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ON SUBSECTIONS OF BLOCKS AND BRAUER PAIRS

MASAFUMI MURAI

(Received May 9, 1997)

Introduction

In Broué [6] and Okuyama [13] relations between blocks of a finite group and those of its subgroups are studied. In Section 1 of the present paper, we show the following theorem. Let $H$ be a subgroup of a group $G$. Let $B$ and $b$ be blocks of $G$ and $H$, respectively. Put $\sigma(B, b) = [G : H \xi_b(1)/\xi(1)]$ for an irreducible character $\xi$ in $B$, where $\xi_b$ is the $b$-component of $\xi_H$.

Theorem 1.7. With the notation above, assume that $d(B) > d(b)$. Let $\chi$ be the character of $G$ afforded by a virtually irreducible $RG$-module in $B$ (in the sense of Knörr [10]). Then

$$\frac{|G : H|\chi_b(a)}{\chi(1)} \equiv \sigma(B, b)\omega_b(a) \mod \pi$$

for all $a \in ZRH$.

Further, the following are equivalent.
(i) $\sigma(B, b) \not\equiv 0 \mod p$.
(ii) $B$ and $b$ are linked, and $B$ and $b$ have a common defect group.
(For notation and the definition of “linked”, see below.) This theorem reformulates some of Broué’s results in [6]. Furthermore, some of Brauer’s results in [2] are immediate consequences of this theorem.

In Section 2, we study the invariant $\sigma(B, b)$ in the case when $b^G = B$ and $d(b) = d(B)$.

In Section 3 we consider the canonical characters of Brauer pairs. If $P$ is a $p$-subgroup of $G$ and $b_P$ is a block of $PC_G(P)$ with defect group $P$, we call $(P, b_P)$ a Brauer pair in $G$. Let $\theta_P$ be the canonical character of $b_P$, i.e. $\theta_P$ is a unique irreducible character in $b_P$ which is trivial on $P$. Let $(P, b_P)$ and $(Q, b_Q)$ be Brauer pairs such that $P \triangleright Q$. Under this condition, a necessary and sufficient condition for $(b_P)^{PC_G(Q)} = (b_Q)^{PC_G(Q)}$ (i.e. $b_P$ and $b_Q$ are linked) involving the multiplicity $(\theta_Q, \theta_P)_{CG(P)}$ is known (Brauer [1, (6D)]). We shall improve that condition slightly and show that if $b_P$ and $b_Q$ are linked, then $(\theta_Q, \theta_P)_{CG(P)} \equiv \pm 1 \mod p$ (Theorem 3.5).
**Notation and terminology**

Let us introduce some notation. Let $G$ be a finite group and $p$ a prime. Let $(K, R, k)$ be a $p$-modular system. We assume that $K$ contains a primitive $|G|$-th root of unity. Let $\pi R$ be the maximal ideal of $R$ and let $v$ be the valuation of $K$ normalized so that $v(p) = 1$. For an irreducible character $\chi$ of $G$, let $\omega_\chi$ be the central character of $KG$ corresponding to $\chi$. Let $B$ be a block of $G$ with defect $d(B)$. Put $\omega_B = \omega_\chi$ for an irreducible character $\chi$ in $B$. Let $\beta_B$ be the block idempotent of $RG$ corresponding to $B$. Let $G_p$ be the set of $p$-elements of $G$. For an $R$-linear combination $\theta$ of irreducible characters or irreducible Brauer characters in $B$, we define, $ht(\theta)$, the height of $\theta$, by $ht(\theta) = v(\theta(1)) - v|G| + d(B)$, and put $\theta^* = \sum \theta(x^{-1})x$, where $x$ runs through $G_p$. For a block $b$ of a subgroup $H$ of $G$ and an $R$-linear combination $\theta$ of irreducible characters of $G$, let $\theta_b$ be the $b$-component of $\theta_H$. An $R$-linear combination of irreducible characters in $B$ is called an $R$-generalized character in $B$.

Let $ZRG$ be the center of $RG$. Put

$$Z_0(B) = \{a \in ZRG| e_B; \omega_B(a) \not\equiv 0 \mod \pi\},$$

where $ZRG_p$ is the $R$-submodule of $ZRG$ spanned by $p$-regular conjugacy class sums. Let $s_H : RG \to RH$ be the $R$-linear map defined by $s_H(x) = x$ if $x \in H$ and $s_H(x) = 0$ if $x \in G - H$. As in [12], we say that $B$ and $b$ are linked if $s_H(Z_0(B))e_b \subseteq Z_0(b)$. Let $Tr_H^G$ be the relative trace map, when $RG$ is considered as a $G$-algebra in the usual way.

Let $(P, b_P)$ be a Brauer pair in $G$. We call $(P, b_P)$ a $B$-Brauer pair if $(b_P)^G = B$. For the canonical character $\theta_P$ of $b_P$, we put $n(P, b_P) = |PC_G(P)|/|P|\theta_P(1)$. Since $\theta_P$ may be regarded as an irreducible character of defect 0 of $PC_G(P)/P$, $n(P, b_P)$ is an integer prime to $p$.

For the definition of virtually irreducible $RG$-modules (lattices) and basic properties of them, see Knörr [10].

1. Blocks of subgroups and some results on subsections

Throughout this section, we use the following notation: Let $G$ be a finite group and $H$ a subgroup of $G$. Let $B$ and $b$ be blocks of $G$ and $H$, respectively.

**Lemma 1.1.** Let $\theta$ be an $R$-generalized character in $B$. Let $\xi$ be an irreducible character of height 0 in $b$. Then, for $a \in ZRH$, $\theta_b(a)/\xi(1)$ lies in $R$ and

$$\frac{\theta_b(a)}{\xi(1)} \equiv \frac{\theta_b(1)}{\xi(1)} \omega_\xi(a) \mod \pi.$$
Proof. Put $\theta_b = \sum r_\eta \eta$, where $r_\eta \in R$ and $\eta$ ranges over the irreducible characters in $b$. Then

$$
\frac{\theta_b(a)}{\xi(1)} = \sum r_\eta \frac{\eta(a) \eta(1)}{\eta(1) \xi(1)} = \sum r_\eta \omega_\xi(a) \frac{\eta(1)}{\xi(1)} \mod \pi \\
= \frac{\theta_b(1)}{\xi(1)} \omega_\xi(a) \mod \pi,
$$

as required. \qed

Lemma 1.2. Let $\theta$ be an $R$-generalized character in $B$. Let $\zeta$ be an irreducible character of height 0 in $B$. Then

$$
\omega_\zeta(\theta^*) = \frac{\theta(1)}{\zeta(1)} \omega_\zeta(\zeta^*) \mod \pi.
$$

Proof. A direct computation shows that $\omega_\zeta(\theta^*) = \theta(\zeta^*)/\zeta(1)$. So the result follows from Lemma 1.1 (with $H = G$, $b = B$, $\xi = \zeta$ and $a = \zeta^*$). \qed

The following proposition is proved in Osima [14, Lemma 1]; the proof below is just a slight modification of Osima's.

Proposition 1.3. Let $\chi$ be an irreducible character of height 0 in $B$. Then

$$
\frac{|G|\chi(1)}{\text{rank}_R RGe_B} \equiv \omega_\chi(\chi^*) \not\equiv 0 \mod p.
$$

Proof. Put $\theta = \sum |G|^{-1} n(\phi) \phi$, where $\phi$ ranges over the irreducible Brauer characters in $B$ and $n(\phi)$ is the degree of the projective cover of a module affording $\phi$. Then $\theta$ is the restriction on $G_{\rho'}$ of an $R$-generalized character in $B$. So, by Lemma 1.2,

$$
\omega_\chi(\theta^*) \equiv \frac{\theta(1)}{\chi(1)} \omega_\chi(\chi^*) \mod \pi.
$$

Also $\nu(\theta(1)/\chi(1)) \geq 0$. Further, we have $e_B = \theta^*$. Thus $\omega_\chi(\theta^*) = 1$. Hence $\nu(\theta(1)/\chi(1)) = 0$ and $\chi(1)/\theta(1) \equiv \omega_\chi(\chi^*) \mod \pi$. Since $\theta(1) = \text{rank}_R RGe_B/|G|$ and $\omega_\chi(\chi^*)$ is a rational integer, the result follows. \qed

Remark 1.4. Theorem 1.3 of [12] follows from Lemma 1.2 and Proposition 1.3. The proof of the following lemma is a simple application of known facts, cf. Broué [6, (P1) in Section 1.1].
Lemma 1.5. Let \( \chi \) be the character afforded by a virtually irreducible RG-module in \( B \) and let \( \zeta \) be an irreducible character in \( B \). Let \( a \in \text{ZRH} \). Then
\[
(i) \quad |G : H| \chi_b(a)/\chi(1) \text{ lies in } R.
(ii) \quad |G : H| \chi_b(a)/\chi(1) \equiv |G : H| \zeta_b(a)/\zeta(1) \mod \pi.
\]
Proof. Clearly we may assume \( a \in \text{ZRH}e_b \). Define \( \omega : RG \to K \) by \( \omega(x) = \chi(x)/\chi(1), x \in RG \). Then, for \( x \in \text{ZR}G \),
\[
(1) \quad \omega(x) \in R,
(2) \quad \omega(x) \equiv \omega(x) \mod \pi,
\]
cf. 1.7 Remark of [10]. Then for \( a \in \text{ZRH}e_b \),
\[
(3) \quad \omega(\text{Tr}^G_H(a)) = |G : H| \chi_b(a)/\chi(1).
\]
So (i) follows from (1). Since a formula similar to (3) holds for \( \omega_\zeta \), (ii) follows from (2).

Lemma 1.6. Assume that \( d(b) = d(B) \). Then the following are equivalent.
\[
(i) \quad B \text{ and } b \text{ are linked.}
(ii) \quad \text{For every } R\text{-generalized character } \theta \text{ in } B \text{ with } \text{ht}(\theta) = 0, \theta_b \text{ is of height } 0.
(iii) \quad \text{For some } R\text{-generalized character } \theta \text{ in } B, \theta_b \text{ is of height } 0.
(iv) \quad \text{For some irreducible character } \zeta \text{ in } B \text{ with } \text{ht}(\zeta) = 0, \zeta_b \text{ is of height } 0.
(v) \quad \text{For some } a \in \text{Z}_0(B), s_H(a)e_b \in \text{Z}_0(b).
\]
Further, if these conditions are satisfied, then \( B \) and \( b \) have a common defect group.

Proof. (i) \( \Leftrightarrow \) (ii) : See [12, Corollary 1.5].
(ii) \( \Rightarrow \) (iii) : Trivial.
(iii) \( \Rightarrow \) (iv) : It follows that there is an irreducible character \( \zeta \) such that \( \text{ht}(\zeta_b) = 0 \). Then, since \( |G : H| \zeta_b(1)/\zeta(1) \text{ lies in } R \), we see \( \text{ht}(\zeta) = 0 \).
(iv) \( \Rightarrow \) (ii) : Put \( \theta = \sum r_\chi \chi \), where \( \chi \) ranges over the irreducible characters in \( B \) and \( r_\chi \in R \). Then, applying Lemma 1.5 with \( a = e_b \), we obtain
\[
\frac{\theta_b(1)|G|}{\zeta(1)|H|} = \sum r_\chi \frac{\chi_b(1)|G|}{\chi(1)|H|} \frac{\chi(1)}{\zeta(1)}
\equiv \frac{\zeta_b(1)|G|}{\zeta(1)|H|} \frac{\theta(1)}{\zeta(1)} \not\equiv 0 \mod \pi.
\]
So \( \theta_b \) is of height 0.
(v) \( \Rightarrow \) (iii) : There is an \( R\text{-generalized character } \theta \) such that \( \theta^* = a \) by [12, Corollary 1.4]. Then \( (\theta_b)^* = s_H(a)e_b \in \text{Z}_0(b) \). So \( \text{ht}(\theta_b) = 0 \) by [12, Theorem 1.3].
(i) \( \Rightarrow \) (v) : Trivial.

If \( B \) and \( b \) are linked, then the standard argument using defect classes of blocks shows that a defect group of \( b \) is contained in a defect group of \( B \). So the last assertion follows. This completes the proof.
For the character $\chi$ of $G$ afforded by a virtually irreducible $RG$-module in $B$, put $\sigma(B, b) = |G : H|\chi_b(1)/\chi(1)$. By Lemma 1.5, $\sigma(B, b)$ lies in $R$, and $\sigma(B, b)$ modulo $p$ is determined uniquely by $B$ and $b$ only (and does not depend on the choice of $\chi$).

The following theorem may be considered as a reformulation of some of Broué’s results in [6], see Remark 1.8 below. See also Okuyama [13, Corollary 1].

**Theorem 1.7.** Assume that $d(B) \geq d(b)$. Let $\chi$ be the character of $G$ afforded by a virtually irreducible $RG$-module in $B$. Then

$$\frac{|G : H|\chi_b(a)}{\chi(1)} \equiv \sigma(B, b)\omega_b(a) \mod \pi$$

for all $a \in ZRH$.

Further, the following are equivalent.

(i) $\sigma(B, b) \not\equiv 0 \mod p$.
(ii) $B$ and $b$ are linked, and $B$ and $b$ have a common defect group.

Proof. To prove the first assertion, we may assume $a \in ZRH_\epsilon b$. Let $\zeta$ be an irreducible character of height 0 in $B$. Then, by Lemma 1.5, we get

$$\frac{|G : H|\chi_b(a)}{\chi(1)} \equiv \frac{|G : H|\zeta_b(a)}{\zeta(1)} \mod \pi.$$ 

Let $\xi$ be an irreducible character of height 0 in $b$. We can write $|G : H|\zeta_b(a)/\zeta(1) = (|G : H|\zeta(1)/\zeta(1)) (\zeta_b(a)/\zeta(1))$. Then $|G : H|\zeta(1)/\zeta(1)$ lies in $R$, since $d(B) \geq d(b)$. Further

$$\frac{\zeta_b(a)}{\xi(1)} \equiv \frac{\zeta_b(1)}{\xi(1)} \omega_{\xi}(a) \mod \pi$$

by Lemma 1.1. So we get

$$\frac{|G : H|\chi_b(a)}{\chi(1)} \equiv \frac{|G : H|\zeta(1)}{\zeta(1)} \frac{\zeta_b(1)}{\xi(1)} \omega_{\xi}(a) \mod \pi$$

$$\equiv \frac{|G : H|\zeta_b(1)}{\zeta(1)} \omega_{\xi}(a) \mod \pi.$$ 

Since $|G : H|\zeta_b(1)/\zeta(1) \equiv \sigma(B, b) \mod \pi$, the first assertion is proved. Further, since

$$\nu \left( \frac{|G : H|\zeta_b(1)}{\zeta(1)} \right) = d(B) - d(b) + ht(\zeta_b),$$

we see that $\sigma(B, b) \not\equiv 0 \mod \pi$ if and only if $d(B) = d(b)$ and $ht(\zeta_b) = 0$. Since $ht(\zeta) = 0$, the last condition is equivalent to (ii) by Lemma 1.6. This completes the proof. \qed
Remark 1.8. (i) Proposition 2.1.1 (b) of Broué [6] states if \( d(B) \geq d(b) \), then

\[ \text{Tr}^G_H(J(ZRH)e_b)e_B \subseteq J(ZRG)e_B. \]

The formula in the above theorem, namely,

\[ \frac{|G : H|\chi_b(a)}{\chi(1)} = \sigma(B, b)\omega_b(a) \mod \pi \quad \text{for all } a \in ZRH, \]

may be considered as a restatement of (1.1). In fact, if \( a = e_b \), then (1.2) is true by definition. On the other hand, if \( a \in J(ZRH)e_b \), then (1.2) yields \(|G : H|\chi_b(a)/\chi(1) = 0 \mod \pi\), which is (1.1).

(ii) By Proposition 2.2.2 (a) of Broué [6], if \( d(B) \geq d(b) \), we obtain

\[ \sigma(B, b) = \lambda(B, b)\omega_b(s_H(e_B)) \mod \pi, \]

where \( \lambda(B, b) = |G : H|^2\text{rank}_R RHe_b/\text{rank}_R RG e_B \). From this, we can obtain the equivalence (i) \( \iff \) (ii) in Theorem 1.7. In fact,

\[ \text{(i) } \iff d(B) = d(b) \quad \text{and} \quad \omega_b(s_H(e_B)) \not\equiv 0 \mod \pi \quad \text{(by (1.3))} \]

\[ \iff \text{(ii) (by Lemma 1.6 (v))}. \]

For the value of \( \sigma(B, b) \), see Section 2 below.

The following corollary (and Lemma 1.5) extends Brauer [2, (3E)(i), (iii), (3F), (4C)] and Okuyama [13, Theorem 1].

Corollary 1.9. Let \( \chi \) be the character of \( G \) afforded by a virtually irreducible \( RG \)-module in \( B \). Then, for \( x \in H \),

\[ v(\chi_b(x)) \geq v|C_H(x)| - d(B) + ht(\chi). \]

If \( d(B) \geq d(b) \), then the equality holds if and only if \( B \) and \( b \) are linked, \( B \) and \( b \) have a common defect group and \( \omega_b(\hat{K}_x) \not\equiv 0 \mod \pi \), where \( \hat{K}_x \) denotes the class sum of the conjugacy class of \( H \) containing \( x \).

Proof. Apply Lemma 1.5 (i) and Theorem 1.7 with \( a = \hat{K}_x \).

The following is a special case of Corollary 1.9. For different proofs, see Broué [5, Proposition 3.4.1], Watanabe [16, Lemma] (see also Corollary 2.6 below).

Corollary 1.10 (Brauer [2, (4C)]). Let \( u \) be a p-element of \( G \). Let \( B \) be a block of \( G \) and let \( b \) be a block of \( C_G(u) \) such that \( b^G = B \) and that \( d(b) = d(B) \); that is, \((u, b)\) is a major subsection associated with \( B \). Let \( \chi \) be an irreducible character in
B. Then

\[ v(\chi_b(u)) = v|C_G(u)| - d(B) + ht(\chi). \]

In particular, \( \chi_b(u) \neq 0. \)

Proof. We apply Corollary 1.9 for \( C_G(u) \) and \( u \) in place of \( H \) and \( x \). Since \( b^G = B \), \( B \) and \( b \) are linked ([12, Proposition 1.6]). So it suffices to show that \( \omega_b(u) \equiv 0 \mod \pi \). But this is verified immediately. \( \square \)

The following extends Brauer [2, (5G), (5H)]. For a different proof of [2, (5H)], see Broué [5, Proposition 3.4.2].

Corollary 1.11. Let \( u \) be a central \( p \)-element of \( H \). Assume that \( b^G \) is defined. Let \( \chi \) be the character of \( G \) afforded by a virtually irreducible \( RG \)-module in \( B \) and let \( \zeta \) be an \( R \)-generalized character in \( B \). Then

\[ v \left( \sum_y \chi_b(uy)\zeta_b(u^{-1}y^{-1}) \right) \geq v|H| - d(B) + ht(\chi), \]

where \( y \) runs through \( H \). Further, if \( d(B) \geq d(b) \), then the equality holds if and only if \( b^G = B \), \( d(b) = d(B) \) and \( ht(\zeta) = 0 \).

Proof. We put \( \psi(y) = \zeta_b(u\psi^*) \), \( y \in H \). Then the left side of the above inequality is \( v(\chi_b(u\psi^*)) \). Applying Lemma 1.5 (i) with \( a = u\psi^* \), we get the inequality. Further, since \( b^G \) is defined, \( b^G = B \) if and only if \( B \) and \( b \) are linked ([12, Proposition 1.6]). Thus, by Theorem 1.7, the result follows if we show that \( \omega_b(u\psi^*) \equiv 0 \mod \pi \) if and only if \( ht(\zeta) = 0 \). Now \( \omega_b(u\psi^*) \equiv \omega_b(\psi^*) \mod \pi \) and

\[ \omega_b(\psi^*) \equiv 0 \mod \pi \]

\[ \iff ht(\psi) = 0 \text{ (by [12, Theorem 1.3], since } \psi \text{ belongs to } b) \]

\[ \iff ht(\zeta_b) = 0 \text{ (by Lemma 1.1)} \]

\[ \iff ht(\zeta) = 0 \text{ (by [12, Proposition 1.7 (ii)]).} \]

This completes the proof. \( \square \)

Corollary 1.12 (Brauer and Feit [4]). Let \( \chi \) and \( \zeta \) be irreducible characters in \( B \). Assume that \( \zeta \) is of height 0. Then

\[ v \left( \sum_y \chi(y)\zeta(y^{-1}) \right) = v|G| - d(B) + ht(\chi), \]
where \( y \) runs through \( G_{p'} \).

**Proof.** In Corollary 1.11, let \( H = G, b = B \) and \( u = 1 \).

As an application of Lemma 1.1, we have the following.

**Proposition 1.13.** Assume that \( b^G \) is defined. Let \( \chi \) be an \( R \)-generalized character in \( B \) and let \( a \in \text{ZRH} \). Then

\[
v(\chi b(a)) \geq v|H| - d(b)
\]

and the equality holds if and only if \( b^G = B, \, \text{ht}(\chi) = 0 \) and \( \omega_b(a) \not\equiv 0 \mod \pi \).

**Proof.** Let \( \xi \) be an irreducible character of height 0 in \( b \). Then by Lemma 1.1, \( \chi_b(a)/\xi(1) \in R \), so the inequality follows. Also

\[
\frac{\chi_b(a)}{\xi(1)} \equiv \frac{\chi_b(1)}{\xi(1)} \omega_b(a) \mod \pi.
\]

Thus the equality holds if and only if \( \text{ht}(\chi_b) = 0 \) and \( \omega_b(a) \not\equiv 0 \mod \pi \). Since \( b^G \) is defined, \( \text{ht}(\chi_b) = 0 \) if and only if \( b^G = B \) and \( \text{ht}(\chi) = 0 \) by [12, Proposition 1.7 (ii)]. This completes the proof. \( \square \)

The following strengthens Broué [7, Proposition 1] (see also Brauer [2, (3B)]).

**Corollary 1.14.** Let \( u \) be a central \( p \)-element of \( H \). Assume that \( b^G \) is defined. Let \( \chi \) be an \( R \)-generalized character in \( B \). Then

\[
v(\chi_b(u)) \geq v|H| - d(b)
\]

and the equality holds if and only if \( b^G = B \) and \( \text{ht}(\chi) = 0 \).

**Proof.** In Proposition 1.13, let \( a = u \). \( \square \)

The following extends [12, Proposition 1.13] and Broué [7, Corollary 2].

**Corollary 1.15.** Let \( u \) be a central \( p \)-element of \( H \). Assume that \( b^G \) is defined. Let \( \chi \) and \( \zeta \) be \( R \)-generalized characters in \( B \). Then

\[
v \left( \sum_y \chi_b(uy)\zeta_b(u^{-1}y^{-1}) \right) \geq v|H| - d(b),
\]

where \( y \) runs through \( H_{p'} \). Further, the equality holds if and only if \( b^G = B, \, \text{ht}(\chi) = 0 \) and \( \text{ht}(\zeta) = 0 \).
Proof. Put $\psi(y) = \zeta_{b}(u^{-1}y)$, $y \in H_p$. The left side of the above inequality is $v(\chi_{b}(u\psi^*))$. So, applying Proposition 1.13 with $a = u\psi^*$, we get the inequality. It remains to show $\omega_{b}(u\psi^*) \neq 0 \mod \pi$ if and only if $ht(\xi) = 0$. This is proved as in the proof of Corollary 1.11. \hfill $\square$

**Proposition 1.16** (Broué [7, (C)]). Let $u$ be a central $p$-element of $H$. Assume that $b^G$ is defined and equal to $B$. Let $\chi$ be an $R$-generalized character in $B$. Then

$$\frac{|G|\chi(1)}{\text{rank}_{R}RGe_B} \equiv \frac{|H|\chi_{b}(u)}{\text{rank}_{R}RHe_b} \mod \pi.$$ 

Proof. If $\chi$ is an irreducible character of positive height in $B$, then both sides are congruent to 0 modulo $\pi$ by Proposition 1.3 and Corollary 1.14. So we may assume that $\chi$ is an irreducible character of height 0 in $B$. Then by Proposition 1.3,

$$\frac{|G|\chi(1)}{\text{rank}_{R}RGe_B} = \omega_{\chi}(\chi^*) \mod \pi.$$ 

On the other hand, if $\xi$ is an irreducible character of height 0 in $b$, then

$$\frac{|H|\chi_{b}(u)}{\text{rank}_{R}RHe_b} \equiv \frac{\chi_{b}(u)}{\xi(1)} \omega_{\xi}(\xi^*) \mod \pi \text{ (by Proposition 1.3)}$$

$$\equiv \frac{\chi_{b}(1)}{\xi(1)} \omega_{\xi}(\xi^*) \mod \pi \text{ (by Lemma 1.1)}$$

$$\equiv \omega_{\xi}(\chi_{b}^*) \mod \pi \text{ (by Lemma 1.2)}$$

$$\equiv \omega_{\chi}(\chi^*) \mod \pi \text{ (since } b^G = B).$$

So the result follows. \hfill $\square$

2. The invariant $\sigma(B, b)$

Let $B$ be a block of a group $G$ with defect group $D$. Let $b$ be a block of a subgroup of $G$. In this section we consider the value of $\sigma(B, b)$ in the case when $b^G = B$ and $d(b) = d(B)$. Of course the most fundamental is the case when $b$ is the Brauer correspondent of $B$ in $N_G(D)$. In this case we have the following, which is a variant of Sylow's Third Theorem (consider the case of principal blocks). We note that this theorem is a consequence of the formula (35) in the proof of Theorem III.8.19 of [11]. Here we give an alternative (character-theoretical) proof.

**Theorem 2.1.** Let $B'$ be the Brauer correspondent of $B$ in $N_G(D)$. Then $\sigma(B, B') \equiv 1 \mod p$.

Proof. Let $\chi$ and $\xi$ be irreducible characters of height 0 in $B$ and $B'$, respec-
tively. Since $B^G = B$, we get by Lemma 1.2,

$$\omega_x(\chi^*) \equiv \omega_\xi((\chi_B)^*) \equiv \frac{\chi_B(1)}{\xi(1)} \omega_\xi(\xi^*) \mod \pi. \tag{2.1}$$

Let $S$ be a set of representatives of the $p'$-conjugacy classes of $G$ with defect group $D$. We choose $S$ so that $C_G(y) \supseteq D$ for $y \in S$. Let $K_y$ be the conjugacy class of $G$ containing $y$, $y \in S$. Then, as is well-known,

$$\omega_x(\chi^*) \equiv \sum_{y \in S} \omega_x(K_y)(\chi(y^{-1}) \mod \pi. \tag{2.2}$$

Put $L_y = K_y \cap C_G(D)$, $y \in S$. Then it is also well-known that $\{L_y ; y \in S\}$ is exactly the set of $p'$-conjugacy classes of $N_G(D)$ with defect group $D$. Then, as in (2.2), we have

$$\omega_\xi(\xi^*) \equiv \sum_{y \in S} \omega_\xi(L_y)(\xi(y^{-1}) \mod \pi. \tag{2.3}$$

On the other hand, we have

$$\omega_x(K_j) \equiv \omega_\xi(L_j) \mod \pi. \tag{2.4}$$

Further, since a formula similar to (2.4) is true for $y^{-1}$ in place of $y$, $y \in S$, we obtain

$$\chi(y^{-1}) \equiv \frac{\chi(1)|N_G(D)|}{\xi(1)|G|} |C_G(y) : C_G(y) \cap N_G(D)| \xi(y^{-1}) \mod \pi \tag{2.5}$$

since $D$ is a $p$-Sylow subgroup of $C_G(y)$. On substituting (2.4) and (2.5) into (2.2), we obtain by (2.3),

$$\omega_x(\chi^*) \equiv \frac{\chi(1)|N_G(D)|}{\xi(1)|G|} \omega_\xi(\xi^*) \mod \pi. \tag{2.6}$$

Comparison of (2.1) and (2.6) shows that

$$\frac{\chi_B(1)|G|}{\chi(1)|N_G(D)|} \equiv 1 \mod \pi,$$

since $\omega_\xi(\xi^*) \not\equiv 0 \mod \pi$ (by Proposition 1.3). This completes the proof. \qed

**Remark 2.2.** (i) In fact, Theorem 2.1 and Corollary 2.4 (i) below follow from Brauer [2, (2D)].
(ii) Still another proof of 2.1 is available; by Remark 1.8 (ii)

\[ \sigma(B, B') \equiv \frac{|G : N_G(D)|^2 \text{rank}_R R N_G(D) e_B}{\text{rank}_R R G e_B} \mod p. \]

As an \( R[G \times G] \)-module, \( R G e_B \) is indecomposable and its Green correspondent with respect to \( (G \times G, \Delta(D), N_G(D) \times N_G(D)) \) is \( R N_G(D) e_B \), where \( \Delta(D) = \{(x, x); x \in D\} \). From this, the result follows.

Let \((D, b_D)\) be a \( B \)-Brauer pair. Let \( e(B) \) be the inertial index of \( B \).

**Corollary 2.3.** We have \( \sigma(B, b_D) \equiv e(B) \mod p \).

**Proof.** Let \( B' \) be the Brauer correspondent of \( B \) in \( N_G(D) \). Let \( T \) be the inertial group of \( b_D \) in \( N_G(D) \). Then it is easy to see

\[ \left( \chi_{B'} \right)_{D C_G(D)} = \sum \chi_{(b_D)^x} = \sum \chi_{b_D)^x} \]

where \( x \) runs through a transversal of \( T \) in \( N_G(D) \). From this we get the result by Theorem 2.1. \( \square \)

**Corollary 2.4.** Let \( \chi \) be an irreducible character of height 0 in \( B \). Let \( \theta_D \) be the canonical character of \( b_D \). Then

(i) As Brauer characters, \( \chi_{b_D} = m \theta_D \), where \( m \) is an integer such that

\[ m \equiv \frac{e(B)|D C_G(D)|\chi(1)}{|G|\theta_D(1)} \mod p. \]

(ii) \( (|G|/|D|\chi(1))\omega(\chi^*) \equiv e(B)n(D, b_D)^2 \mod p \).

**Proof.** (i) This follows from Corollary 2.3.

(ii) Put \( \theta = \theta_D \). Since \( (b_D)^G = B \), we get by Lemma 1.2

\[ \omega(\chi^*) = \omega_{b_D}(\chi_{b_D})^* \equiv \frac{\chi_{b_D}(1)}{\theta(1)} \omega_{b_D}(\theta^*) \mod \pi. \]

Then, since \( \omega_{b_D}(\theta^*) = |D C_G(D)|/|D|\theta(1) \), the result follows from (i). \( \square \)

**Proposition 2.5.** Let \( b \) be a block of a subgroup \( H \) of \( G \). Assume that \( b^G \) is defined and equal to \( B \) and that \( b \) has defect group \( D \). Let \((D, \beta_D)\) be a \( b \)-Brauer pair in \( H \). Then

\[ \sigma(B, b) \equiv \frac{e(B)n(D, b_D)^2}{e(b)n(D, \beta_D)^2} \mod p, \]

where \( n(D, \beta_D) \) is an integer defined in a manner similar to \( n(D, b_D) \).
Proof. Let \( \chi \) and \( \xi \) be irreducible characters of height 0 in \( B \) and \( b \), respectively. Since \( b^G = B \), we get by Lemma 1.2

\[
\omega_\chi(\chi^*) \equiv \omega_\xi((\chi_b)^*) \equiv \frac{\chi_b(1)}{\xi(1)} \omega_\xi(\xi^*) \mod \pi.
\]

Then, applying Corollary 2.4 (ii) twice, we get

\[
|G|^{-1}D|\chi(1)e(B)n(D, b_D)^2 \equiv |H|^{-1}D|\chi_b(1)e(b)n(D, \beta_D)^2 \mod \pi.
\]

This yields the result. \( \square \)

The following extends Brauer \[2, (4B)\] and Watanabe \[16, Lemma\].

**Corollary 2.6.** Let \( b \) be a block of a subgroup \( H \) of \( G \). Let \( Q \) be a defect group of \( b \). Assume that \( C_G(Q) \leq H \) and that \( b^G = B \). Let \( b_0 \) be a root of \( b \) in \( QC_G(Q) \) and let \( T \) be the inertial group of \( b_0 \) in \( N_G(Q) \). Let \( \chi \) be the character of \( G \) afforded by a virtually irreducible RG-module in \( B \). Then, for any \( x \in H \), we have

\[
\frac{\chi_b(x)|G|}{\chi(1)|C_H(x)|} \equiv \frac{|T|}{|T \cap H|} \omega_b(\bar{K}_x) \mod \pi,
\]

where \( \bar{K}_x \) is the class sum of conjugacy class of \( H \) containing \( x \).

Proof. Applying Theorem 1.7 with \( a = \bar{K}_x \), we get

\[
\frac{\chi_b(x)|G|}{\chi(1)|C_H(x)|} \equiv \sigma(B, b)\omega_b(\bar{K}_x) \mod \pi.
\]

So it suffices to show

\[
(2.1) \quad \sigma(B, b) \equiv \frac{|T|}{|T \cap H|} \mod p.
\]

If \( d(b) = d(B) \), (2.1) follows from Proposition 2.5. On the other hand, if \( d(b) < d(B) \), then \( \sigma(B, b) \equiv 0 \mod p \) by Theorem 1.7. Thus it suffices to prove \(|T|/|T \cap H|\) is divisible by \( p \). Assume the contrary. Then, since \(|T \cap H|/|QC_G(Q)|\) is prime to \( p \), \(|T|/|QC_G(Q)|\) is prime to \( p \). This yields \( d(b) = d(B) \), a contradiction. Thus (2.1) is proved and the proof is complete. \( \square \)

3. Canonical characters of Brauer pairs

Let \( G \) be a group. Let \( (P, b_P) \) and \( (Q, b_Q) \) be Brauer pairs in \( G \) such that \( P \triangleright Q \). Under this condition, a necessary and sufficient condition for \((b_P)^{PC(Q)} = (b_Q)^{PC(Q)}\) involving the multiplicity \((\theta_Q, \theta_P)_{C(P)}\) is known (Brauer \[1, (6D)\]). (In this section,
$C(P)$ and $C(Q)$ will denote $C_G(P)$ and $C_G(Q)$, respectively.) For alternative proofs, see Passman [15, Theorem 7], Feit [8, Theorem V.5.4 (iii)]. Here we give a slightly improved condition and a congruence for the multiplicity.

For two Brauer pairs $(P, b_P)$ and $(Q, b_Q)$, we write $(P, b_P) \triangleright (Q, b_Q)$ if they are linked, and $(P, b_P) \supseteq (Q, b_Q)$, if there exist Brauer pairs $(P_i, b_{P_i})$, $1 \leq i \leq n$, such that $(P, b_P) = (P_1, b_{P_1}) \triangleright (P_2, b_{P_2}) \triangleright \cdots \triangleright (P_n, b_{P_n}) = (Q, b_Q)$.

**Lemma 3.1.** Let $(P, b_P)$ and $(Q, b_Q)$ be Brauer pairs such that $P \geq Q \geq Z(P)$. Then $(\theta_Q, \theta_P)_{C(P)}$ equals the multiplicity of $(\theta_P)_{C(P)}$ in $(\theta_Q)_{C(P)}$ as Brauer characters.

**Proof.** Let $b_P^0$ be the block of $C(P)$ covered by $b_P$. So $Z(P)$ is a defect group of $b_P^0$. Since $Z(P) \leq Z(Q)$ and $\theta_Q$ is trivial on $Z(Q)$, $(\theta_Q)_{C(P)}$ is trivial on $Z(P)$. Then, since $b_P^0$ contains a unique block of $C(P)/Z(P)$ (of defect 0), we see $(\theta_Q)_{b_P^0}$ is a multiple of $(\theta_P)_{C(P)}$. So the result follows. □

**Corollary 3.2.** Let $(P, b_P)$ and $(Q, b_Q)$ be Brauer pairs such that $(P, b_P) \supseteq (Q, b_Q)$. Then $(\theta_Q, \theta_P)_{C(P)}$ equals the multiplicity of $(\theta_P)_{C(P)}$ in $(\theta_Q)_{C(P)}$ as Brauer characters.

**Proof.** By the Brauer-Olsson theorem [3, (4K)], we get $P \geq Q \geq Z(P)$. So Lemma 3.1 yields the result. □

**Proposition 3.3.** Let $(P, b_P)$ and $(Q, b_Q)$ be Brauer pairs such that $P \triangleright Q$ and $b_Q$ is $P$-invariant. Then the following are equivalent.

(i) $(b_P)_{PC(Q)} = (b_Q)_{PC(Q)}$.

(ii) $C_P(Q) \leq Q$ and $(\theta_Q, \theta_P)_{C(P)}$ is prime to $p$.

(iii) $Z(P) \leq Q$ and $(\theta_Q, \theta_P)_{C(P)}$ is prime to $p$.

**Proof.** Since (i) implies $C_P(Q) \leq Q$ [3, (3A)] (and hence $Z(P) \leq Q$), in order to prove the assertion, it suffices to show that

\begin{equation}
\text{if } Z(P) \leq Q, \text{ then } (b_P)_{PC(Q)} = (b_Q)_{PC(Q)} \text{ if and only if } (\theta_Q, \theta_P)_{C(P)} \text{ is prime to } p.
\end{equation}

Assume $Z(P) \leq Q$, then by Lemma 3.1, $(\theta_Q, \theta_P)_{C(P)}$ equals the multiplicity of $(\theta_P)_{C(P)}$ in $(\theta_Q)_{C(P)}$ as Brauer characters. Then the conclusion of (3.1) follows from the proof of [12, Proposition 1.9]. This completes the proof. □

**Proposition 3.4.** Let $(P, b_P)$ and $(Q, b_Q)$ be Brauer pairs such that $(P, b_P) \supseteq (Q, b_Q)$. Then

\[ n(P, b_P) \equiv \pm n(Q, b_Q) \mod p. \]
Proof. By induction on \(|P : Q|\), we may assume \((P, b_P) \triangleright (Q, b_Q)\). Put \(b^* = (b_P)^{PC(Q)}\). So \(b^* = (b_Q)^{PC(Q)}\) and \(P\) is a defect group of \(b^*\). We claim that \(\theta_Q\) extends to \(PC(Q)\). In fact, let \(b_1\) be a unique block of \(QC(Q)/Q\) contained in \(b_Q\). Then \(b_1\) has defect \(0\) and \(\theta_Q\) is a unique irreducible character in \(b_1\). Since \(\theta_Q\) is \(PC(Q)/Q\)-invariant, any irreducible character of height \(0\) in the block of \(PC(Q)/Q\) covering \(b_1\) is an extension of \(\theta_Q\) to \(PC(Q)/Q\). So the claim follows. Let \(\chi\) be such an extension. Since \(\chi\) lies in \(b^*\) and \(ht(\chi) = 0\), applying Corollary 2.4 (ii) for \(PC(Q), b^*, b_P\) in place of \(G, B, b_D\), we see that

\[
\frac{|PC(Q)|}{|P|} \omega_{\chi}(\chi^*) \equiv e(b^*)n(P, b_P)^2 \mod p.
\]

We claim \((PC(Q) \cap N_G(P))/PC(P)\) is a \(p\)-group. In fact, let \(x\) be a \(p'\)-element of \(PC(Q) \cap N_G(P)\). Then \(x\) centralizes \(Q\). On the other hand, \(Q\) is self-centralizing in \(P\) by [3, (3A)]. Hence \(x\) centralizes \(P\), cf. [9, X.1.2]. So the claim is proved. Thus, in particular, \(e(b^*) = 1\). (This last fact also follows from the fact that \(b^*\) is a nilpotent block.) Now \(|PC(Q)|/|P| = |QC(Q)|/|Q|\), since \(C_P(Q) \leq Q\). Further, we have \(\omega_{\chi}(\chi^*) = n(Q, b_Q)\). In fact,

\[
\omega_{\chi}(\chi^*) = \omega_{\theta_Q}((\theta_Q)^*) = \frac{|QC_G(Q)|}{|Q|\theta_Q(1)} = n(Q, b_Q).
\]

Thus (3.1) yields

\[
n(Q, b_Q)^2 \equiv n(P, b_P)^2 \mod p
\]

and the result follows. This completes the proof.

For the multiplicity we have the following.

**Theorem 3.5.** Let \((P, b_P)\) and \((Q, b_Q)\) be Brauer pairs such that \((P, b_P) \triangleright (Q, b_Q)\). Then

\[
(\theta_Q, \theta_P)_{C(P)} \equiv \frac{n(P, b_P)}{n(Q, b_Q)} \equiv \pm 1 \mod p.
\]

Proof. Put \(b^* = (b_Q)^{PC(Q)}\). As in the proof of Proposition 3.4, there is an extension \(\chi\) of \(\theta_Q\) to \(PC(Q)\). Applying Corollary 2.4 (i) for \(PC(Q), b^*, b_P, \theta_P\) in place of \(G, B, b_D, \theta_D\), we see that if \(m\) is the multiplicity of \(\theta_P\) in \(\chi_{C(P)}\) as Brauer characters, then

\[
m \equiv \frac{e(b^*)|PC(P)|\chi(1)}{|PC(Q)|\theta_P(1)} \mod p.
\]

As we have seen in the proof of Proposition 3.4, \(e(b^*) = 1\). Then, since \(|PC(Q)| = |QC(Q)||P : Q|\), the right side of (3.1) equals \(n(P, b_P)/n(Q, b_Q)\). So, by Proposition
3.4, we get \( m \equiv \pm 1 \mod p \). Since \( m \) equals the multiplicity of \( (\theta_p)_{C(P)} \) in \( (\theta_Q)_{C(P)} \) as Brauer characters, we get \( m = (\theta_Q, \theta_P)_{C(P)} \) by Corollary 3.2. Thus the result follows.

\[ \square \]

**Remark 3.6.** (i) From the proof of Passman [15, Theorem 7] (or Brauer [1, (6D)], Feit [8, Theorem V.5.4 (iii)]), we see that

\[ (\theta_Q, \theta_P)_{C(P)} \equiv \frac{n(Q, b_Q)}{n(P, b_P)} \mod p, \]

which also yields Theorem 3.5 by Proposition 3.4.

(ii) In Theorem 3.5, both values \( \pm 1 \mod p \) are possible in general.

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**References**
