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TAUTOLOGICAL SHEAVES: STABILITY, MODULI SPACES AND RESTRICTIONS TO GENERALISED KUMMER VARIETIES

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Abstract

Results on stability of tautological sheaves on Hilbert schemes of points are extended to higher dimensions and to the restriction of tautological sheaves to generalised Kummer varieties. This provides a big class of new examples of stable sheaves on higher dimensional irreducible symplectic manifolds. Some aspects of deformations of tautological sheaves are studied.

Contents

0. Introduction

In the theory of compact Ricci flat manifolds one kind of basic building blocks for these manifolds are irreducible holomorphic symplectic (short: irreducible symplectic) manifolds. These are compact kähler manifolds that are simply connected and admit a—up to scalar—unique everywhere non-degenerate holomorphic two-form. One of the fundamental aspects in the theory of irreducible symplectic manifolds is the fact that only few examples have been constructed. Up to now, we are only aware of the

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existence of two infinite series of deformation classes due to Beauville ([1]) and two more sporadic examples found by O'Grady ([15, 16]). Moduli spaces of sheaves on symplectic surfaces play an important role in the construction of irreducible symplectic manifolds. The fundamental result of Mukai ([14]) states that the moduli space of stable sheaves on a $K3$ or abelian surface is a smooth symplectic variety. In fact, under mild technical conditions these spaces turn out to be irreducible symplectic manifolds.

A natural question that arises is the following: Can we iterate this process? That is, can we construct new examples of irreducible symplectic manifolds using moduli spaces of sheaves on known higher dimensional irreducible symplectic manifolds such as Hilbert schemes of points on $K3$ surfaces or generalised Kummer varieties? Certainly it is difficult to answer this question in this generality. On the other hand almost no examples of stable sheaves on higher dimensional irreducible symplectic manifolds had been encountered. In [18] the first example of a rank two stable vector bundle on the Hilbert scheme of two points on a $K3$ surface was found. In [19] this result was drastically generalised continuing along the following concept: Start with a stable sheaf on a $K3$ surface, transfer this sheaf to the Hilbert schemes of points using the universal property of the latter and obtain what is called a tautological sheaf and, finally, prove its stability.

This article is to be understood as a sequel to [19]. We further extend its results and transfer them to the case of generalised Kummer varieties. The latter are closed subvarieties of the Hilbert schemes of points on abelian surfaces. We study the restriction of tautological sheaves to Kummer varieties and obtain non-trivial examples of stable sheaves on these manifolds.

Summary of the results. Let X be a regular (i.e. $h^1(X, \mathcal{O}_X) = 0$) smooth projective surface. We study the stability of tautological sheaves with respect to an ample class H_N on the Hilbert scheme which is naturally associated with an ample class H on the underlying surface and depending on an integer $N \gg 0$.

Proposition (Proposition 2.4)**.** *Let $\mathcal{F}$ be a torsion-free μ_H-stable sheaf on X. Assume that its reflexive hull $\mathcal{F}^{\vee\vee} \ncong \mathcal{O}_X$. Then the tautological sheaf $\mathcal{F}^{[n]}$ on $X^{[n]}$ does not contain μ_{H_N}-destabilising subsheaves of rank one for all $N \gg 0$.*

From this we can deduce:

Theorem (Theorem 2.5)**.** *Let $\mathcal{F}$ be a torsion-free rank one sheaf on X satisfying* $\det \mathcal{F} \ncong \mathcal{O}_X$. *Then for all sufficiently large N the associated rank three sheaf $\mathcal{F}^{[3]}$ on $X^{[3]}$ is μ_{H_N}-stable.*

If X is abelian, then inside the Hilbert scheme $X^{[n]}$ there is the generalised Kummer variety $K_n(X)$. Let us denote the embedding by j. On X we have the natural involution

ι from the group structure. A sheaf $\mathcal{H}$ is called *symmetric* if $\iota^\star\mathcal{H} \cong \mathcal{H}$. We have the following results:

Theorem (Theorems 3.7, 3.9 and 3.12)**.** *Let $\mathcal{F}$ be a μ_H-stable sheaf on X such that* $\det\mathcal{F} \ncong \mathcal{O}_X$. *There is a polarisation on $K_3(X)$ such that if $\mathcal{F}$ is of rank one, $j^\star\mathcal{F}^{[3]}$ is μ-stable of rank three. Furthermore, there is a polarisation on $K_2(X)$* (*the Kummer surface associated with X*) *such that if* $\det\mathcal{F}$ *is not symmetric and $\mathcal{F}$ is of rank one* (*rank two*)*, the restriction $j^\star\mathcal{F}^{[2]}$ is μ-stable of rank two* (*rank four*)*.*

Furthermore, we have the following relation between moduli spaces of sheaves on $K3$ surfaces and moduli spaces of tautological sheaves:

Proposition (Proposition 4.4)**.** *Let $\mathcal{F}$ be a stable sheaf on a $K3$ surface X of Mukai vector v such that $\mathcal{F}^{[2]}$ is stable* (*of Mukai vector $v^{[2]}$*)*. We have an embedding of moduli spaces $\mathcal{M}^s(v) \hookrightarrow \mathcal{M}^s(v^{[2]})$.*

Structure of the paper. The paper is organised as follows: We begin in Section 1.1 by collecting known results on the geometry of Hilbert schemes of points on a surface and prove a few technical lemmata which will be needed later. Next, in Section 1.2 we introduce the main objects of this article, the tautological sheaves. In Section 1.3 we introduce polarisations on the Hilbert schemes and compute the slopes of tautological sheaves with respect to these polarisations. In Section 2 we analyse destabilising subsheaves of tautological sheaves on Hilbert schemes of three or more points. The case of generalised Kummer varieties is treated in Section 3. We prove the stability of the restriction of certain tautological sheaves to the Kummer surface (Section 3.1) and the generalised Kummer variety of dimension four (Section 3.2). We conclude the paper by studying deformations of tautological sheaves in Section 4. We show that the moduli spaces of tautological sheaves can be singular in Section 4.2 and investigate in which way we may deform tautological sheaves together with the underlying manifold in Section 4.3.

Notations and conventions.

- The base field of all varieties and schemes is the field of complex numbers.
- All functors such as pushforward, pullback, local and global homomorphisms and tensor product are not derived unless mentioned otherwise.
- By $\mathcal{M}^s(v)$ we denote the moduli space of μ-stable sheaves with numerical invariants v. (We assume a polarisation has been fixed.)

1. Preliminaries

Throughout this chapter we consider a smooth projective surface X together with a polarisation $H \in \mathrm{NS}(X)$.

1.1. Geometric considerations. For $n = 2$ the geometry of the Hilbert scheme points on a surface is very well accessible: In fact, $X^{[2]}$ is the blowup of the symmetric square S^2X along the diagonal. If $n > 2$, the situation is a little bit more complicated but by [8, Proposition 2.6] the Hilbert scheme is still the blowup along the big diagonal. This is important, especially if we want to determine the Picard group of the Hilbert scheme. Let us summarise the most important results.

Following [4, Section 1], we consider the following diagram:

$$\begin{tikzcd}
 & X^{[n-1,n]} \arrow[rr, "w"'] \arrow[d, "\sigma"] \arrow[drr, "\psi"] & & \Xi_n \arrow[r, "q"'] \arrow[d, "p"'] & X & \\
 & \Xi_{n-1} \subset X \times X^{[n-1]} \arrow[dl, "p"] \arrow[dr, "q"] & & X^{[n]} \arrow[dr, "\rho"'] & & X^n \arrow[dl] \\
X^{[n-1]} & & X & & S^n X. &
\end{tikzcd}$$

Here we denote by

$$\Xi_n := \{(x, \xi) \mid x \in \xi\} \subset X \times X^{[n]}$$

the universal subscheme and by

$$X^{[n-1,n]} := \{(\xi', \xi) \mid \xi' \subset \xi\} \subset X^{[n-1]} \times X^{[n]}$$

the so-called *nested Hilbert scheme*.

We have the flat degree n covering $p\colon \Xi_n \to X^{[n]}$ which is, in fact, the restriction of the second projection $X \times X^{[n]} \to X^{[n]}$. Furthermore, $X^{[n-1,n]}$ is isomorphic to the blowup of $X \times X^{[n-1]}$ along the universal subscheme Ξ_{n-1}. Denote this blowup morphism by σ and the projections from $X \times X^{[n-1]}$ to $X^{[n-1]}$ and X by p and q, respectively. By [5, Proposition 2.1] the second projection $\psi\colon X^{[n-1,n]} \to X^{[n]}$ factors through Ξ_n and from [9, Proposition 3.5.3] it follows that w is an isomorphism outside codimension four subschemes. Thus the morphism ψ is flat outside codimension four. Finally, we have $q \circ \sigma = q \circ w$ and $w^\star \omega_p \cong \omega_\sigma$.

We have

$$\operatorname{Pic}^0 X^{[n]} \cong \operatorname{Pic}^0 X$$

and embeddings

$$(-)_{X^{[n]}}\colon \operatorname{NS} X \hookrightarrow \operatorname{NS} X^{[n]}, \quad l \mapsto l_{X^{[n]}} := \rho^\star (l^{\boxplus n})^{\mathfrak{S}_n}$$

and

$$(-)_{X^{[n]}}\colon \operatorname{Pic} X \hookrightarrow \operatorname{Pic} X^{[n]}.$$

Furthermore, there is a class $\delta \in \operatorname{NS} X^{[n]}$, such that 2δ is the class of the divisor consisting of all non-reduced subschemes $\xi \subset X$.

Now, using Section 2 of [4] and the proof of Theorem 4.2 of [13], we can deduce the following formulas:

Lemma 1.1. *Let D be the exceptional divisor of σ and let l be a class in* $\mathrm{NS}\,X$*. We have*

$$\psi^\star\delta = [D] + \sigma^\star p^\star\delta$$

and

$$\psi^\star l_{X^{[n]}} = \sigma^\star(p^\star l_{X^{[n-1]}} + q^\star l).$$

If X is regular (i.e. $h^1(X, \mathcal{O}_X) = 0$), we have

$$\mathrm{NS}\,X^{[n]} \cong \mathrm{NS}\,X \oplus \mathbb{Z}\delta$$

by a result of Fogarty (cf. [6]).

Finally, let us briefly introduce the generalised Kummer varieties. If one mimics the construction of Hilbert schemes to the case of abelian surfaces, one again obtains Ricci flat manifolds. But they are not simply connected and contain an additional factor in the Beauville–Bogomolov decomposition. To get rid of this factor we consider (for an abelian surface A) the fibre

$$K_n(A) := m^{-1}(0)$$

and call it *generalised Kummer variety*. It is a $(2n-2)$-dimensional irreducible symplectic manifold (cf. [1]). In the case $n = 2$ this just gives the Kummer surface $\mathrm{Km}\,A$.

Let us denote the inclusion $K_n(A) \hookrightarrow A^{[n]}$ by j. It is a well known fact (again cf. [6]) that we have:

$$\mathrm{NS}(K_n(A)) \cong j^\star\,\mathrm{NS}(A) \oplus \mathbb{Z}\, j^\star\delta,$$

where we embedded, as before, $\mathrm{NS}(A)$ into $\mathrm{NS}(A^{[n]})$.

1.2. Tautological sheaves. Let us give the definition of tautological sheaves, the objects of main interest in this article. Fix a sheaf $\mathcal{F}$ on X and recall that there is the universal subscheme $\Xi_n \subset X \times X^{[n]}$. Furthermore we have the two projections $p\colon X \times X^{[n]} \to X^{[n]}$ and $q\colon X \times X^{[n]} \to X$.

DEFINITION 1.2. The *tautological sheaf associated with* $\mathcal{F}$ is defined as

$$\mathcal{F}^{[n]} := p_\star(q^\star\mathcal{F} \otimes \mathcal{O}_{\Xi_n}).$$

REMARK 1.3. Very important for the study of tautological sheaves is the following observation: The universal subscheme Ξ_n and the nested Hilbert scheme $X^{[n-1,n]}$

are isomorphic outside codimension four subschemes (cf. Section 1.1). Let U denote the open subset where they are actually isomorphic. The restrictions of $q^\star\mathcal{F}$ and $\sigma^\star q^\star\mathcal{F}$ to U are naturally isomorphic. Thus the restriction of $\mathcal{F}^{[n]}$ to the image $p(U)$ in $X^{[n]}$ is isomorphic to $\widetilde{\mathcal{F}^{[n]}} := \psi_\star\sigma^\star q^\star\mathcal{F}$ (restricted to $\psi(U) = p(U)$). Hence we can use $\widetilde{\mathcal{F}^{[n]}}$ instead of $\mathcal{F}^{[n]}$ as long as we want to study properties that are not sensible with respect to modifications in codimension four. In the case $n = 2$ we, in fact, have $\widetilde{\mathcal{F}^{[2]}} \cong \mathcal{F}^{[2]}$.

The restriction of p to Ξ_n is a flat covering of degree n. Hence the tautological sheaf $\mathcal{F}^{[n]}$ is locally free whenever $\mathcal{F}$ is. For the tautological sheaf associated with the dual sheaf $\mathcal{F}^\vee$ we have the following formulas which will be important later.

Lemma 1.4.

$$(\mathcal{F}^\vee)^{[n]} \cong p_\star\, \mathcal{H}om(q^\star\mathcal{F}, \mathcal{O}_\Xi) \tag{1}$$

and

$$(\mathcal{F}^{[n]})^\vee \cong p_\star\, \mathcal{H}om(q^\star\mathcal{F}, \omega_p). \tag{2}$$

Again, from [13] we deduce:

Lemma 1.5. *We have the following formula for the first Chern class of* $\mathcal{F}^{[n]}$:

$$c_1(\mathcal{F}^{[n]}) = c_1(\mathcal{F})_{X^{[n]}} - \mathrm{rk}(\mathcal{F})\delta.$$

Next, we want to summarise the results of Scala and Krug about global sections and extensions of tautological sheaves. These formulas turn out to be a powerful tool to analyse stability and deformations of these sheaves.

Theorem 1.6. *For every sheaf $\mathcal{F}$ and every line bundle $\mathcal{L}$ on X we have*

$$\mathrm{H}^*(X^{[n]}, \mathcal{F}^{[n]} \otimes \mathcal{L}_{X^{[n]}}) \cong \mathrm{H}^*(X, \mathcal{F} \otimes \mathcal{L}) \otimes S^{n-1}\mathrm{H}^*(X, \mathcal{L}).$$

Proof. [17, Corollary 4.5], [12, Theorem 6.17]. □

We continue by stating Krug's formula for the extension groups of tautological sheaves:

Theorem 1.7. *Let $\mathcal{F}$ and $\mathcal{E}$ be sheaves and $\mathcal{L}$ and $\mathcal{M}$ be line bundles on X. We have*

$$\begin{aligned}
&\mathrm{Ext}^*_{X^{[n]}}(\mathcal{E}^{[n]} \otimes \mathcal{L}_{X^{[n]}}, \mathcal{F}^{[n]} \otimes \mathcal{M}_{X^{[n]}})\\
&\cong \mathrm{Ext}^*_X(\mathcal{E} \otimes \mathcal{L}, \mathcal{F} \otimes \mathcal{M}) \otimes S^{n-1}\,\mathrm{Ext}^*_X(\mathcal{L}, \mathcal{M})\\
&\quad \oplus \mathrm{Ext}^*_X(\mathcal{E} \otimes \mathcal{L}, \mathcal{M}) \otimes \mathrm{Ext}^*_X(\mathcal{L}, \mathcal{F} \otimes \mathcal{M}) \otimes S^{n-2}\,\mathrm{Ext}^*_X(\mathcal{L}, \mathcal{M}).
\end{aligned} \tag{3}$$

Proof. [12, Theorem 6.17]. $\square$

Krug also gave a description how to compute Yoneda products on these extension groups (cf. [12, Section 7]). The general formulas are extremely long. We will give a more detailed account on them as needed.

Let us finish this section by deriving a special case of formula (3).

Corollary 1.8. *Let X be a $K3$ surface and let $\mathcal{F}$ be a sheaf on X satisfying $h^2(\mathcal{F}) = 0$. Then we have*

$$\begin{aligned} &\mathrm{Hom}_{X^{[n]}}(\mathcal{F}^{[2]}, \mathcal{F}^{[2]}) \cong \mathrm{Hom}_X(\mathcal{F}, \mathcal{F}), \\ &\mathrm{Ext}^1_{X^{[n]}}(\mathcal{F}^{[2]}, \mathcal{F}^{[2]}) \cong \mathrm{Ext}^1_X(\mathcal{F}, \mathcal{F}) \oplus \mathrm{H}^0(X, \mathcal{F}) \otimes \mathrm{H}^1(X, \mathcal{F})^\vee. \end{aligned} \tag{4}$$

REMARK 1.9. From these equations we can deduce that tautological sheaves $\mathcal{F}^{[2]}$ associated with stable sheaves $\mathcal{F} \ncong \mathcal{O}_X$ are always simple: By Serre duality a stable sheaf $\mathcal{F} \ncong \mathcal{O}_X$ on a $K3$ surface satisfies either $h^2(\mathcal{F}) = 0$ or $h^0(\mathcal{F}) = 0$ and by twisting with a suitable line bundle we may assume that $h^2(\mathcal{F}) = 0$. This is a first indication that tautological sheaves might be stable.

1.3. Polarisations and slopes. In this section we shall talk about polarisations on the Hilbert scheme of points on a surface. In general the ample cone of these varieties is not completely known. Nevertheless, if we fix a polarisation H on our surface X, we will define polarisations H_N on $X^{[n]}$, depending on H and an integer N. Furthermore, we shall derive and discuss the slopes of tautological sheaves with respect to these polarisations. This will be important when we want to study the stability of these sheaves in Sections 2 and 3.

Fix a smooth projective surface X and an ample class $H \in \mathrm{NS}\, X$. For any integer N we consider the class

$$H_N := N H_{X^{[n]}} - \delta \in \mathrm{NS}\, X^{[n]}.$$

Using induction one easily shows that H_N is ample for large N. Thus we have a natural candidate for a polarisation of the Hilbert scheme and, as it turns out, in many cases tautological sheaves are stable with respect to these polarisations (cf. [19]) for $n = 2$.

Next, we want to compute slopes of tautological sheaves also in the case $n > 2$. Hence we need to compute intersection numbers. We have the following general result:

Lemma 1.10. *Let l be a class in* $\mathrm{NS}\, X$. *We have*

$$l_{X^{[n]}} \cdot H^{2n-1}_{X^{[n]}} = \frac{(2n-1)!}{(n-1)!\, 2^{n-1}} (l \cdot H)(H^2)^{n-1} \tag{5}$$

and

$$\delta \cdot H_{X^{[n]}}^{2n-1} = 0, \tag{6}$$

where on the right hand side of (5) *we consider the intersection in* NS X.

We will abbreviate the factor $(2n-1)!/((n-1)!\,2^{n-1})$ by c_n.

Proof of Lemma 1.10. Note that (6) holds trivially since $H_{X^{[n]}}$ is a pullback along the Hilbert–Chow morphism. Let us prove (5). We pull back $l_{X^{[n]}}$ and $H_{X^{[n]}}$ along the $n!$-fold covering $X^n \to S^n X$ and obtain the classes $l^{\boxplus n}$ and $H^{\boxplus n}$, respectively. We have

$$\begin{aligned} l_{X^{[n]}} \cdot H_{X^{[n]}}^{2n-1} &= \frac{1}{n!}(l^{\boxplus n})(H^{\boxplus n})^{2n-1} = \frac{1}{n!}\binom{2n-1}{1, 2, \dots, 2} n(l \cdot H)(H^2)^{n-1} \\ &= \frac{(2n-1)!}{(n-1)!\,2^{n-1}}(l \cdot H)(H^2)^{n-1}. \end{aligned}$$ □

Corollary 1.11. *Let $\mathcal{L}$ be a line bundle on X with first Chern class l and $\mathcal{F}$ a sheaf of rank r and first Chern class f. We have the following expansions for the slopes of $\mathcal{F}^{[n]}$ and $\mathcal{L}$ with respect to H_N:*

$$\mu_{H_N}(\mathcal{L}_{X^{[n]}}) = N^{2n-1} c_n (l \cdot H)(H^2)^{n-1} + O(N^{2n-2})$$

and

$$\mu_{H_N}(\mathcal{F}^{[n]}) = N^{2n-1} c_n \frac{1}{nr}(f \cdot H)(H^2)^{n-1} + O(N^{2n-2}).$$

2. Higher *n*

In this chapter we try to generalise the results on destabilising line subbundles in [19, Section 3] to higher n. From this generalisation we will be able to prove the stability of rank three tautological sheaves on $X^{[3]}$. In this chapter we fix a polarised regular surface (X, H).

Let $\mathcal{F}$ be a torsion-free μ_H-stable sheaf on X. Denote its rank by r and its first Chern class by f. We want to show that the associated tautological sheaf $\mathcal{F}^{[n]}$ on $X^{[n]}$ has no destabilising subsheaves of rank one. We will first assume that $\mathcal{F}$ is reflexive, i.e. locally free. Thus we may assume that a destabilising rank one subsheaf of $\mathcal{F}^{[n]}$ is also reflexive, that is, a line bundle.

Proposition 2.1. *For sufficiently large N, there are no μ_{H_N}-destabilising line subbundles in $\mathcal{F}^{[n]}$ of the form $\mathcal{L}_{X^{[n]}}$, ($\mathcal{L} \in \operatorname{Pic} X$), except the case $r = 1$ and $\mathcal{L} \cong \mathcal{F} \cong \mathcal{O}_X$.*

Proof. Denote the first Chern class of $\mathcal{L}$ by l. Using Scala's calculations of cohomology groups of tautological sheaves with twists as stated in Theorem 1.6 we can immediately deduce the following formula for homomorphisms from line bundles of the form $\mathcal{L}_{X^{[n]}}$ to tautological sheaves $\mathcal{F}^{[n]}$:

$$\mathrm{Hom}_{X^{[n]}}(\mathcal{L}_{X^{[n]}}, \mathcal{F}^{[n]}) \cong \mathrm{Hom}_X(\mathcal{L}, \mathcal{F}) \otimes S^{n-1}\,\mathrm{Hom}_X(\mathcal{L}, \mathcal{O}_X).$$

Let us first assume $r > 1$. Since $\mathcal{F}$ is μ_H-stable, we have necessary conditions for the existence of a line subbundle of $\mathcal{F}^{[n]}$:

$$l \cdot H < \frac{f \cdot H}{r} \quad \text{and} \quad l \cdot H \leq 0. \tag{7}$$

The first inequality is due to the stability of $\mathcal{F}$ and the second comes from the fact that if a line bundle has a section, its first Chern class has non-negative intersection with any ample class H. If $\mathcal{L}_{X^{[n]}} \subset \mathcal{F}^{[n]}$ is destabilising, by Corollary 1.11 we must have

$$l \cdot H \geqslant \frac{f \cdot H}{nr}.$$

But this is certainly a contradiction to (7).

If $r = 1$, we can proceed as above but additionally have to consider the special case $\mathcal{L} \cong \mathcal{F}$, i.e. $l \cdot H = f \cdot H$. The destabilising condition together with $l \cdot H \leq 0$ immediately yields $l \cdot H = 0$. But now $\mathrm{Hom}_X(\mathcal{L}, \mathcal{O}_X)$ can only be nontrivial if $\mathcal{L} \cong \mathcal{O}_X$. □

We will need the analogue of Proposition 3.1 in [19] which allows to reduce the general case to Proposition 2.1 above. We therefore first look at the following more general set-up which will be useful later, too. The proof is a straightforward induction.

Lemma 2.2. *Let $\tau \colon Y \to Z$ be a blow-up morphism of a smooth variety in a smooth codimension two center. Then*

$$R^0\sigma_\star(\omega_\sigma^{\otimes a}) \subseteq \mathcal{O}_Z$$

for all $a \in \mathbb{Z}$.

Applying this to our situation, we easily find:

Lemma 2.3. *Let $\mathcal{L}_{X^{[n]}} \otimes \mathcal{O}(a\delta)$ be a line bundle on $X^{[n]}$ ($a \in \mathbb{Z}$ arbitrary), then for any locally free sheaf $\mathcal{F}$ on X we have*

$$\mathrm{Hom}_{X^{[n]}}(\mathcal{L}_{X^{[n]}} \otimes \mathcal{O}(a\delta), \mathcal{F}^{[n]}) \subseteq \mathrm{Hom}_{X^{[n]}}(\mathcal{L}_{X^{[n]}}, \mathcal{F}^{[n]}).$$

We can thus deduce the first main result of this section.

Proposition 2.4. *Let $\mathcal{F}$ be a torsion-free μ_H-stable sheaf on X. Assume that its reflexive hull $\mathcal{F}^{\vee\vee} \ncong \mathcal{O}_X$. Then $\mathcal{F}^{[n]}$ does not contain μ_{H_N}-destabilising subsheaves of rank one for all $N \gg 0$.*

Proof. We can easily reduce to the case where $\mathcal{F}$ is locally free and then apply Proposition 2.1 and Lemma 2.3 above. □

Since the tautological sheaf on $X^{[3]}$ associated with a rank one sheaf has rank three, the above proposition is enough to show that these sheaves are stable (except $\mathcal{O}_X^{[3]}$, of course).

Theorem 2.5. *Let $\mathcal{F}$ be a torsion-free rank one sheaf on X satisfying* $\det \mathcal{F} \ncong \mathcal{O}_X$. *Then for all sufficiently large N the associated rank three sheaf $\mathcal{F}^{[3]}$ on $X^{[3]}$ is μ_{H_N}-stable.*

Proof. As usual we can reduce to the case that $\mathcal{F}$ is locally free. We have seen that $\mathcal{F}^{[3]}$ cannot contain destabilising subsheaves of rank one. But any destabilising subsheaf of rank two yields a rank one destabilising subsheaf of the dual sheaf. It is now enough to prove that for any line bundle $\mathcal{L}$ on $X^{[3]}$ we have

$$\mathrm{Hom}(\mathcal{L}, (\mathcal{F}^{[3]})^\vee) \subseteq \mathrm{Hom}(\mathcal{L}, (\mathcal{F}^\vee)^{[3]}).$$

To show this formula we use equations (1) and (2), then adjunction $p^\star \dashv p_\star$ and finally we use Lemma 2.2, keeping in mind that w is an automorphism outside codimension four and $w^\star \omega_p \cong \omega_\sigma$. □

3. Restriction to generalised Kummer varieties

In this section we study the stability the restrictions of tautological sheaves to the associated generalised Kummer varieties.

3.1. Restriction to the Kummer surface.

In this section we shall prove the stability of the restriction of certain tautological sheaves from the Hilbert scheme of two points on an abelian surface to the associated Kummer surface. Throughout this section we fix a polarised abelian surface (A, H).

Let b: $\tilde{A} \to A$ denote the simultaneous blowup of all fixed points of the involution ι on A and denote by $E_1, \ldots, E_{16}$ the exceptional divisors. On $\tilde{A}$ we still have an involution which fixes the E_l pointwise. We consider the quotient $\tau\colon \tilde{A} \to \mathrm{Km}\, A$ which is a degree two covering onto the associated Kummer surface. By [2, VIII Proposition 5.1] we have a monomorphism

$$\alpha = \tau_! b^\star \colon \mathrm{H}^2(A, \mathbb{Z}) \to \mathrm{H}^2(\mathrm{Km}\, A, \mathbb{Z})$$

satisfying

$$\alpha(x)\alpha(y) = 2xy \quad \text{for all} \quad x, y \in \mathrm{H}^2(A, \mathbb{Z}).$$

We set $N_l = \tau(E_l)$. It is well known that $E_l^2 = -1$ and $N_l^2 = -2$. Furthermore, the class $\sum_l N_l$ is 2-divisible and we have $\tau^\star\big((1/2)\sum_l N_l\big) = \sum_l E_l$ and $\tau^\star N_l = 2E_l$.

Finally, we have

$$\mathrm{NS}\,\tilde{A} \cong b^\star\,\mathrm{NS}\,A \oplus \bigoplus_{l=1}^{16} \mathbb{Z}E_l \quad \text{and} \quad \mathrm{Pic}^0\,\tilde{A} \cong b^\star\,\mathrm{Pic}^0\,A. \tag{8}$$

We define the class

$$H_N := N\alpha(H) - \frac{1}{2}\sum_l N_l$$

on Km A, which is ample for sufficiently large N. (This is the restriction to Km A of the class H_N defined on $A^{[2]}$ in Section 1.1.)

DEFINITION 3.1. Let $\mathcal{F}$ be a sheaf on A. We set

$$\mathcal{F}^{\mathrm{Km}} := \tau_\star b^\star \mathcal{F}.$$

One easily shows:

Lemma 3.2. *The sheaf $\mathcal{F}^{\mathrm{Km}}$ is the restriction of the tautological sheaf $\mathcal{F}^{[2]}$ along the inclusion $j\colon \mathrm{Km}\,A \hookrightarrow A^{[2]}$:*

$$j^\star\mathcal{F}^{[2]} \simeq \mathcal{F}^{\mathrm{Km}}.$$

Now we want to prove the stability of $\mathcal{F}^{\mathrm{Km}}$ in the case that $\mathcal{F}$ is of rank one or two. As in the previous cases we begin with the analysis of line subbundles in the pullback $b^\star\mathcal{F}$:

Proposition 3.3. *Let $\mathcal{F}$ be a μ_H-stable sheaf on A of rank r and first Chern class $f \in \mathrm{NS}\,A$. Then $b^\star\mathcal{F}$ does not contain any line bundle $\mathcal{L}' = b^\star\mathcal{G} \otimes \mathcal{O}\big(\sum_l a_l E_l\big)$ with $\mathcal{G} \in \mathrm{Pic}(A)$, $c_1(\mathcal{G}) = g$ satisfying*

$$H \cdot g \geqslant \frac{1}{r} H \cdot f$$

but in the case $r = 1$, $\mathcal{G} \simeq \mathcal{F}$.

Proof. As usual we may assume that $\mathcal{F}$ is locally free. We want to show that

$$\mathrm{Hom}_{\tilde{A}}\left(b^\star\mathcal{G} \otimes \mathcal{O}\left(\sum_l a_l E_l\right), b^\star\mathcal{F}\right)$$

vanishes. Using adjunction ($b^\star \dashv b_\star$) and a similar induction argument as in the proof of [19, Proposition 3.1], we see that it is enough to prove that

$$\operatorname{Hom}_A(\mathcal{G}, \mathcal{F}) = 0.$$

This easily follows from the stability of $\mathcal{F}$ if $\mathcal{F} \not\simeq \mathcal{G}$. □

Next, we will show that Proposition 3.3 implies that there are no destabilising line subbundles in $\mathcal{F}^{\mathrm{Km}}$. We only need to calculate slopes.

Lemma 3.4. *Let $\mathcal{F}$ be a sheaf on A of rank r and first Chern class f. We have*

$$c_1(\tau_\star b^\star \mathcal{F}) = \alpha(f) - \frac{r}{2} \sum_l N_l.$$

Proof. We have $c_1(\omega_{\tilde{A}}) = \sum_l E_l$. Thus the Grothendieck–Riemann–Roch theorem reads

$$\begin{aligned} \operatorname{ch}(\tau_\star b^\star \mathcal{F}) = \tau_\star(\operatorname{ch}(b^\star \mathcal{F})\operatorname{td}_\tau) &= \tau_\star\left((r, b^\star f, \ldots)\left(1, -\frac{1}{2}\sum_l E_l, \ldots\right)\right) \\ &= \tau_\star\left(r, b^\star f - \frac{r}{2}\sum_l E_l, \ldots\right). \end{aligned}$$ □

Let $\mathcal{L}$ be a line bundle on $\operatorname{Km} A$. By equation (8) there is a line bundle $\mathcal{G}$ on A and integers a_l such that

$$\tau^\star \mathcal{L} \simeq b^\star \mathcal{G} \otimes \mathcal{O}\left(\sum_l a_l E_l\right).$$

Set $g := c_1(\mathcal{G})$. Note that since $\mathcal{L}$ comes from $\operatorname{Km} A$, the line bundle $\mathcal{G}$ has to be symmetric, i.e. $\iota^\star \mathcal{G} \simeq \mathcal{G}$.

Corollary 3.5. *Let $\mathcal{L}$ be a line bundle on* Km *A as above. We have*

$$\mu_{H_N}(\mathcal{F}^{\mathrm{Km}}) = \frac{1}{r} NH.f - 4$$

and

$$\mu_{H_N}(\mathcal{L}) = NH \cdot g + \frac{1}{2}\sum_l a_l.$$

Proof. We pullback all classes to $\tilde{A}$: Note that $\tau^\star\big((1/2)\sum_l N_l\big) = \sum_l E_l$ and $\tau^\star \alpha(f) = 2b^\star f$ for all $f \in \operatorname{NS} A$. Thus we have $\alpha(f) \cdot \alpha(H) = 2 f \cdot H$. Furthermore, we have $\left(\sum_l E_l\right)^2 = 16 \cdot (-1) = -16$ and $\left(\sum_l E_l\right)\left(\sum_l a_l E_l\right) = -\sum_l a_l$. Finally, we have to divide everything by two because we pulled back along a degree two covering. □

Corollary 3.6. *Let $\mathcal{F}$ be a non-symmetric (i.e. $\iota^\star\mathcal{F} \not\simeq \mathcal{F}$) μ_H-stable sheaf on A. Then $\mathcal{F}^{\mathrm{Km}}$ does not contain $\mu_{H_N^{\mathrm{Km}}}$-destabilising line subbundles for all $N \gg 0$.*

Proof. Let $\mathcal{L}$ be a destabilising line subbundle of $\mathcal{F}^{\mathrm{Km}}$. Again, we can write $\tau^\star\mathcal{L} \simeq b^\star\mathcal{G} \otimes \mathcal{O}\left(\sum_l a_l E_l\right)$ for a symmetric line bundle $\mathcal{G} \in \operatorname{Pic} A$. The destabilising condition yields

$$H \cdot g \geqslant \frac{1}{r} H.f.$$

As usual we use adjunction $\tau^\star \dashv \tau_\star$ to obtain a homomorphism $\tau^\star\mathcal{L} \to b^\star\mathcal{F}$. This gives a contradiction to Proposition 3.3 but in the case $r = 1$, $\mathcal{G} \simeq \mathcal{F}$. But this cannot be since $\mathcal{F}$ was chosen not to be symmetric. □

We immediately deduce:

Theorem 3.7. *Let $\mathcal{F}$ be a non-symmetric rank one torsion-free sheaf on A. Then for all N sufficiently large, $\mathcal{F}^{\mathrm{Km}} = \tau_\star b^\star\mathcal{F}$ is a rank two $\mu_{H_N^{\mathrm{Km}}}$-stable sheaf.*

EXAMPLE 3.8. We apply the theorem to the case $c_1(\mathcal{F}) = 0$. Denote by $\hat{A}$ the dual abelian variety and by $\hat{A}[2]$ its two-torsion points. The assignment $\operatorname{Pic}^0 A \ni \mathcal{F} \mapsto \mathcal{F}^{\mathrm{Km}}$ gives a map

$$\hat{A} \setminus \hat{A}[2] \to \mathcal{M},$$

where $\mathcal{M} := \mathcal{M}_{H_N^{\mathrm{Km}}}(v)$ is the moduli space of H_N^{Km}-stable sheaves with

$$v = \left(2, -\frac{1}{2}\sum_l N_l, -2\right).$$

Note that $v^2 = 0$ and the first Chern class $-(1/2)\sum_l N_l$ is primitive. Hence $\mathcal{M}$ is smooth of dimension two. Since $\mathcal{F}^{\mathrm{Km}} \simeq (\iota^\star\mathcal{F})^{[\mathrm{Km}]}$, this map is two-to-one. Furthermore, let us consider the case that $\mathcal{F}$ is symmetric, i.e. $\mathcal{F} \in \hat{A}[2]$. We concentrate on the case $\mathcal{F} = \mathcal{O}_A$. We have extensions

$$0 \to \mathcal{O}\left(-\frac{1}{2}\sum_l N_l\right) \to \mathcal{E} \to \mathcal{O} \to 0.$$

The sheaf $\mathcal{O}_X^{\mathrm{Km}}$ is isomorphic to the trivial extension $\mathcal{O}\left(-(1/2)\sum_l N_l\right) \oplus \mathcal{O}$ (cf. [2, Lemma 17.2]), which is not stable. On the other hand one can show that every non-trivial extension is μ_H-stable. The vector space of extensions $\mathcal{E}$ is two-dimensional and thus we have a $\mathbb{P}^1 \subset \mathcal{M}$ parametrising the $\mathcal{E}$. Altogether we see that $\mathcal{M}$ is isomorphic to the Kummer surface $\operatorname{Km}\hat{A}$ of the dual abelian surface $\hat{A}$.

If $\mathcal{F}$ has nontrivial first Chern class $f \in \operatorname{NS} A$, we may choose a symmetric line bundle $\mathcal{L}$ satisfying $c_1(\mathcal{L}) = -f$. Then $\mathcal{F} \otimes \mathcal{L}$ is in $\operatorname{Pic}^0(A)$ and

$$(\mathcal{F} \otimes \mathcal{L})^{[\mathrm{Km}]} \simeq \mathcal{F}^{\mathrm{Km}} \otimes \mathcal{O}(\alpha(-f)).$$

Thus the moduli space containing $\mathcal{F}^{\mathrm{Km}}$ is isomorphic to $\operatorname{Km}\hat{A}$, too. Note that by [7, Theorem 1.5] the Kummer surfaces $\operatorname{Km} A$ and $\mathcal{M} \cong \operatorname{Km}\hat{A}$ are isomorphic.

We finish the section by proving the analogue of Theorem 4.4 of [19].

Theorem 3.9. *Let $\mathcal{F}$ be a μ_H-stable rank two sheaf on A such that* $\det \mathcal{F}$ *is not symmetric. Then $\mathcal{F}^{\mathrm{Km}}$ is a μ_{H_N}-stable rank four sheaf on* $\operatorname{Km} A$.

Proof. We exactly imitate the proof of [19, Theorem 4.4]. Assume that $\mathcal{F}$ is locally free and let $f := c_1(\mathcal{F})$. Let $\mathcal{E}$ be a reflexive semistable rank two subsheaf of $\mathcal{F}^{\mathrm{Km}}$ and write $c_1(\mathcal{E}) = \alpha(e) + \sum_l a_l N_l$. The destabilising condition thus implies

$$2H \cdot e \geqslant H \cdot f.$$

We have a homomorphism $\beta\colon \tau^\star\mathcal{E} \to b^\star\mathcal{F}$. Again, the only difficult case is when $\ker\beta = 0$: If the first Chern class of the $\mathcal{Q} := \operatorname{coker}\beta$ is trivial, we see that the homological dimension of $\mathcal{Q}$ is 2. Since $b^\star\mathcal{F}$ is locally free, this would contradict the fact that $\tau^\star\mathcal{E}$ is reflexive. Thus $\mathcal{Q} = 0$ and β has to be an isomorphism. But since $\tau^\star\mathcal{E}$ is symmetric and $\mathcal{F}$ is not, we are done.

If there is an effective divisor with first Chern class $c_1(\mathcal{Q})$, the line bundle

$$b^\star \det \mathcal{F} \otimes \tau^\star \det \mathcal{E}^\vee \left(-\sum_l a_l N_l\right)$$

must have a section. Hence either $a_l < 0\ \forall l$ and $\det \mathcal{F} \simeq \mathcal{O}_A$ (which we excluded) or $a_l \leq 0\ \forall l$ and

$$H \cdot f > 2H \cdot e$$

which contradicts the stability condition. □

3.2. Generalised Kummer varieties of dimension four. Let (A, H) be a polarised abelian surface. In this section we prove some results concerning the stability of the restriction of tautological sheaves from the Hilbert scheme of three points on A to the four dimensional generalised Kummer variety $j\colon K_3(A) \hookrightarrow A^{[3]}$.

We have an isomorphism

$$\mathrm{NS}(K_3(A) \cong j^\star\,\mathrm{NS}(A) \oplus \mathbb{Z}\,j^\star\delta$$

and we define a polarisation on $K_3(A)$

$$H_N := j^\star H_N = Nj^\star H_{A^{[3]}} - j^\star\delta.$$

Lemma 3.10. *We have*

$$(H_N)^3 \cdot j^\star\delta = 0 + O(N^4).$$

Proof. By definition of H_N we have $(H_N)^3 \cdot j^\star\delta = j^\star((H_N)^3\delta)$. Now the lemma follows from equation (6) in Lemma 1.10. $\square$

Proposition 3.11. *Let $\mathcal{F}$ be a μ_H-stable sheaf on A of rank r and first Chern class f. If $\mathcal{F}^{\vee\vee} \ncong \mathcal{O}_A$, then for N sufficiently large $\mathcal{F}^{K_3} := j^\star\mathcal{F}^{[3]}$ does not contain any μ_{H_N}-destabilising subsheaves of rank one.*

Proof. We may assume that $\mathcal{F}$ is locally free. Since all line bundles on $K_3(A)$ come from $A^{[3]}$, we may assume that a destabilising line bundle is of the form $j^\star\mathcal{M}'$ with $c_1(\mathcal{M}') \in \mathrm{NS}(A) \oplus \mathbb{Z}\delta$. By a similar reasoning as in [19, Proposition 3.1] we can reduce to the case where we have no contribution of the δ-summand neither. Thus there is, in fact, a line bundle $\mathcal{M}$ on A such that $\mathcal{M}' \cong \mathcal{M}_{A^{[3]}}$. Let $m := c_1(\mathcal{M})$.

We denote the projections $A^3 \to A$ by π_i. Furthermore let us denote the inclusion of the zero fibre $s^{-1}(0) \subset A^3$ of the group law by i. Using adjunctions, flatness and faithfulness of the pullback along finite coverings, it is straightforward to prove

$$\begin{aligned}\mathrm{Hom}(j^\star\mathcal{M}, j^\star\mathcal{F}^{[3]}) &\subseteq \mathrm{Hom}(i^\star\mathcal{M}^{\boxtimes 3}, i^\star\pi_1^\star\mathcal{F})\\ &\cong \mathrm{H}^0(i^\star(\mathcal{M}^{\vee\boxtimes 3} \otimes \pi_1^\star\mathcal{F})).\end{aligned}$$

In order to proceed, we choose an isomorphism $s^{-1}(0) \cong A^2$ by sending (x, y, z) to (x, z) and denote the projections $A^2 \to A$ by $\hat{\pi}_i$. In this picture we have the identifications: $\pi_1 \circ i = \hat{\pi}_1$, $\pi_2 \circ i \mathrel{\hat{=}} \iota \circ s$ and $\pi_3 \circ i \mathrel{\hat{=}} \hat{\pi}_2$. (Recall that ι denotes -1 on A.) Thus pushing forward along π_1 (π_2 in the second line), we have

$$\mathrm{H}^0(i^\star(\mathcal{M}^{\vee\boxtimes 3} \otimes \pi_1^\star\mathcal{F})) \cong \mathrm{H}^0(\mathcal{F} \otimes \mathcal{M}^\vee \otimes \hat{\pi}_{1\star}(\hat{\pi}_2^\star\mathcal{M}^\vee \otimes s^\star\iota^\star\mathcal{M}^\vee)) \tag{9}$$

$$\cong \mathrm{H}^0(\mathcal{M}^\vee \otimes \hat{\pi}_{2\star}(\hat{\pi}_1^\star(\mathcal{F} \otimes \mathcal{M}^\vee) \otimes s^\star\iota^\star\mathcal{M}^\vee)). \tag{10}$$

By Lemma 3.10 the destabilising condition of $j^\star\mathcal{M}$ in $j^\star\mathcal{F}^{[3]}$ implies

$$m \cdot H \geqslant \frac{f \cdot H}{3r}.$$

If $m \cdot H \geqslant 0$, we see that the right hand side of (9) vanishes but in the case $\mathcal{M} \cong \mathcal{O}_A \cong \mathcal{F}$.

If $m \cdot H < 0$, the destabilising condition implies

$$2m \cdot H > \frac{f \cdot H}{r}.$$

In this case the right hand side of (10) has to vanish. □

As usual, from Proposition 3.11 we can deduce the stability of rank three restricted tautological sheaves associated with rank one sheaves:

Theorem 3.12. *Let $\mathcal{F}$ be a torsion-free rank one sheaf on A. Assume* $\det \mathcal{F} \ncong \mathcal{O}_A$. *Then for all sufficiently large N the sheaf $j^\star \mathcal{F}^{[3]}$ is μ_N^K-stable.*

4. Deformations and moduli spaces of tautological sheaves

This chapter collects a few results on different aspects of the behaviour of tautological sheaves under deformations.

4.1. Deformations of tautological sheaves.

In this section we will make the following general assumption:

ASSUMPTION. X is a $K3$ surface and $\mathcal{F}$ a stable sheaf on X with invariants v such that for every (μ-stable sheaf $\mathcal{G} \in \mathcal{M}^s(v)$ the associated tautological sheaf $\mathcal{G}^{[n]}$ is also stable.

Note that in the cases where the stability of tautological sheaves has been explicitly proven the tautological sheaf associated with a sheaf $\mathcal{F}$ is stable if and only if it is true for every other (μ-stable) $\mathcal{G}$ in the same moduli space. (We are only considering sheaves on $K3$ surfaces.)

Denote by $v^{[n]} \in \mathrm{H}^*(X^{[n]}, \mathbb{Q})$ the Mukai vector of $\mathcal{F}^{[n]}$. The assignment

$$\mathcal{F} \mapsto \mathcal{F}^{[n]}$$

yields a morphism

$$[-]^{[n]} \colon \mathcal{M}^s(v) \to \mathcal{M}^s(v^{[n]}).$$

We shall mainly discuss the case $n = 2$. Let us prove the following lemma which shows that $[-]^{[2]}$ is injective on closed points.

Lemma 4.1. *For every sheaf $\mathcal{F}$ on X we have*

$$\mathcal{F} \cong \mathcal{T}or^1_{\mathcal{O}_{X \times X}}(\mathcal{O}_\Delta, \sigma_\star \psi^\star \mathcal{F}^{[2]}).$$

Thus we can reconstruct the original sheaf $\mathcal{F}$ from the tautological sheaf $\mathcal{F}^{[2]}$.

Proof. Recall that we have an exact sequence on $X \times X$:

$$0 \to \sigma_\star \psi^\star \mathcal{F}^{[2]} \to \mathcal{F}^{\boxplus 2} \to \Delta_\star \mathcal{F} \to 0. \tag{11}$$

We tensor this sequence with the structure sheaf of the diagonal $\Delta \subset X \times X$. Of course we have

$$\pi_1^\star \mathcal{F}|_\Delta \cong \Delta^\star \pi_1^\star \mathcal{F} \cong \mathcal{F}$$

and the higher Tors $\mathcal{T}or^i_{\mathcal{O}_{X\times X}}(\mathcal{O}_\Delta, \pi_1^\star \mathcal{F})$ vanish. Therefore we have an isomorphism

$$\mathcal{T}or^1_{\mathcal{O}_{X\times X}}(\mathcal{O}_\Delta, \sigma_\star \psi^\star \mathcal{F}^{[2]}) \cong \mathcal{T}or^2_{\mathcal{O}_{X\times X}}(\mathcal{O}_\Delta, \Delta_\star \mathcal{F}).$$

By Proposition 11.8 in [11] we find

$$\mathcal{T}or^i_{\mathcal{O}_{X\times X}}(\mathcal{O}_\Delta, \Delta_\star \mathcal{F}) = \begin{cases} \mathcal{F} & i = 0, 2 \quad \text{and} \\ \mathcal{F} \otimes \Omega_X & i = 1. \end{cases} \qquad \square$$

REMARK 4.2. If we tensor (11) with $\mathcal{O}_\Delta$ as above, the first terms of the resulting long exact Tor-sequence yield a short exact sequence

$$0 \to \mathcal{F} \otimes \Omega_X \to \sigma_\star \psi^\star \mathcal{F}^{[2]}|_\Delta \to \mathcal{F} \to 0.$$

It is not clear if this exact sequence is split or if it is equivalent to the natural extension corresponding to the Atiyah class of $\mathcal{F}$.

Let us consider a stable sheaf $\mathcal{F}$ on a $K3$ surface X. The stability implies that either $h^0(X, \mathcal{F})$ or $h^2(X, \mathcal{F}) = h^0(X, \mathcal{F}^\vee)$ vanishes. Let us assume the former is the case. (The case $h^2(X, \mathcal{F}) = 0$ can be treated in exactly the same way.) Corollary 1.8 shows that we have a natural monomorphism

$$[-]^{[2]} \colon \mathrm{Ext}^1(\mathcal{F}, \mathcal{F}) \hookrightarrow \mathrm{Ext}^1(\mathcal{F}^{[2]}, \mathcal{F}^{[2]}),$$

which maps an infinitesimal deformation of $\mathcal{F}$ to its induced deformation of $\mathcal{F}^{[2]}$.

DEFINITION 4.3. We call an infinitesimal deformation of $\mathcal{F}^{[2]}$, the class of which lies in the image of $[-]^{[2]}$ above, a *surface deformation*. Deformations lying in the other summand of equation (4) in Corollary 1.8 are referred to as *additional deformations*.

We conclude:

Proposition 4.4. *We have an embedding of moduli spaces*

$$\mathcal{M}^s(v) \hookrightarrow \mathcal{M}^s(v^{[2]}).$$

The additional deformations are isomorphic to $\mathrm{H}^0(X, \mathcal{F}) \otimes \mathrm{H}^1(X, \mathcal{F})^\vee$.

Corollary 4.5. *Let $\mathcal{F}$ be such that $h^1(X, \mathcal{F}) = 0$. Then we have a local isomorphism of the corresponding moduli spaces.*

Corollary 4.6. *Let $\mathcal{F}$ be such that $\mathcal{M}^s(v)$ is compact and $h^1(X, \mathcal{G}) = 0$ for all $\mathcal{G} \in \mathcal{M}^s(v)$. Then we have an isomorphism of $\mathcal{M}^s(v)$ with a connected component of $\mathcal{M}^s(v^{[2]})$.*

4.2. The additional deformations and singular moduli spaces. In the last section we have seen that the surface deformations of tautological sheaves are unobstructed. This is not true for all deformations. Indeed, in this section we will give an explicit construction of an example of a sheaf $\mathcal{F}$ on an elliptically fibred $K3$ surface such that $\mathcal{F}^{[2]}$ is stable and the corresponding point in the moduli space is singular.

To prove this statement let us recall the most basic properties of the Kuranishi map: The general idea of the deformation theory of a stable sheaf $\mathcal{F}$ is that infinitesimal deformations are parametrised by $\mathrm{Ext}^1(\mathcal{F},\mathcal{F})$ and the obstructions lie in $\mathrm{Ext}^2(\mathcal{F},\mathcal{F})$. This is formalised by the so-called Kuranishi map. More precisely it can be shown that there is a map $\kappa\colon \mathrm{Ext}^1(\mathcal{F}, \mathcal{F}) \to \mathrm{Ext}^2(\mathcal{F}, \mathcal{F})$ such that the completion of the local ring of the point of the moduli space corresponding to $\mathcal{F}$ is isomorphic to the local ring of $\kappa^{-1}(0)$ in 0. In general there is no direct geometric description of the Kuranishi map but it is known that the constant and linear terms of the power series expansion of κ vanish and that its quadratic part is given by $\kappa_2\colon \mathrm{Ext}^1(\mathcal{F}, \mathcal{F}) \to \mathrm{Ext}^2(\mathcal{F}, \mathcal{F})$, $e \mapsto (1/2)(e \circ e)$.

For a $K3$ surface this quadratic term always vanishes since it is exactly the Serre duality pairing which is known to be alternating. But if we consider a tautological sheaf $\mathcal{F}^{[2]}$ the quadratic part of the Kuranishi map may be non-trivial. This would correspond to the existence of a quadratic part in the equation of the tangent cone of the point in the moduli space corresponding to $\mathcal{F}^{[2]}$. Consequently, the tangent cone would be strictly smaller than the tangent space and we would end up with a singularity.

EXAMPLE 4.7. Let X be an elliptically fibred $K3$ surface with fibre class E and section C. Consider the line bundle $\mathcal{G} := \mathcal{O}(kF)$, $k \geqslant 2$. We have $h^0(\mathcal{G}) = k + 1$ and $h^1(\mathcal{G}) = k - 1$. Certainly $\mathcal{G}$ is stable and the moduli space is a reduced point. The rank two tautological sheaf $\mathcal{G}^{[2]}$ is also stable and the tangent space of its moduli space at the point corresponding to $\mathcal{G}^{[2]}$ is isomorphic to $\mathrm{H}^0(X, \mathcal{G}) \otimes \mathrm{H}^1(X, \mathcal{G})^\vee$, which has dimension $k^2 - 1$. The quadratic term of the Kuranishi map vanishes identically but it is not clear if we can deform $\mathcal{G}^{[2]}$ along any of these infinitesimal directions.

EXAMPLE 4.8. We continue with the same elliptic $K3$ as above. From [3] we learn that the linear system of the line bundle $\mathcal{L}$ with first Chern class $C + kE$ has C as a base component for $k \geqslant 2$. We have $h^0(\mathcal{G}) = k + 1$ and $h^1(\mathcal{G}) = 0$. Now let p be a point on the curve C and denote by $\mathcal{I}_p$ the corresponding ideal sheaf. We set $\mathcal{F} := \mathcal{L} \otimes \mathcal{I}_p$. Certainly $\mathcal{F}$ is a torsion-free rank one sheaf with nonvanishing first Chern class. Hence $\mathcal{F}^{[2]}$ is stable by Theorem [19, Theorem 4.2].

Theorem 4.9. *The point in the moduli space corresponding to $\mathcal{F}^{[2]}$ is singular.*

By the above considerations we have to prove the following lemma:

Lemma 4.10. *For the example $\mathcal{F}^{[2]} = (\mathcal{L} \otimes \mathcal{I}_p)^{[2]}$ the quadratic part of the Kuranishi map does not vanish.*

Proof. We have to analyse the Yoneda square

$$\operatorname{Ext}^1(\mathcal{F}^{[2]}, \mathcal{F}^{[2]}) \to \operatorname{Ext}^2(\mathcal{F}^{[2]}, \mathcal{F}^{[2]}),$$
$$x \mapsto x \circ x.$$

Therefore let us use Krug's formula (4) in Corollary 1.8 to write down the extension groups explicitly. Note that $h^2(\mathcal{F}) = 0$.

$$\operatorname{Ext}^1(\mathcal{F}^{[2]}, \mathcal{F}^{[2]}) \cong \operatorname{Ext}^1(\mathcal{F}, \mathcal{F}) \oplus \mathrm{H}^1(\mathcal{F})^\vee \otimes \mathrm{H}^0(\mathcal{F}),$$
$$\operatorname{Ext}^2(\mathcal{F}^{[2]}, \mathcal{F}^{[2]}) \cong \operatorname{Ext}^2(\mathcal{F}, \mathcal{F}) \oplus \mathrm{H}^0(\mathcal{F})^\vee \otimes \mathrm{H}^0(\mathcal{F}) \oplus \mathrm{H}^1(\mathcal{F})^\vee \otimes \mathrm{H}^1(\mathcal{F}).$$

According to this decomposition we can decompose the Yoneda square as well following the detailed formulas in [12, Section 7]:

$$\begin{array}{ccccccccc} \operatorname{Ext}^1(\mathcal{F}, \mathcal{F}) & \oplus & \mathrm{H}^1(\mathcal{F})^\vee \otimes \mathrm{H}^0(\mathcal{F}) & \to & \operatorname{Ext}^2(\mathcal{F}, \mathcal{F}) & \oplus & \mathrm{H}^0(\mathcal{F})^\vee \otimes \mathrm{H}^0(\mathcal{F}) & \oplus & \mathrm{H}^1(\mathcal{F})^\vee \otimes \mathrm{H}^1(\mathcal{F}), \\ e & + & a \otimes b & \mapsto & \underbrace{e \circ e}_{=0} & + & (a \circ e) \otimes b & + & a \otimes (e \circ b). \end{array}$$

Hence we need to show that the map

$$\operatorname{Ext}^1(\mathcal{F}, \mathcal{F}) \times \mathrm{H}^0(\mathcal{F}) \to \mathrm{H}^1(\mathcal{F})$$

is not the zero map. The geometric interpretation of this map is the following: Let $e \in \operatorname{Ext}^1(\mathcal{F}, \mathcal{F})$ be an infinitesimal deformation of $\mathcal{F}$ and $\varphi \in \mathrm{H}^0(\mathcal{F})$ be a global section. Then $\varphi \circ e$ is zero if and only if we can deform the section φ along e.

It is time to return to the geometry of our example. Since p is on the base curve C, we have $\mathrm{H}^0(\mathcal{F}) \cong \mathrm{H}^0(\mathcal{L})$. The deformations of $\mathcal{F}$ are those of $\mathcal{I}_p$, which correspond to deforming the point p in X. Now if we deform p into a direction normal to C, the space of global sections will shrink since the point will fail to be a base point of $\mathcal{L}$ and thus we can find a section $\varphi \in \mathrm{H}^0(\mathcal{F})$ that does not deform with e. □

The Zariski tangent space is $(k+3)$-dimensional and we can explicitly derive the quadratic equation of the tangent cone. It is equivalent to the intersection of a plane (corresponding to the surface deformations) and a hyperplane (the additional deformations and the curve C) in a line (the curve C).

4.3. Deformations of the manifold $X^{[n]}$. A question which has not been touched so far, is the following: The manifold $X^{[n]}$ has an unobstructed deformation theory. Does the tautological sheaf $\mathcal{F}^{[n]}$ deform with $X^{[n]}$?

The technique to answer this question is presented in [10]. We can summarise as follows:

Theorem 4.11 (Huybrechts–Thomas)**.** *Let Y be a projective manifold and $\mathcal{E}$ a sheaf on Y. Let $\kappa \in \mathrm{H}^1(Y, \mathcal{T}_Y) \cong \mathrm{Ext}^1(\Omega_Y, \mathcal{O}_Y)$ be the Kodaira–Spencer class of an infinitesimal deformation of Y and denote by $\mathrm{At}(\mathcal{E}) \in \mathrm{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \Omega_Y)$ the Atiyah class of $\mathcal{E}$. The sheaf $\mathcal{E}$ can be deformed along κ if and only if*

$$0 = \mathrm{ob}(\kappa, \mathcal{E}) := (\kappa \otimes \mathrm{id}_{\mathcal{E}}) \circ \mathrm{At}(\mathcal{E}) \in \mathrm{Ext}^2(\mathcal{E}, \mathcal{E}).$$

For every sheaf $\mathcal{E}$ on Y there are natural trace maps

$$\mathrm{tr}\colon \mathrm{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \Omega_Y) \to \mathrm{H}^1(Y, \Omega_Y)$$

and

$$\mathrm{tr}\colon \mathrm{Ext}^2(\mathcal{E}, \mathcal{E}) \to \mathrm{H}^2(Y, \mathcal{O}_Y),$$

which—up to a sign—commute with the Yoneda product. Furthermore, it is well known that

$$\mathrm{tr}(\mathrm{At}(\mathcal{E})) = c_1(\mathcal{E})$$

and

$$\mathrm{tr}(\mathrm{ob}(\kappa, \mathcal{E})) = \mathrm{ob}(\kappa, \det \mathcal{E}).$$

Applying this theorem to our situation, we get the following picture: The tangent space of the Kuranishi space at the point corresponding to $X^{[n]}$ is isomorphic to

$$\mathrm{H}^1(X^{[n]}, \mathcal{T}_{X^{[n]}}) \cong \mathrm{H}^1(X^{[n]}, \Omega_{X^{[n]}}) \cong \mathrm{H}^1(X, \Omega_X) \oplus \mathbb{C}\delta_n.$$

We write a class in $\mathrm{H}^1(X^{[n]}, \Omega_{X^{[n]}})$ as (κ, a) with $\kappa \in \mathrm{H}^1(X, \Omega_X)$ the class of an infinitesimal deformation of the surface X and $a \in \mathbb{C}$. Unfortunately there is no decomposition of the Atiyah class $\mathrm{At}(\mathcal{F}^{[n]})$ at hand. But we can at least study its trace $\mathrm{tr}(\mathrm{At}(\mathcal{F}^{[n]})) = c_1(\mathcal{F}^{[n]}) = c_1(\mathcal{F})_{X^{[n]}} - r\delta_n$, where we set $r := \mathrm{rk}\,\mathcal{F}$. We have:

$$\mathrm{tr}(\mathrm{ob}((\kappa, a), \mathcal{F}^{[n]})) = \mathrm{ob}(\kappa, \det \mathcal{F}) - ra\delta_n^2 \in \mathrm{H}^2(X^{[n]}, \mathcal{O}_{X^{[n]}}).$$

But we have $\mathrm{ob}(\kappa, \det \mathcal{F}) = \kappa \cdot c_1(\mathcal{F})$ and $\delta_n^2 = 2(1-n)$, where we consider the Beauville–Bogomolov pairing. Thus we see:

- If $\mathcal{F}$ deforms along κ, then surely the tautological sheaf $\mathcal{F}^{[n]}$ deforms along $(\kappa, 0)$.
- If the determinant line bundle $\det \mathcal{F}$ does not deform along κ, then $\mathcal{F}^{[n]}$ does not deform along $(\kappa, 0)$.
- If $\kappa \cdot c_1(\mathcal{F}) \neq 2(1-n)ra$, the tautological sheaf $\mathcal{F}^{[n]}$ does not deform along (κ, a).

Thus there is an interesting hyperplane inside the space of infinitesimal deformations of $X^{[n]}$ consisting of all pairs (κ, a) such that $\kappa \cdot c_1(\mathcal{F}) = 2(1-n)ra$: It is an open question if the tautological sheaf deforms along these directions.

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