_G(u)$ and u in place of H and x. Since $b^G = B$, B and b are linked ([12, Proposition 1.6]). So it suffices to show that $\omega_b(u) \not\equiv 0 \bmod \pi$. But this is verified immediately. □

The following extends Brauer [2, (5G), (5H)]. For a different proof of [2, (5H)], see Broué [5, Proposition 3.4.2].

Corollary 1.11. *Let u be a central p-element of H. Assume that b^G is defined. Let χ be the character of G afforded by a virtually irreducible RG-module in B and let ζ be an R-generalized character in B. Then*

$$\nu\left(\sum_y \chi_b(uy)\zeta_b(u^{-1}y^{-1})\right) \geq \nu|H| - d(B) + ht(\chi),$$

where y runs through $H_{p'}$. Further, if $d(B) \geq d(b)$, then the equality holds if and only if $b^G = B$, $d(b) = d(B)$ and $ht(\zeta) = 0$.

Proof. We put $\psi(y) = \zeta_b(u^{-1}y)$, $y \in H_{p'}$. Then the left side of the above inequality is $\nu(\chi_b(u\psi^*))$. Applying Lemma 1.5 (i) with $a = u\psi^*$, we get the inequality. Further, since b^G is defined, $b^G = B$ if and only if B and b are linked ([12, Proposition 1.6]). Thus, by Theorem 1.7, the result follows if we show that $\omega_b(u\psi^*) \not\equiv 0 \bmod \pi$ if and only if $ht(\zeta) = 0$. Now $\omega_b(u\psi^*) \equiv \omega_b(\psi^*) \bmod \pi$ and

$$\begin{aligned} &\omega_b(\psi^*) \not\equiv 0 \mod \pi \\ \Longleftrightarrow\ & ht(\psi) = 0 \text{ (by [12, Theorem 1.3], since } \psi \text{ belongs to } b) \\ \Longleftrightarrow\ & ht(\zeta_b) = 0 \text{ (by Lemma 1.1)} \\ \Longleftrightarrow\ & ht(\zeta) = 0 \text{ (by [12, Proposition 1.7 (ii)]).} \end{aligned}$$

This completes the proof. □

Corollary 1.12 (Brauer and Feit [4]). *Let χ and ζ be irreducible characters in B. Assume that ζ is of height 0. Then*

$$\nu\left(\sum_y \chi(y)\zeta(y^{-1})\right) = \nu|G| - d(B) + ht(\chi),$$

where y runs through $G_{p'}$.

Proof. In Corollary 1.11, let $H = G$, $b = B$ and $u = 1$. □

As an application of Lemma 1.1, we have the following.

Proposition 1.13. *Assume that* b^G *is defined. Let* χ *be an R-generalized character in* B *and let* $a \in ZRH$. *Then*

$$\nu(\chi_b(a)) \geq \nu|H| - d(b)$$

and the equality holds if and only if $b^G = B$, $ht(\chi) = 0$ *and* $\omega_b(a) \not\equiv 0 \bmod \pi$.

Proof. Let ξ be an irreducible character of height 0 in b. Then by Lemma 1.1, $\chi_b(a)/\xi(1) \in R$, so the inequality follows. Also

$$\frac{\chi_b(a)}{\xi(1)} \equiv \frac{\chi_b(1)}{\xi(1)}\omega_b(a) \mod \pi.$$

Thus the equality holds if and only if $ht(\chi_b) = 0$ and $\omega_b(a) \not\equiv 0 \bmod \pi$. Since b^G is defined, $ht(\chi_b) = 0$ if and only if $b^G = B$ and $ht(\chi) = 0$ by [12, Proposition 1.7 (ii)]. This completes the proof. □

The following strengthens Broué [7, Proposition 1] (see also Brauer [2, (3B)]).

Corollary 1.14. *Let* u *be a central p-element of* H. *Assume that* b^G *is defined. Let* χ *be an R-generalized character in* B. *Then*

$$\nu(\chi_b(u)) \geq \nu|H| - d(b)$$

and the equality holds if and only if $b^G = B$ *and* $ht(\chi) = 0$.

Proof. In Proposition 1.13, let $a = u$. □

The following extends [12, Proposition 1.13] and Broué [7, Corollary 2].

Corollary 1.15. *Let* u *be a central p-element of* H. *Assume that* b^G *is defined. Let* χ *and* ζ *be R-generalized characters in* B. *Then*

$$\nu\left(\sum_y \chi_b(uy)\zeta_b(u^{-1}y^{-1})\right) \geq \nu|H| - d(b),$$

where y runs through $H_{p'}$. *Further, the equality holds if and only if* $b^G = B$, $ht(\chi) = 0$ *and* $ht(\zeta) = 0$.

Proof. Put $\psi(y) = \zeta_b(u^{-1}y)$, $y \in H_{p'}$. The left side of the above inequality is $\nu(\chi_b(u\psi^*))$. So, applying Proposition 1.13 with $a = u\psi^*$, we get the inequality. It remains to show $\omega_b(u\psi^*) \not\equiv 0 \bmod \pi$ if and only if $ht(\zeta) = 0$. This is proved as in the proof of Corollary 1.11. □

Proposition 1.16 (Broué [7, (C)]). *Let u be a central p-element of H. Assume that b^G is defined and equal to B. Let χ be an R-generalized character in B. Then*

$$\frac{|G|\chi(1)}{\operatorname{rank}_R RGe_B} \equiv \frac{|H|\chi_b(u)}{\operatorname{rank}_R RHe_b} \mod \pi.$$

Proof. If χ is an irreducible character of positive height in B, then both sides are congruent to 0 modulo π by Proposition 1.3 and Corollary 1.14. So we may assume that χ is an irreducible character of height 0 in B. Then by Proposition 1.3,

$$\frac{|G|\chi(1)}{\operatorname{rank}_R RGe_B} \equiv \omega_\chi(\chi^*) \mod \pi.$$

On the other hand, if ξ is an irreducible character of height 0 in b, then

$$\begin{aligned}\frac{|H|\chi_b(u)}{\operatorname{rank}_R RHe_b} &\equiv \frac{\chi_b(u)}{\xi(1)}\omega_\xi(\xi^*) \mod \pi \text{ (by Proposition 1.3)}\\ &\equiv \frac{\chi_b(1)}{\xi(1)}\omega_\xi(\xi^*) \mod \pi \text{ (by Lemma 1.1)}\\ &\equiv \omega_\xi((\chi_b)^*) \mod \pi \text{ (by Lemma 1.2)}\\ &\equiv \omega_\chi(\chi^*) \mod \pi \text{ (since } b^G = B).\end{aligned}$$

So the result follows. □

2. The invariant $\sigma(B, b)$

Let B be a block of a group G with defect group D. Let b be a block of a subgroup of G. In this section we consider the value of $\sigma(B, b)$ in the case when $b^G = B$ and $d(b) = d(B)$. Of course the most fundamental is the case when b is the Brauer correspondent of B in $N_G(D)$. In this case we have the following, which is a variant of Sylow's Third Theorem (consider the case of principal blocks). We note that this theorem is a consequence of the formula (35) in the proof of Theorem III.8.19 of [11]. Here we give an alternative (character-theoretical) proof.

Theorem 2.1. *Let B' be the Brauer correspondent of B in $N_G(D)$. Then $\sigma(B, B') \equiv 1 \bmod p$.*

Proof. Let χ and ξ be irreducible characters of height 0 in B and B', respectively. Since $B'^G = B$, we get by Lemma 1.2,

$$\omega_\chi(\chi^*) \equiv \omega_\xi((\chi_{B'})^*) \equiv \frac{\chi_{B'}(1)}{\xi(1)}\omega_\xi(\xi^*) \mod \pi. \tag{2.1}$$

Let S be a set of representatives of the p'-conjugacy classes of G with defect group D. We choose S so that $C_G(y) \geq D$ for $y \in S$. Let K_y be the conjugacy class of G containing y, $y \in S$. Then, as is well-known,

$$\omega_\chi(\chi^*) \equiv \sum_{y \in S} \omega_\chi(\hat{K}_y)\chi(y^{-1}) \mod \pi. \tag{2.2}$$

Put $L_y = K_y \cap C_G(D)$, $y \in S$. Then it is also well-known that $\{L_y; y \in S\}$ is exactly the set of p'-conjugacy classes of $N_G(D)$ with defect group D. Then, as in (2.2), we have

$$\omega_\xi(\xi^*) \equiv \sum_{y \in S} \omega_\xi(\hat{L}_y)\xi(y^{-1}) \mod \pi. \tag{2.3}$$

On the other hand, we have

$$\omega_\chi(\hat{K}_y) \equiv \omega_\xi(\hat{L}_y) \mod \pi. \tag{2.4}$$

Further, since a formula similar to (2.4) is true for y^{-1} in place of y, $y \in S$, we obtain

$$\begin{aligned} \chi(y^{-1}) &\equiv \frac{\chi(1)|N_G(D)|}{\xi(1)|G|}|C_G(y) : C_G(y) \cap N_G(D)|\xi(y^{-1}) \mod \pi \\ &\equiv \frac{\chi(1)|N_G(D)|}{\xi(1)|G|}\xi(y^{-1}) \mod \pi, \end{aligned} \tag{2.5}$$

since D is a p-Sylow subgroup of $C_G(y)$. On substituting (2.4) and (2.5) into (2.2), we obtain by (2.3),

$$\omega_\chi(\chi^*) \equiv \frac{\chi(1)|N_G(D)|}{\xi(1)|G|}\omega_\xi(\xi^*) \mod \pi. \tag{2.6}$$

Comparison of (2.1) and (2.6) shows that

$$\frac{\chi_{B'}(1)|G|}{\chi(1)|N_G(D)|} \equiv 1 \mod \pi,$$

since $\omega_\xi(\xi^*) \not\equiv 0 \mod \pi$ (by Proposition 1.3). This completes the proof. □

REMARK 2.2. (i) In fact, Theorem 2.1 and Corollary 2.4 (i) below follow from Brauer [2, (2D)].

(ii) Still another proof of 2.1 is available; by Remark 1.8 (ii)

$$\sigma(B, B') \equiv \frac{|G : N_G(D)|^2 \operatorname{rank}_R RN_G(D)e_{B'}}{\operatorname{rank}_R RGe_B} \mod p.$$

As an $R[G \times G]$-module, RGe_B is indecomposable and its Green correspondent with respect to $(G \times G, \Delta(D), N_G(D) \times N_G(D))$ is $RN_G(D)e_{B'}$, where $\Delta(D) = \{(x, x); x \in D\}$. From this, the result follows.

Let (D, b_D) be a B-Brauer pair. Let $e(B)$ be the inertial index of B.

Corollary 2.3. *We have $\sigma(B, b_D) \equiv e(B) \mod p$.*

Proof. Let B' be the Brauer correspondent of B in $N_G(D)$. Let T be the inertial group of b_D in $N_G(D)$. Then it is easy to see

$$(\chi_{B'})_{DC_G(D)} = \sum \chi_{(b_D)^x} = \sum (\chi_{b_D})^x$$

where x runs through a transversal of T in $N_G(D)$. From this we get the result by Theorem 2.1. □

Corollary 2.4. *Let χ be an irreducible character of height 0 in B. Let θ_D be the canonical character of b_D. Then*

(i) *As Brauer characters, $\chi_{b_D} = m\theta_D$, where m is an integer such that*

$$m \equiv \frac{e(B)|DC_G(D)|\chi(1)}{|G|\theta_D(1)} \mod p.$$

(ii) $(|G|/|D|\chi(1))\omega_\chi(\chi^*) \equiv e(B)n(D, b_D)^2 \mod p.$

Proof. (i) This follows from Corollary 2.3.

(ii) Put $\theta = \theta_D$. Since $(b_D)^G = B$, we get by Lemma 1.2

$$\omega_\chi(\chi^*) \equiv \omega_\theta((\chi_{b_D})^*) \equiv \frac{\chi_{b_D}(1)}{\theta(1)}\omega_\theta(\theta^*) \mod \pi.$$

Then, since $\omega_\theta(\theta^*) = |DC_G(D)|/|D|\theta(1)$, the result follows from (i). □

Proposition 2.5. *Let b be a block of a subgroup H of G. Assume that b^G is defined and equal to B and that b has defect group D. Let (D, β_D) be a b-Brauer pair in H. Then*

$$\sigma(B, b) \equiv \frac{e(B)n(D, b_D)^2}{e(b)n(D, \beta_D)^2} \mod p,$$

where $n(D, \beta_D)$ is an integer defined in a manner similar to $n(D, b_D)$.

Proof. Let χ and ξ be irreducible characters of height 0 in B and b, respectively. Since $b^G = B$, we get by Lemma 1.2

$$\omega_\chi(\chi^*) \equiv \omega_\xi((\chi_b)^*) \equiv \frac{\chi_b(1)}{\xi(1)}\omega_\xi(\xi^*) \mod \pi.$$

Then, applying Corollary 2.4 (ii) twice, we get

$$|G|^{-1}|D|\chi(1)e(B)n(D, b_D)^2 \equiv |H|^{-1}|D|\chi_b(1)e(b)n(D, \beta_D)^2 \mod \pi.$$

This yields the result. □

The following extends Brauer [2, (4B)] and Watanabe [16, Lemma].

Corollary 2.6. *Let b be a block of a subgroup H of G. Let Q be a defect group of b. Assume that $C_G(Q) \leq H$ and that $b^G = B$. Let b_0 be a root of b in $QC_G(Q)$ and let T be the inertial group of b_0 in $N_G(Q)$. Let χ be the character of G afforded by a virtually irreducible RG-module in B. Then, for any $x \in H$, we have*

$$\frac{\chi_b(x)|G|}{\chi(1)|C_H(x)|} \equiv \frac{|T|}{|T \cap H|}\omega_b(\hat{K}_x) \mod \pi,$$

where $\hat{K}_x$ is the class sum of conjugacy class of H containing x.

Proof. Applying Theorem 1.7 with $a = \hat{K}_x$, we get

$$\frac{\chi_b(x)|G|}{\chi(1)|C_H(x)|} \equiv \sigma(B, b)\omega_b(\hat{K}_x) \mod \pi.$$

So it suffices to show

$$\sigma(B, b) \equiv \frac{|T|}{|T \cap H|} \mod p. \tag{2.1}$$

If $d(b) = d(B)$, (2.1) follows from Proposition 2.5. On the other hand, if $d(b) < d(B)$, then $\sigma(B, b) \equiv 0 \bmod p$ by Theorem 1.7. Thus it suffices to prove $|T|/|T \cap H|$ is divisible by p. Assume the contrary. Then, since $|T \cap H|/|QC_G(Q)|$ is prime to p, $|T|/|QC_G(Q)|$ is prime to p. This yields $d(b) = d(B)$, a contradiction. Thus (2.1) is proved and the proof is complete. □

3. Canonical characters of Brauer pairs

Let G be a group. Let (P, b_P) and (Q, b_Q) be Brauer pairs in G such that $P \rhd Q$. Under this condition, a necessary and sufficient condition for $(b_P)^{PC(Q)} = (b_Q)^{PC(Q)}$ involving the multiplicity $(\theta_Q, \theta_P)_{C(P)}$ is known (Brauer [1, (6D)]). (In this section,

$C(P)$ and $C(Q)$ will denote $C_G(P)$ and $C_G(Q)$, respectively.) For alternative proofs, see Passman [15, Theorem 7], Feit [8, Theorem V.5.4 (iii)]. Here we give a slightly improved condition and a congruence for the multiplicity.

For two Brauer pairs (P, b_P) and (Q, b_Q), we write $(P, b_P) \rhd (Q, b_Q)$, if they are linked, and $(P, b_P) \supseteq (Q, b_Q)$, if there exist Brauer pairs (P_i, b_{P_i}), $1 \leq i \leq n$, such that $(P, b_P) = (P_1, b_{P_1}) \rhd (P_2, b_{P_2}) \rhd \cdots \rhd (P_n, b_{P_n}) = (Q, b_Q)$.

Lemma 3.1. *Let (P, b_P) and (Q, b_Q) be Brauer pairs such that $P \geq Q \geq Z(P)$. Then $(\theta_Q, \theta_P)_{C(P)}$ equals the multiplicity of $(\theta_P)_{C(P)}$ in $(\theta_Q)_{C(P)}$ as Brauer characters.*

Proof. Let ${b_P}^0$ be the block of $C(P)$ covered by b_P. So $Z(P)$ is a defect group of ${b_P}^0$. Since $Z(P) \leq Z(Q)$ and θ_Q is trivial on $Z(Q)$, $(\theta_Q)_{C(P)}$ is trivial on $Z(P)$. Then, since ${b_P}^0$ contains a unique block of $C(P)/Z(P)$ (of defect 0), we see $(\theta_Q)_{{b_P}^0}$ is a multiple of $(\theta_P)_{C(P)}$. So the result follows. □

Corollary 3.2. *Let (P, b_P) and (Q, b_Q) be Brauer pairs such that $(P, b_P) \supseteq (Q, b_Q)$. Then $(\theta_Q, \theta_P)_{C(P)}$ equals the multiplicity of $(\theta_P)_{C(P)}$ in $(\theta_Q)_{C(P)}$ as Brauer characters.*

Proof. By the Brauer-Olsson theorem [3, (4K)], we get $P \geq Q \geq Z(P)$. So Lemma 3.1 yields the result. □

Proposition 3.3. *Let (P, b_P) and (Q, b_Q) be Brauer pairs such that $P \rhd Q$ and b_Q is P-invariant. Then the following are equivalent.*
(i) $(b_P)^{PC(Q)} = (b_Q)^{PC(Q)}$.
(ii) *$C_P(Q) \leq Q$ and $(\theta_Q, \theta_P)_{C(P)}$ is prime to p.*
(iii) *$Z(P) \leq Q$ and $(\theta_Q, \theta_P)_{C(P)}$ is prime to p.*

Proof. Since (i) implies $C_P(Q) \leq Q$ [3, (3A)] (and hence $Z(P) \leq Q$), in order to prove the assertion, it suffices to show that

$$\text{if } Z(P) \leq Q, \text{ then } (b_P)^{PC(Q)} = (b_Q)^{PC(Q)} \text{ if and only if } (\theta_Q, \theta_P)_{C(P)} \text{ is prime to } p. \tag{3.1}$$

Assume $Z(P) \leq Q$, then by Lemma 3.1, $(\theta_Q, \theta_P)_{C(P)}$ equals the multiplicity of $(\theta_P)_{C(P)}$ in $(\theta_Q)_{C(P)}$ as Brauer characters. Then the conclusion of (3.1) follows from the proof of [12, Proposition 1.9]. This completes the proof. □

Proposition 3.4. *Let (P, b_P) and (Q, b_Q) be Brauer pairs such that $(P, b_P) \supseteq (Q, b_Q)$. Then*

$$n(P, b_P) \equiv \pm n(Q, b_Q) \mod p.$$

Proof. By induction on $|P : Q|$, we may assume $(P, b_P) \triangleright (Q, b_Q)$. Put $b^* = (b_P)^{PC(Q)}$. So $b^* = (b_Q)^{PC(Q)}$ and P is a defect group of b^*. We claim that θ_Q extends to $PC(Q)$. In fact, let b_1 be a unique block of $QC(Q)/Q$ contained in b_Q. Then b_1 has defect 0 and θ_Q is a unique irreducible character in b_1. Since θ_Q is $PC(Q)/Q$-invariant, any irreducible character of height 0 in the block of $PC(Q)/Q$ covering b_1 is an extension of θ_Q to $PC(Q)/Q$. So the claim follows. Let χ be such an extension. Since χ lies in b^* and $ht(\chi) = 0$, applying Corollary 2.4 (ii) for $PC(Q)$, b^*, b_P in place of G, B, b_D, we see that

$$\frac{|PC(Q)|}{|P|\chi(1)}\omega_\chi(\chi^*) \equiv e(b^*)n(P, b_P)^2 \mod p. \tag{3.1}$$

We claim $(PC(Q) \cap N_G(P))/PC(P)$ is a p-group. In fact, let x be a p'-element of $PC(Q) \cap N_G(P)$. Then x centralizes Q. On the other hand, Q is self-centralizing in P by [3, (3A)]. Hence x centralizes P, cf. [9, X.1.2]. So the claim is proved. Thus, in particular, $e(b^*) = 1$. (This last fact also follows from the fact that b^* is a nilpotent block.) Now $|PC(Q)|/|P| = |QC(Q)|/|Q|$, since $C_P(Q) \leq Q$. Further, we have $\omega_\chi(\chi^*) = n(Q, b_Q)$. In fact,

$$\omega_\chi(\chi^*) = \omega_{\theta_Q}((\theta_Q)^*) = \frac{|QC_G(Q)|}{|Q|\theta_Q(1)} = n(Q, b_Q).$$

Thus (3.1) yields

$$n(Q, b_Q)^2 \equiv n(P, b_P)^2 \mod p$$

and the result follows. This completes the proof. □

For the multiplicity we have the following.

Theorem 3.5. *Let (P, b_P) and (Q, b_Q) be Brauer pairs such that $(P, b_P) \triangleright (Q, b_Q)$. Then*

$$(\theta_Q, \theta_P)_{C(P)} \equiv \frac{n(P, b_P)}{n(Q, b_Q)} \equiv \pm 1 \mod p.$$

Proof. Put $b^* = (b_Q)^{PC(Q)}$. As in the proof of Proposition 3.4, there is an extension χ of θ_Q to $PC(Q)$. Applying Corollary 2.4 (i) for $PC(Q)$, b^*, b_P, θ_P in place of G, B, b_D, θ_D, we see that if m is the multiplicity of θ_P in $\chi_{C(P)}$ as Brauer characters, then

$$m \equiv \frac{e(b^*)|PC(P)|\chi(1)}{|PC(Q)|\theta_P(1)} \mod p. \tag{3.1}$$

As we have seen in the proof of Proposition 3.4, $e(b^*) = 1$. Then, since $|PC(Q)| = |QC(Q)||P : Q|$, the right side of (3.1) equals $n(P, b_P)/n(Q, b_Q)$. So, by Proposition

3.4, we get $m \equiv \pm 1 \bmod p$. Since m equals the multiplicity of $(\theta_P)_{C(P)}$ in $(\theta_Q)_{C(P)}$ as Brauer characters, we get $m = (\theta_Q, \theta_P)_{C(P)}$ by Corollary 3.2. Thus the result follows. □

REMARK 3.6. (i) From the proof of Passman [15, Theorem 7] (or Brauer [1, (6D)], Feit [8, Theorem V.5.4 (iii)]), we see that

$$(\theta_Q, \theta_P)_{C(P)} \equiv \frac{n(Q, b_Q)}{n(P, b_P)} \mod p,$$

which also yields Theorem 3.5 by Proposition 3.4.
(ii) In Theorem 3.5, both values $\pm 1 \bmod p$ are possible in general.

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