Papageorgiou, N.S. and Smyrlis, G.
Osaka J. Math.
53 (2016), 489–514

NONLINEAR ELLIPTIC EQUATIONS WITH SINGULAR REACTION

NIKOLAOS S. PAPAGEORGIOU and GEORGE SMYRLIS

(Received July 24, 2014, revised March 16, 2015)

Abstract

We study a nonlinear elliptic equation with a singular term and a continuous perturbation. We look for positive solutions. We prove three multiplicity theorems producing at least two positive solutions. The first multiplicity theorem concerns equations driven by a nonhomogeneous in general differential operator. Also, two of the theorems have a superlinear perturbation (but without the Ambrosetti–Rabinowitz condition), while the third has a sublinear perturbation. Our approach is variational together with suitable truncation and comparison techniques.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following nonlinear, nonhomogeneous Dirichlet problem with a singular term:

$$(1) \qquad \begin{cases} -\operatorname{div} a(Du(z)) = \beta(z)u(z)^{-\gamma} + f(z, u(z)) & \text{in} \quad \Omega, \\ u|_{\partial\Omega} = 0, \quad u \geq 0, \quad \gamma \in (0, 1). \end{cases}$$

In (1) the map $a\colon \mathbb{R}^N \to \mathbb{R}^N$ involved in the definition of the differential operator is strictly monotone and satisfies certain other regularity conditions. The precise hypotheses on $a(\cdot)$ are gathered in $H(a)$ below. They are general enough to incorporate as special cases important differential operators such as the p-Laplacian ($1 < p < \infty$), the (p, q)-differential operator ($1 < q < p < \infty$, $p \geq 2$) and the generalized p-mean curvature differential operator ($2 \leq p < \infty$). In general the differential operator is not homogeneous (in contrast to the special case of the p-Laplacian). The perturbation $f(z, x)$ is a continuous function on $\Omega \times \mathbb{R}$ which exhibits $(p-1)$-superlinear growth near $+\infty$. However, to express the $(p-1)$-superlinearity of $f(z, \cdot)$, we do not employ the usual in such cases Ambrosetti–Rabinowitz condition (the AR-condition for short). Here instead, we use a more general "superlinearity" condition which incorporates in our framework perturbations with "slower" growth near $+\infty$. We prove three multiplicity theorems producing at least two positive solutions. The second multiplicity result concerns equations driven by the p-Laplace differential operator and a perturbation which is $(p-1)$-sublinear near $+\infty$.

2010 Mathematics Subject Classification. 35J20, 35J25, 35J67.

Equations involving the combined effects of singular and superlinear terms, were studied by Coclite–Palmieri [4], Ghergu–Rădulescu [10], Hirano–Saccon–Shioji [13], Lair–Shaker [16], Sun–Wu–Long [24] (semilinear equations driven by the Laplacian) and by Gasiński–Papageorgiou [9], Giacomoni–Schindler–Takáč [11], Kyritsi–Papageorgiou [14], Perera–Zhang [22] (nonlinear equations driven by the p-Laplacian). All the aforementioned works deal with equations which have a parametric singular term and prove multiplicity of solutions for all small values of the parameter. We stress that in our case the differential operator is nonhomogeneous and this is a source of difficulties in the analysis of problem (1).

2. Mathematical background—hypotheses

In this section we recall some definitions and facts from critical point theory which we will use in the sequel and also we introduce the hypotheses on the data of (1).

Let X be a Banach space and X^* its topological dual. By $\langle\,\cdot\,,\,\cdot\,\rangle$ we denote the duality brackets for the pair (X^*, X). Let $\varphi \in C^1(X)$. We say that $c \in \mathbb{R}$ is a critical value of φ, if there exists $x \in X$ s.t. $\varphi'(x) = 0$ and $\varphi(x) = c$. We say that φ satisfies the "Cerami condition" (the "C-condition" for short), if the following is true:

"Every sequence $\{x_n\}_{n\geq 1} \subset X$ s.t. $\{\varphi(x_n)\}_{n\geq 1} \subset \mathbb{R}$ is bounded and

$$(1 + \|x_n\|)\varphi'(x_n) \to 0 \quad \text{in} \quad X^* \quad \text{as} \quad n \to \infty,$$

admits a strongly convergent subsequence".

This compactness-type condition is in general weaker than the usual Palais–Smale condition. Nevertheless, the C-condition suffices to prove a deformation theorem and from it derive the minimax theory of certain critical values of $\varphi \in C^1(X)$. In particular, we can state the following theorem, known in the literature as the "mountain pass theorem".

Theorem 1. *If $\varphi \in C^1(X)$ satisfies the C-condition, $x_0, x_1 \in X$, $\|x_0 - x_1\| > \rho > 0$,*

$$\max\{\varphi(x_0), \varphi(x_1)\} < \inf[\varphi(x) \colon \|x - x_0\| = \rho] = \eta_\rho$$

and

$$c = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} \varphi(\gamma(t)) \quad \textit{where} \quad \Gamma = \{\gamma \in C([0,1], X) \colon \gamma(0) = x_0,\ \gamma(1) = x_1\},$$

then $c \geq \eta_\rho$ and c is a critical value of φ.

In this work, in addition to the Sobolev space $W_0^{1,p}(\Omega)$, we will also use the Banach space $C_0^1(\bar{\Omega}) = \{u \in C^1(\bar{\Omega}) \colon u|_{\partial\Omega} = 0\}$. This is an ordered Banach space with positive cone

$$C_+ = \{u \in C_0^1(\bar{\Omega}) \colon u(z) \geq 0, \text{ for all } z \in \bar{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{ u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega,\ \frac{\partial u}{\partial n}(z) < 0 \text{ for all } z \in \partial\Omega \right\},$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$.

By $\|\cdot\|$ we denote the norm of the Sobolev space $W_0^{1,p}(\Omega)$. By virtue of Poincare inequality, we have

$$\|u\| = \|Du\|_p \quad \text{for all} \quad u \in W_0^{1,p}(\Omega).$$

By $\|\cdot\|$ we will also denote the $\mathbb{R}^N$-norm. However, no confusion is possible since it will always be clear from the context which norm we use.

For $x \in \mathbb{R}$, we set $x^\pm = \max\{\pm x, 0\}$ and then for $u \in W_0^{1,p}(\Omega)$ we define $u^\pm(\cdot) = u(\cdot)^\pm$. We know that

$$u^\pm \in W_0^{1,p}(\Omega), \quad |u| = u^+ + u^- \quad \text{and} \quad u = u^+ - u^-.$$

By $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$.

For $h\colon \Omega \times \mathbb{R} \to \mathbb{R}$ a measurable function (for example a Carathéodory function), we define

$$N_h(u)(\cdot) = h(\cdot, u(\cdot)) \quad \text{for all} \quad u \in W_0^{1,p}(\Omega)$$

(the Nemytskii or superposition operator corresponding to h).

Now, let $\vartheta \in C^1(0,\infty)$ be such that

$$\begin{aligned} &0 < \frac{t\vartheta'(t)}{\vartheta(t)} \le c_0 \quad \text{for all} \quad t > 0 \quad \text{and some} \quad c_0 > 0 \\ &\text{and} \quad c_1 t^{p-1} \le \vartheta(t) \le c_2(1 + t^{p-1}) \quad \text{for all} \quad t > 0 \quad \text{and some} \quad c_1, c_2 > 0. \end{aligned} \tag{2}$$

Below we have gathered the hypotheses on the data $a(y)$, $\beta(z)$, $f(z,x)$ of problem (1) which will be used in this work.

The hypotheses on the map $y \to a(y)$ involved in the differential operator are the following:

H(a). $a(y) = a_0(\|y\|)y$ for all $y \in \mathbb{R}^N$ with $a_0(t) > 0$ for all $t > 0$ and
(i) $a_0 \in C^1(0,\infty)$, $t \to a_0(t)t$ is strictly increasing, $a_0(t) \to 0$ as $t \to 0^+$ and $\lim_{t\to 0^+} t a_0'(t)/a_0(t) = c > -1$;
(ii) $\|\nabla a(y)\| \le c_3 \vartheta(\|y\|)/\|y\|$ for all $y \in \mathbb{R}^N \setminus \{0\}$ and some $c_3 > 0$;
(iii) $(\nabla a(y)\xi, \xi)_{\mathbb{R}^N} \ge (\vartheta(\|y\|)/\|y\|)\|\xi\|^2$ for all $y \in \mathbb{R}^N \setminus \{0\}$, all $\xi \in \mathbb{R}^N$.
(iv) if $G_0(t) = \int_0^t a_0(s)s\,ds$ for all $t \ge 0$, then

$$pG_0(t) \ge a_0(t)t^2 - \hat{c} \quad \text{for all} \quad t \ge 0 \quad \text{and some} \quad \hat{c} > 0.$$

The hypotheses on the weight function $\beta(\cdot)$ are the following:

H(β). $\beta \in C(\Omega) \cap L^\infty(\Omega)$, $\beta(z) \geq 0$ for all $z \in \Omega$, $\beta \neq 0$.

The hypotheses on the perturbation $f(z, x)$ are the following:

H(f). $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function s.t. for all $z \in \Omega$, $f(z, 0) = 0$, $f(z, x) \geq 0$ for all $x \geq 0$ and

(i) $f(z, x) \leq \alpha(z)(1 + x^{r-1})$ for all $z \in \Omega$, all $x \geq 0$ and with $\alpha \in L^\infty(\Omega)_+$,

$$p < r < p^* = \begin{cases} \dfrac{Np}{N-p} & \text{if} \quad p < N, \\ +\infty & \text{if} \quad p \geq N; \end{cases}$$

(ii) if $F(z, x) = \int_0^x f(z, s)\, ds$, then

$$\lim_{x \to +\infty} \frac{F(z, x)}{x^p} = +\infty \quad \text{uniformly for all} \quad z \in \Omega;$$

(iii) there exist $\tau \in ((r - p)\max\{1, N/p\}, p^*)$ and $\beta_0 > 0$ s.t.

$$\beta_0 \leq \liminf_{x \to +\infty} \frac{f(z, x)x - pF(z, x)}{x^\tau} \quad \text{uniformly for all} \quad z \in \Omega;$$

(iv) there exists $\eta \in C(\Omega)$, $\eta(z) \geq 0$ for all $z \in \Omega$, with $\eta(z) \leq (c_1/(p-1))\hat{\lambda}_1(p)$ for all $z \in \Omega$, $\eta \neq (c_1/(p-1))\hat{\lambda}_1(p)$ and

$$\limsup_{x \to 0^+} \frac{f(z, x)}{x^{p-1}} \leq \eta(z) \quad \text{uniformly for all} \quad z \in \Omega.$$

In Section 4, we consider equations driven by the p-Laplacian with a $(p-1)$-sublinear perturbation $f(z,x)$. In that case, our conditions on $f(z,x)$ are the following:

H(f)′. $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function s.t. $f(z, 0) = 0$ for all $z \in \Omega$ and

(i) for every $\rho > 0$, there exists $\alpha_\rho \in L^\infty_+(\Omega)$ s.t.

$$|f(z, x)| \leq \alpha_\rho(z) \quad \text{for a.a.} \quad z \in \Omega, \quad \text{all} \quad 0 \leq x \leq \rho;$$

(ii) there exist $\eta \in L^\infty_+(\Omega)$ and $\hat{\eta} > 0$ s.t.

$$\eta(z) \geq \hat{\lambda}_1(p) \quad \text{a.e. in} \quad \Omega, \quad \eta \neq \hat{\lambda}_1(p),$$
$$\eta(z) \leq \liminf_{x \to +\infty} \frac{f(z, x)}{x^{p-1}} \leq \limsup_{x \to +\infty} \frac{f(z, x)}{x^{p-1}} \leq \hat{\eta} \quad \text{uniformly for a.a.} \quad z \in \Omega.$$

(iii) there exist $0 < \delta_0 < \xi_0$ s.t.

$$0 \leq f(z, x) \quad \text{for all} \quad z \in \Omega, \quad \text{all} \quad x \in [0, \delta_0],$$
$$\beta(z)\xi_0^{-\gamma} + f(z, \xi_0) < 0 \quad \text{for all} \quad z \in \Omega;$$

(iv) for every $\rho > 0$, there exists $\hat{\xi}_\rho > 0$ s.t. for a.a. $z \in \Omega$, $x \to f(z, x) + \hat{\xi}_\rho x^{p-1}$ is nondecreasing on $[0, \rho]$.

REMARK. Evidently $G_0(\,\cdot\,)$ is strictly convex and strictly increasing. We set $G(y) = G_0(\|y\|)$ for all $y \in \mathbb{R}^N$. We have

$$\nabla G(y) = G_0'(\|y\|)\frac{y}{\|y\|} = a_0(\|y\|)y = a(y) \quad \text{for all} \quad y \in \mathbb{R}^N \setminus \{0\}.$$

Therefore, $G(\,\cdot\,)$ is the primitive of $a(\,\cdot\,)$, it is convex and $G(0) = 0$. Hence

$$G(y) \le (a(y), y)_{\mathbb{R}^N} \quad \text{for all} \quad y \in \mathbb{R}^N. \tag{3}$$

From hypotheses $H(a)$ and (2), (3), we obtain easily the following lemma which summarizes the main properties of the map $a(\,\cdot\,)$.

Lemma 2. *If hypotheses $H(a)$ hold, then*
(a) $y \to a(y)$ *is maximal monotone and strictly monotone*;
(b) $\|a(y)\| \le c_4(1 + \|y\|^{p-1})$ *for all* $y \in \mathbb{R}^N$ *and some* $c_4 > 0$;
(c) $(a(y), y)_{\mathbb{R}^N} \ge (c_1/(p-1))\|y\|^p$ *for all* $y \in \mathbb{R}^N$.

From this lemma and the integral form of the mean value theorem, we deduce the following growth properties of the primitive $G(\,\cdot\,)$.

Corollary 3. *If hypotheses $H(a)$ hold, then*

$$\frac{c_1}{p(p-1)}\|y\|^p \le G(y) \le c_5(1 + \|y\|^p) \quad \textit{for all} \quad y \in \mathbb{R}^N \quad \textit{and some} \quad c_5 > 0.$$

EXAMPLE. The following maps satisfy hypotheses $H(a)$:
(a) $a(y) = \|y\|^{p-2}y$ with $1 < p < \infty$.
Then the corresponding differential operator is the p-Laplacian

$$\Delta_p u = \operatorname{div}(\|Du\|^{p-2}Du) \quad \text{for all} \quad u \in W_0^{1,p}(\Omega).$$

(b) $a(y) = \|y\|^{p-2}y + \|y\|^{q-2}y$ with $1 < q < p$, $p \ge 2$.
Then the corresponding differential operator is the sum of a p-Laplacian and a q-Laplacian (a (p, q)-differential operator)

$$\Delta_p u + \Delta_q u \quad \text{for all} \quad u \in W_0^{1,p}(\Omega).$$

This operator arises in quantum physics (see Benci–D'Avenia–Fortunato–Pisani [1]) and in plasma physics (see Cherfils–Il'yasov [3]).

(c) $a(y) = (1 + \|y\|^2)^{(p-2)/2} y$ with $2 \le p < \infty$.

Then the corresponding differential operator is the generalized p-mean curvature operator defined by

$$\operatorname{div}[(1 + \|Du\|^2)^{(p-2)/2} Du] \quad \text{for all} \quad u \in W_0^{1,p}(\Omega).$$

(d) $a(y) = \|y\|^{p-2} y + \ln(1 + \|y\|^{p-2}) y$ with $1 < p < \infty$.

Let $A\colon W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*$ $(1/p + 1/p' = 1)$ be the nonlinear map defined by

$$\langle A(u), y \rangle = \int_\Omega (a(Du), Dy)_{\mathbb{R}^N}\, dz \quad \text{for all} \quad u, y \in W_0^{1,p}(\Omega). \tag{4}$$

From Papageorgiou–Rocha–Staicu [21], we have:

Proposition 4. *The nonlinear map* $A\colon W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ *defined by* (4) *is bounded* (*maps bounded sets to bounded sets*), *continuous and strictly monotone* (*hence maximal monotone too*) *and of type* $(S)_+$, *i.e., if* $u_n \xrightarrow{w} u$ *in* $W_0^{1,p}(\Omega)$ *and*

$$\limsup_{n\to+\infty} \langle A(u_n), u_n - u \rangle \le 0,$$

then $u_n \to u$ *in* $W_0^{1,p}(\Omega)$.

REMARK. Since our aim is to produce positive solutions and the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0, +\infty)$, by truncating $f(x, \cdot)$ if necessary, we may and will assume that $f(z, x) = 0$ for all $z \in \Omega$ and all $x \le 0$. From hypotheses $H(f)$ (ii), (iii) it follows that

$$\limsup_{x\to+\infty} \frac{f(z,x)}{x^{p-1}} = +\infty \quad \text{uniformly for all} \quad z \in \Omega.$$

This means that the perturbation $f(z, \cdot)$ is $(p-1)$-superlinear near $+\infty$. However, note that we do not employ the usual in such cases Ambrosetti–Rabinowitz condition (the AR-condition for short). We recall that the AR-condition (unilateral version) says that there exist $\mu > p$ and $M > 0$ s.t.

$$0 < \mu F(z,x) \le f(z,x)x \quad \text{for all} \quad z \in \Omega, \quad \text{all} \quad x \ge M \tag{5}$$

and $\inf_\Omega F(\cdot, M) > 0$.

A direct integration of (5), leads to the following growth estimate

$$c_6 x^\mu \le F(z,x) \quad \text{for all} \quad z \in \Omega, \quad \text{all} \quad x \ge M, \quad \text{and some} \quad c_6 > 0. \tag{6}$$

Evidently (6) implies that hypothesis $H(f)$ (ii) holds. Also, if the AR-condition holds, we may assume that $\mu > (r-p)\max\{1, N/p\}$. Then we have

$$\begin{aligned}\frac{f(z,x)x - pF(z,x)}{x^{\mu}} &= \frac{f(z,x)x - \mu F(z,x)}{x^{\mu}} + (\mu - p)\frac{F(z,x)}{x^{\mu}}\\ &\geq (\mu - p)c_6 \quad \text{(see (5), (6))}\\ \Rightarrow \liminf_{x\to+\infty} \frac{f(z,x)x - pF(z,x)}{x^{\mu}} &\geq (\mu - p)c_6 \quad \text{uniformly for all} \quad z \in \Omega.\end{aligned}$$

So, hypothesis $H(f)$ (ii) holds. Hence our "superlinearity" condition is more general than the AR-condition and permits the use of superlinear perturbations with "slower" growth near $+\infty$. We mention, that similar conditions were also employed by Costa–Magalhães [5], Fei [6] and Li–Wu–Zhou [19].

EXAMPLE. The following functions satisfy hypotheses $H(f)$ (for the sake of simplicity we drop the z-dependence):

$$\begin{aligned}&f_1(x) = \vartheta x^{p-1} + x^{\tau-1} \quad \text{for all} \quad x \geq 0 \quad \text{with} \quad \vartheta \in (0, \hat{\lambda}_1(p)) \quad \text{and} \quad \tau \in (p, p^*),\\ &f_2(x) = x^{p-1}\ln(1+x) \quad \text{for all} \quad x \geq 0.\end{aligned}$$

Note that f_2 does not satisfy the AR-condition.

REMARK. In the case of hypotheses $H(f)'$, again without any loss of generality, we assume that $f(z,x) = 0$ for all $z \in \Omega$, all $x \leq 0$. Hypothesis $H(f)'$ (ii) classifies the perturbation as $(p-1)$-sublinear. Hypothesis $H(f)'$ (iii) expresses the oscillatory behavior near zero.

EXAMPLE. The following function satisfies hypotheses $H(f)'$. As before, for the sake of simplicity, we drop the z-dependence

$$f(x) = \begin{cases} 0 & \text{if} \quad x < 0,\\ x^{p-1} - cx^{\vartheta-1} & \text{if} \quad 0 \leq x \leq 1,\\ \eta x^{p-1} - \hat{c}x^{q-1} & \text{if} \quad 1 < x \end{cases}$$

with $1 < q < p < \vartheta$, $c \geq \|\beta\|_{\infty} + 1$, $\eta > \hat{\lambda}_1(p)$ and $\hat{c} = \eta + c - 1 > 0$.

Next, let us recall some facts about the spectrum of $(-\Delta_p, W_0^{1,p}(\Omega))$. So, let $m \in L^{\infty}(\Omega)_+$, $m \neq 0$ and consider the following weighted nonlinear eigenvalue problem

$$\begin{cases} -\Delta_p u(z) = \lambda m(z)|u(z)|^{p-2}u(z) \quad \text{a.e. in} \quad \Omega,\\ u|_{\partial\Omega} = 0. \end{cases}$$

There is a smallest eigenvalue $\hat{\lambda}_1(p, m)$ which is positive, isolated (i.e., there exists $\varepsilon > 0$ s.t. $(\hat{\lambda}_1(p, m), \hat{\lambda}_1(p, m) + \varepsilon)$ contains no eigenvalue) and simple (i.e., if u, v are eigenfunctions corresponding to $\hat{\lambda}_1(p,m) > 0$, then $u = \xi v$, $\xi \neq 0$). Also, $\hat{\lambda}_1(p,m) > 0$ admits the following variational characterization

$$\hat{\lambda}_1(p, m) = \inf\left[\frac{\|Du\|_p^p}{\int_\Omega m|u|^p\,dz} : u \in W_0^{1,p}(\Omega),\ u \neq 0\right]. \tag{7}$$

The infimum in (7) is realized on the corresponding one-dimensional eigenspace. From (7) it is clear that the eigenfunctions corresponding to $\hat{\lambda}_1(p,m)$ do not change sign. By $\hat{u}_1(p,m)$ we denote the positive L^p-normalized eigenfunction (i.e., $\|\hat{u}_1(p,m)\|_p = 1$). From the nonlinear regularity theory and the nonlinear maximum principle (see Gasiński–Papageorgiou [8] (pp. 737–738)), we have $\hat{u}_1(p, m) \in \operatorname{int} C_+$. Note that $\lambda_1(p, m)$ is the only eigenvalue, with eigenfunctions of constant sign. If $m(z) \leq m'(z)$ a.e. in Ω, $m \neq m'$, then $\hat{\lambda}_1(p, m') < \hat{\lambda}_1(p, m)$. Finally, if $m \equiv 1$, then we write $\hat{\lambda}_1(p, 1) = \hat{\lambda}_1(p)$ and $\hat{u}_1(p, 1) = \hat{u}_1(p)$.

3. The nonhomogeneous problem

We consider the following auxiliary Dirichlet problem:

$$-\operatorname{div} a(Du(z)) = \beta(z)u(z)^{-\gamma} \quad \text{in} \quad \Omega, \quad u|_{\partial\Omega} = 0, \quad u \geq 0, \quad \gamma \in (0, 1). \tag{8}$$

Proposition 5. *If hypotheses $H(a)$, $H(\beta)$ hold, then problem* (8) *has a solution* $\underline{u} \in \operatorname{int} C_+$.

Proof. For every $n \geq 1$, we consider the following perturbed version of problem (8)

$$-\operatorname{div} a(Du_n(z)) = \beta(z)\left(u_n(z) + \frac{1}{n}\right)^{-\gamma} \quad \text{in} \quad \Omega, \quad u_n|_{\partial\Omega} = 0, \quad u_n \geq 0, \tag{9}$$

$n \geq 1$, $\gamma \in (0, 1)$.

First we solve problem (9). To this end, let $w \in L^p(\Omega)$ and let $y = E(w)$ be the unique solution of the following Dirichlet problem

$$-\operatorname{div} a(Dy(z)) = \beta(z)\left(|w(z)| + \frac{1}{n}\right)^{-\gamma} \quad \text{in} \quad \Omega, \quad y|_{\partial\Omega} = 0, \quad y \geq 0, \tag{10}$$

$\gamma \in (0, 1)$.

From the nonlinear regularity theory (see Ladyzhenskaya–Ural'tseva [15] (p. 286)) and Lieberman [18] (p. 320)), we have that $y \in C_+ \setminus \{0\}$. In fact the nonlinear strong maximum principle of Pucci–Serrin [23] (p. 111), implies that $y(z) > 0$ for all $z \in \Omega$.

Therefore, we can apply the nonlinear boundary point theorem of Pucci–Serrin [23] (p. 120) and we infer that $y \in \operatorname{int} C_+$. On (10) we act with y and using Lemma 2 (c), we obtain

$$\frac{c_1}{p-1}\|Dy\|_p^p \le \int_\Omega \frac{\beta y}{(|w|+1/n)^\gamma}\,dz \le n^\gamma\|\beta\|_\infty \int_\Omega y\,dz \quad (\text{see } H(\beta))$$
$$\Rightarrow \frac{c_1}{p-1}\hat{\lambda}_1(p)\|y\|_p^p \le c_7 n^\gamma \|y\|_p \quad \text{for some} \quad c_7 > 0 \quad (\text{see (7)})$$
$$\Rightarrow \|y\|_p \le \hat{\rho}_n \quad \text{for some} \quad \hat{\rho}_n > 0, \quad n \ge 1. \tag{11}$$

Let $\bar{B}^{L^p}_{\hat{\rho}_n} = \{u \in L^p(\Omega) : \|u\|_p \le \hat{\rho}_n\}$ and consider the map $E : \bar{B}^{L^p}_{\hat{\rho}_n} \to \bar{B}^{L^p}_{\hat{\rho}_n}$ (see (11)). Using the Sobolev embedding theorem and the previous calculations, we see that E is compact. Then the Schauder fixed point theorem implies that for every $n \ge 1$, we can find $u_n \in \bar{B}^{L^p}_{\hat{\rho}_n}$ s.t. $u_n = E(u_n)$ for all $n \ge 1$. We have

$$-\operatorname{div} a(Du_n(z)) = \beta(z)\left(u_n(z) + \frac{1}{n}\right)^{-\gamma} \quad \text{in} \quad \Omega, \quad u_n|_{\partial\Omega} = 0, \quad u_n \ge 0,$$
$$\gamma \in (0,1)$$
$$\Rightarrow u_n \in \operatorname{int} C_+ \quad (\text{as above}).$$

Claim. $\{u_n\}_{n\ge1} \subset \operatorname{int} C_+$ *is an increasing sequence.*

For every $n \ge 1$, we have

$$A(u_n) = \beta\left(u_n + \frac{1}{n}\right)^{-\gamma} \le \beta\left(u_n + \frac{1}{n+1}\right)^{-\gamma} \quad \text{in} \quad W^{-1,p'}(\Omega). \tag{12}$$

So, for every $n \ge 1$, we have

$$\begin{aligned} &A(u_n) - A(u_{n+1}) \\ &\le \beta\left[\frac{1}{(u_n + 1/(n+1))^\gamma} - \frac{1}{(u_{n+1} + 1/(n+1))^\gamma}\right] \quad (\text{see (12)}) \\ &= \beta\left[\frac{(u_{n+1} + 1/(n+1))^\gamma - (u_n + 1/(n+1))^\gamma}{(u_n + 1/(n+1))^\gamma (u_{n+1} + 1/(n+1))^\gamma}\right] \quad \text{in} \quad W^{-1,p'}(\Omega). \end{aligned} \tag{13}$$

On (13) we act with $(u_n - u_{n+1})^+ \in W_0^{1,p}(\Omega)$ and obtain

$$0 \le \langle A(u_n) - A(u_{n+1}), (u_n - u_{n+1})^+\rangle \quad (\text{see Lemma 2 (a)})$$
$$= \int_\Omega \beta\left[\frac{(u_{n+1} + 1/(n+1))^\gamma - (u_n + 1/(n+1))^\gamma}{(u_n + 1/(n+1))^\gamma (u_{n+1} + 1/(n+1))^\gamma}\right](u_n - u_{n+1})^+\,dz$$
$$\Rightarrow |\{u_n > u_{n+1}\}|_N = 0$$
$$\Rightarrow u_n \le u_{n+1} \quad \text{for all} \quad n \ge 1.$$

This proves the claim.

By virtue of the claim, we have

$$A(u_n) = \beta\left(u_n + \frac{1}{n}\right)^{-\gamma} \leq \beta u_1^{-\gamma} \quad \text{in} \quad W^{-1,p'}(\Omega) \quad \text{for all} \quad n \geq 1. \tag{14}$$

Since $u_1 \in \operatorname{int} C_+$, we can find $t \in (0,1)$ small s.t. $(t\hat{u}_1(p)^{1/q}) \leq u_1$ (see Filippakis–Kristaly–Papageorgiou [7], Lemma 3.3). Then for $q > \max\{N/p, 1\}$, we have

$$\beta u_1^{-\gamma} \leq t^{-\gamma}\beta(\hat{u}_1(p)^{1/q})^{-\gamma} \leq t^{-\gamma}\|\beta\|_\infty(\hat{u}_1(p)^{1/q})^{-\gamma} \in L^q(\Omega) \tag{15}$$

(see Lazer–McKenna [17]).

From (14), (15) and Ladyzhenskaya–Ural'tseva [15] (p. 286), we know that we can find $M_1 > 0$ s.t. $\|u_n\|_\infty \leq M_1$ for all $n \geq 1$. Then from Lieberman [18], we can find $M_2 > 0$ and $\eta \in (0, 1)$ s.t.

$$u_n \in C_0^{1,\eta}(\bar{\Omega}) \quad \text{and} \quad \|u_n\|_{C_0^{1,\eta}(\bar{\Omega})} \leq M_2 \quad \text{for all} \quad n \geq 1. \tag{16}$$

Exploiting the compact embedding of $C_0^{1,\eta}(\bar{\Omega})$ into $C_0^1(\bar{\Omega})$, from (16) and the claim we have

$$u_n \to \underline{u} \quad \text{in} \quad C_0^1(\bar{\Omega}) \quad \text{and} \quad \underline{u} \in \operatorname{int} C_+. \tag{17}$$

Recall that

$$A(u_n) = \beta\left(u_n + \frac{1}{n}\right)^{-\gamma} \quad \text{for all} \quad n \geq 1.$$

So, passing to the limit as $n \to \infty$ and using (17), we obtain

$$\begin{aligned} &A(\underline{u}) = \beta\underline{u}^{-\gamma} \\ &\Rightarrow \underline{u} \in \operatorname{int} C_+ \quad \text{is a solution of (8).} \end{aligned}$$

□

Since $f \geq 0$ (see $H(f)$), we have

$$A(\underline{u}) \leq \beta\underline{u}^{-\gamma} + N_f(\underline{u}) \quad \text{in} \quad W^{-1,p'}(\Omega). \tag{18}$$

Next note that by virtue of hypotheses $H(f)$ (i), (iv), given $\varepsilon > 0$, we can find $\xi_\varepsilon > 0$ s.t.

$$f(z, x) < (\eta(z) + \varepsilon)x^{p-1} + \xi_\varepsilon x^{r-1} \quad \text{for all} \quad z \in \Omega, \quad \text{all} \quad x > 0. \tag{19}$$

From (15) and (17), we have that $\beta \underline{u}^{-\gamma} \in L^q(\Omega)$. Therefore, we consider the following auxiliary Dirichlet problem

$$
\left\{\begin{array}{l} - \operatorname{div} a(Du(z)) = \beta(z)\underline{u}(z)^{-\gamma} + (\eta(z)+\varepsilon)|u(z)|^{p-2}u(z) \\ + \xi_\varepsilon |u(z)|^{r-2}u(z) \quad \text{in} \quad \Omega, \\ u|_{\partial\Omega} = 0, \quad u \geq 0, \quad \gamma \in (0,1). \end{array}\right\} \tag{20}
$$

Proposition 6. *If hypotheses* $H(a)$, $H(\beta)$ *hold, then for* $\varepsilon > 0$ *and* $\|\beta\|_\infty$ *small, problem* (20) *has a solution* $\bar{u} \in \operatorname{int} C_+$.

Proof. Let $\psi\colon W_0^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$
\begin{aligned}
\psi(u) = \int_\Omega G(Du(z))\,dz - \frac{1}{p}\int_\Omega (\eta(z)+\varepsilon)u^+(z)^p\,dz - \frac{\xi_\varepsilon}{r}\|u^+\|_r^r \\
- \int_\Omega \frac{\beta(z)}{\underline{u}(z)^\gamma} u^+(z)\,dz, \quad \text{for all} \quad u \in W_0^{1,p}(\Omega).
\end{aligned}
$$

In problem (20) the reaction $\sigma(z,x)$ is the continuous on $\Omega\times\mathbb{R}$ function defined by

$$
\sigma(z,x) = (\eta(z)+\varepsilon)(x^+)^{p-1} + \xi_\varepsilon (x^+)^{r-1} + \beta(z)\underline{u}(z)^{-\gamma}.
$$

Clearly this function satisfies the unilateral AR-condition (see (5)) and so it follows easily that

$$
\psi \text{ satisfies the C-condition.} \tag{21}
$$

By virtue of Corollary 3, we have

$$
\begin{aligned}
\psi(u) &\geq \frac{c_1}{p(p-1)}\|Du\|_p^p - \frac{1}{p}\int_\Omega (\eta(z)+\varepsilon)|u|^p\,dz - \frac{\xi_\varepsilon}{r}\|u\|_r^r - \int_\Omega \frac{\beta(z)}{\underline{u}(z)^\gamma}|u|\,dz \\
&\geq \frac{1}{p}\left(\xi^* - \frac{\varepsilon}{\hat{\lambda}_1(p)}\right)\|u\|^p - c_8(\|u\|^r + \|\beta\|_\infty \|u\|)
\end{aligned}
$$

for some ξ^*, $c_8 > 0$.

Here we have used Lemma 5.1.3 (p. 356) of Papageorgiou–Kyritsi-Yiallourou [20], the fact that $\underline{u}^{-\gamma} \in L^q(\Omega)$ (see (15), (17)) and the claim in the proof of Proposition 5). Choosing $\varepsilon \in (0, \xi^*\hat{\lambda}_1(p))$, we have

$$
\begin{aligned}
\psi(u) &\geq c_9\|u\|^p - c_8(\|u\|^r + \|\beta\|_\infty\|u\|) \quad \text{for some} \quad c_9 > 0 \\
&= [c_9 - c_8(\|u\|^{r-p} + \|\beta\|_\infty \|u\|^{1-p})]\|u\|^p.
\end{aligned} \tag{22}
$$

Let $\mu(t) = t^{r-p} + \|\beta\|_\infty t^{1-p}$, $t > 0$. Evidently, $\mu \in C^1(0,\infty)$ and since $1 < p < r$, we see that

$$
\mu(t) \to +\infty \quad \text{as} \quad t \to 0^+ \quad \text{and} \quad \mu(t) \to +\infty \quad \text{as} \quad t \to +\infty.
$$

Therefore we can find $t_0 \in (0, +\infty)$ s.t.

$$\begin{aligned} &\mu(t_0) = \inf[\mu(t)\colon t \geq 0] \\ &\Rightarrow \mu'(t_0) = (r-p)t_0^{r-p-1} + (1-p)\|\beta\|_\infty t_0^{-p} = 0 \\ &\Rightarrow t_0 = \left[\frac{(p-1)\|\beta\|_\infty}{r-p}\right]^{1/(r-1)}. \end{aligned}$$

Then $\mu(t_0) \to 0$ as $\|\beta\|_\infty \to 0^+$. So, for $\|\beta\|_\infty$ small we have

$$\mu(t_0) < \frac{c_9}{c_8}$$

and this by virtue of (22) implies that

$$\psi(u) \geq \xi_0 > 0 \quad \text{for all} \quad \|u\| = t_0. \tag{23}$$

Finally hypothesis $H(f)$ (iii) implies that

$$\psi(t\hat{u}_1(p)) \to -\infty \quad \text{as} \quad t \to +\infty. \tag{24}$$

From (21), (23) and (24) we see that we can apply Theorem 1 (the mountain pass theorem) and find $\bar{u} \in W_0^{1,p}(\Omega)$ s.t. $\xi_0 \leq \psi(\bar{u})$ and

$$\begin{aligned} &\psi'(\bar{u}) = 0 \\ &\Rightarrow A(\bar{u}) = \beta\underline{u}^{-\gamma} + (\vartheta + \varepsilon)(\bar{u}^+)^{p-1} + \xi_\varepsilon(\bar{u}^+)^{r-1} \\ &\Rightarrow \bar{u} \neq 0. \end{aligned} \tag{25}$$

On (25) we act with $\bar{u} \in W_0^{1,p}(\Omega)$ and obtain $\bar{u} \geq 0$, $\bar{u} \neq 0$.
Note that

$$\begin{aligned} A(\underline{u}) &= \beta\underline{u}^{-\gamma} \\ &\leq \beta\underline{u}^{-\gamma} + (\vartheta + \varepsilon)\bar{u}^{p-1} + \xi_\varepsilon\bar{u}^{r-1} = A(\bar{u}) \quad \text{in} \quad W^{-1,p}(\Omega), \\ \Rightarrow &\langle A(\underline{u}) - A(\bar{u}), (\underline{u} - \bar{u})^+\rangle \leq 0 \\ \Rightarrow &\underline{u} \leq \bar{u} \quad \text{(see Lemma 2 (a)).} \end{aligned}$$

From Ladyzhenskaya–Ural'tseva [15] (p.286), we have $\bar{u} \in L^\infty(\Omega)$. Let $d(z) = d(z, \partial\Omega)$. We can find $c_{10} > 0$ s.t.

$$0 \leq \bar{u}(z) \leq c_{10}d(z) \quad \text{for all} \quad z \in \bar{\Omega} \quad \text{(see Guo [12]).}$$

Recall that $\underline{u} \in \operatorname{int} C_+$. So, we can find $c_{11} > 0$ s.t.

$$c_{11}d(z) \leq \underline{u}(z) \quad \text{for all} \quad z \in \bar{\Omega}.$$

We have

$$\underline{u}(z)^{-\gamma} \leq \frac{1}{c_{11}^{\gamma}} d(z)^{-\gamma} \quad \text{for all} \quad z \in \bar{\Omega}.$$

Then from Giacomoni–Schindler–Takáč [11] we infer that

$$\bar{u} \in C_+ \setminus \{0\}.$$

Finally, invoking the boundary point theorem of Pucci–Serrin [23] (p. 120), we conclude that $\bar{u} \in \operatorname{int} C_+$. □

By virtue of (19) and since $\beta\bar{u}^{-\gamma} \leq \beta\underline{u}^{-\gamma}$, we have

$$A(\bar{u}) \geq \beta\bar{u}^{-\gamma} + N_f(\bar{u}) \quad \text{in} \quad W^{-1,p'}(\Omega). \tag{26}$$

Now we are ready for the first multiplicity theorem.

Theorem 7. *If hypotheses $H(a)$, $H(\beta)$ and $H(f)$ hold, then for $\|\beta\|_\infty$ small, problem* (1) *has at least two nontrivial solutions*

$$u_0, u_1 \in \operatorname{int} C_+, \quad u_0 \leq u_1, \quad u_0 \neq u_1.$$

Proof. We consider the following truncation of the reaction in problem (1):

$$k(z,x) = \begin{cases} \beta(z)\underline{u}(z)^{-\gamma} + f(z,\underline{u}(z)) & \text{if} \quad x < \underline{u}(z), \\ \beta(z)x^{-\gamma} + f(z,x) & \text{if} \quad \underline{u}(z) \leq x \leq \bar{u}(z), \\ \beta(z)\bar{u}(z)^{-\gamma} + f(z,\bar{u}(z)) & \text{if} \quad \bar{u}(z) < x. \end{cases} \tag{27}$$

Evidently $k(z,x)$ is continuous on $\Omega \times \mathbb{R}$. Set $K(z,x) = \int_0^x k(z,s)\,ds$ and consider the functional $\sigma : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\sigma(u) = \int_\Omega G(Du(z))\,dz - \int_\Omega K(z,u(z))\,dz \quad \text{for all} \quad u \in W_0^{1,p}(\Omega).$$

Claim 1. $\sigma \in C^1(W_0^{1,p}(\Omega))$ *and* $\sigma'(u) = A(u) - N_k(u)$ *for all* $u \in W_0^{1,p}(\Omega)$.

To establish Claim 1, it suffices to show that $\sigma_0 \in C^1(W_0^{1,p}(\Omega))$, where

$$\sigma_0(u) = \int_\Omega K_0(z,u(z))\,dz \quad \text{for all} \quad u \in W_0^{1,p}(\Omega)$$

where $K_0(z,x) = \int_0^x k_0(z,s)\,ds$ with

$$k_0(z,x) = \begin{cases} \beta(z)\underline{u}(z)^{-\gamma} & \text{if} \quad x < \underline{u}(z), \\ \beta(z)x^{-\gamma} & \text{if} \quad \underline{u}(z) \le x \le \bar{u}(z), \\ \beta(z)\bar{u}(z)^{-\gamma} & \text{if} \quad \bar{u}(z) < x \end{cases}$$

and that $\sigma_0'(u) = N_{k_0}(u)$ for all $u \in W_0^{1,p}(\Omega)$.

To this end, let $u, y \in W_0^{1,p}(\Omega)$ and $\lambda \neq 0$. From the integral form of the mean value theorem, we have

$$\frac{1}{\lambda}[\sigma_0(u+\lambda y) - \sigma_0(u)] = \int_\Omega \int_0^1 k_0(z, u+s\lambda y)\,ds\, y\, dz. \tag{28}$$

We know that

$$\int_0^1 k_0(z, u+s\lambda y)\,ds \to k_0(z,u) \quad \text{for a.a.} \quad z \in \Omega, \quad \text{as} \quad \lambda \to 0^+. \tag{29}$$

For $|\lambda|$ small, we have

$$\begin{aligned}
&\int_0^1 k_0(z, u+s\lambda y)\,ds \\
&\le 2\|\beta\|_\infty \underline{u}(z)^{-\gamma} + \int_0^1 |u(z)+s\lambda y(z)|^{-\gamma}\,ds\, \mathcal{X}_{\{\underline{u}\le u \le \bar{u}\}}(z) \\
&\le 2\|\beta\|_\infty \underline{u}(z)^{-\gamma} + c_{12}\left[\max_{0\le s\le 1} |u(z)+s\lambda y(z)|^{-\gamma}\right]\mathcal{X}_{\{\underline{u}\le u \le \bar{u}\}}(z) \\
&\qquad \text{for some} \quad c_{12} > 0 \quad \text{(see Takáč [25] (p. 233))} \\
&\le c_{13}\underline{u}(z)^{-\gamma} \quad \text{for some} \quad c_{13} > 0 \\
&\le c_{14} d(z)^{-\gamma} \quad \text{for some} \quad c_{14} > 0, \quad \text{all} \quad z \in \Omega.
\end{aligned} \tag{30}$$

Note that

$$c_{14} d(z)^{-\gamma} y(z) = c_{14} d(z)^{1-\gamma} \frac{y(z)}{d(z)} \le c_{15} \frac{y(z)}{d(z)} \tag{31}$$

for all $z \in \Omega$ and some $c_{15} > 0$.

Using Hardy's inequality (see Brezis [2] (p. 313)), we have that

$$\frac{y}{d} \in L^p(\Omega).$$

Then from (29), (30), (31) we see that we can apply the Lebesgue dominated convergence theorem and obtain

$$\begin{aligned}
&\langle \sigma_0'(u), y\rangle = \int_\Omega k_0(z,u)y\,dz \quad \text{for all} \quad y \in W_0^{1,p}(\Omega) \\
&(\text{see (28) and recall that } C_0^1(\bar\Omega) \text{ is dense in } W_0^{1,p}(\Omega)) \\
&\Rightarrow \sigma_0'(u) = N_{k_0}(u) \quad \text{for all} \quad u \in W_0^{1,p}(\Omega).
\end{aligned}$$

This proves Claim 1.

From (27) and since $\beta(\cdot)/\bar u(\cdot)^\gamma \le \beta(\cdot)/\underline{u}(\cdot)^\gamma \in L^q(\Omega)$, it follows that the functional $\sigma(\,\cdot\,)$ is coercive. Also, using the Sobolev embedding theorem, we can easily see that $\sigma(\,\cdot\,)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $u_0 \in W_0^{1,p}(\Omega)$ s.t.

$$\sigma(u_0) = \inf[\sigma(u) : u \in W_0^{1,p}(\Omega)]. \tag{32}$$

From (32) and Claim 1, it follows that

$$\begin{aligned}
&\sigma'(u_0) = 0 \\
&\Rightarrow A(u_0) = N_k(u_0).
\end{aligned} \tag{33}$$

On (33) we act with $(\underline{u} - u_0)^+ \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned}
\langle A(u_0), (\underline{u} - u_0)^+\rangle &= \int_\Omega k(z,u_0)(\underline{u}-u_0)^+\,dz \\
&= \int_\Omega [\beta \underline{u}^{-\gamma} + f(z,\underline{u})](\underline{u}-u_0)^+\,dz \quad (\text{see (27)}) \\
&\ge \langle A(\underline{u}), (\underline{u}-u_0)^+\rangle \quad (\text{see (8) and recall } f \ge 0) \\
\Rightarrow \int_{\{\underline{u}\ge u_0\}} &(a(D\underline{u}) - a(Du_0), D\underline{u} - Du_0)_{\mathbb{R}^N}\,dz \le 0 \\
\Rightarrow |\{\underline{u} \ge u_0\}|_N &= 0 \quad (\text{see Lemma 2 (a)}), \text{ hence } \underline{u} \le u_0.
\end{aligned}$$

Next on (33) we act with $(u_0 - \bar u)^+ \in W_0^{1,p}(\Omega)$. We have

$$\begin{aligned}
\langle A(u_0), (u_0 - \bar u)^+\rangle &= \int_\Omega k(z,u_0)(u_0-\bar u)^+\,dz \\
&= \int_\Omega [\beta \bar u^{-\gamma} + f(z,\bar u)](u_0-\bar u)^+\,dz \quad (\text{see (27)}) \\
&\le \langle A(\bar u), (u_0-\bar u)^+\rangle \quad (\text{see (26)}) \\
\Rightarrow \int_{\{u_0 > \bar u\}} &(a(Du_0) - a(D\bar u), Du_0 - D\bar u)_{\mathbb{R}^N}\,dz \le 0
\end{aligned}$$

$$\Rightarrow |\{u_0 > \bar{u}\}|_N = 0 \quad \text{(see Lemma 2(a)), hence} \quad u_0 \le \bar{u}.$$

So we have proved that

$$u_0 \in [\underline{u}, \bar{u}] = \{u \in W_0^{1,p}(\Omega) \colon \underline{u}(z) \le u(z) \le \bar{u}(z) \text{ a.e. in } \Omega\}.$$

Then from (27) and (33) we infer that u_0 is a nontrivial positive solution of problem (1). As before (see the proof of Proposition 6), using the regularity result of [18] we have that $u_0 \in [\underline{u}, \bar{u}] \cap \operatorname{int} C_+$.

Using $u_0 \in \operatorname{int} C_+$, we introduce the following truncation of the reaction in problem (1)

$$e(z, x) = \begin{cases} \beta(z) u_0(z)^{-\gamma} + f(z, u_0(z)) & \text{if} \quad x < u_0(z), \\ \beta(z) x^{-\gamma} + f(z, x) & \text{if} \quad u_0(z) \le x. \end{cases} \tag{34}$$

Evidently $e(z,x)$ is continuous on $\Omega \times \mathbb{R}$. We set $E(z,x) = \int_0^x e(z,s)ds$ and consider the functional $\mu \colon W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\mu(u) = \int_\Omega G(Du(z))\, dz - \int_\Omega E(z, u(z))\, dz \quad \text{for all} \quad u \in W_0^{1,p}(\Omega).$$

As in the proof of Claim 1, we show that $\mu \in C^1(W_0^{1,p}(\Omega))$ and

$$\mu'(u) = A(u) - N_e(u) \quad \text{for all} \quad u \in W_0^{1,p}(\Omega). \tag{35}$$

Claim 2. *The functional μ satisfies the C-condition.*

Let $\{u_n\}_{n \ge 1} \subset W_0^{1,p}(\Omega)$ be a sequence s.t.

$$|\mu(u_n)| \le M_3 \quad \text{for some} \quad M_3 > 0, \quad \text{all} \quad n \ge 1 \tag{36}$$

and

$$(1 + \|u_n\|)\mu'(u_n) \to 0 \quad \text{in} \quad W^{-1,p'}(\Omega) \quad \text{as} \quad n \to \infty. \tag{37}$$

From (37) we have

$$|\langle \mu'(u_n), h\rangle| \le \frac{\varepsilon_n \|h\|}{1 + \|u_n\|} \quad \text{for all} \quad h \in W_0^{1,p}(\Omega) \quad \text{with} \quad \varepsilon_n \downarrow 0^+$$

$$\Rightarrow \left| \langle A(u_n), h\rangle - \int_\Omega e(z, u_n) h\, dz \right| \le \frac{\varepsilon_n \|h\|}{1 + \|u_n\|} \quad \text{for all} \quad n \ge 1 \tag{38}$$

(see (35)).

In (38), we choose $h = -u_n^- \in W_0^{1,p}(\Omega)$. Then using Lemma 2 (c), we have

$$\frac{c_1}{p-1}\|Du_n^-\|_p^p \leq c_{16}\|u_n^-\| \quad \text{for some} \quad c_{16} > 0, \quad \text{all} \quad n \geq 1$$

(note that $u_0^{-\gamma} \leq \underline{u}^{-\gamma} \in L^q(\Omega)$). Therefore

$$\{u_n^-\}_{n\geq 1} \subset W_0^{1,p}(\Omega) \quad \text{is bounded.} \tag{39}$$

Next in (38) we choose $h = u_n^+ \in W_0^{1,p}(\Omega)$. Then

$$-\int_\Omega (a(Du_n^+), Du_n^+)_{\mathbb{R}^N}\, dz + \int_\Omega e(z, u_n^+)u_n^+\, dz \leq \varepsilon_n, \quad \text{for all} \quad n \geq 1. \tag{40}$$

On the other hand, from (36) and (39), we have

$$\int_\Omega pG(Du_n^+)\, dz - \int_\Omega pE(z, u_n^+)\, dz \leq M_4, \tag{41}$$

for some $M_4 > 0$, all $n \geq 1$.

We add (40) and (41) and obtain

$$\begin{aligned}
&\int_\Omega [pG(Du_n^+) - (a(Du_n^+), Du_n^+)_{\mathbb{R}^N}]\, dz \\
&+ \int_\Omega [e(z, u_n^+)u_n^+ - pE(z, u_n^+)]\, dz \leq M_5 \quad \text{for some} \quad M_5 > 0, \quad \text{all} \quad n \geq 1 \\
&\Rightarrow \int_{\{u_n \geq u_0\}} [e(z, u_n^+)u_n^+ - pE(z, u_n^+)]\, dz \leq M_6 \\
&\qquad\qquad \text{for some } M_6 > 0, \text{ all } n \geq 1 \text{ (see (34) and hypothesis } H(a) \text{ (iv))} \\
&\Rightarrow \int_{\{u_n \geq u_0\}} [f(z, u_n^+)u_n^+ - pF(z, u_n^+)]\, dz \leq M_7, \qquad (42)
\end{aligned}$$

for some $M_7 > 0$, all $n \geq 1$ (see (34) and recall that $u_0^{-\gamma} \leq \underline{u}^{-\gamma} \in L^q(\Omega)$).

By virtue of hypotheses $H(f)$ (i), (iii), we can find $\beta_1 \in (0, \beta_0)$ and $c_{17} > 0$ s.t.

$$\beta_1 x^\tau - c_{17} \leq f(z, x)x - pF(z, x) \quad \text{for a.a.} \quad z \in \Omega, \quad \text{all} \quad x \geq 0. \tag{43}$$

Using (43) in (42), we obtain

$$\begin{aligned}
&\beta_1 \int_{\{u_n \geq u_0\}} (u_n^+)^\tau\, dz \leq M_8, \quad \text{for some} \quad M_8 > 0, \quad \text{all} \quad n \geq 1 \\
&\Rightarrow \{u_n^+\}_{n\geq 1} \subset L^\tau(\Omega) \quad \text{is bounded.} \qquad (44)
\end{aligned}$$

From hypothesis $H(f)$ (iii) it is clear that without any loss of generality, we may assume that $\tau \leq r < p^*$.

First suppose $p \neq N$. We can find $t \in [0, 1)$ s.t.

$$\frac{1}{r} = \frac{1-t}{\tau} + \frac{t}{p^*}. \tag{45}$$

By virtue of the interpolation inequality (see, for example, Gasiński–Papageorgiou [8] (p. 905)), we have

$$\begin{aligned} &\|u_n^+\|_r \leq \|u_n^+\|_\tau^{1-t} \cdot \|u_n^+\|_{p^*}^t \\ &\Rightarrow \|u_n^+\|_r^r \leq M_9 \|u_n^+\|_{p^*}^{tr}, \quad \text{for some} \quad M_9 > 0, \quad \text{all} \quad n \geq 1 \quad \text{(see (44))}. \end{aligned} \tag{46}$$

In (38) we choose $h = u_n^+ \in W_0^{1,p}(\Omega)$ and using Lemma 2 (c) and (34), we obtain

(47)
$$\begin{aligned} \frac{c_1}{p-1}\|Du_n^+\|_p^p &\leq M_{10} + \int_{\{u_n \geq u_0\}} f(z, u_n^+)u_n^+ \, dz \quad \text{for some} \quad M_{10} > 0, \quad \text{all} \quad n \geq 1 \\ &\leq M_{11}(1 + \|u_n^+\|_r^r) \quad \text{for some} \quad M_{11} > 0, \quad \text{all} \quad n \geq 1 \quad \text{(see } H(f) \text{ (i))} \\ &\leq M_{12}(1 + \|u_n^+\|^{tr}) \quad \text{for some} \quad M_{12} > 0, \quad \text{all} \quad n \geq 1 \end{aligned}$$

(see (46)).

The choice of τ (see $H(f)$ (iii)) and (45), imply that $tr < p$. Hence, from (47) it follows that

$$\{u_n^+\}_{n \geq 1} \subset W_0^{1,p}(\Omega) \quad \text{is bounded.} \tag{48}$$

If $p = N$, then $p^* = +\infty$ and by the Sobolev embedding theorem, $W_0^{1,p}(\Omega) \hookrightarrow L^s(\Omega)$ for all $s \in [1, +\infty)$. So, the above argument works and we reach (48), if we replace p^* by $s > r$ large.

From (39) and (48) we infer that

$$\{u_n\}_{n \geq 1} \subset W_0^{1,p}(\Omega) \quad \text{is bounded.}$$

Therefore, we may assume that

$$u_n \xrightarrow{w} u \quad \text{in} \quad W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \to u \quad \text{in} \quad L^s(\Omega) \tag{49}$$

with $s = r$ if $N \leq p$ and $s > \max\{r, N/(N-p)\}$, if $N > p$. In (38) we choose $h = u_n - u \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (49). Then

$$\begin{aligned} &\lim_{n \to +\infty} \langle A(u_n), u_n - u \rangle = 0 \\ &\Rightarrow u_n \to u \quad \text{in} \quad W_0^{1,p}(\Omega) \quad \text{as} \quad n \to \infty \quad \text{(see Proposition 4).} \end{aligned}$$

This proves Claim 2.

From (34) and hypothesis $H(f)$ (ii), we have

$$\mu(t\hat{u}_1(p)) \to -\infty \quad \text{as} \quad t \to +\infty. \tag{50}$$

Recall that $u_0 \leq \bar{u}$. We may assume that there is no solution of (1) distinct from u_0 in the order interval $[u_0, \bar{u}] = \{u \in W_0^{1,p}(\Omega)\colon u_0(z) \leq u(z) \leq \bar{u}(z)$ a.e. in $\Omega\}$. Otherwise, we already have the desired second positive solution of (1) and so we are done.

We introduce the following truncation of $e(z, \cdot)$:

$$e_0(z, x) = \begin{cases} e(z, x) & \text{if} \quad x \leq \bar{u}(z), \\ e(z, \bar{u}(z)) & \text{if} \quad \bar{u}(z) < x. \end{cases} \tag{51}$$

This is a continuous function. We set $E_0(z, x) = \int_0^x e_0(z, s)\,ds$ and consider the C^1-functional $\mu_0\colon W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\mu_0(u) = \int_\Omega G(Du(z))\,dz - \int_\Omega E_0(z, u(z))\,dz \quad \text{for all} \quad u \in W_0^{1,p}(\Omega).$$

From (51) it is clear that $\mu_0(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\hat{u}_0 \in W_0^{1,p}(\Omega)$ s.t.

$$\begin{aligned} &\mu_0(\hat{u}_0) = \inf[\mu_0(u)\colon u \in W_0^{1,p}(\Omega)] \\ &\Rightarrow \mu_0'(\hat{u}_0) = 0 \\ &\Rightarrow A(\hat{u}_0) = N_{e_0}(\hat{u}_0). \end{aligned} \tag{52}$$

Reasoning as in the first part of the proof, using (26) and the fact that u_0 is a solution of (1), we show that

$$\begin{aligned} &\hat{u}_0 \in [u_0, \bar{u}] \\ &\Rightarrow \hat{u}_0 \text{ is a solution of (1) (see (51) and (52))} \\ &\Rightarrow \hat{u}_0 = u_0 \quad \text{(see (53))}. \end{aligned} \tag{53}$$

Next we show that $u_0 \in \operatorname{int}_{C_0^1(\bar{\Omega})}[0, \bar{u}]$. To this end, we have the following inequalities

$$\begin{aligned} &-\operatorname{div} a(Du_0(z)) - \beta(z)u_0(z)^{-\gamma} \\ &= f(z, u_0(z)) \\ &< (\eta(z) + \varepsilon)u_0(z)^{p-1} + \xi_\varepsilon u_0(z)^{r-1} \quad \text{(see (19))} \\ &= -\operatorname{div} a(D\bar{u}(z)) - \beta(z)\bar{u}(z)^{-\gamma} \quad \text{a.e. in} \quad \Omega. \end{aligned}$$

Invoking the strong comparison principle of Giacomoni–Schindler–Takáč [11] (Theorem 2.3) we have

$$\bar{u} - u_0 \in \operatorname{int} C_+$$
$$\Rightarrow u_0 \in \operatorname{int}_{C_0^1(\bar{\Omega})}[0, \bar{u}].$$

Note that

$$\mu|_{[0,\bar{u}]} = \mu_0|_{[0,\bar{u}]} \quad \text{(see (34) and (51))}$$
$$\Rightarrow u_0 \text{ is a local } C_0^1(\bar{\Omega})\text{-minimizer of } \mu$$
$$\Rightarrow u_0 \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \mu \quad \text{(see [11])}.$$

So, we can find $\rho > 0$ s.t.

$$\mu(u_0) < \inf[\mu(u)\colon \|u - u_0\| = \rho] = \eta_\rho \quad \text{(see [7])}. \tag{54}$$

Then (50), (54) and Claim 2, permit the use of Theorem 1 (the mountain pass theorem). So, we can find $u_1 \in W_0^{1,p}(\Omega)$ s.t.

$$\mu'(u_1) = 0 \quad \text{and} \quad \eta_\rho \leq \mu(u_1). \tag{55}$$

From (54) and (55), we see that $u_1 \neq u_0$. Also, from (55), we have

$$A(u_1) = N_e(u_1).$$

Acting with $(u_0 - u_1)^+ \in W_0^{1,p}(\Omega)$ and using (34), we show that

$$u_1 \in [u_0) = \{u \in W_0^{1,p}(\Omega)\colon u_0(z) \leq u(z) \text{ a.e. in } \Omega\}$$
$$\Rightarrow u_1 \text{ is a solution of (1) (see (34)) and } u_1 \geq u_0.$$

As before, we show that $u_1 \in \operatorname{int} C_+$. $\square$

4. The homogeneous problem

In this section we consider problem (1) with the general nonhomogeneous differential operator replaced by the p-Laplacian (which is $(p-1)$-homogeneous). So, the problem under consideration, is now the following:

$$-\Delta_p u(z) = \beta(z)u(z)^{-\gamma} + f(z, u(z)) \quad \text{in} \quad \Omega, \quad u|_{\partial\Omega} = 0, \quad u \geq 0, \quad \gamma \in (0,1). \tag{56}$$

For this problem, we will consider a $(p-1)$-sublinear perturbation $f(z, x)$ which can have partial interaction with $\hat{\lambda}_1(p) > 0$ at $+\infty$ (nonuniform nonresonance). Also, in this case, we do not require the positivity of f and instead for the reaction we assume an oscillatory behavior near zero. Finally, in the multiplicity theorem, we do not impose any restriction on $\|\beta\|_\infty$.

The new hypotheses on the perturbation $f(z, x)$ are $H(f)'$ (see Section 2, p. 492).

Theorem 8. *If hypotheses* $H(\beta)$, $H(f)'$ *hold, then problem* (56) *admits at least two positive solutions*

$$u_0, u_1 \in \operatorname{int} C_+, \quad u_0 \leq u_1, \quad u_0 \neq u_1.$$

Proof. Let $\underline{u} \in \operatorname{int} C_+$ be the solution of the auxiliary problem (8) produced in Proposition 5. Let $t \in (0, 1)$ be small s.t. $t\underline{u}(z) \leq \delta_0$ for all $z \in \bar{\Omega}$, with $\delta_0 > 0$ as in hypothesis $H(f)'$ (iii). We set $\hat{\underline{u}} = t\underline{u} \in \operatorname{int} C_+$ and we have

$$\begin{aligned} -\Delta_p \hat{\underline{u}}(z) &= -t^{p-1}\Delta_p \underline{u}(z) = t^{p-1}\beta(z)\underline{u}(z)^{-\gamma} \quad \text{(see Proposition 5)} \\ &\leq \beta(z)\underline{u}(z)^{-\gamma} + f(z, \hat{\underline{u}}(z)) \quad \text{a.e. in} \quad \Omega \end{aligned} \tag{57}$$

(see $H(f)'$ (iii) and recall that $t \in (0, 1)$).

Also, we have

$$-\Delta_p \xi_0 = 0 \geq \beta(z)\xi_0^{-\gamma} + f(z, \xi_0) \quad \text{a.e. in (see } H(f)' \text{ (iii))} \tag{58}$$

and $\hat{\underline{u}}(z) < \xi_0$ for all $z \in \bar{\Omega}$.

We consider the following truncation of the reaction in problem (56):

$$g(z, x) = \begin{cases} \beta(z)\hat{\underline{u}}(z)^{-\gamma} + f(z, \hat{\underline{u}}(z)) & \text{if} \quad x < \hat{\underline{u}}(z), \\ \beta(z)x^{-\gamma} + f(z, x) & \text{if} \quad \hat{\underline{u}}(z) \leq x \leq \xi_0, \\ \beta(z)\xi_0^{-\gamma} + f(z, \xi_0) & \text{if} \quad \xi_0 < x. \end{cases} \tag{59}$$

This is a continuous function. We set $G(z, x) = \int_0^x g(z, s)\, ds$ and consider the functional $\psi\colon W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi(u) = \frac{1}{p}\|Du\|_p^p - \int_\Omega G(z, u(z))\, dz \quad \text{for all} \quad u \in W_0^{1,p}(\Omega).$$

As in Claim 1 in the proof of Theorem 7, we can check that

$$\psi \in C^1(W_0^{1,p}(\Omega)) \quad \text{and} \quad \psi'(u) = A(u) - N_g(u) \quad \text{for all} \quad u \in W_0^{1,p}(\Omega).$$

From (59) it is clear that $\psi(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. Therefore, we can find $u_0 \in W_0^{1,p}(\Omega)$ s.t.

$$\begin{aligned} &\psi(u_0) = \inf[\psi(u)\colon u \in W_0^{1,p}(\Omega)] \\ &\Rightarrow \psi'(u_0) = 0 \\ &\Rightarrow A(u_0) = N_g(u_0). \end{aligned} \tag{60}$$

On (60) first we act with $(\underline{\hat{u}}-u_0)^+ \in W_0^{1,p}(\Omega)$ and then with $(u_0-\xi_0)^+ \in W_0^{1,p}(\Omega)$. Using (57) and (58) and the nonlinear regularity result of Lieberman [18], we have

$$u_0 \in [\underline{u}, \xi_0] = \{u \in W_0^{1,p}(\Omega) \colon \underline{u}(z) \le u(z) \le \xi_0 \text{ a.e. in } \Omega\}, \quad u_0 \in \operatorname{int} C_+.$$

Let $\rho = \xi_0$ and let $\hat{\xi}_\rho > 0$ be as postulated by hypothesis $H(f)'$ (iv). We have

$$\begin{aligned} &-\Delta_p u_0(z) - \beta(z)u_0(z)^{-\gamma} + \hat{\xi}_\rho u_0(z)^{p-1} \\ &= f(z, u_0(z)) + \hat{\xi}_\rho u_0(z)^{p-1} \\ &\le f(z, \xi_0) + \hat{\xi}_\rho \xi_0^{p-1} \quad (\text{see } H(f)' \text{ (iv) and recall } u_0(z) \le \xi_0 \text{ for all } z \in \bar{\Omega}) \\ &< -\beta(z)\xi_0^{-\gamma} + \hat{\xi}_\rho \xi_0^{p-1} \quad \text{a.e. in} \quad \Omega. \end{aligned} \tag{61}$$

Let $D_0 = \{z \in \Omega \colon u_0(z) = \xi_0\}$ and $D_1 = \{z \in \Omega \colon Du_0(z) = 0\}$. Let $w = \xi_0 - u_0 \in C^1(\bar{\Omega})$. Then $w(z) \ge 0$ for all $z \in \bar{\Omega}$.

Let $\hat{z} \in D_0$. Then $w(\cdot)$ attains its minimum at $\hat{z}$ and so $Dw(\hat{z}) = 0 \Rightarrow Du_0(\hat{z}) = 0$, hence $\hat{z} \in D_1$. So, we have proved that $D_0 \subseteq D_1$.

Since $u_0 \in \operatorname{int} C_+$, it follows that D_1 is a compact subset of Ω. The set D_0 being a closed subset of the compact set D_1 is itself compact. Hence, we can find $\Omega_1 \subseteq \Omega$ open s.t.

$$D_0 \subseteq \Omega_1 \subseteq \bar{\Omega}_1 \subseteq \Omega. \tag{62}$$

Let $h_1(z) = f(z,u_0(z)) + \hat{\xi}_\rho u_0(z)^{p-1}$ and $h_2(z) = -\beta(z)\xi_0^{-\gamma} + \hat{\xi}_\rho \xi_0^{p-1}$. Then $h_1, h_2 \in C(\Omega)$ and $h_1(z) < h_2(z)$ for all $z \in \Omega$ (see (61)). So, we can find $\varepsilon \in (0, 1)$ small s.t.

(63)
$$u_0(z) + \varepsilon \le \xi_0 \quad \text{for all} \quad z \in \partial\Omega_1 \quad (\text{see (62)}), \quad h_1(z) + \varepsilon \le h_2(z) \quad \text{for all} \quad z \in \bar{\Omega}_1.$$

We choose $\delta = \delta(\varepsilon) \in (0, 1)$ s.t.

$$\xi |s^{p-1} - (s')^{p-1}| \le \frac{\varepsilon}{2} \quad \text{and} \quad \|\beta\|_\infty \left| \frac{1}{s^\gamma} - \frac{1}{(s')^\gamma} \right| \le \frac{\varepsilon}{2} \tag{64}$$

for all $s, s' \in [\min_{\bar{\Omega}_1} u_0, \xi_0]$ with $|s - s'| \le \delta$ (recall that $u_0 \in \operatorname{int} C_+$ and so $\min_{\bar{\Omega}_1} u_0 > 0$ and this implies that $s \to \|\beta\|_\infty / s^\gamma$ is uniformly continuous on $[\min_{\bar{\Omega}_1} u_0, \xi_0]$). Then we have

$$\begin{aligned} &-\Delta_p(u_0 + \delta) - \beta(z)(u_0 + \delta)^{-\gamma} + \hat{\xi}_\rho(u_0 + \delta)^{p-1} \\ &= -\Delta_p u_0 - \beta(z)(u_0 + \delta)^{-\gamma} + \hat{\xi}_\rho(u_0 + \delta)^{p-1} \\ &\le \beta(z)u_0^{-\gamma} - \beta(z)(u_0 + \delta)^{-\gamma} + f(z, u_0) + \hat{\xi}_\rho(u_0 + \delta)^{p-1} \\ &\le \|\beta\|_\infty |u_0^{-\gamma} - (u_0 + \delta)^{-\gamma}| + h_1(z) + \hat{\xi}_\rho |(u_0 + \delta)^{p-1} - u_0^{p-1}| \end{aligned}$$

$$\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + h_1(z) \quad \text{(see (63), (64))}$$
$$\le h_2(z) = -\Delta_p \xi_0 - \beta(z)\xi_0^{-\gamma} + \hat{\xi}_\rho \xi_0^{p-1} \quad \text{a.e. in} \quad \Omega_1$$
$$\Rightarrow u_0(z) + \delta \le \xi_0 \quad \text{for} \quad z \in \Omega_1 \quad \text{(by the weak comparison principle, see [23])}$$
$$\Rightarrow D_0 = \emptyset \quad \text{(see (62))}$$
$$\Rightarrow u_0(z) < \xi_0 \quad \text{for all} \quad z \in \bar{\Omega}.$$

So, we have proved that

$$u_0 \in \operatorname{int}_{C_0^1(\bar{\Omega})}[0, \xi_0]. \tag{65}$$

Using $u \in \operatorname{int} C_+$, we introduce the following truncation of the reaction

$$g_0(z, x) = \begin{cases} \beta(z)u_0(z)^{-\gamma} + f(z, u_0(z)) & \text{if} \quad x < u_0(z), \\ \beta(z)x^{-\gamma} + f(z, x) & \text{if} \quad u_0(z) \le x. \end{cases} \tag{66}$$

This is a continuous function on $\Omega \times \mathbb{R}$. Let $G_0(z, x) = \int_0^x g_0(z, s)\, ds$ and consider the C^1-functional $\psi_0 \colon W_0^{1,p}(\Omega) \to \mathbb{R}$ (see Claim 1 in the proof of Theorem 7) defined by

$$\psi_0(u) = \frac{1}{p}\|Du\|_p^p - \int_\Omega G_0(z, u(z))\, dz \quad \text{for all} \quad u \in W_0^{1,p}(\Omega).$$

Claim. *ψ_0 satisfies the C-condition.*

Let $\{u_n\}_{n\ge1} \subset W_0^{1,p}(\Omega)$ be a sequence s.t. $\{\psi_0(u_n)\}_{n\ge1} \subset \mathbb{R}$ is bounded and

$$(1 + \|u_n\|)\psi_0'(u_n) \to 0 \quad \text{in} \quad W^{-1,p'}(\Omega). \tag{67}$$

From (67) we have

$$|\langle \psi_0'(u_n), h\rangle| \le \frac{\varepsilon_n \|h\|}{1 + \|u_n\|} \quad \text{for all} \quad h \in W_0^{1,p}(\Omega) \quad \text{with} \quad \varepsilon_n \downarrow 0^+$$
$$\Rightarrow \left| \langle A(u_n), h\rangle - \int_\Omega g_0(z, u_n)h\, dz \right| \le \frac{\varepsilon_n \|h\|}{1 + \|u_n\|} \quad \text{for all} \quad n \ge 1 \tag{68}$$

(see Claim 1 in the proof of Theorem 7).

In (68) we choose $h = -u_n^- \in W_0^{1,p}(\Omega)$. Then

$$\|Du_n^-\|_p^p \le M_{13} \quad \text{for some} \quad M_{13} > 0, \quad \text{all} \quad n \ge 1 \quad \text{(see (66))}$$
$$\Rightarrow \{u_n^-\}_{n\ge1} \subset W_0^{1,p}(\Omega) \quad \text{is bounded.} \tag{69}$$

Suppose that $\|u_n^+\| \to \infty$. Let $y_n = u_n^+/\|u_n^+\|$ for all $n \geq 1$. Then $\|y_n\| = 1$, $y_n \geq 0$ for all $n \geq 1$. So, we may assume that

$$y_n \xrightarrow{w} y \quad \text{in} \quad W_0^{1,p}(\Omega) \quad \text{and} \quad y_n \to y \quad \text{in} \quad L^s(\Omega), \quad y \geq 0, \tag{70}$$

where $s = p$ if $N \leq p$ and $s > \max\{p, N/(N-p)\}$ if $N > p$. From (68) and (69) we have

$$\left| \langle A(y_n), h\rangle - \int_\Omega \frac{g_0(z, u_n^+)}{\|u_n^+\|^{p-1}} h\, dz \right| \leq \varepsilon_n' \|h\| \quad \text{for all} \quad h \in W_0^{1,p}(\Omega) \tag{71}$$

with $\varepsilon_n' \to 0^+$.

Hypothesis $H(f)'$ (ii) implies that

$$\frac{N_{g_0}(u_n^+)}{\|u_n^+\|^{p-1}} \xrightarrow{w} \eta_0 y^{p-1} \quad \text{in} \quad L^s(\Omega) \quad \text{with} \quad \eta(z) \leq \eta_0(z) \leq \hat{\eta} \tag{72}$$

a.e. in Ω (see [7]).

Also, if in (71) we choose $h = y_n - y \in W_0^{1,p}(\Omega)$, then using (70) we have

$$\begin{aligned} &\lim_{n\to\infty} \langle A(y_n), y_n - y\rangle = 0 \\ &\Rightarrow y_n \to y \quad \text{in} \quad W_0^{1,p}(\Omega) \quad \text{(see Proposition 4), hence} \quad \|y\| = 1, \quad y \geq 0. \end{aligned} \tag{73}$$

So, if in (71) we pass to the limit as $n \to \infty$ and use (72), (73), then

$$\begin{aligned} &\langle A(y), h\rangle = \int_\Omega \eta_0 y^{p-1} h\, dz \quad \text{for all} \quad h \in W_0^{1,p}(\Omega) \\ &\Rightarrow A(y) = \eta_0 y^{p-1} \\ &\Rightarrow -\Delta_p y(z) = \eta_0(z) y(z)^{p-1} \quad \text{a.e. in} \quad \Omega,\ y|_{\partial\Omega} = 0. \end{aligned} \tag{74}$$

We have

$$\begin{aligned} &\hat{\lambda}_1(p, \eta_0) \leq \hat{\lambda}_1(p, \eta) < \hat{\lambda}_1(p, \hat{\lambda}_1(p)) = 1 \\ &\Rightarrow y \text{ must be nodal (see (74)), a contradiction to (73).} \end{aligned}$$

This proves that

$$\begin{aligned} &\{u_n^+\}_{n\geq 1} \subset W_0^{1,p}(\Omega) \quad \text{is bounded} \\ &\Rightarrow \{u_n\}_{n\geq 1} \subset W_0^{1,p}(\Omega) \quad \text{is bounded (see (69)).} \end{aligned}$$

From this as in the proof of Theorem 7 (see Claim 2), via Proposition 4, we conclude that ψ_0 satisfies the C-condition. This proves the claim.

As in the proof of Theorem 7, by truncating $g_0(z,\cdot)$ at ξ_0 and using (65), we show that u_0 is a local minimizer of ψ_0. So, we can find $\rho \in (0,1)$ s.t.

$$\psi_0(u_0) < \inf[\psi_0(u)\colon \|u-u_0\| = \rho] = \tilde{\eta}_0. \tag{75}$$

Hypothesis $H(f)'$ (ii) implies that

$$\psi_0(t\hat{u}_1(p)) \to -\infty \quad \text{as} \quad t \to +\infty. \tag{76}$$

Then from (75), (76) and the claim, we see that we can apply Theorem 1 (the mountain pass theorem) and find $\hat{u} \in W_0^{1,p}(\Omega)$ s.t.

$$\psi_0'(\hat{u}) = 0 \quad \text{and} \quad \tilde{\eta}_0 \le \psi_0(\hat{u}). \tag{77}$$

From (76) and (77) we have $u_0 \ne \hat{u}$, $u_0 \le \hat{u}$, $\hat{u} \in \operatorname{int} C_+$ and solves problem (56). □

Evidently, combining the proof of Theorem 7 with the first part of the proof of Theorem 8, we can have the following multiplicity theorem for p-Laplacian equations with the combined effects of singular and superlinear terms. We emphasize that no restriction on $\|\beta\|_\infty$ is imposed and so our result is in this respect an improvement over all the previous singular p-Laplacian equations.

Theorem 9. *If hypotheses $H(\beta)$ and $H(f)$ hold, then problem* (56) *has at least two positive solutions*

$$u_0, \hat{u} \in \operatorname{int} C_+, \quad u_0 \le \hat{u}, \quad u_0 \ne \hat{u}.$$

ACKNOWLEDGEMENT. The authors wish to thank a knowledgeable referee for his/her corrections and remarks.

References

[1] V. Benci, P. D'Avenia, D. Fortunato and L. Pisani: *Solitons in several space dimensions*: *Derrick's problem and infinitely many solutions*, Arch. Ration. Mech. Anal. **154** (2000), 297–324.
[2] H. Brezis: Functional Analysis, Sobolev Spaces and Partial Differential Equations, Springer, New York, 2011.
[3] L. Cherfils and Y. Il'yasov: *On the stationary solutions of generalized reaction diffusion equations with p&q-Laplacian*, Commun. Pure Appl. Anal. **4** (2005), 9–22.
[4] M.M. Coclite and G. Palmieri: *On a singular nonlinear Dirichlet problem*, Comm. Partial Differential Equations **14** (1989), 1315–1327.
[5] D.G. Costa and C.A. Magalhães: *Existence results for perturbations of the p-Laplacian*, Nonlinear Anal. **24** (1995), 409–418.

[6] G. Fei: *On periodic solutions of superquadratic Hamiltonian systems*, Electron. J. Differential Equations (2002), 12 pp.
[7] M. Filippakis, A. Kristály and N.S. Papageorgiou: *Existence of five nonzero solutions with exact sign for a p-Laplacian equation*, Discrete Contin. Dyn. Syst. **24** (2009), 405–440.
[8] L. Gasiński and N.S. Papageorgiou: Nonlinear Analysis, Chapman & Hall/CRC, Boca Raton, FL, 2006.
[9] L. Gasiński and N.S. Papageorgiou: *Nonlinear elliptic equations with singular terms and combined nonlinearities*, Ann. Henri Poincaré **13** (2012), 481–512.
[10] M. Ghergu and V.D. Rădulescu: Singular Elliptic Problems: Bifurcation and Asymptotic Analysis, Oxford Univ. Press, Oxford, 2008.
[11] J. Giacomoni, I. Schindler and P. Takáč: *Sobolev versus Hölder local minimizers and existence of multiple solutions for a singular quasilinear equation*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **6** (2007), 117–158.
[12] Z. Guo: *On the number of positive solutions for quasilinear elliptic eigenvalue problems*, Nonlinear Anal. **27** (1996), 229–247.
[13] N. Hirano, C. Saccon and N. Shioji: *Brezis–Nirenberg type theorems and multiplicity of positive solutions for a singular elliptic problem*, J. Differential Equations **245** (2008), 1997–2037.
[14] S.Th. Kyritsi and N.S. Papageorgiou: *Pairs of positive solutions for singular p-Laplacian equations with a p-superlinear potential*, Nonlinear Anal. **73** (2010), 1136–1142.
[15] O.A. Ladyzhenskaya and N.N. Ural'tseva: Linear and quasilinear elliptic equations, Academic Press, New York, 1968.
[16] A.V. Lair and A.W. Shaker: *Classical and weak solutions of a singular semilinear elliptic problem*, J. Math. Anal. Appl. **211** (1997), 371–385.
[17] A.C. Lazer and P.J. McKenna: *On a singular nonlinear elliptic boundary-value problem*, Proc. Amer. Math. Soc. **111** (1991), 721–730.
[18] G.M. Lieberman: *The natural generalization of the natural conditions of Ladyzhenskaya and Ural'tseva for elliptic equations*, Comm. Partial Differential Equations **16** (1991), 311–361.
[19] S. Li, S. Wu and H.-S. Zhou: *Solutions to semilinear elliptic problems with combined nonlinearities*, J. Differential Equations **185** (2002), 200–224.
[20] N.S. Papageorgiou and S.Th. Kyritsi-Yiallourou: Handbook of Applied Analysis, Springer, New York, 2009.
[21] N.S. Papageorgiou, E.M. Rocha and V. Staicu: *A multiplicity theorem for hemivariational inequalities with a p-Laplacian-like differential operator*, Nonlinear Anal. **69** (2008), 1150–1163.
[22] K. Perera and Z. Zhang: *Multiple positive solutions of singular p-Laplacian problems by variational methods*, Bound. Value Probl. (2005), 377–382.
[23] P. Pucci and J. Serrin: The Maximum Principle, Birkhäuser, Basel, 2007.
[24] Y. Sun, S. Wu and Y. Long: *Combined effects of singular and superlinear nonlinearities in some singular boundary value problems*, J. Differential Equations **176** (2001), 511–531.
[25] P. Takáč: *A variational approach to the Fredholm alternative for the p-Laplacian near the first eigenvalue*, J. Dynam. Differential Equations **18** (2006), 693–765.

Nikolaos S. Papageorgiou
National Technical University of Athens
Department of Mathematics
Zografou Campus, Athens 157 80
Greece
e-mail: npapg@math.ntua.gr

George Smyrlis
National Technical University of Athens
Department of Mathematics
Zografou Campus, Athens 157 80
Greece
e-mail: gsmyrlis@math.ntua.gr