COMPACT HOMOGENEOUS
LOCALLY CONFORMALLY KÄHLER MANIFOLDS

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Abstract

In this paper we show as main results two structure theorems of a compact homogeneous locally conformally Kähler (or shortly l.c.K.) manifold, a holomorphic structure theorem asserting that it has a structure of holomorphic principal fiber bundle over a flag manifold with fiber a 1-dimensional complex torus, and a metric structure theorem asserting that it is necessarily of Vaisman type. We also discuss and determine l.c.K. reductive Lie groups and compact locally homogeneous l.c.K. manifolds of reductive Lie groups.

Introduction

A *locally conformally Kähler structure* (or shortly *l.c.K. structure*) on a differentiable manifold M is a Hermitian structure h on M with its associated fundamental form Ω satisfying $d\Omega = \theta \wedge \Omega$ for some closed 1-form θ (which is so called Lee form). A differentiable manifold M is called a *locally conformal Kähler manifold* (or shortly *l.c.K. manifold*) if M admits an l.c.K. structure. Note that l.c.K. structure Ω is globally conformally Kähler (or Kähler) if and only if θ is exact (or 0 respectively); and a compact l.c.K. manifold of non-Kähler type (i.e. the Lee form is neither 0 nor exact) never admits a Kähler structure (compatible with the complex structure).

There have been recently extensive studies on l.c.K. manifolds (cf. [18], [5], [12], [2], [7]). In this paper we are concerned with l.c.K. structures on homogeneous and locally homogeneous spaces of Lie groups. There exist many examples of compact non-Kähler l.c.K. manifolds which are homogeneous or locally homogeneous spaces of certain Lie groups, such as Hopf surfaces, Inoue surfaces, Kodaira surfaces, or some class of elliptic surfaces (cf. [2], [8]). Their l.c.K. structures are *homogeneous* or *locally homogeneous* in the sense we will explicitly define in this paper (Definitions 1 or 2 respectively). Note that homogeneous l.c.K. structures on Lie groups are nothing but left-invariant l.c.K. structures, which can be considered as l.c.K. structures on their Lie algebras.

In this paper we show as main results two structure theorems of a compact homogeneous l.c.K. manifold: a holomorphic structure theorem asserting that it is a holomorphic principal fiber bundle over a flag manifold with fiber a 1-dimensional complex

torus (Theorem 1), and a metric structure theorem asserting that it is of *Vaisman type*, that is, the Lee form is parallel with respect to the Hermitian metric (Theorem 2). It should be noted that the same structure theorem was proved by Vaisman ([17]) for compact homogeneous l.c.K. manifolds of Vaisman type. As a simple application of the theorem, we can show that only compact homogeneous l.c.K. manifolds of complex dimension 2 are Hopf surfaces of homogeneous type (Theorem 3), and that there exist no compact homogeneous complex l.c.K. manifolds; in particular, no complex parallelizable manifolds admit their compatible l.c.K. structures (Corollary 4).

We will take the following key strategies to prove the main theorems. A compact homogeneous l.c.K. manifold M is expressed as $M = G/H$, where G is a compact Lie group and H is a closed subgroup of G. Since the Lie algebra $\mathfrak{g}$ of G is a reductive, $\mathfrak{g}$ can be written as $\mathfrak{g} = \mathfrak{t} + \mathfrak{s}$, where $\mathfrak{t}$ is the center of $\mathfrak{g}$ and $\mathfrak{s} = [\mathfrak{g}, \mathfrak{g}]$ is a semi-simple ideal of $\mathfrak{g}$. Our first observation is (1) $\mathfrak{g}$ must satisfies $1 \leq \dim \mathfrak{t} \leq 2$. As the second observation, applying a result of Hochschild and Serre, (2) we can express an l.c.K. form Ω as $\Omega = -\theta \wedge \psi + d\psi$, where θ is the Lee form and ψ is a 1-form. Let $\xi \in \mathfrak{g}$ be the Lee field (the associated vector field to θ with respect to h). We put $\xi = t + s$ ($t \in \mathfrak{t}$, $s \in \mathfrak{s}$). We define the vector field $\eta = J\xi$ (Reeb field) for the complex structure J, and the Reeb form ϕ (the associated 1-form to η with respect to h). We will see as the third observation (3) under the condition Ω is Jt-invariant, we have $\psi = \phi$ and $\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$, where $\mathfrak{p} = \langle t, \eta \rangle = \langle t, Jt \rangle = \langle \xi, \eta \rangle$, and $\mathfrak{k} = \ker \theta \cap \ker \phi$. In particular we can express $\Omega = -\theta \wedge \phi + d\phi$ with $\phi \in \wedge^2 \mathfrak{k}^*$. As the fourth observation, since the closure K of the 1-parameter subgroup of G generated by Jt is compact, (4) we can use the averaging method to make Ω on M invariant by $\mathrm{Ad}(K)$: $\bar{\Omega} = \int_K \mathrm{Ad}(x)^* \Omega$ while preserving the complex structure J.

Our fifth observation is (5) we can consider a compact homogeneous l.c.K. manifold M up to holomorphic isometry as $M = G/H$ with a homogeneous l.c.K. structure (Ω, J), satisfying $\mathfrak{g} = \mathfrak{t} + \mathfrak{s}$ ($\dim \mathfrak{t} = 1$); and up to biholomorphism, as such with a Jt-invariant l.c.K. form $\bar{\Omega}$. In particular we can express $M = S^1 \times_\Gamma S/H_0$, where S is a simply connected semi-simple Lie group, H_0 is the connected component of H and Γ is a finite abelian group. These observations lead to Theorem 1. As for the proof of Theorem 2, we have the sixth observation (6) the Lee form θ and the Reeb field η are stable under the averaging by K. In order to show it we need the seventh observation (7) we have a compact subgroup $S^1 \times N_S(H_0)/H_0$ imbedded in $G/H_0 = S^1 \times S/H_0$ as an l.c.K. manifold. We also need a classification of l.c.K. compact Lie algebras. We will see as the eighth observation (8) a reductive Lie algebra admits an l.c.K. structure if and only if $\dim \mathfrak{t} = 1$ and $\mathrm{rank}\, \mathfrak{s} = 1$. In particular a compact Lie algebra admits a homogeneous l.c.K. structure if and only if it is $\mathfrak{u}(2)$; and any homogeneous l.c.K. structure on a compact Lie group is of Vaisman type (Theorem 4).

1. Preliminaries

In this section we review some terminologies and basic results in the field of homogeneous spaces and l.c.K. geometry, relevant to our arguments on homogeneous and locally homogeneous l.c.K. structures in this paper.

DEFINITION 1. A homogeneous locally conformally Kähler (or shortly *homogeneous l.c.K.*) manifold M is a homogeneous Hermitian manifold with its homogeneous Hermitian structure h, defining a locally conformally Kähler structure Ω on M.

DEFINITION 2. If a simply connected homogeneous l.c.K. manifold $M = G/H$, where G is a connected Lie group and H a closed subgroup of G, admits a free action of a discrete subgroup Γ of G on the left, then we call a double coset space $\Gamma \backslash G/H$ *a locally homogeneous l.c.K.* manifold.

A homogeneous manifold M can be written as G/H, where G is a connected Lie group with closed Lie subgroup H. If we take the universal covering Lie group $\hat{G}$ of G with the projection $p \colon \hat{G} \to G$ and the pull-back $\hat{H} = p^{-1}(H)$ of H, then we have the universal covering $\hat{M} = \hat{G}/H_0$ of M, where H_0 is the connected component of the identity of $\hat{H}$; and $\Gamma = \hat{H}/H_0$ is the fundamental group of M acting on the right. In case G is compact, $\hat{G}$ is of the form $\mathbf{R}^k \times S$ ($k \geq 0$), where S is a simply connected compact semi-simple Lie group. It is also known that G has a finite normal covering $\tilde{G}$ of the form $T^k \times S$ with the projection $\tilde{p} \colon \tilde{G} \to G$; and a compact homogeneous manifold $M = G/H$ can be expressed as $\tilde{G}/\tilde{H} = T^k \times_\Gamma S/\tilde{H}_0$, where $\tilde{H}_0$ is the connected component of the identity of $\tilde{H} = \tilde{p}^{-1}H$ and $\Gamma = \tilde{H}/\tilde{H}_0$ is a finite group acting on $\tilde{M} = T^k \times S/\tilde{H}_0$ on the right.

In case M is a homogeneous l.c.K. manifold, $\hat{M}$ is also a homogeneous l.c.K. manifold; and since the Lee form $\hat{\theta} = p^{-1}\theta$ is exact the fundamental form $\hat{\Omega} = p^{-1}\Omega$ is globally conformal to a Kähler structure ω. The Lie group $\hat{G}$ acts holomorphically and homothetically on $(\hat{M}, \omega)$ on the left; and the fundamental group Γ acts likewise on $(\hat{M}, \omega)$ on the right. Conversely, a Kähler structure ω on $\hat{M} = \hat{G}/H_0$ with holomorphic and homothetic action of $\hat{G}$ on the left and Γ on the right defines a homogeneous l.c.K. structure Ω on $M = G/H$, where $H = H_0 \rtimes \Gamma$ with $\Gamma \cap H_0 = \{0\}$ and $\Gamma \subset N_{\hat{G}}(H_0)$. If Γ is a discrete subgroup of $\hat{G}$ acting properly discontinuously and freely on $\hat{G}/H_0$ on the left, then we can define a locally homogeneous l.c.K. structure on $\Gamma \backslash \hat{G}/H_0$. In particular, for a simply connected Lie group G with a left invariant l.c.K. structure Ω and a discrete subgroup Γ of G, Ω induces a locally homogeneous l.c.K. structure $\tilde{\Omega}$ on $\Gamma \backslash G$.

Let $M = G/H$ be a homogeneous space of a connected Lie group G with closed subgroup H. Then the tangent space of M is given as a G-bundle $G \times_H \mathfrak{g}/\mathfrak{h}$ over $M = G/H$ with fiber $\mathfrak{g}/\mathfrak{h}$, where the action of H on the fiber is given by $\mathrm{Ad}(x)$ ($x \in$

H). A vector field on M is a section of this bundle; and a p-form on M is a section of G-bundle $G \times_H \bigwedge^p (\mathfrak{g}/\mathfrak{h})^*$, where the action of H on the fiber is given by $\mathrm{Ad}(x)^*$ ($x \in H$). An invariant vector field (respectively p-form), the one which is invariant by the left action of G, is canonically identified with an element of $(\mathfrak{g}/\mathfrak{h})^H$ (respectively $\left(\bigwedge^p (\mathfrak{g}/\mathfrak{h})^* \right)^H$), which is the set of elements of $\mathfrak{g}/\mathfrak{h}$ (respectively $\bigwedge^p (\mathfrak{g}/\mathfrak{h})^*$) invariant by the adjoint action of H. A complex structure J on M is likewise considered as an element J of $\mathrm{Aut}(\mathfrak{g}/\mathfrak{h})$ such that $J^2 = -1$ and $\mathrm{Ad}(x)J = J \mathrm{Ad}(x)$ ($x \in H$). Note that we may also consider an invariant p-form as an element of $\bigwedge^p \mathfrak{g}^*$ vanishing on $\mathfrak{h}$ and invariant by the action $\mathrm{Ad}(x)^*$ ($x \in H$).

We recall that $\mathfrak{g}$ is *decomposable* with respect to H if there is a direct sum decomposition of $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{m} + \mathfrak{h},$$

for a subspace $\mathfrak{m}$ of $\mathfrak{g}$ and $\mathrm{Ad}(x)(\mathfrak{m}) \subset \mathfrak{m}$ for any $x \in H$. This is the case, for instance, when H is a reductive Lie group. In case $\mathfrak{g}$ is decomposable, the tangent space of $M = G/H$ is given by the G-bundle $G \times_H \mathfrak{m}$ over $M = G/H$, identifying $\mathfrak{g}/\mathfrak{h}$ with $\mathfrak{m}$. An invariant vector field (respectively p-form) on M is identified with an element of $\mathfrak{m}^H$ (respectively $\left(\bigwedge^p (\mathfrak{m})^* \right)^H$), which is the set of elements of $\mathfrak{m}$ (respectively $\bigwedge^p (\mathfrak{m})^*$) invariant by the adjoint action of H. A complex structure J on M can be considered as an element J of $\mathrm{Aut}(\mathfrak{m})$ such that $J^2 = -1$ on $\mathfrak{m}$ and $\mathrm{Ad}(x)J = J \mathrm{Ad}(x)$ ($x \in H$). It is also convenient to consider a complex structure J on M as an element J of $\mathrm{End}(\mathfrak{g})$ such that $J^2 = -1$ on $\mathfrak{m}$, $J\mathfrak{h} = 0$ and $\mathrm{Ad}(x)J = J \mathrm{Ad}(x)$ ($x \in H$) (cf. [11]).

An invariant vector field $X \in \mathfrak{m}^H$ generates a global 1-parameter group of diffeomorphisms on $M = G/H$ given by the right action of $\exp tX$:

$$\phi \colon \mathbf{R} \times G/H \to G/H, \quad \phi(t, gH) = g(\exp tX)H.$$

Since the closure K of the 1-parameter subgroup of G generated by X is compact, we can use the averaging method to make differential forms ω on M invariant by $\mathrm{Ad}(K)$:

$$\int_K \mathrm{Ad}(x)^* \omega.$$

For an l.c.K. form Ω with its Lee from θ, we can average Ω to make a $\mathrm{Ad}(K)$-invariant l.c.K. form $\bar{\Omega}$ under the condition that the action is compatible with the complex structure J. Note that we have the Lee form $\hat{\theta}$ identical with θ, but since the metric $\bar{h}$ is in general different from h its associated Lee field $\bar{\xi}$ is in general different from ξ.

For a $\mathfrak{g}$-module M, we can define p-cochains as the p-linear alternating functions on $\mathfrak{g}^p$, which are $\mathfrak{g}$-modules defined by

$$(\gamma f)(x_1, x_2, \ldots, x_p) = \gamma f(x_1, x_2, \ldots, x_p)$$
$$- \sum_{i=1}^p f(x_1, \ldots, x_{i-1}, [\gamma, x_i], x_{i+1}, \ldots, x_p),$$

where $\gamma \in \mathfrak{g}$ and f is a p-cochain (cf. [10]). The coboundary operator is defined by

$$
\begin{aligned}
(df)(x_0, x_1, \ldots, x_p) = &\sum_{i=0}^{p} (-1)^i x_i f(x_0, \ldots, \hat{x}_i, \ldots, x_p) \\
&+ \sum_{j<k} (-1)^{j+k} f([x_j, x_k], x_0, \ldots, \hat{x}_j, \ldots, \hat{x}_k, \ldots, x_p).
\end{aligned}
$$

We are interested in the case when a $\mathfrak{g}$-module is defined by the representation of $\mathfrak{g}$ on $\mathbf{R}$, assigning $X \in \mathfrak{g}$ to $-\theta(X)$ for the Lee form θ on an l.c.K. Lie algebra $\mathfrak{g}$. The corresponding coboundary operator is given by

$$
d_\theta : w \rightarrow -\theta \wedge w + dw,
$$

and its cohomology group $H_\theta^p(\mathfrak{g}, \mathbf{R})$ is called the *p-th twisted cohomology group* with respect to the Lee form θ. The condition of l.c.K. structure Ω on $\mathfrak{g}$ is expressed by $d_\theta \Omega = 0$. We know ([10]) that for a reductive Lie algebra $\mathfrak{g}$, all of the cohomology groups $H_\theta^p(\mathfrak{g}, \mathbf{R})$ ($p \geq 0$) vanish; and in particular we have $\Omega = -\theta \wedge \psi + d\psi$ for some 1-form ψ.

2. A holomorphic structure theorem of compact homogeneous l.c.K. manifolds

In this section we prove a structure theorem of compact homogeneous l.c.K. manifolds, which asserts that such a compact complex manifold is biholomorphic to a holomorphic principal bundle over a flag manifold with fiber a 1-dimensional complex torus. This result may be compared with the well-known theorem (due to Matsushima [13]) that a compact homogeneous Kähler manifold is biholomorphic to a Kählerian product of a complex torus and a flag manifold.

Let M be a compact homogeneous l.c.K. manifold of dimension $(2m + 2)$, $m \geq 1$, with its associated fundamental form Ω and Lee form θ, satisfying $d\Omega = \theta \wedge \Omega$. M can be written as G/H, where G is a connected holomorphic isometry group of the Hermitian manifold (M, h) and H a compact subgroup of G which contains no normal Lie subgroups of G. Since G is a closed subgroup of the isometry group of (M, h), it is a compact Lie group; in particular G is *reductive*, that is, the Lie algebra $\mathfrak{g}$ of G can be written as

$$
\mathfrak{g} = \mathfrak{t} + \mathfrak{s}
$$

where $\mathfrak{t}$ is the center of $\mathfrak{g}$ and $\mathfrak{s}$ is a semi-simple Lie algebra. Let $\mathfrak{h}$ be the Lie algebra of H. Then $\mathfrak{g}$ also admits a decomposition:

$$
\mathfrak{g} = \mathfrak{m} + \mathfrak{h}
$$

satisfying $\mathrm{Ad}(x)(\mathfrak{m}) \subset \mathfrak{m}$ ($x \in H$) for a subspace $\mathfrak{m}$ of $\mathfrak{g}$. Note that we have also $\mathfrak{t} \cap \mathfrak{h} = 0$. Since the Lee form θ is invariant, its associated vector field ξ (which is called *Lee*

field) with respect to the metric h is also invariant; and thus ξ may be taken as an element of $\mathfrak{m}$ invariant by $\mathrm{Ad}(x)$ for any $x \in H$.

Any invariant form on M can be considered as an element of $\bigwedge^p \mathfrak{g}^*$ vanishing on $\mathfrak{h}$ and invariant by the action $\mathrm{Ad}(x)^*$ ($x \in H$). In particular, we consider Ω, θ as the elements of $\bigwedge \mathfrak{g}^*$ satisfying these conditions and

$$d\Omega = \theta \wedge \Omega.$$

From now on we assume M is of non-Kähler type; and thus θ is a non-zero, closed but not exact form on $\mathfrak{g}$. Note that since $\mathfrak{s} = [\mathfrak{g}, \mathfrak{g}]$ and θ is a non-zero closed form, $\theta([X,Y]) = -d\theta(X,Y) = 0$ for all $X, Y \in \mathfrak{g}$ and thus θ vanishes on $\mathfrak{s}$. In particular we must have $\dim \mathfrak{t} \geq 1$ and $\theta \in \mathfrak{t}^*$.

The Lee field $\xi \in \mathfrak{m}$ may be expressed as $\xi = t + s$, $t \in \mathfrak{t}$ ($t \neq 0$), $s \in \mathfrak{s}$, where ξ is normalized, satisfying $h(\xi, \xi) = 1$ and thus $\theta(\xi) = \theta(t) = 1$. We define the Reeb field $\eta \in \mathfrak{m}$ as $\eta = J\xi$ with its associated 1-form ϕ satisfying $\phi(\eta) = 1$. We can express $\mathfrak{g}$ as

$$\mathfrak{g} = \langle \xi, \eta \rangle + \mathfrak{k},$$

where $\langle \xi, \eta \rangle$ is the 2-dimensional subspace of $\mathfrak{g}$ generated by ξ and η over $\mathbf{R}$, and $\mathfrak{k} = \ker \theta \cap \ker \phi$ with $\mathfrak{k} \supset \mathfrak{h}$. Note that $h(\xi, \eta) = \Omega(\eta, \eta) = 0$ and $\langle \xi, \eta \rangle$ is orthogonal to $\mathfrak{k}$ with respect to h.

It is known (due to Hochschild and Serre [10]) that there exists a 1-form $\psi \in \mathfrak{g}^*$ such that

$$\Omega = -\theta \wedge \psi + d\psi,$$

where ψ defines an invariant 1-form on M: ψ vanishes on $\mathfrak{h}$ since we have $\psi(\mathfrak{h}) = \Omega(\mathfrak{h}, t) = 0$; and ψ is $\mathrm{Ad}(x)$-invariant for $x \in H$ since we have $\psi([\mathfrak{h}, Y]) = -d\psi(\mathfrak{h}, Y) = -\Omega(\mathfrak{h}, Y) = 0$. We set $\psi_c = \psi - c\theta$ for $c \in \mathbf{R}$. Note that we have $d\psi_c = d\psi$; and

$$\Omega = -\theta \wedge \psi_c + d\psi_c.$$

Lemma 1. *There exists $\sigma \in \mathfrak{g}$ and $c \in \mathbf{R}$ such that*

$$\psi_c(\sigma) = 1, \quad \psi_c(t) = 0, \quad \theta(t) = 1, \quad \theta(\sigma) = 0,$$

and $d\psi_c(\sigma, Y) = 0$ for all $Y \in \mathfrak{g}$.

Proof. As already seen we have $\theta(t) = 1$. Since θ and ψ are linearly independent, we can take an element σ' such that $\psi(\sigma') = 1$ and $\theta(\sigma') = 0$. If $\psi(t) \neq 0$, then take $\psi_c = \psi - c\theta$ for $c = \psi(t)$ satisfying $\psi_c(t) = 0$. Then we have $\psi_c(\sigma') = 1$, $\theta(t) = 1$, $\psi_c(t) = \theta(\sigma') = 0$. Note that since $d\psi_c(t, \sigma') = -\psi_c([t, \sigma']) = 0$, we have $\Omega(\sigma', t) = 1$; in particular $\sigma' \notin \mathfrak{h}$.

Recall that for a skew-symmetric bilinear form Φ on a vector space V,

$$\mathrm{Rad}\,\Phi = \{u \in V \mid \Phi(u, v) = 0 \text{ for any } v \in V\}.$$

Let $\mathfrak{p}' = \langle t, \sigma' \rangle$ and $\mathfrak{q} = \mathrm{Ker}\,\theta \cap \mathrm{Ker}\,\psi_c = \mathrm{Ker}\,\theta \cap \mathrm{Ker}\,\psi$ with $\mathfrak{q} \supset \mathfrak{h}$. Then we have an orthogonal direct sum with respect to Ω:

$$\mathfrak{g} = \mathfrak{p}' + \mathfrak{q}, \quad \mathfrak{p}' \cap \mathfrak{q} = \{0\}.$$

We first note that $\Omega|\mathfrak{q} = d\psi_c$ is non-degenerate on $\mathfrak{q}$ (mod $\mathfrak{h}$). In fact, suppose that there exists a non-zero element $v \in \mathfrak{q}$ such that $d\psi_c(\mathfrak{q}, v) = 0$. Then for $v' = at + v$ with $a = -d\psi_c(\sigma', v)$, we have

$$\Omega(\sigma', v') = -(\theta \wedge \psi_c)(\sigma', v') + d\psi_c(\sigma', v') = a + d\psi_c(\sigma', v) = 0.$$

Since we also have $\Omega(t, v') = 0$ and $\Omega(\mathfrak{q}, v') = 0$, we have $\Omega(\mathfrak{g}, v') = 0$, contradicting the non-degeneracy of Ω on $\mathfrak{g}$ (mod $\mathfrak{h}$).

Let χ be a 1-form defined on $\mathfrak{q}$ by $\chi(X) = d\psi_c(\sigma', X)$. Since $d\psi_c$ is non-degenerate on $\mathfrak{q}$, there exists $\tau \in \mathfrak{q}$ such that $\chi(X) = d\psi_c(\tau, X)$; and thus $d\psi_c(\sigma' - \tau, X) = 0$ for all $X \in \mathfrak{q}$. Let $\sigma = \sigma' - \tau$ and $\mathfrak{p} = \langle t, \sigma \rangle$, then we have an orthogonal direct sum with respect to Ω:

$$\mathfrak{g} = \mathfrak{p} + \mathfrak{q}, \quad \mathfrak{p} \cap \mathfrak{q} = \{0\}.$$

and $\psi_c(\sigma) = 1$, $\theta(\sigma) = 0$ ($\sigma \notin \mathfrak{h}$). Since $d\psi_c(\sigma, t) = -\psi_c([\sigma, t]) = 0$, we have

$$\mathrm{Rad}\,d\psi_c = \mathfrak{p} \quad (\text{mod } \mathfrak{h}).$$

This completes the proof of Lemma 1. $\qquad\square$

From now on we write ψ_c simply as ψ.

Corollary 1. *We have $J\xi = \sigma$ (mod $\mathfrak{h}$); and thus $\eta = \sigma$ (mod $\mathfrak{h}$).*

Proof. By the definition, the Lee field ξ satisfies that $h(\xi, X) = \theta(X)$; and thus $\Omega(J\xi, X) = \theta(X)$. By Lemma 1 we have $\mathfrak{g} = \mathfrak{p} + \mathfrak{q}$ where $\mathfrak{p} = \langle t, \sigma \rangle$ and $\mathfrak{q} = \mathrm{Ker}\,\theta \cap \mathrm{Ker}\,\psi$. Hence we have $\Omega(J\xi, X) = 0$ for all $X \in \mathfrak{q}$, $\Omega(J\xi, t) = 1$ and $\Omega(J\xi, \sigma) = 0$. On the other hand, since we have $\Omega = \psi \wedge \theta + d\psi$, we get $\Omega(\sigma, X) = \psi(\sigma)\theta(X) - \psi(X)\theta(\sigma) + d\psi(\sigma, X) = 0$ for all $X \in \mathfrak{q}$, and $\Omega(\sigma, t) = 1$. Hence we have $J\xi = \sigma$ (mod $\mathfrak{h}$); and thus $\eta = \sigma$ (mod $\mathfrak{h}$), where $\eta = J\xi$ is the Reeb field by definition. $\qquad\square$

Corollary 2. *We have $\mathcal{L}_\sigma \Omega = 0$.*

Proof. We write $\Omega = -\theta \wedge \psi + d\psi$. Since we have $\psi(\sigma) = 1$ and $d\psi(\sigma, X) = 0$ for all $X \in \mathfrak{g}$, we get $\mathcal{L}_\sigma \psi = d\iota_\sigma \psi + \iota_\sigma d\psi = 0$. Since we have $\mathcal{L}_\sigma(\theta \wedge \psi) = (\mathcal{L}_\sigma \theta) \wedge \psi - \theta \wedge \mathcal{L}_\sigma \psi = (\mathcal{L}_\sigma \theta) \wedge \psi$ and $\mathcal{L}_\sigma \theta = d\iota_\sigma \theta + \iota_\sigma d\theta = 0$, we get $\mathcal{L}_\sigma \Omega = 0$. $\square$

Corollary 3. *We have* $1 \leq \dim \mathfrak{t} \leq 2$, $\mathfrak{t} \subset \langle t, \sigma \rangle + \mathfrak{h}$.

Proof. We have seen in Lemma 1 that $d\psi$ is non-degenerate on $\mathfrak{q}$ (mod $\mathfrak{h}$). For any $X \in \mathfrak{t}$ written as $X = at + b\sigma + Z$ ($a, b \in \mathbf{R}$, $Z \in \mathfrak{q}$) and any $Y \in \mathfrak{q}$, we have $d\phi(Z, Y) = \Omega(Z, Y) = \Omega(X, Y) = 0$; and thus $Z \in \mathfrak{h}$. In particular, we have $\mathfrak{t} \cap \mathfrak{q} = \mathfrak{t} \cap \mathfrak{h} = \{0\}$. Since $\dim \mathfrak{q} = n - 2$, we must have $1 \leq \dim \mathfrak{t} \leq 2$. $\square$

Lemma 2. *Suppose that the l.c.K. form* Ω *is* Jt-*invariant. Then* $\mathfrak{p}$ *as in* Lemma 1 *is generated by* $\{t, Jt\}$ *or* $\{\xi, \sigma\}$:

$$\mathfrak{p} = \langle t, \sigma \rangle = \langle t, Jt \rangle = \langle \xi, \sigma \rangle.$$

Proof. Let $\bar{\mathfrak{q}}$ be the orthogonal complement of $\langle t, Jt \rangle$ with respect to Ω. We show first that $\mathfrak{q} = \bar{\mathfrak{q}} = \text{Ker}\,\theta \cap \text{Ker}\,\psi$; and thus $\mathfrak{p} = \langle t, Jt \rangle$. For $X \in \bar{\mathfrak{q}}$, we have

$$d\Omega(X, Jt, t) = \theta(X)\Omega(Jt, t) = \theta(X)h(t, t).$$

On the other hand, we have

$$d\Omega(X, Jt, t) = \Omega([Jt, X], t) + \Omega(X, [Jt, t]) = 0,$$

due to the invariance of Ω by $\text{Ad}(\exp Jt)$. Hence we have $X \in \ker \theta$. For $X \in \bar{\mathfrak{q}}$, we also have $0 = \Omega(X, t) = \psi(X)$; and thus $X \in \ker \psi$. Since $\mathfrak{q} \subset \bar{\mathfrak{q}}$ and $\dim \mathfrak{q} = \dim \bar{\mathfrak{q}}$, we must have $\mathfrak{q} = \bar{\mathfrak{q}}$. Note that since $\mathfrak{p}$ is J-invariant $\mathfrak{q}$ is also the orthogonal complement with respect to h.

We show that $\xi = t + b\sigma$ for $b \in \mathbf{R}$; and thus $\mathfrak{p} = \langle \xi, \sigma \rangle$. We have

$$h(\xi, X) = \theta(X) = \Omega(\sigma, X) = 0$$

for $X \in \mathfrak{q}$; and thus $\xi \in \mathfrak{p}$. If we write $\xi = at + b\sigma$, then $a = \theta(\xi) = 1$. $\square$

Lemma 3. *If* Ω *is* Jt-*invariant, we have* $\Omega = -\theta \wedge \phi + d\phi$, $d\phi \in \bigwedge^2 \mathfrak{k}^*$.

Proof. We have shown that $\mathfrak{p}$ is generated by $\{\xi, \sigma\}$; and $\mathfrak{q}$ is the orthogonal complement of $\mathfrak{p}$ with respect to both Ω and h. Since $d\psi$ is non-degenerate on $\mathfrak{q}$ (mod $\mathfrak{h}$), there exist $X_i, Y_j \in \mathfrak{q}$, $i, j = 1, 2, \ldots, k$ ($k \leq m$) which are linearly independent and $d\psi = \sum \rho_i \wedge \tau_i$, where ρ_i, τ_i are the dual forms corresponding to X_i, Y_i. Since $\sigma \in \text{Rad}\,d\psi$, we have

$$\Omega(X, \sigma) = -(\theta \wedge \psi)(X, \sigma) = -\theta(X)$$

for any $X \in \mathfrak{g}$. Hence we have

$$\Omega(J\sigma, \sigma) = -\theta(J\sigma) = -h(\xi, J\sigma) = -\Omega(\xi, \sigma) = 1.$$

Since $h(\xi, \xi) = \Omega(J\xi, \xi) = 1$, we can see $J\xi = \sigma$. In fact, we can set $J\xi = \sigma + Z$ and $J\sigma = -\xi + Z'$ for $Z \in \langle \xi, X_i, Y_j \rangle$, $Z' \in \langle \sigma, X_i, Y_j \rangle$, $i, j = 1, 2, \ldots, k$; and thus we have $Z' = -JZ$. Then we have

$$\Omega(\xi, J\xi) = \Omega(\sigma + Z, J\sigma + JZ) = \Omega(\sigma, J\sigma) + \Omega(Z, JZ),$$

$$\Omega(\sigma, J\sigma) = \Omega(-\xi + Z', -J\xi + JZ') = \Omega(\xi, J\xi) + \Omega(Z', JZ'),$$

from which we get $h(Z, Z) + h(Z', Z') = 0$; and thus $Z = Z' = 0$. Since $\eta = J\xi$ by definition we must have $\sigma = \eta$; and thus $\mathfrak{q} = \mathfrak{k}$ and $\psi = \phi$. We can also see that $JX_i = Y_i$, $i, j = 1, 2, \ldots, k$. $\qquad \square$

We have seen, under the assumption that Ω is Jt-invariant, that ξ can be written as $\xi = t + b\eta$. We have $\mathfrak{t} = \langle \xi, \eta \rangle \pmod{\mathfrak{h}}$ for the case $\dim \mathfrak{t} = 2$. For the case $\dim \mathfrak{t} = 1$, we have $\mathfrak{g} = \mathfrak{t} + \mathfrak{s}$ with $\mathfrak{s} = \langle \eta \rangle + \mathfrak{k}$, and t is a generator of $\mathfrak{t}$. Note that the complex structure J may be expressed with respect to a basis $\{t, \eta\}$ as $Jt = bt + (1 + b^2)\eta$, $J\eta = -t - b\eta$; and $\theta = t^*$, $\phi = \eta^* - bt^*$ $(t^*, \eta^* \in \mathfrak{g}^*)$.

Lemma 4. *Under the condition that Ω is Jt-invariant, we can reduce the case* $\dim \mathfrak{t} = 2$ *to the case* $\dim \mathfrak{t} = 1$.

Proof.   In case $\dim \mathfrak{t} = 2$ we have by Corollary 3 that $\mathfrak{t} = \langle \xi, \eta \rangle \pmod{\mathfrak{h}}$, $\eta \notin \mathfrak{t}$. It follows that $\mathfrak{s} = [\mathfrak{g}, \mathfrak{g}] = [\mathfrak{k}, \mathfrak{k}]$, and $\mathfrak{g} = \langle \xi, \eta \rangle + \mathfrak{s}$. We will show that $\eta \in \mathfrak{s} \pmod{\mathfrak{h}}$, $\eta \notin \mathfrak{h}$. Since $\theta(\xi) = 1$ and θ vanishes on $\langle \eta \rangle + \mathfrak{s}$, we have $\mathfrak{h} \subset \langle \eta \rangle + \mathfrak{s}$, $\mathfrak{h} \not\subset \mathfrak{s}$. Hence we get $\eta \in \mathfrak{s} \pmod{\mathfrak{h}}$.

Let $\mathfrak{g}'$ be the subalgebra of $\mathfrak{g}$ generated by ξ and $\mathfrak{s}$, and G' the Lie subgroup of G corresponding to $\mathfrak{g}'$. Note that $\mathfrak{g}'$ is a proper subalgebra of $\mathfrak{g}$ and $\mathfrak{g}/\mathfrak{h} \cong \mathfrak{g}'/\mathfrak{h}'$ where $\mathfrak{h}' = \mathfrak{g}' \cap \mathfrak{h}$ is a proper subalgebra of $\mathfrak{h}$. Then since we have $\eta \in \mathfrak{s} \pmod{\mathfrak{h}}$, G' acts on M transitively; and M can be written as G'/H' with its isotropy subgroup $H' = H \cap G'$. It is clear that the center $\mathfrak{t}'$ of $\mathfrak{g}'$ is generated by t, and thus $\dim \mathfrak{t}' = 1$. The canonical injection $G' \hookrightarrow G$ induces a holomorphic isometry from G'/H' to G/H. $\qquad \square$

Since Jt is an invariant vector field compatible with J, satisfying $\mathrm{ad}(Jt)J = J\,\mathrm{ad}(Jt)$, we can apply the averaging method to make an l.c.K. form $\bar{\Omega}$ invariant by $\mathrm{Ad}(\exp Jt)$; in particular, we have

$$\bar{\Omega}([Jt, X], Y) + \bar{\Omega}(X, [Jt, Y]) = 0$$

for all $X, Y \in \mathfrak{g}$, where $\bar{\Omega}$ defines an l.c.K. structure on M compatible with the original complex structure J. By Lemma 4 we can express $M = G'/H'$ with $\mathfrak{g}' = \mathfrak{t}' + \mathfrak{s}$

(dim $\mathfrak{t}' = 1$). Since G' is a subgroup of G, G' preserves the original l.c.K. structure (Ω, J) on M as well as the averaged l.c.K. structure $(\bar{\Omega}, J)$ on M. Therefore, we have the following key observation.

REMARK 1. We may consider a compact homogeneous l.c.K. manifold M up to holomorphic isometry as $M = G/H$ with a homogeneous l.c.K. structure (Ω, J), satisfying $\mathfrak{g} = \mathfrak{t} + \mathfrak{s}$ (dim $\mathfrak{t} = 1$); and up to biholomorphism, as such with a Jt-invariant l.c.K. form $\bar{\Omega}$.

Proposition 1. *A compact homogeneous l.c.K. manifold M admits a holomorphic flow, which is a Lie group homomorphism from $\mathbf{C}^1$ to the holomorphic automorphism group of M.*

Proof. Let Aut(M) be the holomorphic automorphism group of M. Then we know that Aut(M) is a complex Lie group with its associated complex Lie algebra $\mathfrak{a}(M)$ consisting of holomorphic vector fields on M. Let Isom(M) be the (maximal connected) isometry group of M. Then we know that Isom(M) is a compact real Lie group with its associated Lie algebras $\mathfrak{i}(M)$ consisting of all Killing vector fields on M. Note that G can be taken as the intersection of Aut(M) and Isom(M) being a compact subgroup of Isom(M),

Since $\xi \in \langle t, Jt \rangle$ by Lemma 2, the Lee field ξ is an infinitesimal automorphism on M; and thus $\xi - \sqrt{-1}J\xi$ is a holomorphic vector field on M. Hence the homomorphism $\bar{\phi}$ of Lie algebras mapping $\xi - \sqrt{-1}J\xi$ to $\mathfrak{a}(M)$ induces a homomorphism ϕ of complex Lie groups mapping $\mathbf{C}$ to Aut(M). $\qquad\square$

Theorem 1. *A compact homogeneous l.c.K. manifold M is, up to biholomorphism, isomorphic to a holomorphic principal fiber bundle over a flag manifold with fiber a 1-dimensional complex torus $T_{\mathbf{C}}^1$.*

To be more precise, M can be written as a homogeneous space form G/H, where G is a compact connected Lie group of holomorphic automorphisms on M which is of the form

$$G = S^1 \times S,$$

where S is a compact simply connected semi-simple Lie group, including the connected component H_0 of H which is a closed subgroup of S. S/H_0 is a compact simply connected homogeneous Sasaki manifold, which is a principal fiber bundle over a flag manifold S/Q with fiber $S^1 = Q/H_0$ for some parabolic subgroup Q of S including H_0. $M = G/H$ can be expressed as

$$M = S^1 \times_\Gamma S/H_0,$$

where $\Gamma = H/H_0$ is a finite abelian group acting holomorphically on the fiber $T_{\mathbf{C}}^1$ of the fibration $G/H_0 \to G/Q$ on the right.

Proof.   We can assume that $\mathfrak{g} = \mathfrak{t} + \mathfrak{s}$ with dim $\mathfrak{t} = 1$; and $\eta \in \mathfrak{s}$. Let $\mathfrak{q} = \langle \eta \rangle + \mathfrak{h}$, then since $[\eta, \mathfrak{h}] \subset \mathfrak{h}$, $\mathfrak{q}$ is a Lie subalgebra of $\mathfrak{s}$; in fact we have $\mathfrak{q} = \{X \in \mathfrak{s} \mid d\phi(X, \mathfrak{s}) = 0\}$. Let S and Q be the corresponding Lie subgroup of G, then Q is a closed subgroup of S since we have $Q = \{x \in S \mid \mathrm{Ad}(x)^*\phi = \phi\}$, which is clearly a closed subset of S; in particular, H_0 is a normal subgroup of Q with $Q/H_0 = S^1$, and η generates an S^1 action on S. (cf. [4]). We have seen in Lemma 3 that $d\phi$ defines a homogeneous symplectic structure on S/Q compatible with the complex structure J, which is a Kähler structure on S/Q (due to Borel [3]); in particular Q is a parabolic subgroup of S.

We have seen that the abelian Lie subalgebra $\langle \xi, \eta \rangle = \langle t, \eta \rangle$ of $\mathfrak{g}$ generates a 2-dimensional torus $T_{\mathbf{R}}^2$ action on M where t is a generator of the center of $\mathfrak{g}$ generating an S^1 action on M; and $\xi - \sqrt{-1}\eta$ generates a holomorphic 1-dimensional complex torus action on $M = G/H$ on the right. We have $M = S^1 \times_\Gamma S/H_0$, where $S/H_0 \to S/Q$ is a principal S^1-bundle over the flag manifold S/Q; and $\hat{M} = S^1 \times S/H_0 \to S/Q$ is a holomorphic principal fiber bundle over the flag manifold S/Q with fiber $T_{\mathbf{C}}^1$. Since $H \subset Q$ and thus the holomorphic action of $\Gamma = H/H_0$ is trivial on the base space S/Q, it actually acts on the fiber $T_{\mathbf{C}}^1$, inducing a holomorphic principal fiber bundle $M \to S/Q$ with fiber $T_{\mathbf{C}}^1$.   □

Corollary 4.   *There exist no compact homogeneous complex l.c.K. manifolds*; *in particular, no complex parallelizable manifolds admit their compatible l.c.K. structures.*

Proof.   We know that only compact complex Lie groups are complex tori, which can not act transitively on compact l.c.K. manifolds.   □

3.   A metric structure theorem of compact homogeneous l.c.K. manifolds

DEFINITION 3.   An l.c.K. manifold (M, h) is of *Vaisman type* if the Lee field ξ is parallel with respect to the Riemannian connection for h.

For a homogeneous l.c.K. manifold $M = G/H$, the Lee field ξ is parallel with respect to the Riemannian connection for h if and only if

$$h(\nabla_X \xi, Y) = h([X, \xi], Y) - h([\xi, Y], X) + h([Y, X], \xi) = 0$$

for all $X, Y \in \mathfrak{g}$. Since the Lee form is closed: $h([Y, X], \xi) = 0$, this condition is equivalent to

$$h([\xi, X], Y) + h(X, [\xi, Y]) = 0$$

for all $X, Y \in \mathfrak{g}$. And this is exactly the case when the Lee field ξ is Killing field. It should be also noted that ξ is Killing if and only if $\mathcal{L}_\xi \Omega = 0$ and $\mathcal{L}_\xi J = 0$ for the l.c.K. form Ω and its compatible complex structure J.

Let σ be an element of $\mathfrak{g}$ obtained in Lemma 1 for the original l.c.K. form Ω. We have the following key lemma.

Lemma 5. *We have $\mathcal{L}_\sigma J = 0$.*

Proof. We have seen (in Remark 1 and Theorem 1) that $M = G/H$ can be expressed as $M = S^1 \times_\Gamma S/H_0$ with the original l.c.K. form Ω, where $\Gamma = H/H_0$ is a finite abelian group. We have a compact Lie group $S^1 \times N_S(H_0)/H_0$ imbedded in $\tilde{M} = S^1 \times S/H_0$; and an l.c.K. structure $(\hat{\Omega}, \hat{J})$ on $S^1 \times N_S(H_0)/H_0$ can be induced from the l.c.K. structure $(\tilde{\Omega}, \tilde{J})$ on $\tilde{M}$ by restriction, where $N_S(H_0)$ denotes the normalizer of H_0 in S. In fact we can define an l.c.K. form $\hat{\Omega}$ just as the restriction on $\mathfrak{t} + \mathfrak{n}_\mathfrak{s}(\mathfrak{h})$ of the l.c.K. form Ω on $\mathfrak{g}$; and since we have $(X)J = J \operatorname{ad}(X)$ $(X \in \mathfrak{h})$ with $J\mathfrak{h} = 0$, we can also define a complex structure $\hat{J}$ on $\mathfrak{t} + \mathfrak{n}_\mathfrak{s}(\mathfrak{h})/\mathfrak{h}$ as the restriction of J on $\mathfrak{t} + \mathfrak{n}_\mathfrak{s}(\mathfrak{h})$. Note that we have $\mathfrak{t} = \langle t \rangle$ and $\sigma, Jt \in \mathfrak{n}_\mathfrak{s}(\mathfrak{h})$.

For the case $\mathfrak{n}_\mathfrak{s}(\mathfrak{h}) \supsetneq \mathfrak{q}$, since $\mathfrak{t} + \mathfrak{n}_\mathfrak{s}(\mathfrak{h})/\mathfrak{h}$ is a compact l.c.K. Lie algebra it must be $\mathfrak{u}(2) = \mathbf{R} \oplus \mathfrak{su}(2)$ by Theorem 4; in particular $\hat{\Omega}$ is Jt-invariant. Applying Lemma 2 we have $\sigma \in \langle t, Jt \rangle$. Since $\mathcal{L}_{Jt} J = 0$ and $\mathcal{L}_Y J = 0$ for all $Y \in \mathfrak{h}$, we get $\mathcal{L}_\sigma J = 0$.

For the case $\mathfrak{n}_\mathfrak{s}(\mathfrak{h}) = \mathfrak{q}$, since we have $\sigma \in \langle Jt \rangle + \mathfrak{h}$, it follows that $\mathcal{L}_\sigma J = 0$. $\square$

Corollary 5. *We have $[\sigma, Jt] = 0$; in particular $\operatorname{Ad}(\exp Jt)_* \sigma = \sigma$.*

Proof. We have $(\mathcal{L}_\sigma J)t = \mathcal{L}_\sigma(Jt) - J\mathcal{L}_\sigma t = 0$ by Lemma 5. Since $[\sigma, t] = 0$, it follows that $[\sigma, Jt] = 0$. $\square$

Theorem 2. *A compact homogeneous l.c.K. manifold (M, h) is necessarily of Vaisman type; that is, the Lee field ξ is a Killing field with respect to any homogeneous l.c.K. metric h on M.*

Proof. We first consider the l.c.K. form $\bar{\Omega}, \bar{\psi}$ on M averaged by the closure K of the 1-parameter subgroup of G generated by Jt. We have $\bar{\psi}(\sigma) = \int_K \operatorname{Ad}(x)^* \psi(\sigma) = \int_K \psi(\sigma) = 1$ by Lemma 1 and Corollary 5. Here we have normalized the volume of K to 1. We also have $d\bar{\psi}(\sigma, Z) = 0$ for any $Z \in \mathfrak{g}$. Hence we have $\bar{\psi}, \bar{\theta} = \theta, \sigma, t$ satisfying the condition of Lemma 1; and thus by Lemma 2 we have

$$\bar{\mathfrak{p}} = \langle t, \sigma \rangle = \langle t, Jt \rangle.$$

Now we show that $\mathcal{L}_\xi \Omega = 0$, $\mathcal{L}_\xi J = 0$ for the original l.c.K. form Ω. Since $Jt \in \langle t, \sigma \rangle$ as shown above, we have $\mathcal{L}_{Jt} \Omega = 0$ by Corollary 2. As $\xi = -J\sigma \pmod{\mathfrak{h}}$ from Corollary 1 and $\sigma \in \langle t, Jt \rangle$, we must have $\xi \in \langle t, Jt \rangle + \mathfrak{h}$. Thus, $\mathcal{L}_\xi \Omega = 0$ and $\mathcal{L}_\xi J = 0$. Hence ξ is a holomorphic Killing field with respect to h. $\square$

REMARK 2. We may prove Theorem 2 separately for the case $\dim \mathfrak{t} = 2$ without reducing to the case $\dim \mathfrak{t} = 1$. In fact, for the case $\dim \mathfrak{t} = 2$ we have $\mathfrak{t} = \langle t, \sigma \rangle$ (mod $\mathfrak{h}$) by Corollary 3; and thus $\sigma \in \mathfrak{t}$ mod $\mathfrak{h}$. Hence we get $\mathcal{L}_\sigma J = 0$, $[\sigma, Jt] = 0$ without applying Lemma 5 and Corollary 5.

4. Compact homogeneous l.c.K. manifolds of complex dimension 2

We know (due to Vaisman [16], Gauduchon–Ornea [7] and Belgun [2]) that there is a class of Hopf surfaces which admit homogeneous l.c.K. structures. We can show, applying the above theorem, that the only compact homogeneous l.c.K. manifolds of complex dimension 2 are Hopf surfaces of homogeneous type (see Theorem 3). We first determine, recalling a result of Sasaki ([14]), all homogeneous complex structures on $G = S^1 \times SU(2)$, or equivalently all complex structures on the Lie algebra $\mathfrak{g} = \mathfrak{u}(2)$.

Proposition 2. *Let $\mathfrak{g} = \mathfrak{u}(2) = \mathbf{R} \oplus \mathfrak{su}(2)$ be a reductive Lie algebra with basis $\{T, X, Y, Z\}$ of $\mathfrak{g}$, where T is a generator of the center $\mathbf{R}$ of $\mathfrak{g}$, and*

$$X = \frac{1}{2}\begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \quad Y = \frac{1}{2}\begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \quad Z = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

such that non-vanishing bracket multiplications are given by

$$[X, Y] = Z, \quad [Y, Z] = X, \quad [Z, X] = Y.$$

Then $\mathfrak{g}$ admits a family of complex structures J_δ, $\delta = c + \sqrt{-1}d$ defined by

$$J_\delta(T - dX) = cX, \quad J_\delta(cX) = -(T - dX), \quad J_\delta Y = \pm Z, \quad J_\delta Z = \mp Y.$$

Conversely, the above family of complex structures exhaust all homogeneous complex structures on $\mathfrak{g}$.

Proof. Let $\mathfrak{g}_\mathbf{C} = \mathfrak{gl}(2, \mathbf{C}) = \mathbf{C} + \mathfrak{sl}(2, \mathbf{C})$ be the complexification of $\mathfrak{g}$, which has a basis $\mathfrak{b}_\mathbf{C} = \{T, U, V, W\}$, where

$$U = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad V = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad W = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

with the bracket multiplication defined by

$$[U, V] = V, \quad [U, W] = -W, \quad [V, W] = \frac{1}{2}U.$$

Here we have

$$U = \sqrt{-1}X, \quad V = \frac{1}{2}(Z - \sqrt{-1}Y), \quad W = -\frac{1}{2}(Z + \sqrt{-1}Y),$$

and their conjugations given by

$$\bar{T} = T, \quad \bar{U} = -U, \quad \bar{V} = -W, \quad \bar{W} = -V.$$

We know that there is a one to one correspondence between complex structures J and complex subalgebras $\mathfrak{h}$ such that $\mathfrak{g}_{\mathbf{C}} = \mathfrak{h} + \bar{\mathfrak{h}}$ and $\mathfrak{h} \cap \bar{\mathfrak{h}} = \{0\}$. Let $\mathfrak{a}$ be the subalgebra of $\mathfrak{g}_{\mathbf{C}}$ generated by T and $\mathfrak{b}$ the subalgebra of $\mathfrak{g}_{\mathbf{C}}$ generated by U, V, W, then we have

$$\mathfrak{g}_{\mathbf{C}} = \mathfrak{a} \oplus \mathfrak{b}$$

where $\mathfrak{a} = \langle T \rangle_{\mathbf{C}}$, $\mathfrak{b} = \langle U, V, W \rangle_{\mathbf{C}}$. Let π be the projection $\pi \colon \mathfrak{g}_{\mathbf{C}} \to \mathfrak{b}$ and $\mathfrak{c}$ the image of $\mathfrak{h}$ by π, then we have

$$\mathfrak{b} = \mathfrak{c} + \bar{\mathfrak{c}},$$

and dim $\mathfrak{c} \cap \bar{\mathfrak{c}} = 1$. We can set a basis η of $\mathfrak{h}$ as $\eta = \{P + Q, R\}$ ($P \in \mathfrak{a}$, $Q, R \in \mathfrak{b}$) such that $Q \in \mathfrak{c} \cap \bar{\mathfrak{c}}$ and $\gamma = \{Q, R\}$ is a basis of $\mathfrak{c}$:

$$\mathfrak{h} = \langle P + Q, R \rangle_{\mathbf{C}}, \quad \mathfrak{c} = \langle Q, R \rangle_{\mathbf{C}}.$$

Furthermore, we can assume that $Q + \bar{Q} = 0$ so that Q is of the form $aU + bV + \bar{b}W$ ($a \in \mathbf{R}$, $b \in \mathbf{C}$).

We first consider the case where $R = qV + rW$ ($q, r \in \mathbf{C}$). Since we have $[\mathfrak{g}_{\mathbf{C}}, \mathfrak{g}_{\mathbf{C}}] = \mathfrak{b}$, there is some $\alpha \in \mathbf{C}$ such that $[Q, R] = \alpha R$. We see by simple calculation that if $b \neq 0$, then $q = sb$, $r = s\bar{b}$ for some non zero constant $s \in \mathbf{C}$. But then $\bar{R} = -(\bar{s}/s)R$, contradicting to the fact that $\beta = \{Q, R, \bar{R}\}$ consists a basis of $\mathfrak{b}$:

$$\mathfrak{b} = \langle Q, R, \bar{R} \rangle_{\mathbf{C}}.$$

Hence we have $b = 0$, and $q \neq 0$, $r = 0$ with $\alpha = a$ or $q = 0$, $r \neq 0$ with $\alpha = -a$. Therefore we can take, as a basis of $\mathfrak{h}$, $\eta = \{T + \delta U, V\}$ or $\{T + \delta U, W\}$ with $\delta = c + \sqrt{-1}d \in \mathbf{C}$:

$$\mathfrak{h} = \langle T + \delta U, V \rangle_{\mathbf{C}}$$

or

$$\langle T + \delta U, W \rangle_{\mathbf{C}}.$$

It should be noted that the latter defines a conjugate complex structure of the former, which are not equivalent but define biholomorphic complex structures on its associated Lie group G.

In the case where $R = pU + qV + rW$, $p, q, r \in \mathbf{C}$ with $p \neq 0$, we show that there exists an automorphism $\hat{\phi}$ on $\mathfrak{g}_{\mathbf{C}}$ which maps $\mathfrak{h}_0$ to $\mathfrak{h}$, preserving the conjugation, where $\mathfrak{h}_0$ is a subalgebra of $\mathfrak{g}_{\mathbf{C}}$ of the first type with $p = 0$. As in the first case, we

must have $[Q, R] = \eta R$ for some non zero constant $\eta \in \mathbf{C}$. We may assume that $p = 1$. We see, by simple calculation that $b, q, r \neq 0$ and

$$(a - \eta)q = b, \quad (a + \eta)r = \bar{b},$$

from which we get

$$a^2 + |b|^2 = \eta^2 \quad (\eta \in \mathbf{R}),$$

and

$$|q|^2 - |r|^2 = \frac{4a\eta}{|b|^2}.$$

Then an automorphism ϕ on $\mathfrak{b}$ defined by

$$\phi(U) = \frac{1}{\eta} Q, \quad \phi(V) = \frac{|b|}{2\eta} R, \quad \phi(W) = -\frac{|b|}{2\eta} \bar{R},$$

extends to the automorphism $\hat{\phi}$ on $\mathfrak{g}_{\mathbf{C}}$ which satisfies the required condition. $\square$

Proposition 3. *Let $G = S^1 \times SU(2)$ (which is, as is well known, diffeomorphic to $S^1 \times S^3$). Then all homogeneous complex structures on G admit their compatible homogeneous l.c.K. structures, defining a primary Hopf surfaces S_λ which are compact quotient spaces of the form W/Γ_λ, where $W = \mathbf{C}^2 \backslash \{0\}$ and Γ_λ is a cyclic group of holomorphic automorphisms on W generated by a contraction $f \colon (z_1, z_2) \to (\lambda z_1, \lambda z_2)$ with $|\lambda| \neq 0, 1$. Furthermore, all of those l.c.K. structures are of Vaisman type.*

Proof. We consider the following canonical diffeomorphism Φ_δ, which turns out to be biholomorphic for each homogeneous complex structure J_δ on $\mathfrak{g}$ and λ_δ:

$$\Phi_\delta \colon \mathbf{R} \times SU(2) \to W$$

defined by

$$(t, z_1, z_2) \to (\lambda_\delta^t z_1, \lambda_\delta^t z_2),$$

where $SU(2)$ is identified with $S^3 = \{(z_1, z_2) \in \mathbf{C} \mid |z_1|^2 + |z_2|^2 = 1\}$ by the correspondence:

$$\begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix} \leftrightarrow (z_1, z_2),$$

and $\lambda_\delta = e^{c + \sqrt{-1}d}$. Then we see that Φ_δ is a biholomorphic map. It is now clear that Φ_δ induces a biholomorphism between $G = S^1 \times SU(2)$ with homogeneous complex structure J_δ and a primary Hopf surface $S_{\lambda_\delta} = W/\Gamma_{\lambda_\delta}$.

Let $t, x, y, z \in \mathfrak{g}^*$ be the Maurer–Cartan forms corresponding to $T, X, Y, Z \in \mathfrak{g}$ in Proposition 2. Then we have

$$dz = -x \wedge y, \quad dx = -y \wedge z, \quad dy = -z \wedge x,$$

and

$$\Omega = -\theta \wedge \phi + d\phi,$$

where $\theta = t$, $\phi = x/c$, defines an l.c.K. form on $\mathfrak{g}$ for the complex structure J_δ in Proposition 2. Note that we have the Lee field $\xi = T - d\eta/c$, which is irregular for an irrational d/c, while the Reeb field $\eta = cX$, which is always regular. The Lee field ξ is a Killing field, since we have

$$h([\xi, U], V) + h(U, [\xi, V]) = -d(h([X, U], V) + h(U, [X, V])) = 0$$

for all $U, V \in \mathfrak{g}$. Hence $(G; \Omega, J_\delta)$ is of Vaisman type.

A secondary Hopf surface with homogeneous l.c.K. structure can be obtained as a quotient space of a primary Hopf surface S_{λ_δ} by some finite subgroup of G. For instance, $U(2)$ is a quotient Lie group of G by the central subgroup $\mathbf{Z}_2 = \{(1, I), (-1, -I)\}$. In general we have a secondary Hopf surface $G/\mathbf{Z}_m = S^1 \times_{\mathbf{Z}_m} SU(2)$, where $\mathbf{Z}_m$ is a finite cyclic subgroup of G generated by c:

$$c = (\xi, \tau), \quad \tau = \begin{pmatrix} \xi^{-1} & 0 \\ 0 & \xi \end{pmatrix}, \quad \xi^m = 1,$$

with homogeneous l.c.K. structures induced from those on G by the averaging method (cf. [8]). A (primary or secondary) Hopf surface defined as above is called a *Hopf surface of homogeneous type*, which is a holomorphic principal bundle over a 1-dimensional projective space $\mathbf{C}P^1$ with fiber a 1-dimensional complex torus $T^1_{\mathbf{C}}$. $\square$

Theorem 3. *Only compact homogeneous l.c.K. manifolds of complex dimension 2 are Hopf surfaces of homogeneous type (up to biholomorphism).*

Proof. It is sufficient to show that any compact homogeneous l.c.K. manifold M of complex dimension 2 is a Hopf surface of homogeneous type as defined in Proposition 3. As we have seen in Theorem 1, a compact homogeneous l.c.K. manifold M of complex dimension 2 can be expressed as $S^1 \times_\Gamma S$, where S is a compact homogeneous contact manifold of real dimension 3 which admits a Hopf fibration over $\mathbf{C}P^1$ with fiber S^1, and Γ is a finite abelian group acting on the fiber $T^1_{\mathbf{C}}$ of the fibration $M \to \mathbf{C}P^1$. These are exactly Hopf surfaces with homogeneous l.c.K. structures as defined in Proposition 3. Conversely a Hopf surface of homogeneous type admits a homogeneous l.c.K. structure as defined in Proposition 3. $\square$

5. Homogeneous l.c.K. structures on reductive Lie groups

A homogeneous l.c.K. structure on a Lie group G is nothing but a left invariant l.c.K. structure on G. Since G can be expressed as $\hat{G}/\Delta$, where Δ is a finite subgroup of the center of $\hat{G}$, G admits an l.c.K. structure Ω if and only if $\hat{G}$ admits an l.c.K. structure $\hat{\Omega}$, or equivalently the Lie algebra $\mathfrak{g}$ of G admits an l.c.K. structure $\tilde{\Omega}$ in $\bigwedge \mathfrak{g}^*$.

Theorem 4. *Let $\mathfrak{g}$ be a reductive Lie algebra of dimension $2m$; that is, $\mathfrak{g} = \mathfrak{t} + \mathfrak{s}$, where $\mathfrak{t}$ is an abelian and $\mathfrak{s}$ a semi-simple Lie subalgebra of $\mathfrak{g}$ with $\mathfrak{s} = [\mathfrak{g}, \mathfrak{g}]$. Then $\mathfrak{g}$ admits an l.c.K. structure if and only if $\dim \mathfrak{t} = 1$ and $\mathrm{rank}\,\mathfrak{s} = 1$. In particular a compact Lie group admits a homogeneous l.c.K. structure if and only if it is $U(2)$, $S^1 \times SU(2) \cong S^1 \times Sp(1)$, or $S^1 \times SO(3)$; and any homogeneous l.c.K. structure on a compact Lie group is of Vaisman type.*

Proof. Suppose that $\mathfrak{g}$ admits an l.c.K. structure Ω. Since we have $\mathfrak{h} = \{0\}$, $\eta \in \mathfrak{s}$ and thus $\dim \mathfrak{t} = 1$. If we apply the proof of Theorem 1 for the case $\mathfrak{h} = \{0\}$, we see that $\mathfrak{q} = \langle \eta \rangle = \{V \in \mathfrak{s} \mid [\eta, V] = 0\}$; and thus $\mathrm{rank}\,\mathfrak{s} = 1$ (cf. [4]). We know all of the reductive Lie algebras $\mathfrak{g} = \mathfrak{t} + \mathfrak{s}$ with $\dim \mathfrak{t} = 1$ and $\mathrm{rank}\,\mathfrak{s} = 1$: $\mathbf{R} \oplus \mathfrak{sl}(2, \mathbf{R})$ and $\mathfrak{u}(2) = \mathbf{R} \oplus \mathfrak{su}(2) = \mathbf{R} \oplus \mathfrak{so}(3)$. We show that all homogeneous l.c.K. structures on $\mathfrak{u}(2)$ are the ones we obtained in Proposition 3: $\Omega = -\theta \wedge \phi + d\phi$; and they are all of Vaisman type. In fact, any l.c.K. form Ω' is of the form

$$\Omega' = -\theta \wedge \psi + d\psi,$$

where we can set $\theta = t$ and $\psi = ax + by + cz$ $(a, b, c \in \mathbf{R})$; and thus $d\psi = -(ay \wedge z + bz \wedge x + cx \wedge y)$. For the complex structure J_δ in Proposition 2, we denote by A the (4×4)-matrix determined by $h'(U, V) = \Omega'(J_\delta U, V)$ for $U, V = T, X, Y, Z$. By the condition that A is a positive-definite symmetric matrix, we can see by calculation that $b = c = 0$; and thus $A = a I_4$. Hence Ω' is equal to the original Ω up to constant multiplication.   □

EXAMPLE 1. We can also consider $M = S^1 \times S^3$ as a compact homogeneous space $\tilde{G}/H$, where $\tilde{G} = S^1 \times U(2)$ with its Lie algebra $\tilde{\mathfrak{g}} = \mathbf{R} \oplus \mathfrak{u}(2)$ and $H = U(1)$ with its Lie algebra $\mathfrak{h}$. Then, we have a decomposition $\tilde{\mathfrak{g}} = \mathfrak{m} + \mathfrak{h}$ for the subspace $\mathfrak{m}$ of $\tilde{\mathfrak{g}}$ generated by S, T, Y, Z and $\mathfrak{h}$ generated by W, where

$$S = \frac{1}{2} \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & \sqrt{-1} \end{pmatrix}, \quad W = \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{-1} \end{pmatrix}.$$

Since we have $S = X + 2W$, we can take $\mathfrak{m}'$ generated by T, X, Y, Z for $\mathfrak{m}$; and homogeneous l.c.K. structures on $\tilde{G}/H$ are the same as those on G. In other words any homogeneous l.c.K. structures on G can be extended as those on $\tilde{G}/H$.

Furthermore, we can construct locally homogeneous l.c.K. manifolds $\Gamma \backslash \hat{G}/H$ for some discrete subgroups Γ of $\hat{G}$, where $\hat{G} = \mathbf{R} \times U(2)$. For instance, let $\Gamma_{p,q}$ $(p,q \neq 0)$ be a discrete subgroup of $\hat{G}$:

$$\Gamma_{p,q} = \left\{ \left(k, \begin{pmatrix} e^{\sqrt{-1}pk} & 0 \\ 0 & e^{\sqrt{-1}qk} \end{pmatrix} \right) \in \mathbf{R} \times U(2) \;\middle|\; k \in \mathbf{Z} \right\}.$$

Then $\Gamma_{p,q} \backslash \hat{G}/H$ is biholomorphic to a Hopf surface $S_{p,q} = W/\Gamma_{\lambda_1, \lambda_2}$, where $\Gamma_{\lambda_1, \lambda_2}$ is the cyclic group of automorphisms on W generated by

$$\phi : (z_1, z_2) \to (\lambda_1 z_1, \lambda_2 z_2)$$

with $\lambda_1 = e^{r + \sqrt{-1}p}$, $\lambda_2 = e^{r + \sqrt{-1}q}$, $r \neq 0$. In fact, if we take a homogeneous complex structure J_r on $\hat{G}/H$ induced from the diffeomorphism $\Phi_r : \hat{G}/H \to W$ defined by $(t, z_1, z_2) \to (e^{rt}z_1, e^{rt}z_2)$, Φ_r induces a biholomorphism between $\Gamma_{p,q} \backslash \hat{G}/H$ and $S_{p,q}$. Note that in case $p = q$, $S_{p,q}$ is biholomorphic to S_λ with $\lambda = r + \sqrt{-1}q$.

We have an example of a compact locally homogeneous l.c.K. manifold of non-compact reductive Lie group which is not of Vaisman type ([1]).

EXAMPLE 2.   There exists a homogeneous l.c.K. structure on $\mathfrak{g} = \mathbf{R} \oplus \mathfrak{sl}(2, \mathbf{R})$ which is not of Vaisman type. Take a basis $\{X, Y, Z\}$ for $\mathfrak{sl}(2, \mathbf{R})$ with bracket multiplication defined by

$$[X, Y] = -Z, \quad [Z, X] = Y, \quad [Z, Y] = -X,$$

and W as a generator of the center $\mathbf{R}$ of $\mathfrak{g}$, where we set

$$W = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Z = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let w, x, y, z, be the Maurer–Cartan forms corresponding to W, X, Y, Z respectively; then we have

$$dw = 0, \quad dx = z \wedge y, \quad dy = x \wedge z, \quad dz = x \wedge y,$$

and a locally conformally Kähler structure $\Omega = z \wedge w + x \wedge y$ compatible with an integrable homogeneous complex structure J on $\mathfrak{g}$ defined by

$$JY = X, \quad JX = -Y, \quad JW = Z, \quad JZ = -W.$$

This locally conformally Kähler structure Ω is of Vaisman type.

We can generalize Ω to a locally conformally Kähler structure of the form

$$\Omega_\psi = \psi \wedge w + d\psi$$

compatible with the above complex structure J on $\mathfrak{g}$, where $\psi = ax + by + cz$ with $a, b, c \in \mathbf{R}$.

We see that the symmetric bilinear form $h_\psi(U, V) = \Omega_\psi(JU, V)$ is represented, with respect to the basis $\{W, X, Y, Z\}$, by the matrix

$$A = \begin{pmatrix} c & -b & a & 0 \\ -b & c & 0 & a \\ a & 0 & c & b \\ 0 & a & b & c \end{pmatrix},$$

which has the characteristic polynomial $\Phi_A = \{(t - c)^2 - (a^2 + b^2)\}^2$, and has only positive eigenvalues if and only if $c > 0$, $c^2 > a^2 + b^2$. The Lee form is $\theta = w$ and the Lee field is

$$\xi = \frac{1}{D}(cW + bX - aY)$$

with $D = c^2 - a^2 - b^2$. We also have

$$h_\psi(\xi, \xi) = \frac{c}{D}.$$

We see that $h_\psi([\xi, U], V) + h_\psi(U, [\xi, V]) \not\equiv 0$ unless $a = b = 0$. In fact for $U = V = Z$,

$$h_\psi([\xi, Z], Z) + h_\psi(Z, [\xi, Z]) = 2h_\psi([\xi, Z], Z) = -\frac{2}{D}(a^2 + b^2),$$

which is 0 if and only if $a = b = 0$. Conversely for $a = b = 0$, it is easy to check that $h_\psi([\xi, U], V) + h_\psi(U, [\xi, V]) \equiv 0$. Therefore we have shown

For J and Ω_ψ defined above, h_ψ defines a (positive definite) l.c.K. metric if and only if $c > 0$, $c^2 > a^2 + b^2$. It is of Vaisman type if and only if $c > 0$, $a = b = 0$. And it is of non-Vaisman type if and only if $c > 0$, $c^2 > a^2 + b^2 > 0$.

Note that for the complex structure J on $\mathfrak{g}$ and any lattice Γ of $G = \mathbf{R} \times \widetilde{SL}(2)$, we get a complex surface $S = \Gamma \backslash G$ (a properly elliptic surface with $b_1 = 1$); and S admits locally homogeneous l.c.K. structures Ω_ψ of both Vaisman type and non-Vaisman type, according to the above condition.

NOTE. There appeared recently a paper [6] on which the authors give a proof for Theorem 2. However, it should be noted that its preprint version [arXiv:1312.6266] was uploaded shortly after the original preprint version of the current paper [arXiv:1312.2202] was uploaded to Mathematics arXiv in December 2013. While

their paper is focusing only on the proof of Theorem 2 (and thus naturally shorter than the proof of ours), the current paper discusses other related results and technical aspects of the topics such as the holomorphic structure theorem (Theorem 1), the twisted cohomology groups, the averaging methods of l.c.K. forms, l.c.K. structures on reductive Lie algebras before and after Theorem 2. There also appeared a paper [9] for the detail discussion and a complete classification of l.c.K. structures on four-dimensional compact homogeneous and locally homogeneous manifolds.

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