Title	A LOGARITHMICALLY IMPROVED REGULARITY CRITERION OF SMOOTH SOLUTIONS FOR THE 3D BOUSSINESQ EQUATIONS
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Citation	Osaka Journal of Mathematics. 2016, 53(2), p. 417-423
Version Type	VoR
URL	https://doi.org/10.18910/58886
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Ye, Z.
Osaka J. Math.
53 (2016), 417–423

A LOGARITHMICALLY IMPROVED REGULARITY CRITERION OF SMOOTH SOLUTIONS FOR THE 3D BOUSSINESQ EQUATIONS

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(Received September 17, 2014, revised January 27, 2015)

Abstract

In this note, we consider the three-dimensional (3D) incompressible Boussinesq equations. We obtain the logarithmically improved regularity criterion of smooth solutions in terms of the velocity field. This result improves some previous works.

1. Introduction

This note is devoted to the study of the regularity criterion of smooth solutions for the 3D Boussinesq equations

$$\begin{cases} \partial_t u + (u\cdot\nabla)u - \nu\Delta u + \nabla p = \theta e_3, & x\in\mathbb{R}^3,\ t>0, \\ \partial_t\theta + (u\cdot\nabla)\theta - \kappa\Delta\theta = 0, & x\in\mathbb{R}^3,\ t>0, \\ \nabla\cdot u = 0, & x\in\mathbb{R}^3,\ t>0, \\ u(x,0)=u_0(x), \quad \theta(x,0)=\theta_0(x), & x\in\mathbb{R}^3, \end{cases} \tag{1.1}$$

where $u = u(x, t) \in \mathbb{R}^3$ is the velocity, $p = p(x, t) \in \mathbb{R}$ is the scalar pressure, $\theta = \theta(x, t) \in \mathbb{R}^3$ is the temperature and $e_3 = (0, 0, 1)^{\mathrm{T}}$. $\nu \geq 0$ denotes the viscosity, $\kappa \geq 0$ denotes the thermal diffusivity. The Boussinesq equations are of relevance to study a number of models coming from atmospheric or oceanographic turbulence where rotation and stratification play an important role (see e.g. [10, 12]).

Local existence and uniqueness theories of solutions to the Boussinesq equations have been studied by many mathematicians and physicists (see, e.g., [1, 2, 9]). But due to the presence of Navier–Stokes equations in the system (1.1) whether this unique local solution can exist globally is an outstanding challenge problem. For this reason, there have been a lot of literatures devoted to finding sufficient conditions to ensure the smoothness of the solutions; see [4, 3, 5, 6, 14, 15, 13, 16, 17, 18, 19] and so forth.

Motivated by the above cited works, our aim is to establish a logarithmically improved regularity criterion of smooth solutions in terms of the velocity field which significantly extends the result in [19]. For the sake of simplicity, we set $\nu = \kappa = 1$. More precisely, we will prove

2010 Mathematics Subject Classification. 35Q35, 35B65, 76W05.

Theorem 1.1. *Assume that* $(u_0, \theta_0) \in H^3(\mathbb{R}^3) \times H^3(\mathbb{R}^3)$. *Let* (u, θ) *be a local smooth solution of the system* (1.1). *If the following condition holds*

$$\int_0^T \frac{\|\nabla u(t)\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{\ln\Big(e + \|\nabla u(t)\|_{\dot{B}^{-1}_{\infty,\infty}}\Big)}\, dt < \infty, \tag{1.2}$$

then the solution pair (u, θ) *remains smooth on* $[0, T]$.

As a consequence of the fact $\|\nabla u\|_{\dot{B}^{-1}_{\infty,\infty}} \approx \|u\|_{\dot{B}^{0}_{\infty,\infty}}$, we have the following result.

Corollary 1.2. *Assume that* $(u_0, \theta_0) \in H^3(\mathbb{R}^3) \times H^3(\mathbb{R}^3)$. *Let* (u, θ) *be a local smooth solution of the system* (1.1). *If the following condition holds*

$$\int_0^T \frac{\|u(t)\|^2_{\dot{B}^{0}_{\infty,\infty}}}{\ln\Big(e + \|u(t)\|_{\dot{B}^{0}_{\infty,\infty}}\Big)}\, dt < \infty, \tag{1.3}$$

then the solution pair (u, θ) *remains smooth on* $[0, T]$.

REMARK 1.3. As the case $\theta = 0$, the system (1.1) reduces to the classical Navier–Stokes equations. It is easy to see that the Corollary 1.2 is a refined improvement of that Theorem 1 in [8] due to the well-known embedding BMO $\hookrightarrow \dot{B}^{0}_{\infty,\infty}$.

2. The proof of the Theorem 1.1

This section is devoted to the proof of the Theorem 1.1. Throughout the paper, C stands for some real positive constants which may be different in each occurrence.

Proof of Theorem 1.1. If (1.2) holds, one can deduce that for any small $\epsilon > 0$, there exists $T_0 = T_0(\epsilon) < T$ such that

$$\int_{T_0}^T \frac{\|\nabla u(t)\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{\ln\Big(e + \|\nabla u(t)\|_{\dot{B}^{-1}_{\infty,\infty}}\Big)}\, dt \le \epsilon. \tag{2.1}$$

Consequently, the main goal of this section is to establish the following a priori estimate

$$\limsup_{t\to T^-}(\|\nabla^3 u(t)\|^2_{L^2} + \|\nabla^3 \theta(t)\|^2_{L^2}) < \infty.$$

Thanks to the divergence-free condition $\nabla \cdot u = 0$, from the temperature θ equation, we immediately have the global a priori bound for θ in any Lebesgue space

$$\|\theta(t)\|_{L^p} \le \|\theta_0\|_{L^p}, \quad \forall p \in [2, \infty], \tag{2.2}$$

for any $t \in [0, T]$.

We also have the following basic L^2 energy estimate

$$\frac{1}{2}\frac{d}{dt}(\|u(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2) + \|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2 \le \|u\|_{L^2}\|\theta\|_{L^2}, \tag{2.3}$$

which together with (2.2) implies that for any $t \in [0, T]$

$$\|u(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2 + \int_0^t (\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2)(\tau)\, d\tau \le C < \infty. \tag{2.4}$$

Multiplying the equation of $(1.1)_1$ and $(1.1)_2$ by Δu and $\Delta\theta$, respectively, integration by parts and taking the divergence free property into account, one concludes that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\nabla u(t)\|_{L^2}^2 + \|\nabla\theta(t)\|_{L^2}^2) + \|\nabla^2 u\|_{L^2}^2 + \|\nabla^2\theta\|_{L^2}^2 \\ &= -\int_{\mathbb{R}^3} \theta e_3 \cdot \Delta u\, dx + \int_{\mathbb{R}^3} (u\cdot\nabla u)\cdot\Delta u\, dx + \int_{\mathbb{R}^3} (u\cdot\nabla\theta)\cdot\Delta\theta\, dx \\ &:= N_1 + N_2 + N_3. \end{aligned} \tag{2.5}$$

Integrating by parts and Young inequality, it yields

$$N_1 \le C\|\nabla u\|_{L^2}\|\nabla\theta\|_{L^2}. \tag{2.6}$$

In order to deal with the terms N_2 and N_3, we need the following interpolation inequality due to Meyer–Gerard–Oru [11]

$$\|f\|_{L^4} \le C\|\nabla f\|_{L^2}^{1/2}\|f\|_{\dot{B}^{-1}_{\infty,\infty}}^{1/2}, \quad \forall f \in \dot{H}^1(\mathbb{R}^3) \cap \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3). \tag{2.7}$$

By the above interpolation inequality (2.7) and Young inequality, we can bound the terms N_2 and N_3 as

$$\begin{aligned} N_2 &= -\int_{\mathbb{R}^3} (\partial_k u\cdot\nabla u)\cdot\partial_k u\, dx \\ &\le C\|\nabla u\|_{L^2}\|\nabla u\|_{L^4}^2 \le C\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\|\nabla u\|_{\dot{B}^{-1}_{\infty,\infty}} \\ &\le \frac{1}{2}\|\nabla^2 u\|_{L^2}^2 + C\|\nabla u\|_{\dot{B}^{-1}_{\infty,\infty}}^2\|\nabla u\|_{L^2}^2, \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} N_3 &= -\int_{\mathbb{R}^3} (\partial_k u\cdot\nabla\theta)\cdot\partial_k\theta\, dx \\ &\le C\|\nabla u\|_{L^2}\|\nabla\theta\|_{L^4}^2 \le C\|\nabla u\|_{L^2}\|\nabla^2\theta\|_{L^2}\|\nabla\theta\|_{\dot{B}^{-1}_{\infty,\infty}} \\ &\le C\|\nabla u\|_{L^2}\|\nabla^2\theta\|_{L^2}\|\theta\|_{\dot{B}^{0}_{\infty,\infty}} \le C\|\nabla u\|_{L^2}\|\nabla^2\theta\|_{L^2}\|\theta\|_{L^\infty} \\ &\le \frac{1}{2}\|\nabla^2\theta\|_{L^2}^2 + C\|\theta\|_{L^\infty}^2\|\nabla\theta\|_{L^2}^2, \end{aligned} \tag{2.9}$$

where we have used the following fact in (2.9)

$$\|\nabla\theta\|_{\dot{B}^{-1}_{\infty,\infty}} \le C\|\theta\|_{\dot{B}^{0}_{\infty,\infty}} \le C\|\theta\|_{L^\infty}.$$

Substituting (2.6), (2.8), and (2.9) into (2.5), we arrive at

$$\begin{aligned} &\frac{d}{dt}(\|\nabla u(t)\|_{L^2}^2 + \|\nabla\theta(t)\|_{L^2}^2) + \|\nabla^2 u\|_{L^2}^2 + \|\nabla^2\theta\|_{L^2}^2 \\ &\le C(1 + \|\theta\|_{L^\infty}^2 + \|\nabla u\|_{\dot{B}^{-1}_{\infty,\infty}}^2)(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2). \end{aligned} \tag{2.10}$$

For any $t \in (T_0, T)$, we denote

$$y(t) := \max_{\tau\in(T_0,t]}(\|u(\tau)\|_{H^3}^2 + \|\theta(\tau)\|_{H^3}^2).$$

It should be noted that $y(t)$ is a nondecreasing function.

Thanks to the Gronwall inequality, it follows from (2.10) that for any $T_0 \le t < T$

$$\begin{aligned} &\|\nabla u(t)\|_{L^2}^2 + \|\nabla\theta(t)\|_{L^2}^2 + \int_{T_0}^t (\|\nabla^2 u(\tau)\|_{L^2}^2 + \|\nabla^2\theta(\tau)\|_{L^2}^2)\,d\tau \\ &\le (\|\nabla u(T_0)\|_{L^2}^2 + \|\nabla\theta(T_0)\|_{L^2}^2)\exp\Big[\tilde{C}\int_{T_0}^t (1 + \|\theta\|_{L^\infty} + \|\nabla u\|_{\dot{B}^{-1}_{\infty,\infty}}^2)(\tau)\,d\tau\Big] \\ &\le M\exp\Big[\tilde{C}\int_{T_0}^t \|\nabla u(\tau)\|_{\dot{B}^{-1}_{\infty,\infty}}^2\,d\tau\Big] \\ &\le M\exp\Bigg[\tilde{C}\int_{T_0}^t \frac{\|\nabla u(\tau)\|_{\dot{B}^{-1}_{\infty,\infty}}^2}{\ln\Big(e + \|\nabla u(\tau)\|_{\dot{B}^{-1}_{\infty,\infty}}\Big)}\ln\Big(e + \|\nabla u(\tau)\|_{\dot{B}^{-1}_{\infty,\infty}}\Big)\,d\tau\Bigg] \\ &\le M\exp\Bigg[\tilde{C}\int_{T_0}^t \frac{\|\nabla u(\tau)\|_{\dot{B}^{-1}_{\infty,\infty}}^2}{\ln\Big(e + \|\nabla u(\tau)\|_{\dot{B}^{-1}_{\infty,\infty}}\Big)}\ln\Big(e + \|u(\tau)\|_{\dot{B}^{0}_{\infty,\infty}}\Big)\,d\tau\Bigg] \\ &\le M\exp\Bigg[\tilde{C}\int_{T_0}^t \frac{\|\nabla u(\tau)\|_{\dot{B}^{-1}_{\infty,\infty}}^2}{\ln\Big(e + \|\nabla u(\tau)\|_{\dot{B}^{-1}_{\infty,\infty}}\Big)}\ln(e + \|u(\tau)\|_{L^\infty})\,d\tau\Bigg] \\ &\le M\exp\Bigg[\tilde{C}\int_{T_0}^t \frac{\|\nabla u(\tau)\|_{\dot{B}^{-1}_{\infty,\infty}}^2}{\ln\Big(e + \|\nabla u(\tau)\|_{\dot{B}^{-1}_{\infty,\infty}}\Big)}\ln(e + \|u(\tau)\|_{H^3})\,d\tau\Bigg] \\ &\le M\exp\Bigg[\tilde{C}\int_{T_0}^t \frac{\|\nabla u(\tau)\|_{\dot{B}^{-1}_{\infty,\infty}}^2}{\ln\Big(e + \|\nabla u(\tau)\|_{\dot{B}^{-1}_{\infty,\infty}}\Big)}\ln(e + y(\tau))\,d\tau\Bigg] \\ &\le M\exp\Bigg[\tilde{C}\int_{T_0}^t \frac{\|\nabla u(\tau)\|_{\dot{B}^{-1}_{\infty,\infty}}^2}{\ln\Big(e + \|\nabla u(\tau)\|_{\dot{B}^{-1}_{\infty,\infty}}\Big)}\,d\tau\,\ln(e + y(t))\Bigg]. \end{aligned} \tag{2.11}$$

We want to state here that from the above observation $\tilde{C}$ is an absolute constant and M depends on $\|\nabla u(T_0)\|_{L^2}$, $\|\nabla\theta(T_0)\|_{L^2}$, T_0, T and θ_0.

It follows from the condition (2.1) that

$$\|\nabla u(t)\|_{L^2}^2 + \|\nabla\theta(t)\|_{L^2}^2 + \int_{T_0}^{t} (\|\nabla^2 u(\tau)\|_{L^2}^2 + \|\nabla^2\theta(\tau)\|_{L^2}^2)\, d\tau \le C(e + y(t))^{\tilde{C}\epsilon}. \tag{2.12}$$

Applying ∇^3 to the equation $(1.1)_1$ and $(1.1)_2$, multiplying the resulting equations by $\nabla^3 u$ and $\nabla^3\theta$ respectively and adding them up, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\nabla^3 u(t)\|_{L^2}^2 + \|\nabla^3\theta(t)\|_{L^2}^2) + \|\nabla^4 u\|_{L^2}^2 + \|\nabla^4\theta\|_{L^2}^2 \\ &= \int_{\mathbb{R}^3} \nabla^3\theta e_3 \cdot \nabla^3 u\, dx - \int_{\mathbb{R}^3} \nabla^3(u\cdot\nabla u)\cdot\nabla^3 u\, dx - \int_{\mathbb{R}^3} \nabla^3(u\cdot\nabla\theta)\cdot\nabla^3\theta\, dx \\ &:= K_1 + K_2 + K_3. \end{aligned} \tag{2.13}$$

It follows from Young inequality that

$$\begin{aligned} K_1 &\le C\|\nabla^3 u\|_{L^2}\|\nabla^3\theta\|_{L^2} \\ &\le C\|\nabla^2 u\|_{L^2}^{1/2}\|\nabla^4 u\|_{L^2}^{1/2}\|\nabla^2\theta\|_{L^2}^{1/2}\|\nabla^4\theta\|_{L^2}^{1/2} \\ &\le \frac{1}{4}\|\nabla^4 u\|_{L^2}^2 + \frac{1}{4}\|\nabla^4\theta\|_{L^2}^2 + C(\|\nabla^2 u\|_{L^2}^2 + \|\nabla^2\theta\|_{L^2}^2), \end{aligned} \tag{2.14}$$

where we have applied the following Gagliardo–Nirenberg inequality

$$\|\nabla^3 f\|_{L^2} \le C\|\nabla^2 f\|_{L^2}^{1/2}\|\nabla^4 f\|_{L^2}^{1/2}.$$

Now we recall the following commutator estimate (see [7])

$$\|[\Lambda^s, f]g\|_{L^p} \le C(\|\nabla f\|_{L^{p_1}}\|\Lambda^{s-1}g\|_{L^{p_2}} + \|\Lambda^s f\|_{L^{p_3}}\|g\|_{L^{p_4}}), \tag{2.15}$$

with $s > 0$, $p_2, p_3 \in (1, \infty)$ such that $1/p = 1/p_1 + 1/p_2 = 1/p_3 + 1/p_4$.

From the divergence-free condition and the commutator estimate (2.15), we obtain

$$\begin{aligned} K_2 &= -\int_{\mathbb{R}^3} [\nabla^3, u\cdot\nabla]u\cdot\nabla^3 u\, dx \\ &\le C\|\nabla u\|_{L^2}\|\nabla^3 u\|_{L^4}^2 \\ &\le C\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}^{1/4}\|\nabla^4 u\|_{L^2}^{7/4} \\ &\le \frac{1}{8}\|\nabla^4 u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^8\|\nabla^2 u\|_{L^2}^2, \end{aligned}$$

where we have used the following Gagliardo–Nirenberg inequality

$$\|\nabla^3 u\|_{L^4} \le C\|\nabla^2 u\|_{L^2}^{1/8}\|\nabla^4 u\|_{L^2}^{7/8}.$$

Similar to the estimate of K_2, the term K_3 can be bounded as

(2.16)

$$\begin{aligned}
K_3 &= -\int_{\mathbb{R}^3} [\nabla^3, u\cdot\nabla]\theta\cdot\nabla^3\theta\,dx \\
&\le C\|\nabla u\|_{L^2}\|\nabla^3\theta\|_{L^4}^2 + \|\nabla\theta\|_{L^2}\|\nabla^3 u\|_{L^4}\|\nabla^3\theta\|_{L^4} \\
&\le C\|\nabla u\|_{L^2}\|\nabla^2\theta\|_{L^2}^{1/4}\|\nabla^4\theta\|_{L^2}^{7/4} + \|\nabla\theta\|_{L^2}\|\nabla^2 u\|_{L^2}^{1/8}\|\nabla^4 u\|_{L^2}^{7/8}\|\nabla^2\theta\|_{L^2}^{1/8}\|\nabla^4\theta\|_{L^2}^{7/8} \\
&\le \frac{1}{8}\|\nabla^4 u\|_{L^2}^2 + \frac{1}{4}\|\nabla^4\theta\|_{L^2}^2 + C(\|\nabla u\|_{L^2}^8 + \|\nabla\theta\|_{L^2}^8)(\|\nabla^2 u\|_{L^2}^2 + \|\nabla^2\theta\|_{L^2}^2).
\end{aligned}$$

Combining the previous estimates, we get

$$\begin{aligned}
&\frac{d}{dt}(\|\nabla^3 u(t)\|_{L^2}^2 + \|\nabla^3\theta(t)\|_{L^2}^2) \\
&\le C(\|\nabla u\|_{L^2}^8 + \|\nabla\theta\|_{L^2}^8)(\|\nabla^2 u\|_{L^2}^2 + \|\nabla^2\theta\|_{L^2}^2) + C(\|\nabla^2 u\|_{L^2}^2 + \|\nabla^2\theta\|_{L^2}^2).
\end{aligned} \tag{2.17}$$

Integrating the inequality (2.17) over (T_0, t), we easily get

(2.18)

$$\begin{aligned}
&\|\nabla^3 u(t)\|_{L^2}^2 + \|\nabla^3\theta(t)\|_{L^2}^2 - (\|\nabla^3 u(T_0)\|_{L^2}^2 + \|\nabla^3\theta(T_0)\|_{L^2}^2) \\
&\le C\int_{T_0}^t (\|\nabla^2 u\|_{L^2}^2 + \|\nabla^2\theta\|_{L^2}^2)\,d\tau + C\int_{T_0}^t (\|\nabla u\|_{L^2}^8 + \|\nabla\theta\|_{L^2}^8)(\|\nabla^2 u\|_{L^2}^2 + \|\nabla^2\theta\|_{L^2}^2)\,d\tau \\
&\le C(e+y(t))^{\tilde{C}\epsilon} + C\int_{T_0}^t (e+y(\tau))^{4\tilde{C}\epsilon}(\|\nabla^2 u\|_{L^2}^2 + \|\nabla^2\theta\|_{L^2}^2)(\tau)\,d\tau \\
&\le C(e+y(t))^{\tilde{C}\epsilon} + C(e+y(t))^{4\tilde{C}\epsilon}\int_{T_0}^t (\|\nabla^2 u\|_{L^2}^2 + \|\nabla^2\theta\|_{L^2}^2)(\tau)\,d\tau \\
&\le C(e+y(t))^{\tilde{C}\epsilon} + C(e+y(t))^{5\tilde{C}\epsilon} \\
&\le C(e+y(t))^{5\tilde{C}\epsilon},
\end{aligned}$$

which immediately implies that

$$e + y(t) \le C_{T_0} + C(e+y(t))^{5\tilde{C}\epsilon}, \quad C_{T_0} = \|\nabla^3 u(T_0)\|_{L^2}^2 + \|\nabla^3\theta(T_0)\|_{L^2}^2.$$

By appropriately selecting $\epsilon < 1/(5\tilde{C})$, the above inequality allows us to show

$$y(t) \le C(\|\nabla^3 u(T_0)\|_{L^2}, \|\nabla^3\theta(T_0)\|_{L^2}, T_0, T) < \infty, \quad \forall t \in [T_0, T].$$

As a consequence, we get the boundness of $H^3 \times H^3$-norm of (u, θ) for all $t \in [0, T]$. Consequently, The proof of Theorem 1.1 is completed. □

ACKNOWLEDGEMENTS. The author would like to thank the referees for their valuable comments and suggestions, which are very helpful for author to improve this paper.

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