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Osaka J. Math.
40 (2003), 449–454

AN UPPER BOUND FOR SMALL EIGENVALUES OF THE LAPLACIAN

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(Received December 11, 2001)

M. Berger gave a curvature free upper bound for the first eigenvalue in terms of injectivity radius and dimension for a compact Riemannian manifold admitting a fixed point free, isometric involution [2]. P. Bérard and G. Besson [1] extended this result to homogeneous and globally harmonic Riemannian manifolds. C. Croke [4] improved Berger's estimate for the Dirichlet problem with a bound in terms of convexity radius which gave as a corollary an upper bound for compact manifolds. In this note we give an estimate (Theorem 3) which is sharper than those mentioned above with the additional hypothesis of an upper curvature bound but without global hypothesis on the injectivity radius. Moreover, this estimate is sharp with equality holding only in the case of spheres of constant curvature and gives bounds for higher eigenvalues if the dimension is at least three. If the manifold is homeomorphic to certain n-dimensional spherical space forms and the bounds on the injectivity radius and sectional curvature hold globally, then we can give an upper bound for the nth eigenvalue (Theorem 4).

A sharp lower bound for the sum of the reciprocals of the first three eigenvalues of S^2 was made by Hersch [10] in terms of area alone. P. Yang and S.-T. Yau [17] generalized this estimate to compact surfaces in terms of genus and area. Both of these results give an upper bound for the first eigenvalue which is sharp in the case of spheres. P. Li and S.-T. Yau [12] reproduced this estimate by employing a conformal invariant, the conformal area. They were also able to give a sharp estimate in the case of the real projective plane and for the conformal class of the square, flat torus also only in terms of area. In higher dimensions their estimates require (M, g) to be conformally equivalent to an immersed, minimal submanifold of the standard sphere of dimension $m \geq n$. Examples of H. Urakawa [15] and J. Dodziuk [5] show that for dimension at least three, there does not exist an upper bound for the first eigenvalue in terms of volume alone. Other upper estimates for λ_1 involve either a lower bound on curvature [3], [7], or hold only for surfaces [6], [9], [10], [14]. See [11] for a survey of eigenvalue bounds.

We will use the following notation. The metric ball of radius r at a point q

The author was partially supported by the Japan Society for the Promotion of Science, Monbusho Grant-in-Aid of Research No. 13099720 and the Graduiertenkolleg of the Mathematisches Institut, Universität Bonn while this paper was in preparation.

will be denoted $B_r(q)$. For Riemannian manifold (M, g), denote $\|v\|_g = \sqrt{g(v, v)}$, by $\mathrm{d}_g(\cdot, \cdot)$ the distance, and μ_g the Riemannian measure. The L^2 inner product and norm with respect to this measure will be written $(\cdot, \cdot)_2$ and $\|\cdot\|_2$, respectively. Let $\sigma(r, q) = \mathrm{Vol}(B_{2r}(q), g) - \mathrm{Vol}(B_r(q), g)$, and $\tau(r, q) = \int_{B_r(q)} \cos^2 \{(\pi/2r)\, \mathrm{d}_g(x, q)\}\, d\mu_g(x)$. In the special case of the simply connected, n-dimensional space form of constant curvature κ, we write $\sigma(\kappa, r, n) = \sigma(r, \cdot)$ and, $\tau(\kappa, r, n) = \tau(r, \cdot)$. The eigenvalues of the Laplacian of a compact, Riemannian manifold (M, g) will be denoted $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$ and corresponding eigenfunctions $\{\varphi_j\}_{j\geq 0}$ are chosen so as to form an orthonormal basis of the Hilbert space $L^2(M, \mu_g)$. For a point $x \in S^m$ we denote its antipode by $-x$.

Lemma 1. *Let (M, g) be a smooth, compact, connected, Riemannian manifold of dimension $n \geq 2$. If for $m \geq 1$ there exists a continuous map $\psi\colon S^m \to M$ such that $\mathrm{d}_g(\psi(x), \psi(-x)) \geq 2r > 0$ for all $x \in S^m$, then there exists $x \in S^m$ such that for $q = \psi(x)$ and $q' = \psi(-x)$,*

$$\lambda_m + \frac{\pi^2}{4r^2} \leq \frac{\pi^2}{4r^2} \cdot \frac{\mathrm{Vol}(B_r(q), g) + \mathrm{Vol}(B_r(q'), g)}{\tau(r, q) + \tau(r, q')}, \tag{1}$$

where $\tau(r, p)$ is defined above.

Proof. As in [2, 4], we use a test function that approximates an eigenfunction (with lowest non-zero eigenvalue) of the Laplacian on spheres of constant curvature. For any $x \in S^m$ and $k = 0, 1$, denote $q_k = q_k(x) = \psi((-1)^k x)$. Let $F(t, r) = \cos\left(\pi t/(2r)\right)$ and consider the function on M,

$$f_x(y) = F\,(\mathrm{d}(y, q_0), r)\, \chi_0 - F\,(\mathrm{d}(y, q_1), r)\, \chi_1,$$

where χ_k is the characteristic function of $B_r(q_k)$. Note that by construction, $f_{-x} = -f_x$.

Except on a set of measure zero,

$$\|df_x\|_g^2\,(y) = \frac{\pi^2}{4r^2}\left[1 - F^2(\mathrm{d}(y, q_0), r)\right]\chi_0 + \frac{\pi^2}{4r^2}\left[1 - F^2(\mathrm{d}(y, q_1), r)\right]\chi_1.$$

Therefore,

$$\begin{aligned}\int_M \|df_v\|_g^2\, d\mu_g \leq\ & \frac{\pi^2}{4r^2}\left[\mathrm{Vol}(B_r(q_0), g) - \int_{B_r(q_0)} F^2(\mathrm{d}(y, q_0), r)\, d\mu_g(y)\right] \\ & + \frac{\pi^2}{4r^2}\left[\mathrm{Vol}(B_r(q_1), g) - \int_{B_r(q_1)} F^2(\mathrm{d}(y, q_1), r)\, d\mu_g(y)\right].\end{aligned} \tag{2}$$

In addition,

$$\int_M f_x^2\, d\mu_g = \int_{B_r(q_0)} F^2(\mathrm{d}(y, q_0), r)\, d\mu_g(y) + \int_{B_r(q_1)} F^2(\mathrm{d}(y, q_1), r)\, d\mu_g(y) \tag{3}$$

Now we will show the existence of a point $x \in S^m$ such that $(f_x, \varphi_j)_2 = 0$ for all $j = 0, \ldots, m-1$. Define a map $H\colon S^m \to \mathbb{R}^m$ by

$$H(x) = \left(h_0(x), h_1(x), \ldots, h_{m-1}(x)\right),$$

where $h_j(x) = \left(f_x, \varphi_j\right)_2$. By construction $H(-x) = -H(x)$. An easy corollary of the Borsuk-Ulam theorem (see, for example [13]) gives that H must have the value zero for some point $x \in S^m$. Hence, $h_0(x) = h_1(x) = \cdots = h_{m-1}(x) = 0$. Since f_x is Lipschitz and has compact support the minimax principle applies, and by (2) and (3), we have (1).

In the case (M, g) is a standard sphere with constant curvature κ, and φ is the identity map, the functions f_x are eigenfunctions of the Laplacian and equality is realized in (1). □

REMARK. The Borsuk-Ulam theorem gives that if there exists a continuous, injective map $h\colon M \to \mathbb{R}^l$ then for any map $\varphi\colon S^l \to M$ there exists a point $x \in S^l$ such that $\varphi(x) = \varphi(-x)$. So Lemma 1 can be applied to estimate λ_m for $m \le l - 1$ at the very most. In particular, a theorem of Whitney[16] shows that $m \le 2n - 1$.

Lemma 2. *Let M be a smooth, compact manifold. If $\psi\colon S^m \to M$ is continuous and has the property that $\psi(x) \neq \psi(-x)$ for all $x \in S^m$, then for any Riemannian metric g on M, and any $r > 0$ with $2r \le \mathrm{inj}(g)$ there exists a map $\overline{\psi}\colon S^m \to M$ homotopic to ψ such that $\mathrm{d}_g(\overline{\psi}(x), \overline{\psi}(-x)) \ge 2r$ for all $x \in S^m$.*

Proof. If $\mathrm{d}_g(\psi(x), \psi(-x)) \ge 2r$ then define $\Psi(x, t) = \psi(x)$ for all $t \in [0, 1]$. Otherwise, there exists a unique, normal, minimal geodesic $\gamma\colon [-r, r] \to M$, and $s \in (0, r)$, with $\gamma(\pm s) = \psi(\pm x)$. Let $\Psi(x, t) = \gamma((1 - t)s + tr)$. The continuity of both ψ and the exponential map ensure that Ψ is continuous. We have that $\Psi(x, 0) = \psi(x)$ and that $\mathrm{d}_g(\Psi(x, 1), \Psi(-x, 1)) \ge 2r$ for all $x \in S^m$. □

Now we apply the lemmas to give an upper bound for small eigenvalues in terms of volume, injectivity radius and an upper bound for sectional curvature.

Theorem 3. *Let (M, g) be a smooth, compact, connected, Riemannian manifold of dimension n. Fix $p_0 \in M$ and for $\rho_0 \geq \mathrm{inj}(p_0)$ let A be the closed ball with centre p_0 and radius ρ_0. Let κ and ρ be, respectively, the maximum of the sectional curvature and the minimum of the injectivity radius on A. If $0 < 2r \leq \rho$ then,*

$$\lambda_{n-1} + \frac{\pi^2}{4r^2} \leq \pi^2 \frac{\mathrm{Vol}(M, g)}{8r^2 \tau(\kappa, r, n)},$$

with equality if and only if $\kappa > 0$, $2r = \pi/\sqrt{\kappa}$ and (M, g) is isometric to the sphere of constant curvature κ.

Proof. As $2r \leq \mathrm{inj}(p_0)$, the map $v \mapsto \exp_{p_0}(rv)$ restricted to the unit tangent sphere at p_0 has the property that images of anti-podal pairs are a distance $2r$ apart. Applying Lemma 1 yields a pair of points q, q' such that (1) applies for $l = n - 1$. Next observe that

$$\mathrm{Vol}(B_r(q), g) + \mathrm{Vol}(B_r(q'), g) \leq \mathrm{Vol}(M, g) \tag{4}$$

and furthermore that

$$\tau(r, q) \geq \tau(\kappa, r, n) \quad \text{and} \quad \tau(r, q') \geq \tau(\kappa, r, n), \tag{5}$$

by comparison of the volume form as expressed in normal coordinates at q and q' (see, for example, the proof of the Bishop-Gunther comparison theorem in [8]). The desired upper bound follows from (4) and (5).

If equality holds for $0 < r \leq \rho$, we must have equality in (4) and (5). The former gives that M is homeomorphic to S^n as follows. The set $B_r(q) \cup B_r(q')$ is of full measure in M. For $r \leq \rho/2$, $\exp_q$ is injective on $B_{3r/2}(q)$, and similarly for q'. Hence the map $\psi = \exp_{q'}^{-1} \circ \exp_q$ restricted to $\{v \in T_qM \mid \|v\|_g = r\}$ is injective. It is also onto $\{w \in T_{q'}M \mid \|w\|_g = r\}$ since if $w \in T_{q'}M$, with $\|w\|_g = r$, $B_\epsilon(\exp_{q'}(1 + 2\epsilon)w) \cap B_r(q) \neq \emptyset$, for all $\epsilon > 0$ sufficiently small. Thus there exists a sequence $\{p_j\} \subset B_r(q)$ with $p_j \to \exp_{q'} w$. Therefore there exists $v \in T_qM$ with $\psi(v) = w$ and $\|v\|_g = r$. It quickly follows that M is homeomorphic to S^n. From equality in (5), we have that the volume form expressed in normal coordinates at q and q' is, up to distance r, equal to that of the metric ball of radius r in the space form with constant curvature κ. Since $K_g \leq \kappa$, by a standard argument, we have that $K_g \equiv \kappa$ on $B_r(q) \cup B_r(q')$, so this holds on all of M and since M is homeomorphic to S^n we must have that $\kappa > 0$ and (M, g) is isomorphic to S^n with constant curvature κ. ☐

In the case that M is homeomorphic to a sphere or has the sphere as universal cover such that the covering map is injective on anti-podal pairs, the following estimate on λ_n holds.

Theorem 4. *Let (M, g) be a smooth, Riemannian manifold of dimension $n \geq 2$ with injectivity radius $\mathrm{inj}(g) \geq 2r > 0$ and sectional curvature bounded above by κ. If M is homeomorphic to S^n/G where G is a discrete group of diffeomorphisms that acts properly and freely on S^n such that no orbit of G contains an anti-podal pair, then*

$$\lambda_n + \frac{\pi^2}{4r^2} \leq \pi^2 \frac{\mathrm{Vol}(M, g) - \sigma(\kappa, r, n)}{4r^2 \tau(\kappa, r, n)}.$$

Equality is realized in the case of standard spheres of constant curvature.

Proof. The quotient map $\psi\colon S^n \to S^n/G$ is continuous and $\psi(x) \neq \psi(-x)$ for all $x \in S^n$. By Lemma 2 there exists a map $\underline{\psi}\colon S^n \to M$ such that $\mathrm{d}_g(\underline{\psi}(x), \underline{\psi}(-x) \geq 2r$ for all $x \in S^n$. Apply Lemma 1 and, as in Theorem 3, a comparison argument on the volume form in normal coordinates gives $\mathrm{Vol}(M, g) - \mathrm{Vol}(B_r(q), g) \geq \mathrm{Vol}(B_{2r}(q), g) - \mathrm{Vol}(B_r(q), g) \geq \sigma(\kappa, r, n)$. Also, $\tau(r, q) \geq \tau(\kappa, r, n)$ and both these estimates hold at q' as well. It is easy to check that equality holds for spheres of constant curvature. □

With global bounds on sectional curvature and injectivity radius, and a set of k points whose pairwise distances are at least inj(g) (for example if $\mathrm{diam}(g) \geq k\,\mathrm{inj}(g)$), we can improve Theorem 3 as follows.

Proposition 5. *If (M^n, g) is compact with sectional curvature bounded above by κ, and we can find k points $x_1, \ldots, x_k$ with $\mathrm{d}_g(x_i, x_j) \geq 2r$ for all $i \neq j$ for some $0 < r \leq \mathrm{inj}(g)/2$, then*

$$\lambda_{n-1} + \frac{\pi^2}{4r^2} \leq \frac{\pi^2 \,\mathrm{Vol}(M, g)}{8k\, r^2 \tau(\kappa, r, n)}.$$

Proof. For each point x_j, by the proof of Lemma 1 we can find $v_j \in T_{x_j}M$ with $\|v_j\|_g = r$ and the following property. If $f_j(y) = F(\mathrm{d}(y, p_j), r)\chi_j - F(\mathrm{d}(y, q_j), r)\overline{\chi}_j$, then $(f_i, \varphi_j)_2 = 0$ for $j = 0, \ldots, n-2$. Here, $p_j = \exp_{x_j}(v_j)$, $q_j = \exp_{x_j}(-v_j)$ and χ_j, $\overline{\chi}_j$ are the characteristic functions of $B_r(p_j)$, $B_r(q_j)$, respectively. Note that $(f_i, f_j)_2 = 0$ for all $i \neq j$. Let $f = f_1 + \cdots + f_k$. We have that $(\varphi_i, f)_2 = 0$ for all $i = 0, \ldots, n-2$. Then,

$$\begin{aligned}\int_M \|df\|_g^2 \, d\mu_g &\leq \frac{\pi^2}{4r^2} \sum_{i=1}^{k} [\mathrm{Vol}(B_r(p_i), g) + \mathrm{Vol}(B_r(q_i), g) - 2\tau(\kappa, r, n)] \\ &\leq \frac{\pi^2}{4r^2} \mathrm{Vol}(M, g) - \frac{k\pi^2}{2r^2} \tau(\kappa, r, n).\end{aligned}$$

Also, $\int_M f^2 \, d\mu_g = \sum_{j=1}^k \|f_j\|_2^2 \geq 2k\tau(\kappa, r, n)$, and the conclusion follows by the minimax principle. □

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