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# BUCHSTABER INVARIANT, MINIMAL NON-SIMPLICES AND RELATED

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**Abstract**

Buchstaber invariant is a numerical characteristic of a simplicial complex (or a polytope), measuring the degree of freeness of the torus action on the corresponding moment-angle complex. Recently an interesting combinatorial theory emerged around this invariant. In this paper we answer two questions, considered as conjectures in [2], [11]. First, Buchstaber invariant of a convex polytope $P$ equals 1 if and only if $P$ is a pyramid. Second, there exist two simplicial complexes with isomorphic bigraded Tor-algebras, which have different Buchstaber invariants. In the proofs of both statements we essentially use the result of N. Erokhovets, relating Buchstaber invariant of simplicial complex $K$ to the distribution of minimal non-simplices of $K$. Gale duality is used in the proof of the first statement. Taylor resolution of a Stanley–Reisner ring is used for the second.

## 1. Introduction

Consider a finite set $[m] = \{1, 2, \dots, m\}$. A collection $K$ of subsets of $[m]$ is called a *simplicial complex* on $[m]$, if it is closed under taking subsets, i.e. $I \in K$, $J \subset I$ imply $J \in K$; and contains the empty set: $\varnothing \in K$. The elements of $K$ are called simplices. The elements of $[m]$ are called the vertices of $K$. If $i \in [m]$ and $\{i\} \notin K$, we call $i$ a ghost vertex of $K$. The dimension of a simplex $I \in K$ is the number $|I| - 1$. The maximal dimension of all simplices of $K$ is called the dimension of $K$ and is denoted $\dim K$.

Let $I$ be a subset of $[m]$, and $A \subset X$ be a pair of topological spaces. Let $(X, A)^I$ denote the subset of $X^m$ defined by $(X, A)^I = Y_1 \times \dots \times Y_m$, where $Y_i = X$ if $i \in I$, and $Y_i = A$ otherwise. Let $K$ be a simplicial complex on the vertex set $[m]$. Certain topological spaces are associated to $K$, called moment-angle complexes.

DEFINITION 1.1 (Moment-angle complex [5, 6]). (1) Let $D^2 \subset \mathbb{C}$ be the unit disk with the boundary circle $S^1$. The *moment-angle complex* of $K$ is the topological space

$$\mathcal{Z}_K = \bigcup_{I \in K} (D^2, S^1)^I \subseteq (D^2)^m.$$

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This subset is preserved by the coordinatewise action of the compact torus $T^m = (S^1)^m$ on $(D^2)^m$, where each component $S^1$ acts on the corresponding $D^2 \subset \mathbb{C}$ by rotations. This determines the action of $T^m$ on $\mathcal{Z}_K$.
(2) Let $D^1 = [-1, 1] \subset \mathbb{R}$ and $S^0 = \partial D^1 = \{-1, 1\}$. The *real moment-angle complex* of $K$ is the topological space

$$\mathbb{R}\mathcal{Z}_K = \bigcup_{I \in K} (D^1, S^0)^I \subseteq (D^1)^m.$$

This subset is preserved by the coordinatewise action of the finite group $\mathbb{Z}_2^m$ on $(D^1)^m$. Here the group $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ acts on $D^1 \subset \mathbb{R}$ by change of sign. This determines the action of $\mathbb{Z}_2^m$ on $\mathbb{R}\mathcal{Z}_K$.

Homotopy types of moment-angle complexes first appeared in the seminal work [8] as an important tool in the study of quasitoric manifolds. The theory of moment-angle complexes was later developed in the works of Buchstaber and Panov ([5], [6], and other); they proposed the name "moment-angle complex" and gave the definition which is used here. Moment-angle complexes have rich topological and geometrical structures, and serve as topological models for some objects in combinatorial commutative algebra. We review some of these facts later in the paper. Besides, moment-angle complexes give rise to interesting and nontrivial combinatorial invariants of simplicial complexes.

It can be easily seen that the action of $T^m$ on $\mathcal{Z}_K$ and $\mathbb{Z}_2^m$ on $\mathbb{R}\mathcal{Z}_K$ are not free if $K$ has at least one nonempty simplex. The main objects of this paper are Buchstaber invariants measuring *the degree of symmetry* of moment-angle complexes.

DEFINITION 1.2 (Buchstaber invariant). (1) The (*ordinary*) *Buchstaber invariant* $s(K)$ of a simplicial complex $K$ is the maximal dimension of toric subgroups $G \subset T^m$ for which the restricted action of $G$ on $\mathcal{Z}_K$ is free.
(2) The *real Buchstaber invariant* $s_{\mathbb{R}}(K)$ is the maximal rank of subgroups $G \subset \mathbb{Z}_2^m$ for which the restricted action of $G$ on $\mathbb{R}\mathcal{Z}_K$ is free.

Several approaches to Buchstaber invariants are developed up to date [18, 19, 10, 12, 15]. We refer to [13] for the comprehensive review of this field.

The definition of Buchstaber invariant can be extended to polytopes by the following construction [2, 3]. Recall, that a facet of a convex polytope $P$ is a face of codimension 1.

DEFINITION 1.3. Let $P \subset \mathbb{R}^n$ be a convex polytope with facets $\mathcal{F}_1, \dots, \mathcal{F}_m$. Consider the simplicial complex $K_P$ on the set $[m]$, such that $I = \{i_1, \dots, i_k\} \in K_P$ if and only if the facets $\mathcal{F}_{i_1}, \dots, \mathcal{F}_{i_k}$ intersect. $K_P$ is called the *nerve-complex* of a polytope $P$.

REMARK 1.4. If $P$ is a simple polytope, then its polar dual $P^*$ is a simplicial polytope, and $K_P$ coincides with its boundary: $K_P = \partial P^*$. In this case, in particular, $K_P$ is a simplicial sphere.

In [3] we showed that for the purposes of toric topology the complex $K_P$ is a nice combinatorial substitute of a polytope $P$, even in the case when $P$ is not simple. This motivated the following definition.

DEFINITION 1.5. Buchstaber invariants of a polytope are the corresponding invariants of its nerve-complex: $s(P) \stackrel{\text{def}}{=} s(K_P)$, $s_{\mathbb{R}}(P) \stackrel{\text{def}}{=} s_{\mathbb{R}}(K_P)$.

In [2] we conjectured that among all polytopes pyramids have the most asymmetric torus actions on moment-angle complexes (i.e. least possible Buchstaber invariants). Recall, that a pyramid is a polytope $P$, which can be represented as a convex hull of some polytope of smaller dimension (the base of $P$) and a point (the apex of $P$). The proof of this statement is the first result of this paper.

**Theorem 1.** *Let $P$ be a convex polytope. The following are equivalent*:
(1) $s(P) = 1$;
(2) $s_{\mathbb{R}}(P) = 1$;
(3) *$P$ is a pyramid.*

In Section 2 we review some known results in the theory of Buchstaber invariants from which follow the equivalence of (1) and (2) and the implication (3) $\Rightarrow$ (1). The nontrivial implication (1) $\Rightarrow$ (3) is proved using Gale diagrams in Section 3.

The second block of questions asks about the relation between Buchstaber invariants and other well-studied invariants. If $\mathcal{A}(\,\cdot\,)$ is an invariant (or a set of invariants) of a simplicial complex, then the general question is:

PROBLEM 1. Does $\mathcal{A}(K) = \mathcal{A}(L)$ imply $s(K) = s(L)$ or $s_{\mathbb{R}}(K) = s_{\mathbb{R}}(L)$?

There are several natural candidates for $\mathcal{A}(\,\cdot\,)$:
- Chromatic number $\gamma(K)$ or its generalizations;
- $f$-vector (or, equivalently, $h$-vector) of $K$;
- Topological characteristics of $K$, e.g. Betti numbers;
- Topological characteristics of the moment-angle complex $\mathcal{Z}_K$.

Classical chromatic number $\gamma(K)$ on itself is too weak invariant for rigidity problem 1 to make sense. On the other hand, Buchstaber invariants can themselves be considered as generalized chromatic invariants (see Section 2). N. Erokhovets [9, 10] proved that Buchstaber invariants are not determined by $f$-vector and chromatic number. He constructed two simple polytopes with equal $f$-vectors and chromatic numbers, but different Buchstaber invariants.

Recall the definition of Stanley–Reisner algebra. Let $\Bbbk$ be a ground field, and $\Bbbk[m] = \Bbbk[v_1, \dots, v_m]$ be the polynomial algebra with the grading $\deg v_i = 2$. The *Stanley–Reisner algebra* (otherwise called the *face ring*) of a simplicial complex $K$ on $m$ vertices is the quotient algebra $\Bbbk[K] = \Bbbk[m]/I_{SR}(K)$, where $I_{SR}(K)$ is the square-free ideal generated by monomials corresponding to non-simplices of $K$:

$$I_{SR}(K) = (v_{i_1} \cdot \dots \cdot v_{i_k} \colon \{i_1, \dots, i_k\} \notin K).$$

The cohomology ring of a moment-angle complex is the subject of intensive study during last fifteen years. It is known [5, 14] that,

$$H^*(\mathcal{Z}_K; \Bbbk) \cong \operatorname{Tor}^{*,*}_{\Bbbk[m]}(\Bbbk[K], \Bbbk) = \bigoplus_{l,j} \operatorname{Tor}^{-l,2j}_{\Bbbk[m]}(\Bbbk[K], \Bbbk), \tag{1.1}$$

the Tor-algebra of a Stanley–Reisner ring $\Bbbk[K]$. The dimensions of graded components

$$\beta^{-l,2j}(K) \stackrel{\text{def}}{=} \dim_{\Bbbk} \operatorname{Tor}^{-l,2j}_{\Bbbk[m]}(\Bbbk[K], \Bbbk). \tag{1.2}$$

are called bigraded Betti numbers of $K$. In general, they may depend on the ground field $\Bbbk$. These invariants represent a lot of information about $K$ [22, 6]. In particular, from bigraded Betti numbers, it is possible to extract: the $h$-vector of $K$; the ordinary Betti numbers of $K$ and the ordinary Betti numbers of $\mathcal{Z}_K$ by the formulas:

$$h_0(K) + h_1(K)t + \dots + h_n(K)t^n = \frac{1}{(1-t)^{m-n}} \sum \beta^{-l,2j}(-1)^l t^j$$

([6, Theorem 7.15]);

$$\dim_{\Bbbk} \tilde{H}^i(K; \Bbbk) = \beta^{-(m-i-1),2m}(K)$$

(part of Hochster's formula [17], [6, Theorem 3.27]);

$$\dim_{\Bbbk} H^i(\mathcal{Z}_K; \Bbbk) = \sum_{-l+2j=i} \beta^{-l,2j}(K) \quad \text{(follows from (1.1))},$$

where $n = \dim K + 1$. Note, that bigraded Betti numbers do not determine the dimension of $K$: e.g. the cone over $K$ has the same bigraded Betti numbers as $K$.

So far, bigraded Betti numbers together with dimension is a very strong set of invariants. Problem 1 makes sense for such choice of $\mathcal{A}(\cdot)$. Still the answer is negative.

**Theorem 2.** *There exist simplicial complexes $K_1$ and $K_2$ such that*
(1) $\beta^{-l,2j}(K_1) = \beta^{-l,2j}(K_2)$ *for all* $l, j$;
(2) $\dim K_1 = \dim K_2$;
(3) $\gamma(K_1) = \gamma(K_2)$;
(4) $s(K_1) \neq s(K_2)$ *and* $s_{\mathbb{R}}(K_1) \neq s_{\mathbb{R}}(K_2)$.

In Section 4 we construct such complexes $K_1$, $K_2$, and prove that Tor-algebras of both $K_1$ and $K_2$ have trivial multiplications. Thus not only the bigraded Betti numbers but also the multiplicative structure of $H^*(\mathcal{Z}_K)$ does not determine Buchstaber invariant in general. The construction of such counterexample relies on the properties of the Taylor resolution of Stanley–Reisner ring.

In the proofs of both Theorems 1 and 2 we use the result of Erokhovets, which describes Buchstaber invariants in terms of the distribution of minimal non-simplices of $K$ in some particular cases. We review his result in the next section.

## 2. Preliminaries

There is a canonical coordinate splitting $T^m = S^1_1 \times \cdots \times S^1_m$ where each $S^1_i$ is a 1-dimensional torus. For each $I \subset [m]$ we can consider a coordinate subtorus $T^I = G_1 \times \cdots \times G_m \subseteq T^m$, where $G_i = S^1_i$ if $i \in I$, and $G_i = \{1\}$ otherwise.

A subgroup $G \subseteq T^m$ acts freely on a moment-angle complex $\mathcal{Z}_K$ if and only if $G$ intersects stabilizers of the action $T^m$ on $\mathcal{Z}_K$ trivially.

**Lemma 2.1.** *Stabilizers of $T^m$ acting on $\mathcal{Z}_K$ are the coordinate subtori $T^I \subseteq T^m$, corresponding to simplices $I \in K$.*

Proof. Let $(a_1, \ldots, a_m) \in (D^2)^m$ be the point with coordinates $a_i = 0$ if $i \in I$, and $a_i = 1$ if $i \notin I$. Then $(a_1, \ldots, a_m) \in (D^2, S^1)^I \subseteq \mathcal{Z}_K$. The action of $T^I$ preserves this point. $\square$

In this section we suppose for simplicity that $K$ does not have ghost vertices. In other words, $i \in [m]$ implies $\{i\} \in K$. Let $G \subset T^m$ be a toric subgroup of rank $s$ acting freely on $\mathcal{Z}_K$. Consider the quotient map $\phi\colon T^m \to T^m/G$, and fix an arbitrary isomorphism $T^m/G \cong T^r$, where $r = m - s$. We get a map $\phi\colon T^m \to T^r$ such that the restriction $\phi|_{T^I}$ to any stabilizer subgroup is injective. For each vertex $i \in [m]$ consider the $i$-th coordinate subgroup $S^1_i \subset T^m$. Since $\{i\} \in K$, the subgroup $\phi(S^1_i) \subset T^r$ is 1-dimensional, therefore $\phi(S^1_i) = (t^{\lambda^1_i}, t^{\lambda^2_i}, \ldots, t^{\lambda^r_i})$, where $t \in T^1$ and $(\lambda^1_i, \lambda^2_i, \ldots, \lambda^r_i) \in \mathbb{Z}^r/\pm$ is a primitive integral vector defined uniquely up to sign. Consider a map: $\Lambda\colon [m] \to \mathbb{Z}^r/\pm$, $\Lambda(i) = (\lambda^1_i, \lambda^2_i, \ldots, \lambda^r_i)$, called *characteristic map* (corresponding to the subgroup $G \subseteq T^m$). Since $\phi|_{T^I}$ is injective for $I \in K$, characteristic map satisfies the condition:

$(*)$ If $I = \{i_1, \ldots, i_k\} \in K$,
then $\Lambda(i_1), \ldots, \Lambda(i_k)$ is a part of some basis of the lattice $\mathbb{Z}^r$.

Vice a versa, any map $\Lambda\colon [m] \to \mathbb{Z}^r/\pm$ satisfying $(*)$ corresponds to some toric subgroup $G \subset T^m$ of rank $s = m - r$ acting freely on $\mathcal{Z}_K$, by reversing the above construction.

The case of real moment-angle complexes is similar. Each subgroup $G \subset \mathbb{Z}_2^m$ of rank $s$ acting freely on $\mathbb{R}\mathcal{Z}_K$ determines a map $\Lambda_{\mathbb{R}}\colon [m] \to \mathbb{Z}_2^r$, $r = m - s$ which satisfies the condition

$$(*_{\mathbb{R}}) \qquad \begin{array}{l} \text{If } I = \{i_1, \dots, i_k\} \in K, \\ \text{then } \Lambda(i_1), \dots, \Lambda(i_k) \text{ are linearly independent in } \mathbb{Z}_2^r. \end{array}$$

These considerations prove the following statement.

**Proposition 2.2** (I. Izmest'ev [19]). *Let $r(K)$ denote the minimal integer $r$ for which there exists a map $\Lambda\colon [m] \to \mathbb{Z}^r/\pm$ satisfying $(*)$. Let $r_{\mathbb{R}}(K)$ denote the minimal integer $r$ for which there exists a map $\Lambda\colon [m] \to \mathbb{Z}_2^r$ satisfying $(*_{\mathbb{R}})$. Then $s(K) = m - r(K)$ and $s_{\mathbb{R}}(K) = m - r_{\mathbb{R}}(K)$.*

For any $r \in \mathbb{N}$ consider a simplicial complex $U_r$ whose vertices are the primitive vectors of $\mathbb{Z}^r/\pm$ and simplices are the unimodular sets of vectors (i.e. $\{\eta_1, \dots, \eta_k\} \in U_r \Leftrightarrow \{\eta_1, \dots, \eta_k\}$ is a part of some basis of the lattice $\mathbb{Z}^r$). Similarly, consider the complexes $\mathbb{R}U_r$ whose vertices are the nonzero binary vectors of length $r$, $\operatorname{Vert}\mathbb{R}U_r = \mathbb{Z}_2^r \setminus \{0\}$, and simplices are linearly independent subsets of vectors. In this notation we can reformulate Proposition 2.2 as follows

**Corollary 2.3.** *Let $m$ be the number of vertices of $K$. Then the number $r(K) = m - s(K)$ coincides with the minimal integer $r$ for which there exists a non-degenerate simplicial map from $K$ to $U_r$. The number $r_{\mathbb{R}}(K) = m - s_{\mathbb{R}}(K)$ is the minimal integer $r$ for which there exists a non-degenerate simplicial map from $K$ to $\mathbb{R}U_r$.*

Thus the numbers $r(K)$ and $r_{\mathbb{R}}(K)$ are the very natural examples of generalized chromatic numbers as defined in [24, Definition 4.11]. By constructing non-degenerate simplicial maps $\Delta^{r-1} \to U_r \to \mathbb{R}U_r$ one can easily prove the estimation

$$m - \gamma(K) \leqslant s(K) \leqslant s_{\mathbb{R}}(K) \leqslant m - \dim K - 1, \tag{2.3}$$

for $K \neq \Delta^{m-1}$. Here $\gamma(K)$ is the chromatic number of $K$, i.e. the minimal number of colors needed to color the vertices of $K$ so that adjacent vertices are of different colors. See [18] and [15] for different explanation of estimation (2.3). Also note, that

$$s(K) \geqslant 1, \tag{2.4}$$

if $K$ has at least one nonempty simplex. This general bound implies the easy part of Theorem 1.

**Lemma 2.4.** *If $P$ is a pyramid, then $s(P) = s_{\mathbb{R}}(P) = 1$.*

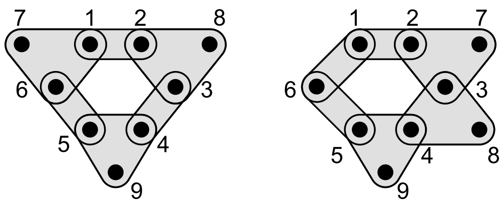


Fig. 1. Collections $\mathcal{C}_1$ and $\mathcal{C}_2$.

Proof. Let $m$ be the number of facets of $P$. Then all its facets except the base intersect in the apex and, consequently, $K_P$ has a simplex with $m-1$ vertices. Thus $\dim K_P = m-2$. Now apply (2.3) and (2.4) to $K_P$. $\square$

N. Erokhovets developed a different approach to Buchstaber invariants in [12, 13]. His description is given in terms of minimal non-simplices of $K$. Recall, that if $K$ is a simplicial complex on the set $[m]$ and $J \subseteq [m]$, then $J$ is called a minimal non-simplex of $K$ if $J \notin K$, but any proper subset of $J$ is a simplex of $K$. The set of all minimal non-simplices of $K$ is denoted $N(K)$.

**Proposition 2.5** (N. Erokhovets [12, 13]). *The following conditions are equivalent*:
(i) $s(K) \geqslant 2$;
(ii) $s_{\mathbb{R}}(K) \geqslant 2$;
(iii) *there exist* $J_1, J_2, J_3 \in N(K)$ *such that* $J_1 \cap J_2 \cap J_3 = \varnothing$. *Sets* $J_i$ *may coincide.*

Thus $s(K) = 1 \Leftrightarrow s_{\mathbb{R}}(K) = 1$ for any simplicial complex, not only the nerve complexes of polytopes.

Erokhovets also proves a criterion, when $s_{\mathbb{R}}(K) \geqslant k$, for any given $k$, in terms of minimal non-simplices, see [12]. We do not need the general statement, but Proposition 2.5 is essential for the proofs of both theorems.

REMARK 2.6. One can see that "minimal non-simplices" in Proposition 2.5 can be replaced by "non-simplices". Indeed, if $J_1', J_2', J_3' \notin K$ satisfy $J_1' \cap J_2' \cap J_3' = \varnothing$, then there exist $J_i \subseteq J_i'$, $J_i \in N(K)$ for $i = 1, 2, 3$, and the same non-intersecting condition holds for $J_i$.

The next example will be used in the proof of Theorem 2.

EXAMPLE 2.7. Let $S_9 \overset{\text{def}}{=} \{1, 2, \ldots, 9\}$. Consider two collections of subsets of $S_9$ shown on Fig. 1. In the first collection there exist $A_1, A_2, A_3 \in \mathcal{C}_1$ such that $A_1 \cup A_2 \cup A_3 = S_9$. As for the second collection, there does not exist $A_1, A_2, A_3 \in \mathcal{C}_2$ such that $A_1 \cup A_2 \cup A_3 = S_9$. Consider simplicial complexes $L_1$ and $L_2$ with $N(L_i) = \{I \subset S_9 \colon S_9 \setminus I \in \mathcal{C}_i\}$ for $i = 1, 2$. The complement of $A_i$ becomes $J_i$ in Proposition 2.5, thus condition (3) of Proposition 2.5 holds for $L_1$, and does not hold for $L_2$. Therefore $s(L_1) > 1$ and $s(L_2) = 1$ (and same for $s_{\mathbb{R}}$).

REMARK 2.8. One can consider collections $\mathcal{C}_1$ and $\mathcal{C}_2$ as simplicial complexes. Then $L_i$ are Alexander duals of $\mathcal{C}_i$ by the definition of combinatorial Alexander duality (see e.g. [6, Example 2.26]).

## 3. Gale diagrams and proof of Theorem 1

We use the properties of Gale diagrams to prove Theorem 1. Let $\mathbb{S}^r$ denote the unit sphere in $\mathbb{R}^{r+1}$ centered at the origin. If $A = (a_1, \dots, a_m)$ is an $m$-tuple of points (in any given space) and $I \subseteq [m]$, then $A(I)$ denotes the sub-array $(a_i : i \in I)$.

Let $Q \subset \mathbb{R}^n$ be a convex polytope, $\dim Q = n$. Let $Y = (y_1, \dots, y_m)$ be the $m$-tuple of all its vertices, $Q = \operatorname{conv} Y$. To each such polytope we can associate its Gale diagram, i.e. an $m$-tuple $X = G(Y) = (x_1, \dots, x_m)$, $x_i \in \mathbb{S}^{m-n-2} \sqcup \{0\}$. The properties of Gale diagrams essential for the proof are listed in the following proposition (see [16, Section 5.4]).

**Proposition 3.1.** *Let $Y$ be the set of vertices of a polytope $Q$ and $X = G(Y)$ be its Gale diagram, $|Y| = |X| = m$.*
(1) *Let $I \subset [m]$. Points $Y(I)$ lie in a common proper face of $Q$ if an only if the points $X([m] \setminus I) \subset \mathbb{S}^{m-n-2} \sqcup \{0\}$ contain the origin in their convex hull.*
(2) *$Q$ is a pyramid if and only if $0 \in X$.*

Let $P$ be a polytope, $\dim P = n$, and $Q = P^*$ be its dual polytope. Facets $\mathcal{F}_{i_1}, \dots, \mathcal{F}_{i_k}$ of $P$ intersect if and only if the corresponding vertices $y_{i_1}, \dots, y_{i_k}$ of $Q$ lie in a common proper face. If we let $X \subset \mathbb{S}^{m-n-2} \sqcup \{0\}$ denote the Gale diagram of $Y = \operatorname{Vert} Q$, as before, then

$$I \in K_P \Leftrightarrow 0 \in \operatorname{conv} X([m] \setminus I).$$

In general, if $A$ is a finite subset of $\mathbb{R}^r$, then the standard separation argument in convex geometry shows that condition $0 \notin \operatorname{conv} A$ is equivalent to the existence of hyperplane $\Pi$ through 0 such that $A$ lies strictly at one side of $\Pi$. This argument proves

**Corollary 3.2.** *Let $J \subseteq [m]$. Then $J \notin K_P$ if and only if there exists a hyperplane $\Pi$ in $\mathbb{R}^{m-n-1}$ such that all points $X([m] \setminus J) \subset \mathbb{S}^{m-n-2} \sqcup \{0\} \subset \mathbb{R}^{m-n-1}$ are located strictly at one side of $\Pi$.*

Now we are ready to prove the rest of Theorem 1. Let $P$ be a polytope (with $m$ facets, $\dim P = n$), and suppose $P$ is not a pyramid. Then its dual $Q = P^*$ is not a pyramid as well. Thus its Gale diagram $X = G(\operatorname{Vert} Q) \subset \mathbb{S}^{m-n-2} \sqcup \{0\}$ does not have points at the origin by Proposition 3.1. Choose a hyperplane $\Pi \subset \mathbb{R}^{m-n-1}$ through the origin such that $\Pi \cap X = \varnothing$. Let $X(J_+)$ (and $X(J_-)$) be the subsets of points of $X$ lying at the right (resp. left) side of $\Pi$. We have $J_+ \cap J_- = \varnothing$ and $J_+ \cup J_- = [m]$. By Corollary 3.2, $J_- = [m] \setminus J_+ \notin K_P$ and $J_+ = [m] \setminus J_- \notin K_P$. Thus $J_+$, $J_-$ are disjoint non-simplices of $K_P$ and Proposition 2.5 shows $s(P) = s(K_P) \geqslant 2$. Theorem is proved.

## 4. Taylor resolutions and proof of Theorem 2

**4.1. Bigraded Betti numbers and Taylor resolution.** First, we review the basics of commutative algebra needed for our goals.

There exists a natural multigrading on the polynomial ring $\Bbbk[m]$ given by $\operatorname{mdeg}(v_1^{n_1} \cdot \ldots \cdot v_m^{n_m}) = (2n_1, \ldots, 2n_m) \in \mathbb{Z}^m$. We denote by $\Bbbk[m]^+$ the maximal graded ideal of $\Bbbk[m]$. The Stanley–Reisner algebra of a simplicial complex $K$ inherits the multigrading. Both $\Bbbk$ and $\Bbbk[K]$ carry the structure of (multi)graded $\Bbbk[m]$-modules via quotient epimorphisms $\Bbbk[m] \to \Bbbk[m]/\Bbbk[m]^+ \cong \Bbbk$ and $\Bbbk[m] \to \Bbbk[K]$. Then $\operatorname{Tor}^{*,*}_{\Bbbk[m]}(\Bbbk[K], \Bbbk)$ is a Tor-functor of (multi)graded modules $\Bbbk[K]$ and $\Bbbk$. Recall its standard construction in homological algebra.

CONSTRUCTION 4.1. To describe $\operatorname{Tor}^{*,*}_{\Bbbk[m]}(\Bbbk[K], \Bbbk)$ do the following:

(1) Take any free resolution of the module $\Bbbk[K]$ by (multi)graded $\Bbbk[m]$-modules:

$$\begin{array}{ccccccccccccc} \cdots & \xrightarrow{d} & R^{-l} & \xrightarrow{d} & R^{-l+1} & \xrightarrow{d} & \cdots & \xrightarrow{d} & R^{-1} & \xrightarrow{d} & R^0 & \xrightarrow{d} & 0 \\ & & & & & & & & & & \downarrow & & \\ & & & & & & & & & & \Bbbk[K] & \longrightarrow & 0 \end{array}$$

(2) apply the functor $\otimes_{\Bbbk[m]} \Bbbk$ to $R^*$;

(3) calculate the cohomology of the resulting complex:

$$\operatorname{Tor}^{*,*}_{\Bbbk[m]}(\Bbbk[K], \Bbbk) \stackrel{\text{def}}{=} H^*(R^* \otimes_{\Bbbk[m]} \Bbbk; d \otimes_{\Bbbk[m]} \Bbbk).$$

The resulting vector space inherits the inner (multi)grading from $R$ and has an additional grading $-l$ called homological. It is well known that $\operatorname{Tor}^{*,*}_{\Bbbk[m]}(\Bbbk[K], \Bbbk) \cong \bigoplus_{(l, \bar{j}) \in \mathbb{Z}^{m+1}} \operatorname{Tor}^{-l, 2\bar{j}}_{\Bbbk[m]}(\Bbbk[K], \Bbbk)$ does not depend on the choice of a free (multi)graded resolution $R^*$. Define the *bigraded Betti numbers* of $K$ as

$$\beta^{-l,2j}(K) \stackrel{\text{def}}{=} \dim_{\Bbbk} \operatorname{Tor}^{-l,2j}_{\Bbbk[m]}(\Bbbk[K], \Bbbk).$$

DEFINITION 4.2 (Minimal resolution). A resolution $R$ is called *minimal* if $\operatorname{im}(d) \subset \Bbbk[m]^+ \cdot R$, or, equivalently, $d \otimes_{\Bbbk[m]} \Bbbk = 0$.

For a minimal resolution $R^*$ step (3) in Construction 4.1 can be skipped. Therefore, if $R$ is minimal, then:

$$\beta^{-l,2j}(K) = \text{the number of generators of the module } R^{-l} \text{ in degree } 2j.$$

Several explicit constructions of free resolutions of $\Bbbk[K]$ are known. In our considerations we use one of the most important and basic constructions: the Taylor resolution. In general, Taylor resolution is defined for any monomial ideal (see [21] or [20]). Here we restrict ourselves to Stanley–Reisner rings, i.e. the case of square-free monomial ideals. The work [23] is also devoted to this particular case and its applications to toric topology.

We use the following convention. A subset $J \subseteq [m]$ determines the vector $\delta_J \in \mathbb{Z}^m$ with $i$-th coordinate equal to 1 if $i \in J$ and 0 otherwise. We simply write $J \in \mathbb{Z}^m$ meaning $\delta_J \in \mathbb{Z}^m$. The monomial $\prod_i (v_i)^{\delta_J^i} \in \Bbbk[m]$ is denoted $v^J$.

CONSTRUCTION 4.3 (Taylor resolution). Consider the set $N(K)$ of minimal non-simplices of $K$. Fix a linear order on $N(K)$. To each $J \in N(K)$ associate a formal variable $w_J$ and construct a free $\Bbbk[m]$-module $R_T^{-l}$, generated by formal expressions $W_\sigma = w_{J_1} \wedge \cdots \wedge w_{J_l}$ for all subsets $\sigma = \{J_1 < \cdots < J_l\} \subseteq N(K)$ of cardinality $l$.

Define the multigrading

$$\operatorname{mdeg}(w_{J_1} \wedge \cdots \wedge w_{J_l}) \stackrel{\text{def}}{=} \left(-l, 2\bigcup_{i=1}^{l} J_i\right) \in \mathbb{Z} \times \mathbb{Z}^m, \tag{4.1}$$

and specialize it to the double grading

$$\operatorname{bideg}(w_{J_1} \wedge \cdots \wedge w_{J_l}) \stackrel{\text{def}}{=} \left(-l, 2\left|\bigcup_{i=1}^{l} J_i\right|\right) \in \mathbb{Z}^2.$$

Define the differential of $\Bbbk[m]$-modules $d_T\colon R_T^{-l} \to R_T^{-l+1}$ by

$$d_T(w_{J_1} \wedge \cdots \wedge w_{J_l}) \stackrel{\text{def}}{=} \sum_{i=1}^{l} (-1)^{i+1} v^{X_{\sigma,J_i}} \cdot w_{J_1} \wedge \cdots \wedge \hat{w}_{J_i} \wedge \cdots \wedge w_{J_l}, \tag{4.2}$$

where $v^{X_{\sigma,J_i}} \in \Bbbk[m]$ is the monomial corresponding to the set

$$X_{\sigma,J_i} \stackrel{\text{def}}{=} J_i \setminus (J_1 \cup \cdots \cup \hat{J}_i \cup \cdots \cup J_l) \subset [m].$$

Define the multiplication on the $\Bbbk[m]$-module $R_T^* = \bigoplus_l R_T^{-l}$. Let $\sigma = \{J_1 < \cdots < J_l\}$, $\tau = \{I_1 < \cdots < I_k\} \subseteq N(K)$.

$$W_\sigma \cdot W_\tau \stackrel{\text{def}}{=} \begin{cases} 0, & \text{if} \quad \sigma \cap \tau \neq \varnothing; \\ \operatorname{sgn}(\sigma, \tau) v^{Y_{\sigma,\tau}} W_{\sigma \sqcup \tau}, & \text{otherwise.} \end{cases} \tag{4.3}$$

Here $v^{Y_{\sigma,\tau}} \in \Bbbk[m]$ is the monomial corresponding to the set

$$Y_{\sigma,\tau} = \left(\bigcup_{i\in\sigma} J_i\right) \cap \left(\bigcup_{i\in\tau} I_i\right).$$

The sign $\operatorname{sgn}(\sigma,\tau)$ is the sign of the permutation needed to sort the ordered set $(J_1,\ldots, J_l, I_1, \ldots, I_k)$.

**Proposition 4.4** ([21], [20]). (1) *$R_T^* = \bigoplus_l R_T^{-l}$ is a differential $\mathbb{Z}^{m+1}$-graded algebra over the ring $\Bbbk[m]$ with respect to multigrading, differential, and multiplication described above. This algebra is skew-commutative with respect to homological grading.*
(2) *$H^{-l}(R_T^*, d) = 0$, if $l > 0$. $H^0(R_T^*, d) \cong \Bbbk[K]$ as $\Bbbk[m]$-algebras.*
*Therefore, $R_T^*$ is a free multiplicative resolution of the Stanley–Reisner algebra $\Bbbk[K]$.*

EXAMPLE 4.5. Let $o_m$ denote the simplicial complex on a set $[m]$ in which all vertices are ghost. We have $\Bbbk[o_m] \cong \Bbbk$ and $N(o_m) = [m]$. The Taylor resolution in this case is given by $R_T^{-l} = \Lambda^l[u_1, \ldots, u_m] \otimes \Bbbk[m]$, where formal variables $u_i$ correspond to elements of $N(o_m) = [m]$ and $\operatorname{bideg} u_i = (-1, 2)$. The general definitions of differential and product imply that $R_T^*$ is isomorphic to $\Lambda[u_1, \ldots, u_m] \otimes \Bbbk[m]$ with the standard Grassmann product, and the differential $du_i = v_i$. In this example we get the multiplicative resolution $\Lambda[u_1, \ldots, u_m] \otimes \Bbbk[m]$ of the $\Bbbk[m]$-module $\Bbbk$. This resolution is widely known as the *Koszul resolution*.

EXAMPLE 4.6. Let $K$ be the boundary of a square. Its maximal simplices are $\{1, 2\}$, $\{2, 3\}$, $\{3, 4\}$, $\{1, 4\}$. In this case $N(K) = \{\{1, 3\}, \{2, 4\}\}$. The Taylor resolution has the form

$$\begin{array}{ccccccc}
\Lambda^{(2)}[w_{\{1,3\}}, w_{\{2,4\}}] \otimes \Bbbk[4] & \xrightarrow{d_2} & \Lambda^{(1)}[w_{\{1,3\}}, w_{\{2,4\}}] \otimes \Bbbk[4] & \xrightarrow{d_1} & \Bbbk[4]\cdot 1 & \twoheadrightarrow & \Bbbk[K] \\
\| & & \| & & & & \\
W_{\{\{1,3\},\{2,4\}\}} \cdot \Bbbk[4] & & w_{\{1,3\}} \cdot \Bbbk[4] \oplus w_{\{2,4\}} \cdot \Bbbk[4] & & & &
\end{array}$$

with the multigrading

$$\begin{aligned}
\operatorname{mdeg}(w_{\{1,3\}}) &= (-1; (2, 0, 2, 0)),\\
\operatorname{mdeg}(w_{\{2,4\}}) &= (-1; (0, 2, 0, 2)),\\
\operatorname{mdeg}(W_{\{\{1,3\},\{2,4\}\}}) &= (-2; (2, 2, 2, 2));
\end{aligned}$$

the differentials

$$\begin{aligned}
d_1(w_{\{1,3\}}) &= v_1v_3 \cdot 1,\\
d_1(w_{\{2,4\}}) &= v_2v_4 \cdot 1,\\
d_2(W_{\{\{1,3\},\{2,4\}\}}) &= v_1v_3 \cdot w_{\{2,4\}} - v_2v_4 \cdot w_{\{1,3\}};
\end{aligned}$$

and the product $w_{\{1,3\}} \times w_{\{2,4\}} = -w_{\{2,4\}} \times w_{\{1,3\}} = W_{\{\{1,3\},\{2,4\}\}}$. Clearly, $\mathrm{im}(d_2) = \ker(d_1)$ and $\mathrm{im}(d_1) = I_{SR}(K)$.

EXAMPLE 4.7. Let $\Delta_M$ denote the simplex on a set $M \neq \varnothing$. Consider $K = \partial\Delta_{M_1} * \cdots * \partial\Delta_{M_k}$. Complex $K$ is a simplicial sphere on the set $M_1 \sqcup \cdots \sqcup M_n$. Then $N(K) = \{M_1, \ldots, M_n\}$. The Taylor resolution of $K$ is a differential algebra

$$\Lambda^*[w_1, \ldots, w_n] \otimes \Bbbk[M_1 \sqcup \cdots \sqcup M_n]$$

with the standard Grassmann product, $\mathrm{bideg}(w_i) = (-1, 2|M_i|)$, and the differential:

$$d_T(w_{i_1} \wedge \cdots \wedge w_{i_l}) = \sum_{k=1}^{l} (-1)^{k+1} v^{M_{i_k}} w_{i_1} \wedge \cdots \wedge \hat{w}_{i_k} \wedge \cdots \wedge w_{i_l}.$$

The Taylor resolution is minimal, therefore $\mathrm{Tor}^{*,*}_{\Bbbk[M_1\sqcup\cdots\sqcup M_n]}(\Bbbk[K];\Bbbk) \cong \Lambda^*[w_1, \ldots, w_n]$. Both previous examples are particular cases of this one.

### 4.2. Multiplication in Tor.

CONSTRUCTION 4.8. There is a standard way to understand the structure of $\mathrm{Tor}^{*,*}_{\Bbbk[m]}(\Bbbk[K];\Bbbk)$ using Koszul resolution. At first, note that $\mathrm{Tor}^{*,*}_{\Bbbk[m]}(\Bbbk[K];\Bbbk) \cong \mathrm{Tor}^{*,*}_{\Bbbk[m]}(\Bbbk;\Bbbk[K])$. By construction,

$$\mathrm{Tor}^{*,*}_{\Bbbk[m]}(\Bbbk;\Bbbk[K]) \cong H^*(R^* \otimes_{\Bbbk[m]} \Bbbk[K]; d \otimes_{\Bbbk[m]} \Bbbk[K]),$$

where $(R^*, d)$ is any graded free resolution of $\Bbbk$ as a $\Bbbk[m]$-module. By taking Koszul resolution $R^{-l} \cong \Lambda[u_1, \ldots, u_m] \otimes \Bbbk[m]$ with grading and differential as described in Example 4.5 we get

$$\mathrm{Tor}^{*,*}_{\Bbbk[m]}(\Bbbk;\Bbbk[K]) \cong H^*(\Lambda[u_1, \ldots, u_m] \otimes \Bbbk[K]; d \otimes_{\Bbbk[m]} \Bbbk[K]). \tag{4.4}$$

The differential complex $\Lambda[u_1,\ldots,u_m] \otimes \Bbbk[K]$ has the structure of a graded differential algebra. Thus $\mathrm{Tor}^{*,*}_{\Bbbk[m]}(\Bbbk;\Bbbk[K])$ has the structure of an algebra as well. The word “Tor-algebra” usually refers to this definition of multiplication.

**Proposition 4.9** ([5, 14])**.** *The cohomology ring $H^*(\mathcal{Z}_K;\Bbbk)$ is isomorphic, as a graded algebra, to the* Tor-*algebra* $\mathrm{Tor}^{*,*}_{\Bbbk[m]}(\Bbbk[K];\Bbbk)$ *with the total grading* $(-i, 2j) \rightsquigarrow 2j - i$.

REMARK 4.10. According to Construction 4.1,

$$\mathrm{Tor}^{*,*}_{\Bbbk[m]}(\Bbbk[K];\Bbbk) \cong H^*(R_T^* \otimes_{\Bbbk[m]} \Bbbk; d_T \otimes_{\Bbbk[m]} \Bbbk), \tag{4.5}$$

where $(R_T^*, d_T)$ is the Taylor resolution of $\Bbbk[K]$. The differential complex $R_T^* \otimes_{\Bbbk[m]} \Bbbk$ obtains the multiplication induced by the multiplication in the Taylor resolution. This, in turn, induces the multiplication on $H^*(R_T^* \otimes_{\Bbbk[m]} \Bbbk; d_T \otimes_{\Bbbk[m]} \Bbbk)$. The question arises: is this multiplication on $\mathrm{Tor}^{*,*}_{\Bbbk[m]}(\Bbbk[K]; \Bbbk)$ the same as the one given by Construction 4.8 or not? Fortunately, this multiplicative structures are indeed the same (see e.g. [1, Construction 2.3.2]). So far the cohomological product in $H^*(\mathcal{Z}_K; \Bbbk)$ can be described in terms of the Taylor resolution (see [23] for examples of such calculations).

### 4.3. Taylor resolutions and minimality.

**Lemma 4.11.** *Let $K$ be a simplicial complex on $[m]$ and $N(K)$ be the set of its minimal non-simplices. The following two conditions are equivalent*:

(1) *The Taylor resolution $(R_T^*, d_T)$ of $\Bbbk[K]$ is minimal.*

(2) *Any minimal non-simplex $J \in N(K)$ is not a subset of the union of others*:

$$J \not\subseteq \bigcup_{I \in N(K), I \neq J} I. \tag{4.6}$$

Proof. By definition, $R_T^*$ is minimal if $d_T(R_T^{-l}) \subseteq \Bbbk[m]^+ \cdot R_T^{-l+1}$ for each $l > 0$. From (4.2) follows that $d_T(R_T^{-l}) \subseteq \Bbbk[m]^+ \cdot R_T^{-l+1}$ if and only if $v^{X_{\sigma,J}} \in \Bbbk[m]^+$ for each $\sigma \subseteq N(K)$ and $J \in \sigma$. This is equivalent to $X_{\sigma,J} \neq \varnothing$. By definition, $X_{\sigma,J} = J \setminus \big(\bigcup_{I \in \sigma, I \neq J} I\big)$. If the Taylor resolution is minimal, then, in particular, $X_{N(K),J} \neq \varnothing$, which is precisely the condition (4.6) of the lemma. On the other hand, $X_{N(K),J} \neq \varnothing$ implies $X_{\sigma,J} \neq \varnothing$ for any $\sigma \subseteq N(K)$. $\square$

**Lemma 4.12.** *If the Taylor resolution of $\Bbbk[K]$ is minimal, then* $\mathrm{Tor}^{*,*}_{\Bbbk[m]}(\Bbbk[K], \Bbbk)$ *has the following description*:

- *It is generated as a vector space over $\Bbbk$ by $W_\sigma$ for $\sigma \subseteq N(K)$*;
- *The multidegree is given by* (4.1);
- *The multiplication is given by*

$$W_\sigma \cdot W_\tau = \begin{cases} \mathrm{sgn}(\sigma, \tau) W_{\sigma \sqcup \tau}, & \textit{if} \quad \sigma \cap \tau = \varnothing \quad \textit{and} \quad \big(\bigcup_{J \in \sigma} J\big) \cap \big(\bigcup_{I \in \tau} I\big) = \varnothing, \\ 0, & \textit{otherwise.} \end{cases} \tag{4.7}$$

The proof follows easily from the definitions. Bigraded Betti numbers of complexes with the minimal Taylor resolution are expressed in combinatorial terms:

$$\beta^{-l,2j}(K) = \#\left\{\sigma \subseteq N(K) \colon |\sigma| = l, \ \left|\bigcup_{J \in \sigma} J\right| = j\right\}. \tag{4.8}$$

**4.4. Proof of Theorem 2.** As a starting point take the complexes $L_1$ and $L_2$ defined in Example 2.7. The outline of the proof is the following:
(1) To upgrade $L_1$ and $L_2$ to the new complexes $K_1$ and $K_2$ satisfying condition (4.6) (Taylor resolutions are minimal);
(2) To prove that $\beta^{-l,2j}(K_1) = \beta^{-l,2j}(K_2)$ using formula (4.8);
(3) To prove that $s(K_1) = 1$ and $s(K_2) \geqslant 2$.
(4) Final technical remarks: $\dim(K_1) = \dim(K_2)$, $\gamma(K_1) = \gamma(K_2)$, and algebra isomorphism $\mathrm{Tor}_{\Bbbk[m]}(\Bbbk[K_1], \Bbbk) \cong \mathrm{Tor}_{\Bbbk[m]}(\Bbbk[K_2], \Bbbk)$.

STEP 1. Let $L$ be any complex on a set $[m]$ with the set of minimal non-simplices $N(L)$. For each $J \in N(L)$ consider a symbol $a_J$. Define the complex $\tilde{L}$ on the set $V = [m] \sqcup \{a_J : J \in N(L)\}$ with the set of minimal non-simplices given by

$$N(\tilde{L}) = \{\tilde{J} = J \sqcup \{a_J\} \subset V : J \in N(L)\} \tag{4.9}$$

The Taylor resolution of the complex $\tilde{L}$ is minimal. Indeed, any $\tilde{J} \in N(\tilde{L})$ contains the vertex $a_J$ which does not belong to other minimal non-simplices of $\tilde{L}$ by construction. Therefore, condition (4.6) holds for $\tilde{L}$.

Now we apply this construction to simplicial complexes $L_1$ and $L_2$ constructed in Example 2.7. Recall that $N(L_i) = \{I \subset S_9 : S_9 \setminus I \in \mathcal{C}_i\}$, for $i = 1, 2$, with collections $\mathcal{C}_1$, $\mathcal{C}_2$ shown on Fig. 1. Set $K_i = \tilde{L}_i$ for $i = 1, 2$. Both $K_1$ and $K_2$ have $9 + 6 = 15$ vertices.

STEP 2. Apply (4.8) to $K_i$:

$$\begin{aligned}\beta^{-l,2j}(K_i) &= \#\left\{\sigma \subseteq N(K_i) \colon |\sigma| = l,\ \left|\bigcup_{\tilde{J}\in\sigma} \tilde{J}\right| = j\right\} \\ &= \#\left\{\sigma \subseteq N(L_i) \colon |\sigma| = l,\ \left|\bigcup_{J\in\sigma} \tilde{J}\right| = j\right\}.\end{aligned} \tag{4.10}$$

The last equality is the consequence of the bijective correspondence between $N(L_i)$ and $N(K_i)$, sending $J \in N(L_i)$ to $\tilde{J} \in N(K_i)$. We have

$$\bigcup_{J\in\sigma} \tilde{J} = \bigcup_{J\in\sigma}(J \sqcup \{a_J\}) = \left(\bigcup_{J\in\sigma} J\right) \sqcup \{a_J : J \in \sigma\},$$

therefore

$$\left|\bigcup_{J\in\sigma} \tilde{J}\right| = \left|\bigcup_{J\in\sigma} J\right| + |\sigma|.$$

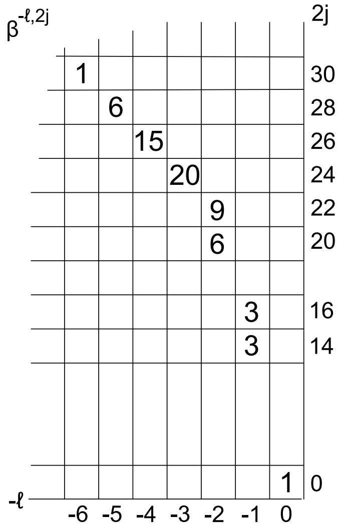


Fig. 2. Bigraded Betti numbers of $K_1$ and $K_2$.

Returning to (4.10),

$$\begin{aligned}\beta^{-l,2j}(K_i) &= \#\left\{\sigma \subseteq N(L_i)\colon |\sigma| = l,\ \left|\bigcup_{J\in\sigma} J\right| = j-l\right\}\\ &= \#\left\{\sigma \subseteq \mathcal{C}_i\colon |\sigma| = l,\ \left|\bigcap_{A\in\sigma} A\right| = 9-(j-l)\right\}.\end{aligned} \tag{4.11}$$

The last equality follows from the definition of $L_i$, since $N(L_i)$ consists of complements to subsets of the collection $\mathcal{C}_i$. By analyzing Fig. 1 we see that for each $l$ and $j$

$$\begin{aligned}&\#\left\{\sigma \subseteq \mathcal{C}_1\colon |\sigma| = l,\ \left|\bigcap_{A\in\sigma} A\right| = 9-(j-l)\right\}\\ &= \#\left\{\sigma \subseteq \mathcal{C}_2\colon |\sigma| = l,\ \left|\bigcap_{A\in\sigma} A\right| = 9-(j-l)\right\}.\end{aligned}$$

Indeed, in both $\mathcal{C}_1$ and $\mathcal{C}_2$ there are 3 subsets of cardinality 2; 3 subsets of cardinality 3; 6 pairwise intersections of cardinality 1; and all other intersections are empty. Therefore, $\beta^{-l,2j}(K_1) = \beta^{-l,2j}(K_2)$. The nonzero bigraded Betti numbers calculated by this method are presented in Fig. 2 (empty cells represent zeroes).

STEP 3. Condition (3) of Proposition 2.5 holds for the complex $L$ whenever it holds for $\tilde{L}$. Indeed, $\tilde{J}_1 \cap \tilde{J}_2 \cap \tilde{J}_3 = (J_1 \sqcup \{a_{J_1}\}) \cap (J_2 \sqcup \{a_{J_2}\}) \cap (J_3 \sqcup \{a_{J_1}\}) = J_1 \cap J_2 \cap J_3$. As observed in Example 2.7 condition (3) holds for $L_1$ and does not hold for $L_2$. Therefore it also holds for $K_1 = \tilde{L}_1$ and does not hold for $K_2 = \tilde{L}_2$. Thus $s(K_1) \neq s(K_2)$ and $s_{\mathbb{R}}(K_1) \neq s_{\mathbb{R}}(K_2)$.

STEP 4. Final remarks.

REMARK 4.13. Let us prove that $\dim K_1 = \dim K_2 = 12$. Consider the complement to the set $\{1, 4\}$ in the set of vertices of $K_1$ (see Fig. 1):

$$S = \{1, 2, \dots, 9, a_1, \dots, a_6\} \setminus \{1, 4\}.$$

Suppose that $S \notin K_1$. Then there exists $\tilde{J} \in N(K_1)$ such that $\tilde{J} \subseteq S$. Therefore, $\{1,4\} \subseteq S_9 \setminus \tilde{J}$. By construction, $S_9 \setminus \tilde{J} \in \mathcal{C}_1$. But $\{1, 4\}$ is not a subset of any $A \in \mathcal{C}_1$, the contradiction. Thus $S \in K_1$ and $\dim K_1 \geqslant |S| - 1 = 12$. Similar reasoning shows that there is no simplex with 14 vertices in $K_1$ (because any singleton lies in some $A \in \mathcal{C}_1$). Therefore, $\dim K_1$ is exactly 12. Similar for $K_2$.

REMARK 4.14. In both complexes $K_1$ and $K_2$ there are no minimal non-simplices of cardinality 1 and 2. Therefore all pairs of vertices in $K_1$ and $K_2$ are connected by edges, so 1-skeletons $K_1^{(1)}$, $K_2^{(1)}$ are complete graphs on 15 vertices. Thus chromatic numbers coincide: $\gamma(K_1) = \gamma(K_2) = 15$.

REMARK 4.15. Tor-algebras of $K_1$ and $K_2$ are isomorphic as algebras. Actually, the products in $\mathrm{Tor}_{\Bbbk[15]}(\Bbbk[K_1], \Bbbk)$ and $\mathrm{Tor}_{\Bbbk[15]}(\Bbbk[K_2], \Bbbk)$ are trivial by dimensional reasons. See Fig. 2: products of nonzero elements hit zero cells.

### 4.5. Other invariants defined from $\mathcal{Z}_K$.

REMARK 4.16. Problem 1 is answered in the negative if $A(\cdot)$ is a bigraded Tor-algebra. We may ask the same question when $A(\,\cdot\,)$ is the collection of multigraded Betti numbers $\beta^{-i,2\bar{j}}(K) \stackrel{\text{def}}{=} \dim \mathrm{Tor}_{\Bbbk[m]}^{-i,2\bar{j}}(\Bbbk[K], \Bbbk)$.

Eventually, this question does not make sense. Multigraded Betti numbers are too strong invariants: $\beta^{-1,2\bar{j}}(K) = \beta^{-1,2\bar{j}}(L)$ implies $K = L$. Indeed, for a subset $J \subseteq [m]$ the condition $\beta^{-1,2J}(K) \neq 0$ is equivalent to $J \in N(K)$ by the construction of the Taylor resolution (also by Hochster's formula [7, Theorem 3.2.9]). Therefore multigraded Betti numbers encode all minimal non-simplices thus determine the complex $K$ uniquely.

REMARK 4.17. Problem 1 may be formulated for an equivariant cohomology ring of $\mathcal{Z}_K$. This task is not interesting as well. Indeed, $H^*_{T^m}(\mathcal{Z}_K; \Bbbk) \cong \Bbbk[K]$ (see [8] or [5]). It is known, that the Stanley–Reisner algebra $\Bbbk[K]$ determines the combinatorics of $K$ uniquely [4]. Therefore multiplicative isomorphism $H^*_{T^m}(\mathcal{Z}_{K_1}; \Bbbk) \cong H^*_{T^m}(\mathcal{Z}_{K_2}; \Bbbk)$ implies $K_1 \cong K_2$ and, in particular, $s(K_1) = s(K_2)$.

## 5. Conclusion and open problem

Constructions of Buchstaber invariants and bigraded Betti numbers are defined for any simplicial complex. Nevertheless, in toric topology the most important ones are simplicial complexes arising from simple polytopes.

If $P$ is a simple polytope with $m$ facets, then the complex $K_P = \partial P^*$ is a simplicial sphere with $m$ vertices. It is known [5, 6] that $\mathcal{Z}_{K_P}$ is a compact orientable manifold in this case. The algebraic version of this fact is Avramov–Golod theorem [7, Theorem 3.4.4]. It states the following. The Tor-algebra $\mathrm{Tor}^{*,*}_{\Bbbk[m]}(\Bbbk[K];\Bbbk)$ is a (multigraded) Poincare duality algebra if and only if the complex $K$ is Gorenstein*. Any simplicial sphere $K$ is Gorenstein* [22, Theorem 5.1]. In particular, for any simple polytope $P$ the complex $K_P$ is Gorenstein*, thus $\mathrm{Tor}^{*,*}_{\Bbbk[m]}(\Bbbk[K_P];\Bbbk)$ is a Poincare duality algebra. This is not surprising since $\mathrm{Tor}^{*,*}_{\Bbbk[m]}(\Bbbk[K_P];\Bbbk) \cong H^*(\mathcal{Z}_{K_P};\Bbbk)$ and $\mathcal{Z}_{K_P}$ is an orientable manifold.

PROBLEM 2. Does an isomorphism of algebras $\mathrm{Tor}^{*,*}_{\Bbbk[m]}(\Bbbk[K_P];\Bbbk) \cong \mathrm{Tor}^{*,*}_{\Bbbk[m]}(\Bbbk[K_Q];\Bbbk)$ imply $s(K_P) = s(K_Q)$ or $s_{\mathbb{R}}(K_P) = s_{\mathbb{R}}(K_Q)$ for simple polytopes $P$ and $Q$?

The complexes $K_1$ and $K_2$ constructed in Section 4 are not simplicial spheres. One can deduce this from the table of bigraded Betti numbers (Fig. 2): if the complexes were spheres, the distribution of bigraded Betti numbers would be symmetric according to (bigraded) Poincare duality.

It is tempting to modify the construction of $K_1$ and $K_2$ of Section 4 to obtain spheres in the output. Unfortunately, this attempt fails due to the following observation.

**Proposition 5.1.** *Let $K$ be a simplicial sphere. The Taylor resolution of $\Bbbk[K]$ is minimal if and only if $K$ is a join of boundaries of simplices.*

REMARK 5.2. For such $K$ holds $s(K) = s_{\mathbb{R}}(K) = m - \dim K - 1$ (see [13]). Thus a counterexample to Problem 2 can not be constructed using minimal Taylor resolutions.

Proof of Proposition 5.1. The “if” part is already verified in Example 4.7. Let us prove the “only if” part. Let $[m]$ be the vertex set of $K$. Any vertex $i \in [m]$ is contained in at least one minimal non-simplex. Otherwise, $K$ is a cone with apex $i$, thus contractible, thus not a sphere. Since the Taylor resolution is minimal, we may apply Lemma 4.12. Complex $K$ is a sphere, thus $\Bbbk[K]$ is Gorenstein* and $\mathrm{Tor}^{*,*}_{\Bbbk[m]}(\Bbbk[K];\Bbbk)$ is a multigraded Poincare duality algebra. There should be a graded component of $\mathrm{Tor}^{*,*}_{\Bbbk[m]}(\Bbbk[K];\Bbbk)$ of maximal total degree which plays the role of fundamental cycle. It is generated by $W_{N(K)}$ in the notation of Lemma 4.12 and has multidegree $(-|N(K)|, (2,2,\dots,2))$. Non-degenerate pairing in Poincare duality algebra $\mathrm{Tor}^{*,*}_{\Bbbk[m]}(\Bbbk[K];\Bbbk)$ yields that for each $\sigma \subseteq N(K)$ there exists $\tau \subseteq N(K)$ such that $W_\sigma \cdot W_\tau = \alpha W_{N(K)}$ with $\alpha \neq 0$.

Taking multigrading into account and applying Lemma 4.12 we get the following condition: for each $\sigma \subseteq N(K)$ the vertex subsets $\bigcup_{J\in\sigma} J$ and $\bigcup_{J\in N(K)\setminus\sigma} J$ are disjoint. In particular, any single non-simplex $J \in N(K)$ is disjoint from the union of others. Therefore, $N(K) = \{J_1, \dots, J_k\}$ and $[m] = J_1 \sqcup \dots \sqcup J_k$. Thus $K = (\partial\Delta_{J_1}) * \dots * (\partial\Delta_{J_k})$ which was to be proved. □

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