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SEIFERT SURGERY ON KNOTS VIA REIDEMEISTER TORSION AND CASSON–WALKER–LESCOP INVARIANT II

Dedicated to Professor Makoto Sakuma for his 60th birthday

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Abstract

For a knot K with $\Delta_K(t) \doteq t^2 - 3t + 1$ in a homology 3-sphere, let M be the result of $2/q$-surgery on K. We show that an appropriate assumption on the Reidemeister torsion of the universal abelian covering of M implies $q = \pm 1$, if M is a Seifert fibered space.

1. Introduction

The first author [2] studied the Reidemeister torsion of Seifert fibered homology lens spaces, and showed the following:

Theorem 1.1 ([2, Theorem 1.4]). *Let K be a knot in a homology 3-sphere Σ such that the Alexander polynomial of K is $t^2 - 3t + 1$. The only surgeries on K that may produce a Seifert fibered space with base S^2 and with $H_1 \neq \{0\}, \mathbb{Z}$ have coefficients $2/q$ and $3/q$, and produce Seifert fibered space with three singular fibers. Moreover*
(1) *if the coefficient is $2/q$, then the set of multiplicities is $\{2\alpha, 2\beta, 5\}$ where $\gcd(\alpha, \beta) = 1$, and*
(2) *if the coefficient is $3/q$, then the set of multiplicities is $\{3\alpha, 3\beta, 4\}$ where $\gcd(\alpha, \beta) = 1$.*

It is conjectured that Seifert surgeries on non-trivial knots are integral (except some cases). We [4] have studied the $2/q$-Seifert surgery, one of the remaining cases of the above theorem, by applying the Reidemeister torsion and the Casson–Walker–Lescop invariant, and have given sufficient conditions to determine the integrality of $2/q$ ([4, Theorems 2.1, 2.3]).

In this paper, we give another condition for the integrality of $2/q$ (Theorem 2.1). Like as in [4], the condition is also suggested by computations for the figure eight knot ([4, Example 2.2]).

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We note two differences of this paper from [4]; one is that the surgery coefficient appears in the condition instead of the Casson–Walker–Lescop invariant, and another is that we need more delicate estimation for the Dedekind sum to prove the result.

(1) Let Σ be a homology 3-sphere, and let K be a knot in Σ. Then $\Delta_K(t)$ denotes the Alexander polynomial of K, and $\Sigma(K; p/r)$ denotes the result of p/r-surgery on K.

(2) The first author [3] introduced the norm of polynomials and homology lens spaces: Let ζ_d be a primitive d-th root of unity. For an element α of $\mathbb{Q}(\zeta_d)$, $N_d(\alpha)$ denotes the norm of α associated to the algebraic extension $\mathbb{Q}(\zeta_d)$ over $\mathbb{Q}$. Let $f(t)$ be a Laurent polynomial over $\mathbb{Z}$. We define $|f(t)|_d$ by

$$|f(t)|_d = |N_d(f(\zeta_d))| = \left|\prod_{i\in(\mathbb{Z}/d\mathbb{Z})^\times} f(\zeta_d^i)\right|.$$

Let X be a homology lens space with $H_1(X) \cong \mathbb{Z}/p\mathbb{Z}$. Then there exists a knot K in a homology 3-sphere Σ such that $X = \Sigma(K; p/r)$ ([1, Lemma 2.1]). We define $|X|_d$ by

$$|X|_d = |\Delta_K(t)|_d,$$

where d is a divisor of p. Then $|X|_d$ is a topological invariant of X (Refer to [3] for details).

(3) Let X be a closed oriented 3-manifold. Then $\lambda(X)$ denotes the Lescop invariant of X ([5]). Note that $\lambda(S^3) = 0$.

2. Result

Let K be a knot in a homology 3-sphere Σ. Let M be the result of $2/q$-surgery on K: $M = \Sigma(K; 2/q)$. Let $\pi\colon X \to M$ be the universal abelian covering of M (i.e. the covering associated to $\mathrm{Ker}(\pi_1(M) \to H_1(M))$). Since $H_1(M) \cong \mathbb{Z}/2\mathbb{Z}$, π is the 2-fold unbranched covering.

In [4], we have defined $|K|_{(q,d)}$ by the following formula, if $|X|_d$ is defined:

$$|K|_{(q,d)} := |X|_d.$$

Assume that the Alexander polynomial of K is $t^2 - 3t + 1$. Then, as noted in [4], $H_1(X) \cong \mathbb{Z}/5\mathbb{Z}$ and $|K|_{(q,5)}$ is defined.

We then have the following.

Theorem 2.1. *Let K be a knot in a homology 3-sphere Σ. We assume the following.*

(2.1) $$\lambda(\Sigma) = 0,$$

(2.2) $$\Delta_K(t) \doteq t^2 - 3t + 1,$$

$$|q| \geq 3, \tag{2.3}$$

$$\sqrt{|K|_{(q,5)}} > 4q^2. \tag{2.4}$$

Then $M = \Sigma(K; 2/q)$ *is not a Seifert fibered space.*

REMARK 2.2. Let K be the figure eight knot in S^3. Note that $\Delta_K(t) \doteq t^2 - 3t + 1$. Then $|K|_{(q,5)} = (5q^2 - 1)^2$ by [4, Example 2.2]. Hence (2.4) holds if $|q| \geq 3$.

REMARK 2.3. Theorem 2.1 seems to suggest studying the asymptotic behavior of $|K|_{(q,d)}$ as a function of q.

3. An inequality for the Dedekind sum

To prove Theorem 2.1, we need the following inequality for the Dedekind sum $s(\cdot\,,\,\cdot)$ ([7]):

Proposition 3.1 ([6, Lemma 3])**.** *For an even integer* $p \geq 8$ *and for an odd integer* q *such that* $3 \leq q \leq p - 3$ *and* $\gcd(p, q) = 1$, *we have*

$$|s(q, p)| < f(2, p)$$

where $f(2, p) = (p - 1)(p - 5)/(24p)$.

By this proposition, we immediately have the following.

Lemma 3.2. *For an even integer* $p \geq 8$ *and for an integer* q_* *such that* $q_* \not\equiv \pm 1 \pmod{p}$ *and* $\gcd(p, q_*) = 1$, *we have*

$$|s(q_*, p)| < \frac{p}{24}.$$

Proof. By assumptions, there exists q such that $q_* \equiv q \pmod{p}$ and $3 \leq q \leq p - 3$. Hence by Proposition 3.1, we have

$$|s(q_*, p)| = |s(q, p)| < \frac{(p-1)(p-5)}{24p} < \frac{p}{24}. \qquad \square$$

REMARK 3.3. The estimation given in Proposition 3.1 has a natural application ([6]).

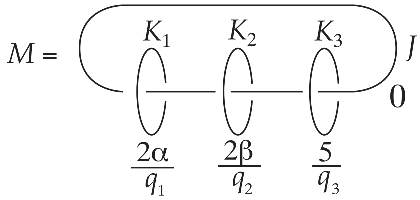

Fig. 1. A framed link presentation of $M = \Sigma(K; 2/q)$.

4. Proof of Theorem 2.1

Suppose that $M = \Sigma(K; 2/q)$ is a Seifert fibered space. Then, as shown in [4], we may assume that

$$(*) \qquad M \text{ has a framed link presentation as in Fig. 1,}$$

where $1 \leq \alpha < \beta$ and $\gcd(\alpha, \beta) = 1$.

Also as shown in [4], $\sqrt{|K|_{(q,5)}} = (\alpha\beta)^2$. Hence by (2.4),

$$(\alpha\beta)^2 > 4q^2 \tag{4.1}$$

By (2.1), (2.2) and [5, 1.5 T2], we have $\lambda(M) = -q$. Hence $(\alpha\beta)^2 > 4\{\lambda(M)\}^2$, and hence

$$|\lambda(M)| < \frac{\alpha\beta}{2}. \tag{4.2}$$

We now consider e defined as follows:

$$e := \frac{q_1}{2\alpha} + \frac{q_2}{2\beta} + \frac{q_3}{5}.$$

According to the sign of e, we treat two cases separately: We first consider the case $e > 0$. Then the order of $H_1(M)$ is $20\alpha\beta e$. Since $H_1(M) \cong \mathbb{Z}/2\mathbb{Z}$, $20\alpha\beta e = 2$, and $e = 1/(10\alpha\beta)$. Hence by $(*)$ and [5, Proposition 6.1.1], we have

$$\lambda(M) = \left(-\frac{4}{5}\right)\alpha\beta + \frac{5\beta}{24\alpha} + \frac{5\alpha}{24\beta} + \frac{1}{120\alpha\beta} - \frac{1}{4} - T \tag{4.3}$$

where $T = s(q_1, 2\alpha) + s(q_2, 2\beta) + s(q_3, 5)$.

By (4.2), we have

$$-\frac{\alpha\beta}{2} < \lambda(M).$$

Hence by (4.3),

$$-\frac{\alpha\beta}{2} < \left(-\frac{4}{5}\right)\alpha\beta + \frac{5\beta}{24\alpha} + \frac{5\alpha}{24\beta} + \frac{1}{120\alpha\beta} - \frac{1}{4} + |T|.$$

Consequently

$$\frac{3}{10}\alpha\beta < -\frac{1}{4} + \frac{5}{24\alpha}\beta + \frac{5}{24}\left(\frac{\alpha}{\beta}\right) + \frac{1}{120\alpha\beta} + |T|. \tag{4.4}$$

As in [4], we show that $\alpha \geq 2$ implies a contradiction: Suppose that $\alpha \geq 2$. Since $\alpha < \beta$, we have $\beta \geq 3$ and $\alpha/\beta < 1$. Hence

$$\frac{3}{5}\beta < -\frac{1}{4} + \frac{5}{24\cdot 2}\beta + \frac{5}{24} + \frac{1}{120\cdot 2\cdot 3} + |T|.$$

Since $|s(q_1, 2\alpha)| \leq 2\alpha/12 < 2\beta/12$, $|s(q_2, 2\beta)| \leq 2\beta/12$, and $|s(q_3, 5)| \leq 1/5$ as in [4], we have

$$|T| \leq |s(q_1, 2\alpha)| + |s(q_2, 2\beta)| + |s(q_3, 5)| \leq \frac{\beta}{3} + \frac{1}{5}.$$

Hence

$$\frac{3}{5}\beta < -\frac{1}{4} + \frac{5}{48}\beta + \frac{5}{24} + \frac{1}{120\cdot 6} + \left(\frac{\beta}{3} + \frac{1}{5}\right).$$

Thus

$$\left(\frac{3}{5} - \frac{5}{48} - \frac{1}{3}\right)\beta < -\frac{1}{4} + \frac{5}{24} + \frac{1}{120\cdot 6} + \frac{1}{5}.$$

Therefore

$$\frac{39}{240}\beta < \frac{1}{240}\left(38 + \frac{1}{3}\right) < \frac{39}{240}.$$

This contradicts $\beta \geq 3$.

We next show that $\alpha = 1$ implies a contradiction: Suppose that $\alpha = 1$. By (4.1), $\beta^2 > 4q^2$. Since $|q| \geq 3$, $\beta^2 > 4\cdot 3^2 = 36$. Hence $\beta > 6$. Since $\alpha = 1$, $e = 1/(10\beta)$. Hence

$$\frac{q_1}{2} + \frac{q_2}{2\beta} + \frac{q_3}{5} = \frac{1}{10\beta}$$

and hence we have the following equation.

$$(5\beta)q_1 + 5q_2 + (2\beta)q_3 = 1. \tag{4.5}$$

Since q_1 and q_2 are odd (see Fig. 1), β must be even. Since $\beta > 6$, we have $\beta \geq 8$. We then have

$$q_2 \not\equiv \pm 1 \pmod{2\beta}. \tag{$\sharp$}$$

In fact, since q_1 is odd, $(5\beta)q_1 \equiv \beta \pmod{2\beta}$. Hence by (4.5),

$$\beta + 5q_2 \equiv 1 \pmod{2\beta}.$$

Now suppose that $q_2 \equiv 1 \pmod{2\beta}$. Then $\beta + 5 \equiv 1 \pmod{2\beta}$. This is impossible since $\beta \geq 8$. Next suppose that $q_2 \equiv -1 \pmod{2\beta}$. Then $\beta - 5 \equiv 1 \pmod{2\beta}$. This is also impossible since $\beta \geq 8$. Thus ($\sharp$) holds.

Substituting $\alpha = 1$ in (4.4),

$$\frac{3}{10}\beta < -\frac{1}{4} + \frac{5}{24}\beta + \frac{5}{24\beta} + \frac{1}{120\beta} + |T|$$

where $T = s(q_2, 2\beta) + s(q_3, 5)$ (since $s(q_1, 2) = 0$). By ($\sharp$) and Lemma 3.2,

$$|s(q_2, 2\beta)| < \frac{2\beta}{24} = \frac{\beta}{12}.$$

Hence

$$|T| \leq |s(q_2, 2\beta)| + |s(q_3, 5)| < \frac{\beta}{12} + \frac{1}{5}.$$

Since $\beta \geq 8$,

$$\frac{3}{10}\beta < -\frac{1}{4} + \frac{5}{24}\beta + \frac{5}{24 \cdot 8} + \frac{1}{120 \cdot 8} + \left(\frac{\beta}{12} + \frac{1}{5}\right).$$

Thus

$$\left(\frac{3}{10} - \frac{5}{24} - \frac{1}{12}\right)\beta < -\frac{1}{4} + \frac{5}{24 \cdot 8} + \frac{1}{120 \cdot 8} + \frac{1}{5}$$

and hence $\beta/120 < 0$. This is a contradiction, and ends the proof in the case $e > 0$.

We finally consider the case $e < 0$. Then $e = -1/(10\alpha\beta)$. By ($*$) and [5, Proposition 6.1.1], we have

$$\lambda(M) = -\left\{\left(-\frac{4}{5}\right)\alpha\beta + \frac{5\beta}{24\alpha} + \frac{5\alpha}{24\beta} + \frac{1}{120\alpha\beta} - \frac{1}{4} + T\right\}.$$

The remaining part of the proof is similar to that in the case $e > 0$.

This completes the proof of Theorem 2.1. $\square$

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